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Lie Superalgebras Based on a 3-Dimensional Real or Complex Lie Algebra

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Abstract. We give a complete classification of real and complex Lie superalgebras $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, for which $\mathfrak{g}_0$ is a 3-dimensional Lie algebra, and $\mathfrak{g}_1$ is $\mathfrak{g}_0$ itself under the adjoint representation.
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1. Introduction

Our purpose is to classify the Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, for which $\mathfrak{g}_0$ is a 3-dimensional (real or complex) Lie algebra, having its action on $\mathfrak{g}_1 = \mathfrak{g}_0$ given by the adjoint representation. In general, a Lie superalgebra structure on a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a triple $([\,\cdot\,,\,\cdot\,], \rho, \Gamma)$ consisting of (see [9], [10], and [11]):

(1) A Lie algebra structure $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}_0$,
(2) A representation $\rho : \mathfrak{g}_0 \to \operatorname{End} \mathfrak{g}_1$,
(3) A symmetric bilinear map $\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$,

satisfying the following *super*-Jacobi identities:

$$\text{(J1)} \qquad [x, \Gamma(u,v)] = \Gamma(\rho(x)u, v) + \Gamma(u, \rho(x)v)\,, \qquad x \in \mathfrak{g}_0\,,\ u, v \in \mathfrak{g}_1$$

and

$$\text{(J2)} \quad (\rho\left(\Gamma(u,v)\right)(w) + (\rho\left(\Gamma(w,u)\right)(v) + (\rho\left(\Gamma(v,w)\right)(u) = 0\,,\ u,v,w \in \mathfrak{g}_1\,.$$

When $\mathfrak{g}_0 = \mathfrak{g}_1$ and $\rho = \mathrm{ad}$, only (J1) is relevant, as (J2) is automatically satisfied (see [9]). Lie superalgebras specified by this data will be called *based on* $\mathfrak{g}_0$, and we shall write $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ for the $\mathbb{F}$-vector space of symmetric, bilinear maps $\Gamma : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$ satisfying (J1) for $\rho = \mathrm{ad}$. Clearly, $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ is the space of equivariant maps $\mathrm{Hom}_{\mathfrak{g}_0}(S^2(\mathfrak{g}_0), \mathfrak{g}_0)$.

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When a Lie superalgebra structure on the $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is viewed as a triple $([\,\cdot\,,\,\cdot\,],\rho,\Gamma)$, the group $\mathrm{GL}(\mathfrak{g}_0)\times\mathrm{GL}(\mathfrak{g}_1)$ acts on the set of such triples, producing isomorphic Lie superalgebras on each orbit: The Lie superalgebras defined on $\mathfrak{g}_0\oplus\mathfrak{g}_1$ by the data $([\,\cdot\,,\,\cdot\,],\rho,\Gamma)$ and $([\,\cdot\,,\,\cdot\,]',\rho',\Gamma')$ are isomorphic if and only if there exists a pair $(T,S)\in\mathrm{GL}(\mathfrak{g}_0)\times\mathrm{GL}(\mathfrak{g}_1)$ such that,

$$\begin{aligned}
[\,\cdot\,,\,\cdot\,]' &= T\,[\,T^{-1}(\,\cdot\,)\,,\,T^{-1}(\,\cdot\,)\,]\,,\\
\rho' &= S\,\circ\,\rho(T^{-1}(\,\cdot\,))\,\circ\,S^{-1}\,,\\
\Gamma' &= T(\Gamma(S^{-1}(\,\cdot\,),S^{-1}(\,\cdot\,)))\,.
\end{aligned}$$

When the Lie algebra structure $[\,\cdot\,,\,\cdot\,]$ of $\mathfrak{g}_0$ is kept fixed, and $\rho = \mathrm{ad}$, we are actually looking at the action of the group

$$G = \{(T,S)\in\mathrm{Aut}(\mathfrak{g}_0)\times\mathrm{GL}(\mathfrak{g}_0)\mid S\circ T^{-1}\circ\mathrm{ad}(\,\cdot\,) = \mathrm{ad}(\,\cdot\,)\,\circ\,S\circ T^{-1}\}$$

since $\mathrm{ad}(T^{-1}(x)) = T^{-1}\circ\mathrm{ad}(x)\,\circ\,T$. To classify the different Lie superalgebras based on $\mathfrak{g}_0$, amounts to parametrize the orbits in $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ under the left action of the group of pairs $(T,S)\in\mathrm{Aut}(\mathfrak{g}_0)\times\mathrm{GL}(\mathfrak{g}_1)$, such that $[T(x),S(y)] = S([x,y])$, given by,

$$\Gamma \quad\mapsto\quad (T,S)\cdot\Gamma = T\Big(\Gamma\big(S^{-1}(\,\cdot\,),\,S^{-1}(\,\cdot\,)\big)\Big)\,.$$

The automorphism group of the Lie superalgebra determined by a given Γ is the isotropy subgroup at Γ of this action. Let Γ and Γ' be two ad-equivariant symmetric bilinear maps $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{g}_0$. We shall say that they are equivalent if and only if they are in the same orbit.

The classification of the Lie superalgebras based on a 3-dimensional Lie algebra will be approached by first dividing the analysis according to the dimension of the derived ideal $\mathfrak{g}_0'$ of $\mathfrak{g}_0$. It turns out that $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ has a fixed dimension for a fixed value of $\dim\mathfrak{g}_0'$ according to,

$\dim\mathfrak{g}_0'$	3	2	1	0
$\dim\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$	0	1	5	18

Table 1

In other words, to specify a Lie superalgebra based on a 3-dimensional Lie algebra, one requires $\dim\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ parameters to build up Γ. That is, $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ gets identified with $\mathbb{F}^{\dim\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)}$, and the G action on $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ translates into the corresponding G action on $\mathbb{F}^{\dim\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)}$ (see §2 below).

Our results can now be summarized in the following general statement whose proof is precisely the aim of this work. Use has been made of the well-known parametrization of 3-dimensional real or complex Lie algebras. We refer the reader to [2] and [7] for details, and to §2 below for the notation to be used throughout this work.

Theorem 1.1. *Up to isomorphism,*
1. There is only one Lie superalgebra based on a 3*-dimensional semisimple Lie algebra* $\mathfrak{g}_0$*, and it has* $\Gamma = 0$.

2. There are 2 *(resp.,* 3*) different Lie superalgebras based on* $\mathfrak{p}(\mathbb{C})$ *(resp.,* $\mathfrak{p}(\mathbb{R})$*). The same is assertion is true for* $\mathfrak{q}_\lambda(\mathbb{C})$ *(resp.,* $\mathfrak{q}_\lambda(\mathbb{R})$*).*

3. There are 3 *different (real) Lie superalgebras based on* $\mathfrak{q}^1_\lambda(\mathbb{R})$.

4. There are 15 *(resp.,* 23*) different Lie superalgebras based on* $\mathfrak{q}_0(\mathbb{C})$ *(resp.,* $\mathfrak{q}_0(\mathbb{R})$*), plus a non-zero-parameter family (resp., two (non-zero)-one-parameter families) of different Lie superalgebras that are also based on* $\mathfrak{q}_0(\mathbb{C})$ *(resp.,* $\mathfrak{q}_0(\mathbb{R})$*).*

5. There are 5 *(resp.,* 6*) different Lie superalgebras based on the Heisenberg Lie algebra* $\mathfrak{h}_2(\mathbb{C})$ *(resp.,* $\mathfrak{h}_2(\mathbb{R})$*).*

6. There are 10 *(resp.,* 18*) different isomorphism classes of Lie superalgebras based on* $\mathfrak{a}(\mathbb{C})$ *(resp.,* $\mathfrak{a}(\mathbb{R})$*), plus a family depending on four complex parameters (resp., two families depending on two real parameters each, plus three families depending on four real parameters each).*

What motivates the problem is the geometric idea that 3-dimensional smooth manifolds are locally modeled on a 3-dimensional vector space. If a given 3-manifold comes equipped with an additional structure —say, a given Lie group acts on it— this structure gets reflected on the local model. In particular, the real 3-dimensional vector space with which we are in close contact in elementary physics and geometry is precisely the Lie algebra of the 3-dimensional rotation group, as we naturally associate to it the geometric operations of 'cross product' (*ie,* the Lie bracket) and 'scalar product' (*ie,* the Cartan-Killing form). What we are obtaining in this work is a list of the *different* 3*-dimensional superspaces* that can be defined naturally on a 3-dimensional space that happens to have a Lie algebra structure; *naturally* means with no additional hypotheses made on the given Lie algebra itself, as all that is required to build up these superspaces is the adjoint representation. This problem came quite naturally to us after giving a physical interpretation of a previous similar result: Say, classify the Lie superalgebras that can be defined from the Lie algebra $\mathfrak{u}_2$ of the group of unitary 2×2 matrices, through the adjoint representation. It turns out that there are ten different such Lie superalgebras (see [8], and [9]), but $\mathfrak{u}_2$ itself is usually identified with Minkowski spacetime; therefore, our approach yield all the possible superspacetime structures that can be defined on top of Minkowski spacetime with no extra hypotheses.

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2. Determination of $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$

The purpose of this section is to prove the following,

Proposition 2.1. *Let $\mathfrak{g}_0$ be a 3-dimensional real or complex Lie algebra, and let $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ be the real or complex vector space of symmetric, bilinear maps $\Gamma : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$ satisfying (J1) for $\rho = \mathrm{ad}$. Then, $\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ only depends on the dimension of the derived ideal $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$, and this dependence is given in Table 1 above.*

We shall start the proof by introducing some notation and fixing some conventions, making use of well-known facts about 3-dimensional Lie algebras.

Let $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ be the derived ideal. Now, $\mathfrak{g}_0' = \mathfrak{g}_0$ when $\mathfrak{g}_0$ is semisimple. It is well-known that, up to isomorphism, there is only one complex semisimple Lie algebra of dimension 3; namely, $\mathfrak{sl}_2$. Over the real field there are two: $\mathfrak{sl}_2$ and $\mathfrak{su}_2$. The Abelian Lie algebra structure —corresponding to $\dim \mathfrak{g}_0' = 0$— is unique over either ground field.

It is well known (*cf,* [7]) that when $\mathfrak{g}_0 = \mathfrak{sl}_2$, the space $S^2(\mathfrak{sl}_2)$ gets decomposed into $V_1 \oplus V_5$ where V_i is the $\mathfrak{sl}_2$-irreducible representation of dimension i. Therefore, $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{sl}_2) = \mathrm{Hom}_{\mathfrak{sl}_2}(V_1 \oplus V_5, V_3) = \{0\}$ by Schur's Lemma, thus proving that there is only one way to define Γ in order to obtain a Lie superalgebra structure on $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$; namely, $\Gamma = 0$. The same is true when $\mathbb{F} = \mathbb{R}$ and $\mathfrak{g}_0 = \mathfrak{su}_2$. Therefore we may concentrate from now on, in those cases with nonsemisimple $\mathfrak{g}_0$.

Let $\mathfrak{g}_0$ be a fixed 3-dimensional nonsemisimple Lie algebra. We shall make a choice of basis in $\mathfrak{g}_0$, say $\{e_1, e_2, e_3\}$, and therefore identify the space $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ with the set of triples $(\Gamma^1, \Gamma^2, \Gamma^3)$ of symmetric bilinear forms, for which $\Gamma(u,v) = \sum \Gamma^i(u,v) e_i$ satisfies (J1). It is easy to see that (J1) translates into,

$$\sum_j \Gamma^j \,\mathrm{ad}(x)_{kj} = \mathrm{ad}(x)^T \Gamma^k + \Gamma^k \,\mathrm{ad}(x)\,, \qquad \text{for all } x \in \mathfrak{g}_0 \tag{2.1}$$

where $\mathrm{ad}(x)_{kj}$ is defined through $[x, e_j] = \sum_k \mathrm{ad}(x)_{kj} e_k$. By letting x run through the basis $\{e_i\}$, we obtain a set of linear equations for the entries $\Gamma^k_{ij} := \Gamma^k(e_i, e_j)$. Thus we need to find a convenient expression for the matrices C_i associated to the linear transformations $\mathrm{ad}(e_i)$, in such a way that we can deal with all the nonsemisimple Lie algebras at once.

Since $\mathfrak{g}_0 \neq \mathfrak{g}_0'$, it can be easily seen that $\mathfrak{g}_0$ contains a 2-dimensional abelian ideal $\mathfrak{a}$. The 3-dimensional nonsemisimple Lie algebras can be classified by choosing

$e_1 \notin \mathfrak{a}$ and bringing $\mathrm{ad}(e_1)|_{\mathfrak{a}}$ into a convenient canonical form (see [2], [4], or [7] for details). Thus, we shall write,

$$\begin{aligned} [e_1, e_2] &= ae_2 + ce_3\,, \\ [e_1, e_3] &= be_2 + de_3\,, \\ [e_2, e_3] &= 0\,, \end{aligned} \qquad \text{so that,} \qquad \mathrm{ad}(e_1)|_{\mathfrak{a}} \leftrightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and will use the following notation:

$$\text{(2.2)} \qquad \begin{array}{cccc} \mathfrak{g}_0 & \dim \mathfrak{g}_0' & A & \text{Constraints} \\ \mathfrak{p}(\mathbb{F}) & 2 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \\ \mathfrak{q}_\lambda(\mathbb{F}) & 2 & \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} & 0 < |\lambda| \le 1 \\ \mathfrak{q}^1_\lambda(\mathbb{R}) & 2 & \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} & \lambda \in \mathbb{R} \\ \mathfrak{q}_0(\mathbb{F}) & 1 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \\ \mathfrak{h}(\mathbb{F}) & 1 & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ \mathfrak{a}(\mathbb{F}) & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \end{array}$$

It is then easy to see that the matrices C_i of $\mathrm{ad}(e_i)$ take the form,

$$\text{(2.3)} \qquad C_1 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad C_2 = -\begin{pmatrix} 0 & 0 \\ A\delta_1 & 0 \end{pmatrix}, \qquad C_3 = -\begin{pmatrix} 0 & 0 \\ A\delta_2 & 0 \end{pmatrix},$$

where $\delta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\delta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We shall now proceed to find $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ by solving (2.1) for the entries of the Γ^k's, specializing each case with the appropriate A.

A. Lie algebras for which A is invertible. When A is invertible, the only nonzero entries of Γ are Γ^2_{12}, Γ^3_{13}, and Γ^1_{11}, and it is easy to see that $\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2}\Gamma^1_{11}$. Thus,

$$\Gamma^1 = a\begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \quad \Gamma^2 = \frac{a}{2}\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}, \quad \Gamma^3 = \frac{a}{2}\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix},$$

and therefore, $\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0) = 1$.

B. A corresponding to the Lie algebra $\mathfrak{q}_0(\mathbb{F})$. In this case we obtain the following relations: $\Gamma^2_{12} = \frac{1}{2}\Gamma^1_{11}$, and $\Gamma^2_{23} = \Gamma^1_{13}$. The parameters Γ^3_{11}, Γ^3_{13}, and Γ^3_{33} can be chosen arbitrarily. Thus,

$$\Gamma^1 = \begin{pmatrix} 2a&0&b \\ 0&0&0 \\ b&0&0 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0&a&0 \\ a&0&b \\ 0&b&0 \end{pmatrix} \qquad \Gamma^3 = \begin{pmatrix} c&0&d \\ 0&0&0 \\ d&0&e \end{pmatrix}$$

and it follows that $\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0) = 5$.

C. *A* **corresponding to the Heisenberg Lie algebra** $\mathfrak{h}(\mathbb{F})$**.** In this case we obtain the relations: $\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2}\Gamma^1_{11}$, and $\Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2}\Gamma^2_{22}$. The parameters Γ^3_{11}, Γ^3_{12}, and Γ^3_{22} can be chosen arbitrarily. Thus,

$$\Gamma^1 = \begin{pmatrix} 2a & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^2 = \begin{pmatrix} 0 & a & 0 \\ a & 2b & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^3 = \begin{pmatrix} c & d & a \\ d & e & b \\ a & b & 0 \end{pmatrix}$$

showing again that $\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0) = 5$.

D. *A* **corresponding to the Abelian Lie algebra** $\mathfrak{a}(\mathbb{F})$**.** Note that in this case Γ is not restricted by equations and therefore the three symmetric matrices Γ^1, Γ^2, and Γ^3 can be chosen arbitrarily. Thus, $\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0) = 18$ in this case.

3. Lie superalgebras based on 3-dimensional nonsemisimple Lie algebras

Let $\mathfrak{g}_0$ be spanned over $\mathbb{F}$ by $\{e_1, e_2, e_3\}$ as before, and let us assume that $\mathfrak{g}_0'$ is neither $\{0\}$ nor $\mathfrak{g}_0$. Let $[\,\cdot\,,\,\cdot\,]$ have one of the canonical forms given in (2.2) above. Write $C_i = \mathrm{ad}(e_i)$ $(i = 1,\, 2,\, 3)$ as before, and let T be an automorphism of $\mathfrak{g}_0$, with $Te_j = \sum_{i=1}^3 T_{ij} e_i$. It is easy to see that

$$T \in \mathrm{Aut}(\mathfrak{g}_0) \quad \Longleftrightarrow \quad T\,C_j = \sum_{i=1}^{3} T_{ij}\, C_i T\,. \tag{3.1}$$

We may write T in block form in the same way as we wrote the C_i's in (2.3); say, $T = \begin{pmatrix} \delta_0 & v^T \\ u & t \end{pmatrix}$. A straightforward computation leads to the following:

Lemma 3.1.

(1) *Let* $\mathfrak{g}_0 = \mathfrak{p}(\mathbb{F})$, $\mathfrak{q}_\lambda(\mathbb{F})$ *(*$\lambda \neq -1$*),* $\mathfrak{q}^1_\lambda(\mathbb{R})$ *(*$\lambda \neq 0$*), or* $\mathfrak{q}_0(\mathbb{F})$. *Then*

$$\mathrm{Aut}(\mathfrak{g}_0) = \left\{ T \in \mathrm{GL}_3(\mathbb{F}) \mid T = \begin{pmatrix} 1 & 0 \\ v & B \end{pmatrix},\ B \in \mathrm{GL}_2(\mathbb{F}),\ AB = BA,\ v \in \mathbb{F}^2 \right\}$$

(2) *Let* $\mathfrak{g}_0 = \mathfrak{q}_{-1}(\mathbb{F})$, *or* $\mathfrak{q}^1_0(\mathbb{R})$. *Then,*

$$\mathrm{Aut}(\mathfrak{g}_0) = \left\{ T \in \mathrm{GL}_3(\mathbb{F}) \mid T = \begin{pmatrix} \pm 1 & 0 \\ v & B \end{pmatrix},\ B \in \mathrm{GL}_2(\mathbb{F}),\ AB \mp BA = 0,\ v \in \mathbb{F}^2 \right\}$$

(3) $\mathrm{Aut}(\mathfrak{h}(\mathbb{F})) = \left\{ T \in \mathrm{GL}_3(\mathbb{F}) \mid T = \begin{pmatrix} B & 0 \\ v^T & \delta \end{pmatrix},\ \delta = \det(B) \right\}$

(4) $\mathrm{Aut}(\mathfrak{a}(\mathbb{F})) = \mathrm{GL}_3(\mathbb{F})$.

Proof.

$$T \in \mathrm{Aut}(\mathfrak{g}_0) \quad \Longleftrightarrow \quad T = \begin{pmatrix} \delta_0 & v^T \\ u & B \end{pmatrix} \quad \text{where,} \quad \begin{aligned} v^T A &= 0\,, \\ Auv^T &= \delta_0 AB - BA\,, \\ v_1 AB\delta_2 &= v_2 AB\delta_1\,. \end{aligned}$$

It is easy to see that if A^{-1} exists, then $v^T = 0$; whence, $\delta_0 AB = BA$. Therefore, $A = \delta_0 B^{-1}AB$. Since $\mathrm{Tr}(A) = \mathrm{Tr}(B^{-1}AB)$ and $\det(A) = \det(B^{-1}AB)$, it follows that, either $\delta_0 = 1$, or $\mathrm{Tr}(A) = 0$ and $\delta_0^2 = 1$. But $\mathrm{Tr}(A) = 0$ implies that $\mathfrak{g}_0$ is either $\mathfrak{q}_{-1}(\mathbb{F})$, or $\mathfrak{q}_0^1(\mathbb{R})$.

If $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ it is easy to see that $\delta_0 = 1$, $v = 0$, u is arbitrary, and B is diagonal.

If $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ then $v^T = (v_1, 0)$, B is lower triangular, say $B = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$, u is arbitrary, and $d = \delta_0 a - u_1 v_1$. ■

Our next task though, is to determine the group G for each of the possible $\mathfrak{g}_0$'s. According to §1, the group G consists of those pairs $(T, S) \in \mathrm{Aut}(\mathfrak{g}_0) \times \mathrm{GL}(\mathfrak{g}_0)$ for which $T^{-1} \circ S$ commutes with $\mathrm{ad}(x)$ for any $x \in \mathfrak{g}_0$. Let $R = T^{-1} \circ S$. We need to find those R's satisfying,

$$[R, C_i] = 0 \qquad i = 1, 2, 3.$$

After a straightforward computation of the possible R's, we may summarize the information in the following table which gives R and T to compute from it $S = RT$.

$\mathfrak{g}_0$	$\mathrm{Aut}(\mathfrak{g}_0)$	Associated R
$\mathfrak{p}(\mathbb{F}), \mathfrak{q}_\lambda(\mathbb{F}), \mathfrak{q}_\lambda^1(\mathbb{F})$	$\left(\begin{smallmatrix} \pm 1 & 0 \\ v & B \end{smallmatrix}\right)$	$r_0 \,\mathrm{Id}_3$
$\mathfrak{q}_0(\mathbb{F})$	$\left(\begin{smallmatrix} 1 & 0 \\ v & \mathrm{Diag}(d_1,d_2) \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} r_0 & 0 \\ r_1\delta_2 & \mathrm{Diag}(r_0,r_2) \end{smallmatrix}\right)$
$\mathfrak{h}(\mathbb{F})$	$\left(\begin{smallmatrix} B & 0 \\ v^T & \delta \end{smallmatrix}\right)$, $\delta = \det(B)$	$\left(\begin{smallmatrix} \mathrm{Diag}(r_0,r_0) & 0 \\ r^T & r_0 \end{smallmatrix}\right)$
$\mathfrak{a}(\mathbb{F})$	$T \in \mathrm{GL}_3$	$R \in \mathrm{GL}_3$

4. The action of G on $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$

Under the identification of $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ with the space of triples $(\Gamma^1, \Gamma^2, \Gamma^3)$ given in §2, the action of G on $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ translates into

$$\begin{aligned}(\Gamma^1, \Gamma^2, \Gamma^3) &\mapsto (T, S) \cdot (\Gamma^1, \Gamma^2, \Gamma^3) = (\Gamma'^1, \Gamma'^2, \Gamma'^3) \\ &= \Big(\sum_{j=1}^3 T_{1j} (S^{-1})^T \Gamma^j S^{-1}, \sum_{j=1}^3 T_{2j} (S^{-1})^T \Gamma^j S^{-1}, \sum_{j=1}^3 T_{3j} (S^{-1})^T \Gamma^j S^{-1}\Big).\end{aligned}$$

Equivalently, using the identification of the space of triples $(\Gamma^1, \Gamma^2, \Gamma^3)$ with $\mathbb{F}^n$ ($n = \dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$) obtained in §2 **A-D**, and changing the left G-action by its right G-action, we have,

$$(a, b, c, \dots) \mapsto (T, S)^{-1} \cdot (a, b, c, \dots) = (a', b', c', \dots)$$

which is the way in which we shall express some of our results below. We shall be able to classify the different Lie superalgebras we are interested in, by letting the parameters $(a', b', c', \dots)$ take some appropriate values that lead to a convenient parametrization of the G-orbits.

Using the classification of the nonsemisimple 3-dimensional Lie algebras given in (2.2) in terms of the matrices A, we shall divide the analysis in four cases according to whether A is invertible, A corresponds to $\mathfrak{q}_0(\mathbb{F})$, A corresponds to the Heisenberg algebra $\mathfrak{h}(\mathbb{F})$, and $A = 0$ corresponding to the Abelian Lie algebra.

A. A **invertible and corresponding to** $\mathfrak{p}(\mathbb{F})$**,** $\mathfrak{q}_\lambda(\mathbb{F})$**, and** $\mathfrak{q}^1_\lambda(\mathbb{R})$**.** The space $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ of triples $(\Gamma^1, \Gamma^2, \Gamma^3)$ where $\Gamma(\,\cdot\,,\,\cdot\,) = \sum_i \Gamma^i(\,\cdot\,,\,\cdot\,)\, e_i$ satisfies (J1) is 1-dimensional, and we have,

$$\begin{aligned}(T,S)^{-1} \cdot &\left(a \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \frac{a}{2} \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \frac{a}{2} \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \right) \\ &= \left(r_0^2 a \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \frac{r_0^2 a}{2} \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \frac{r_0^2 a}{2} \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \right)\end{aligned}$$

for any choice of T, where use has been made of the explicit forms for S and T given in Tables 2 and 3 above. We could have anticipated this result because the action of G on a 1-dimensional space must be given by a scalar. Therefore, when the underlying field is $\mathbb{C}$, there are two different orbits represented by the parameter values $a = 0$, and $a = 1$. When the underlying field is $\mathbb{R}$ we have three different orbits, as $a = 1$ and $a = -1$ are now separated.

Remark . Note that we could have made the identification $(\Gamma^1, \Gamma^2, \Gamma^3) \leftrightarrow (a)$ right from the start, and could have written the corresponding action as $(T, r_0 T)^{-1} \cdot (a) = \big(r_0^2 a\big)$.

B. A **corresponding to** $\mathfrak{q}_0(\mathbb{F})$**.** We shall make the identification $(\Gamma^1, \Gamma^2, \Gamma^3) \leftrightarrow (a, b, c, d, e)$, according to the prescription given in §2 **B**. Writing

$$(T, S)^{-1} \cdot (\Gamma^1, \Gamma^2, \Gamma^3) = (\Gamma^{1'}, \Gamma^{2'}, \Gamma^{3'}) \leftrightarrow (a', b', c', d', e')$$

we get,

$$
\begin{aligned}
a' &= r_0^2 a + r_0 d_2 \left(r_1 + v_2 \frac{r_0}{d_2} \right) b\,, \\
b' &= r_0 r_2 d_2 b\,, \\
c' &= -2 v_2 \frac{r_0^2}{d_2} a - 2 v_2 r_0 \left(r_1 + v_2 \frac{r_0}{d_2} \right) b \\
&\quad + \frac{r_0^2}{d_2} c + 2 r_0 \left(r_1 + v_2 \frac{r_0}{d_2} \right) d + d_2 \left(r_1 + v_2 \frac{r_0}{d_2} \right)^2 e \\
d' &= -v_2 r_0 r_2 b + r_0 r_2 d + d_2 r_2 \left(r_1 + v_2 \frac{r_0}{d_2} \right) e\,, \\
e' &= r_2^2 d_2 e\,.
\end{aligned} \tag{4.1}
$$

We may now use the arbitrariness of v in T and of the r_i's in S to make appropriate choices that let us select a complete set of representatives for the G-orbits.

Cases with $b = e = 0$. The trivial subcase is $a = c = d = 0$. Let us first assume $a = d = 0$, and $c \neq 0$. All what we are left with from Eqns. (4.1) is $c' = \frac{r_0^2}{d_2} c$, and the choice $d_2 = r_0^2 c$ make us obtain the orbit representative $(a', b', c', d', e') = (0, 0, 1, 0, 0)$.

Let us now assume that $a \neq 0$, and $d = 0$. Choosing $v_2 = \frac{c}{2a}$, and $r_0^2 = \pm\frac{1}{a}$, we see that the orbit representatives would correspond to the parameter values $(a', b', c', d', e') = (1, 0, 0, 0, 0)$ when the ground field is $\mathbb{C}$, and to $(\pm 1, 0, 0, 0, 0)$ when the ground field is $\mathbb{R}$.

Now assume $a = 0$, with $d \neq 0$. Since $d_2 \neq 0$, may then choose r_1 so as to make $c' = 0$ regardless of the value of c. Then we choose the product $r_0 r_2$ in the expression for d' to make it equal to 1. Therefore, the orbit representative in this case would correspond to the parameter values $(a', b', c', d', e') = (0, 0, 0, 1, 0)$.

For the next subcase assume $ad \neq 0$. The resulting equation for c' shows that we may choose r_1 so as to make $c' = 0$, regardless of the value of c. The equations for a' and d' then show that r_0 and r_2 may be chosen so as to bring (a', b', c', d', e') to the parameter values $(\pm 1, 0, 0, 1, 0)$ over $\mathbb{R}$, and to $(1, 0, 0, 1, 0)$ over $\mathbb{C}$.

Cases with $b = 0$, and $e \neq 0$. Regardless of the value of d, we can make $d' = 0$ by letting $r_1 + v_2 \frac{r_0}{d_2} = -\frac{r_0 d}{e d_2}$, in which case $c' = \frac{r_0^2}{d_2}\left(c - 2 a v_2 - \frac{d^2}{e} \right)$; v_2 can be expressed in terms of r_1, it follows that when $a \neq 0$, r_1 can be chosen so as to make $c' = 0$, too so that the orbit representatives for those cases having $a \neq 0$, will be $(a', b', c', d', e') = (1, 0, 0, 0, 1)$ over the complex field, and $(\pm 1, 0, 0, 0, 1)$ over the real field. If, on the other hand, $a = 0$ we can still make $d' = 0$ as before with $r_1 + v_2 \frac{r_0}{d_2} = -\frac{r_0 d}{e d_2}$, but then $c' = (r_0 r_2)^2 (ec - d^2)$, and an appropriate choice of r_0 will further make c' be equal to 1 or 0 over the complex field, and ± 1 or 0 over the real field depending only on whether $ec - d^2$ is or is not equal to cero;

altogether these are the orbit representatives $(0,0,1,0,1)$, and $(0,0,0,0,1)$ over $\mathbb{C}$, and $(0,0,\pm 1,0,1)$, and $(0,0,0,0,1)$ over $\mathbb{R}$, respectively.

Cases with $b \neq 0$. In (4.1) above we may first choose $v_2 = -\frac{r_1 d_2}{r_0}$, and using the fact that $d_2 \neq 0$ and $b \neq 0$ we may also choose $r_1 = -\frac{d r_0}{d d_2}$, so as to bring (a', b', c', d', e') to the parameter values,

$$\left(r_0^2 a \,,\, r_0 r_2 d_2 b \,,\, \frac{r_0^2}{b d_2} \left(bc - 2ad \right) \,,\, 0 \,,\, r_2^2 d_2 e \right)$$

regardless of the values of d and e. It is then easy to see that the orbit representatives will be given by $(a', 1, c', 0, e')$ where the possibilities for a', c', and e' are given by the following table, where $\Delta = bc - 2ad$, $\varepsilon = 0, 1$ depending on whether e is or is not equal to zero, and ν is either ± 1 or 1, depending on whether the ground field is $\mathbb{R}$ or $\mathbb{C}$:

	a'	c'	e'
$\Delta = 0$ and $a = 0$	0	0	ε
$\Delta = 0$ and $a \neq 0$	ν	0	ε
$\Delta \neq 0$ and $a = 0$	0	1	$\nu\varepsilon$
$\Delta \neq 0$ and $a \neq 0$	ν	1	$\dfrac{\varepsilon}{a^2 b \Delta}$

In the final count of orbits for $\mathfrak{q}_0(\mathbb{F})$, the value $\varepsilon(a^2 b\Delta)^{-1}$ appears as a nonzero parameter. Adding up all the possibilities for the orbit representatives we find that there are 23 different orbits, plus two one-non-zero-parameter families of orbits when $\mathbb{F} = \mathbb{R}$, and 15 different orbits, plus one non-zero-parameter family of orbits when $\mathbb{F} = \mathbb{C}$.

C. A corresponding to the Heisenberg Lie algebra $\mathfrak{h}_2$. We shall make the identification $(\Gamma^1, \Gamma^2, \Gamma^3) \leftrightarrow (a, b, c, d, e)$, according to the prescription given in §2 **C**. The group G is now given by

$$G = \left\{ \begin{pmatrix} t & 0 \\ \tau^T & \delta \end{pmatrix}, \lambda \begin{pmatrix} t & 0 \\ \sigma^T & \delta \end{pmatrix} \;\middle|\; t \in \mathrm{GL}_2,\ \delta = \det t \neq 0,\ \lambda \in \mathbb{F} - \{0\},\ \sigma, \tau \in \mathbb{F}^2 \right\}.$$

We shall use the following block notation for the Γ^k's given in §2 **C**:

$$\Gamma^1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} \gamma_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} \gamma_3 & g \\ g^T & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 2a & b \\ b & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & a \\ a & 2b \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} c & d \\ d & e \end{pmatrix}, \quad g = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The equations for the left G-action can be reduced to

$$\lambda^2 t^T \hat{\gamma}_j t = t_{j1}\gamma_1 + t_{j2}\gamma_2 \,, \qquad j = 1, 2. \tag{4.2}$$

(4.3) $$\lambda^2 t^T \hat{g} = g$$

(4.4) $$\lambda^2(t^T\hat{\gamma}_3 t + \sigma(t^T\hat{g})^T + t^T\hat{g}\sigma^T) = \tau_1\gamma_1 + \tau_2\gamma_2 + \delta\gamma_3\,,$$

where $\tau^T = (\tau_1, \tau_2)$.

Case 1: $\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \neq 0$. Choose,

$$t = \begin{pmatrix} t_{11} & -\lambda^2\delta\hat{b} \\ t_{21} & \lambda^2\delta\hat{a} \end{pmatrix}\,, \quad \text{with} \quad 1 = \lambda^2(t_{11}\hat{a} + t_{21}\hat{b})$$

to obtain $g^T = (1, 0)$. Then Eqns. (4.2) are automatically satisfied with this choice of t, and Eqn. (4.4) yields

$$\lambda^2 t^T\hat{\gamma}_3 t + \begin{pmatrix} 2(\sigma_1 - \tau_1) & \sigma_2 - \tau_2 \\ \sigma_2 - \tau_2 & 0 \end{pmatrix} = \delta\gamma_3\,.$$

A straightforward computation of the 22 entry in the LHS yields

$$\lambda^6\delta^2(\hat{a}^2\hat{e} - 2\hat{a}\hat{b}\hat{d} + \hat{b}^2\hat{c})\,.$$

We can choose $\sigma_1 - \tau_1$ and $\sigma_2 - \tau_2$ to always anihilate the 11 and 12 entries of γ_3, and end up with the following canonical forms:

$$\gamma_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \gamma_3 = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}\,, \quad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where e is either 0 or 1, depending on whether $\hat{a}^2\hat{e} - 2\hat{a}\hat{b}\hat{d} + \hat{b}^2\hat{c}$ is zero or nonzero.

Case 2: $\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = 0$. The LHS of Eqns. (4.3) is identically zero. Therefore, the canonical forms for γ_1 and γ_2 are $\gamma_1 = \gamma_2 = 0$. Then, Eqn. (4.3) becomes a trivial identity with $g = 0$, and Eqn. (4.4) reduces to $\delta^{-1}\lambda^2 t^T\hat{\gamma}_3 t = \gamma_3$. The possible canonical forms for γ_3 over the real numbers are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\,.$$

It is now clear that we end up with six different orbits when the ground field is $\mathbb{R}$. When the ground field is $\mathbb{C}$ there are only three possible canonical forms for γ_3, giving thus rise to 5 different orbits.

5. Lie superalgebras based on the 3-dimensional Abelian Lie algebra

Notation 5.1.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,,$$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall also use the following special notation for the standard Pauli matrices:

$$H := \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad iJ := \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad K := \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We shall denote by $1_{2\times 2}$ the 2×2 identity matrix, and we let

$$L := 1_{2\times 2} + K - H = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

The purpose of this section is to prove Thm.**5.4.** below. In order to be systematic along its proof we give the following:

Definition 5.2.

(1) $a(\Gamma) = \max\{ \text{ \# of diagonal matrices in } (S,T)\cdot(\Gamma^1,\Gamma^2,\Gamma^3) \mid S,T \in \mathrm{GL}_3 \}$.
(2) $b(\Gamma) = \dim \mathrm{Span}_{\mathbb{F}} \{\Gamma^1,\Gamma^2,\Gamma^3\}$.
(3) $c(\Gamma) = \max\{ \text{ \# of diagonal matrices in } (S,I)\cdot(\Gamma^1,\Gamma^2,\Gamma^3) \mid S \in \mathrm{GL}_3 \}$.

Observation 5.3. Note that $a(\Gamma)$ and $b(\Gamma)$ are invariant along each orbit. The number $c(\Gamma)$ in (3) above corresponds to the number of matrices that can be simultaneously diagonalized in a given triple $(\Gamma^1,\Gamma^2,\Gamma^3)$. It is clear that $a(\Gamma) \geq c(\Gamma)$.

Theorem 5.4. *Let $\mathcal{S}$ be the set of triples of 3×3 real symmetric matrices, and let $G = \mathrm{GL}_3 \times \mathrm{GL}_3$ act on $\mathcal{S}$ on the left by means of,*

$$(S,T)\cdot(\Gamma^1,\Gamma^2,\Gamma^3) = (\sum T_{1i}(S\,\Gamma^i S^T),\, \sum T_{2i}(S\,\Gamma^i S^T),\, \sum T_{3i}(S\,\Gamma^i S^T))$$

and write $[\Gamma^1,\Gamma^2,\Gamma^3]$ for the G-orbit of the triple $(\Gamma^1,\Gamma^2,\Gamma^3)$. Then, the following is a complete list of orbit representatives in $\mathcal{S}$:

(1) **Orbits for which $(\Gamma^1,\Gamma^2,\Gamma^3)$ can be simultaneously diagonalized:**

(1.i) **Orbits for which** $\dim \mathrm{Span}_{\mathbb{F}} \{\Gamma^1,\Gamma^2,\Gamma^3\} = 3$*:*

$$[I, I_1 + I_2, I_1].$$

(1.ii) **Orbits for which** $\dim \mathrm{Span}_{\mathbb{F}} \{\Gamma^1,\Gamma^2,\Gamma^3\} = 2$*:*

$$[I, I_1 + yI_2, 0], \quad y = 1, -1, \qquad [I_{2,1}, I_1 + yI_2, 0], \quad y = 0, -1. \qquad [I_1 + I_2, I_1, 0],$$

(1.iii) **Orbits for which** $\dim \operatorname{Span}_{\mathbb{F}} \{\Gamma^1, \Gamma^2, \Gamma^3\} = 1$*:*

$$[I, 0, 0]\,, \quad [I_{2,1}, 0, 0]\,, \quad [I_1 + I_2, 0, 0]\,, \quad [I_1 - I_2, 0, 0]\,, \quad [I_1, 0, 0]\,.$$

(1.iv) **Orbits for which** $\dim \operatorname{Span}_{\mathbb{F}} \{\Gamma^1, \Gamma^2, \Gamma^3\} = 0$*:*

$$[0, 0, 0]\,.$$

(2) **Orbits for which** $(\Gamma^1, \Gamma^2, \Gamma^3)$ **cannot be simultaneously diagonalized:**

(2.i) **Orbits having a pair of matrices that can be simultaneously diagonalized:**

(2.i.1) Orbits with $\dim \operatorname{Span}_{\mathbb{F}} \{\Gamma^1, \Gamma^2, \Gamma^3\} = 2$*:*

$$[I_1 - I_2, 0, \Gamma^2]\,, \qquad \Gamma^2 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad X = K, L\,.$$

$$[I_{2,1}, 0, \Gamma^2]\,, \qquad \Gamma^2 = \begin{pmatrix} 0 & w^T \\ w & X \end{pmatrix}, \quad X = K, L, Y\,,$$

with $w \in \mathbb{R}^2$ *arbitrary, and* Y *a diagonal matrix.*

(2.i.2) Orbits with $\dim \operatorname{Span}_{\mathbb{F}} \{\Gamma^1, \Gamma^2, \Gamma^3\} = 3$*:*

$$[I_1 + I_2, I_2, \Gamma^3]\,, \qquad \Gamma^3 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

$$[I, I_1 + yI_2, \Gamma^3]\,, \quad y = 1, -1; \qquad \text{and} \qquad [I_{2,1}, I_1 + yI_2, \Gamma^3]\,, \quad y = 0, -1$$

with Γ^3 *having at least two diagonal entries equal to zero, but otherwise arbitrary.*

(2.ii) **Orbits having no pair of matrices that can be simultaneously diagonalized:**

$$[I_{2,1}, \Gamma^2, \Gamma^3]\,, \qquad \Gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \quad X = K, L\,, \qquad \Gamma^3 = \begin{pmatrix} 0 & w^T \\ w & Y \end{pmatrix},$$

with $w \in \mathbb{R}^2$ *arbitrary, and* Y *a diagonal matrix.*

Remark 5.5. When the ground field is $\mathbb{C}$, the statements made in Thm.**5.4.** above get simplified since the nonzero entries of the different diagonal forms that arise in the analysis can always be chosen equal to $+1$. Therefore, the statements over the complex numbers are the following:

(1) **Orbit representatives for which** $(\Gamma^1, \Gamma^2, \Gamma^3)$ **can be simultaneously diagonalized:**
$[I, I_1{+}I_2, I_1]$, $[I, I_1{+}I_2, 0]$, $[I_1{+}I_2, I_1, 0]$, $[I, 0, 0]$, $[I_1{+}I_2, 0, 0]$, $[I_1, 0, 0]$, and $[0, 0, 0]$.

(2) **Orbit representatives for which** $(\Gamma^1, \Gamma^2, \Gamma^3)$ **cannot be simultaneously diagonalized:**
$[I_1 + I_2, \Gamma^2, 0]$, with Γ^2 as in the first case of (2.i.1) of Thm.**5.4** above; $[I_1, I_2, \Gamma^3]$, with Γ^3 as in (2.i.2); and $[I, I_1 + I_2, \Gamma^3]$, with Γ^3 having at least two diagonal entries equal to zero, but otherwise arbitrary.

All together we have 10 different orbits, plus a 4-parameter family of orbits.

Observation 5.6. We may always act on a triple $(\Gamma^1, \Gamma^2, \Gamma^3)$ with a pair (T, I), so as to further arrange it in such a way that $\operatorname{rk}\Gamma^1 \geq \operatorname{rk}\Gamma^2 \geq \operatorname{rk}\Gamma^3$.

Strategy for proving Thm. 5.4

We may organize the proof by filling in the table below the different orbits that can have a given pair of values for $a(\Gamma)$ and $b(\Gamma)$:

$(a(\Gamma)$=3,$b(\Gamma)$=3);	$(a(\Gamma)$=3,$b(\Gamma)$=2);	$(a(\Gamma)$=3,$b(\Gamma)$=1);
$(a(\Gamma)$=2,$b(\Gamma)$=3);	$(a(\Gamma)$=2,$b(\Gamma)$=2);	$(a(\Gamma)$=2,$b(\Gamma)$=1); Void case by Lemma 5.7 (2)
$(a(\Gamma)$=1,$b(\Gamma)$=3);	$(a(\Gamma)$=1,$b(\Gamma)$=2); Void case by Lemma 5.7 (5)	$(a(\Gamma)$=1,$b(\Gamma)$=1); Void case by Lemma 5.7 (5)

We shall first deal with orbits having $a(\Gamma) = 3$. We then look at the other extreme; namely, orbits having $a(\Gamma) = 1$. We shall finally look at the cases for which $a(\Gamma) = 2$. In dealing with the cases having $a(\Gamma) = 3$ we shall choose orbit representatives in such a way that $\operatorname{rk}\Gamma^1 \geq \operatorname{rk}\Gamma^2 \geq \operatorname{rk}\Gamma^3$ with maximal $\operatorname{rk}\Gamma^1$. For the cases having $a(\Gamma) \leq 2$ our orbit representatives will be chosen in such a way that the first two matrices are those that we can simultaneously diagonalize with $\operatorname{rk}\Gamma^1$ maximal. If, at the end of this procedure there is still a $\Gamma^i = 0$, we act with a permutation matrix T so that we present the orbit representatives of this sort with 0's at the end.

We may immediately prove the following Lemma whose value is to organize the strategy indicated above.

Lemma 5.7.

(1) *If* $b(\Gamma) = 2$ *there is a pair* $(S, T) \in \mathrm{GL}_3 \times \mathrm{GL}_3$ *such that* $(S, T) \cdot (\Gamma^1, \Gamma^2, \Gamma^3) = (\tilde{\Gamma}^1, \tilde{\Gamma}^2, 0)$ *having* $\operatorname{rk}\tilde{\Gamma}^1 \geq \operatorname{rk}\tilde{\Gamma}^2$. *Therefore* $a(\Gamma) \geq 2$.

(2) *If* $b(\Gamma) = 1$ *there is a pair* $(S, T) \in \mathrm{GL}_3 \times \mathrm{GL}_3$ *such that* $(S, T) \cdot (\Gamma^1, \Gamma^2, \Gamma^3) = (\tilde{\Gamma}^1, 0, 0)$. *Therefore* $a(\Gamma) = 3$.

(3) *If* $b(\Gamma) = 0$. *Then* $a(\Gamma) = 3$.

(4) *If* Γ^1, Γ^2 *and* Γ^3 *are linearly dependent. Then* $a(\Gamma) \geq 2$.

(5) *If* $a(\Gamma) = 1$. *Then* $b(\Gamma) = 3$.

We may now start with the strategy stated:

Orbits with $a(\Gamma) = 3$.

It is easily proved that $a(\Gamma) = 3$ if and only if $c(\Gamma) = 3$, and that $c(\Gamma) = 3$ if and only if for any pair $(S, T) \in \mathrm{GL}_3 \times \mathrm{GL}_3$, $c((S, T) \cdot \Gamma) = 3$. It is then easy to see that we may start with a triple of diagonal matrices $(\Gamma^1, \Gamma^2, \Gamma^3)$ satisfying $\operatorname{rk}\Gamma^1 \geq \operatorname{rk}\Gamma^2 \geq \operatorname{rk}\Gamma^3$. We then proceed to give a case-by-case analysis subdivided according to $\operatorname{rk}\Gamma^1$: Say either $\operatorname{rk}\Gamma^1 = 3$, 2, 1 or 0, under specific hypotheses on $b(\Gamma)$; say, either $b(\Gamma) = 3$, 2, 1 or 0. The way to go is by starting with a clever choice of Γ^1 and then acting on the triple $(\Gamma^1, \Gamma^2, \Gamma^3)$ with (S, I), having $S \in \mathrm{O}(\Gamma^1)$, the isotropy group of Γ^1.

Orbits with $a(\Gamma) = 1$.

It is similarly proved that $a(\Gamma) = 1$ if and only if $c(\Gamma) = 1$, and that $c(\Gamma) = 1$ if and only if for any pair $(S, T) \in \mathrm{GL}_3 \times \mathrm{GL}_3$, $c((S,T) \cdot \Gamma) = 1$. One can further show that the hypotheses $a(\Gamma) = 1$ and $b(\Gamma) = 3$ imply that there must be scalars x_1, x_2, x_3 such that $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) \neq 0$. Indeed, we have the following result whose proof is given in the Appendix:

Lemma 5.8. *Let $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ with $b(\Gamma) = 3$. Then, $[\Gamma^1, \Gamma^2, \Gamma^3] = [I_1, I_2, \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right)]$ if and only if for any choice of scalars x_1, x_2, and x_3, $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) = 0$. In particular, under any of these equivalent hypotheses, $a(\Gamma) = 2$.*

Corollary 5.9. *If $a(\Gamma) = 1$, there exist a triple $(\Gamma^1, \Gamma^2, \Gamma^3)$ in the orbit and scalars x_1, x_2, x_3 such that $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) \neq 0$. Furthermore, the diagonal form of $x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3$ is $I_{2,1}$.*

Proof. If the diagonal form of $x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3$ was $\pm I$, then $a(\Gamma) \geq 2$ contrary to the assumption $a(\Gamma) = 1$.

By statement (5) in Lemma 5.7, $a(\Gamma) = 1$ implies $b(\Gamma) = 3$. If $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) = 0$ for all $x_1, x_2, x_3 \in \mathbb{R}$, then $a(\Gamma) = 2$, contrary to the assumption. ∎

We then conclude the following (see the Appendix for its proof) :

Proposition 5.10. *The* **only orbits** *having $a(\Gamma) = 1$ and $b(\Gamma) = 3$ are,*

$$\left[I_{2,1}, \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} 0 & w^T \\ w & \mathrm{Diag}(c,d) \end{pmatrix} \right]$$

where $X = K, L$, and $\mathbb{R}^2 \ni w \neq 0$.

Orbits with $a(\Gamma) = 2$.

Note that $b(\Gamma) = 1$ would produce a triple of the form $(\tilde{\Gamma}^1, 0, 0)$ in the orbit; whence, $a(\Gamma) = 3$. Therefore, either $b(\Gamma) = 2$, or $b(\Gamma) = 3$. In any case we can look separately at the subcases for which $\det\left(\sum x_i\Gamma^i\right) = 0$ for all $x_i \in \mathbb{F}$, and those for which there exist $x_i \in \mathbb{F}$ such that $\det\left(\sum x_i\Gamma^i\right) \neq 0$, $i = 1, 2, 3$.

Orbits with $a(\Gamma) = 2$, $b(\Gamma) = 2$, and $\det\left(\sum x_i\Gamma^i\right) = 0$ for all x_i.

We may start with $\Gamma = (\Gamma^1, \Gamma^2, 0)$ and $\mathrm{rk}\,\Gamma^1 \geq \mathrm{rk}\,\Gamma^2$. Now, $\mathrm{rk}\,\Gamma^1$ cannot be 3 because $\det\left(\sum x_i\Gamma^i\right)$ must be zero for any triple $(x_1, x_2, x_3) \in \mathbb{F}^3$.

If $\mathrm{rk}\,\Gamma^1 = 1$ a suitable $T \in \mathrm{GL}_3$ can be found to produce a rank-2 matrix in the triple $(I, T) \cdot (\Gamma^1, \Gamma^2, 0)$, while still keeping the third matrix equal to 0; otherwise, Γ^2 would have to be a scalar multiple of Γ^1 and $a(\Gamma) = 3$ contrary to our assumptions.

Therefore $2 = \operatorname{rk}\Gamma^1 \geq \operatorname{rk}\Gamma^2$, and the triple we start with can be brought to the form $(I_1 \pm I_2, \tilde{\Gamma}^2, 0)$.

Suppose there is some $S \in \mathrm{O}(I_1 \pm I_2)$ such that $S\,\tilde{\Gamma}^2 S^T$ has a nonzero entry in its third row, say. Then, scalars x_1 and x_2 can be found such that $\det(x_1(I_1 \pm I_2) + x_2\tilde{\Gamma}^2) \neq 0$. Therefore, $\tilde{\Gamma}^2$ must have a 2×2 symmetric matrix in its upper left block and zero entries in its third row (and hence in its third column). If $\tilde{\Gamma}^1 = I_1 + I_2$, then the symmetric matrix in the upper left block of $\tilde{\Gamma}^2$ can be further diagonalized with an element of $\mathrm{O}(I_1 + I_2)$, thus implying that $a(\Gamma) = 3$. So, $\tilde{\Gamma}^1$ must be $I_1 - I_2$, and the 2×2 symmetric matrix in the upper left block of $\tilde{\Gamma}^2$ must be either K or L. The proof is given in the Appendix (see Lemma A1).

Orbits with $a(\Gamma) = 2$, $b(\Gamma) = 3$, and $\det\big(\sum x_i\Gamma^i\big) = 0$ for all x_i.

According to the Lemma 5.7 above there is only one orbit having $b(\Gamma) = 3$ and $\det(\sum x_i\Gamma^i) = 0$ which necessarily has $a(\Gamma) = 2$.

Orbits with $a(\Gamma) = 2$, $b(\Gamma) = 2$, and $\det\big(\sum x_i\Gamma^i\big) \neq 0$.

Since $b(\Gamma) = 2$ we may assume that we have started with a triple $(\Gamma^1, \Gamma^2, 0)$ having $3 = \operatorname{rk}\Gamma^1 \geq \Gamma^2$. Acting with (S, I) for a suitable $S \in \mathrm{GL}_3$ we may further assume that $\Gamma^1 = I$, or $\Gamma^1 = I_{2,1}$. If $\Gamma^1 = I$ then $S \in \mathrm{O}(I) = \mathrm{O}_3$ can be found so as to further diagonalize the second matrix in the triple $(S, I)\cdot(I, \Gamma^2, 0)$; thus, $a(\Gamma) = 3$, contrary to our assumptions. Whence $\Gamma^1 = I_{2,1}$. Since Γ^2 cannot be diagonalized by an element from the Lorentz group $S \in \mathrm{O}(I_{2,1})$, it follows that Γ^2 must be of the form, (see Lemma A.4.3 in the Appendix)

$$\Gamma^2 = \begin{pmatrix} 0 & w \\ w^T & X \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2, \qquad X = K, L, \mathrm{Diag}(d_1, d_2).$$

Orbits with $a(\Gamma) = 2$, $b(\Gamma) = 3$, and $\det\big(\sum x_i\Gamma^i\big) \neq 0$.

We may assume that $\operatorname{rk}\Gamma^1 = 3 \geq \operatorname{rk}\Gamma^2$, with Γ^1 and Γ^2 diagonal. Therefore, we can find a pair (T, S) such that either $\Gamma^1 = I$, and $\Gamma^2 = I_1 + yI_2$ with $y = \pm 1$, or else $\Gamma^1 = I_{2,1}$, and $\Gamma^2 = I_1 + yI_2$ with $y = 0$ or $y = -1$. In either case Γ^3 can be chosen with two diagonal entries equal to zero.

Appendix

The purpose of this section is to outline the proofs of the Lemmas from which the statements in Thm. follow. For any real symmetric 3×3 matrix X, we shall write $\mathrm{O}(X)$ for the isotropy group $\{g \in \mathrm{GL}_3 \mid gXg^T = X\}$, regardless of the rank of X, and we shall keep using the notation I, $I_{2,1}$, I_1, $\ldots$, for the matrices introduced in **5.1**.

A.1 Case $a(\Gamma) = 2$ and $b(\Gamma) = 2$.

A.1.1 $\operatorname{rk}\Gamma^1 = 1$**.** Let us write $\Gamma^2 = \left(\begin{smallmatrix} a & w^T \\ w & B \end{smallmatrix}\right)$, with $B^T = B$, $w \in \mathbb{R}^2$, and $a \in \mathbb{R}$. An explicit description for $g \in \mathrm{O}(I_1)$ can be written in the form,

$$g = \begin{pmatrix} \pm 1 & v^T \\ 0 & A \end{pmatrix}, \qquad A \in \mathrm{GL}_2 \quad \text{and} \quad v \in \mathbb{R}^2 .$$

It is then easy to see that Γ^2 can be simultaneously diagonalized with I_1 if and only if $w \in \mathrm{Im}(B)$. We are interested precisely in the complementary case. In particular, B should not be invertible and we may analize the cases $B = 0$ and $\operatorname{rk} B = 1$ with $w \notin \operatorname{Im} B$ separately.

Let us first assume $B = 0$. By adding $-aI_1$ to Γ^2 we may assume from the start that the $(1,1)$-entry of Γ^2 is equal to zero. We then choose $v = 0$, and the invertibility of A in $g \in \mathrm{O}(I_1)$ in order to have $Aw = \delta_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and therefore, $[I_1, \Gamma^2, 0] = \left[I_1, \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right]$. Acting on the last triple with (I, h), where $h = \left(\begin{smallmatrix} r & -r & 0 \\ r & r & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, and $r = \frac{1}{\sqrt{2}}$, we get,

$$\left[(I,h)\cdot\left(I_1, \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right)\right] = \left[\left(\begin{smallmatrix} H & 0 \\ 0 & 0 \end{smallmatrix}\right), r^2\left(\begin{smallmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), 0\right] = \left[\left(\begin{smallmatrix} H & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right]$$

where we have used the fact that: If X cannot be simultaneously diagonalized with H then via $\mathrm{O}(H)$, X is equivalent to K or L. (see [8], Prop. in §6.1)

Let us now assume that $\operatorname{rk} B = 1$ and $w \notin \operatorname{Im} B$. We may again assume from the start that the $(1,1)$-entry of Γ^2 is equal to zero. Now choose $g \in \mathrm{O}(I_1)$ with $v = 0$ and A such that $ABA^T = \mathrm{Diag}(0,1)$. Then,

$$[I_1, \Gamma^2, 0] = \left[I_1, \left(\begin{smallmatrix} 0 & w^T A^T \\ Aw & \mathrm{Diag}(0,1) \end{smallmatrix}\right), 0\right].$$

Now choose $g = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & r & s \end{smallmatrix}\right)$ ($s^2 = 1$, and $ps \neq 0$), so that

$$g \left(\begin{smallmatrix} 0 & w^T A^T \\ Aw & \mathrm{Diag}(0,1) \end{smallmatrix}\right) g^T = \begin{cases} \left(\begin{smallmatrix} K & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if} \quad \delta_1^T A w \neq 0, \\ \left(\begin{smallmatrix} 0 & \pm y\delta_1 \\ \pm y\delta_1^T & 1 \end{smallmatrix}\right) & \text{if} \quad \delta_1^T A w = 0. \end{cases}$$

Note that we can proceed as before to get

$$\left[I_1, \left(\begin{smallmatrix} K & 0 \\ 0 & 1 \end{smallmatrix}\right), 0\right] = \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & H \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & K \end{smallmatrix}\right), 0\right].$$

On the other hand,

$$\left[I_1, \left(\begin{smallmatrix} 0 & \pm y\delta_1 \\ \pm y\delta_1^T & 1 \end{smallmatrix}\right), 0\right] = \left[\left(\begin{smallmatrix} 0 & 0 \\ 0 & H \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right), 0\right] = \left[\left(\begin{smallmatrix} H & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right), 0\right]$$

where $X = K, L$ (See [8] Prop. in §6.1). Altogether we obtain the following:

Lemma A.1.2. *Write* Γ^2 *be as before, and assume* $a(I_1,\Gamma^2,0)=2$. *Then either* $\left[I_1,\Gamma^2,0\right]=\left[\begin{pmatrix} H & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix},0\right]$ *with* $X=K$ *or* L, *or else* $\left[I_1,\Gamma^2,0\right]=\left[I_{2,1},\begin{pmatrix} 1 & 0\\ 0 & K\end{pmatrix},0\right]$ *depending on whether* $B=0$ *or* $\operatorname{rk}B=1$ *with* $w\notin \operatorname{Im}B$.

A.2 Case $\operatorname{rk}\Gamma^1=2$ **and** $g\Gamma^1g^T=I_1+I_2$**.** We shall now write $\Gamma^2=\begin{pmatrix} B & w\\ w^T & b\end{pmatrix}$ with $B^T=B$, $w\in\mathbb{R}^2$, and $b\in\mathbb{R}$. An explicit computation of $\mathrm{O}(I_1+I_2)$ yields:

$$\mathrm{O}(I_1+I_2)\ni g \quad\Longleftrightarrow\quad g=\begin{pmatrix} A & v\\ 0 & a\end{pmatrix}, \quad A\in \mathrm{O}_2\,,\ v\in\mathbb{R}^2\,,\ \text{and, } a\in\mathbb{R}-\{0\}\,.$$

It is easy to see that Γ^2 and I_1+I_2 can be simultaneously diagonalized if and only if $b\neq 0$. We are therefore interested in the case $b=0$. Choosing $g\in\mathrm{O}(I_1+I_2)$ with $v=0$, A such that ABA^T becomes diagonal, and a in such a way that $\|aAw\|=1$, we see that either $[I_1+I_2,\Gamma^2,0]=[I_1+I_2,\begin{pmatrix} 0 & \delta_1\\ \delta_1^T & 0\end{pmatrix},0]$ when $B=0$, or else $[I_1+I_2,\Gamma^2,0]=\left[I_1+I_2,I_1+yI_2+\begin{pmatrix} 0 & u\\ u^T & 0\end{pmatrix},0\right]$ ($\|u\|=1$), when $B\neq 0$, and the possible choices for y are -1, 0, or 1 with $y\neq 0$ if and only if $\operatorname{rk}B=2$.

A.2.1 Remark. Note that in this case we may always find real scalars x_1, and x_2 such that $\operatorname{rk}(x_1(I_1+I_2)+x_2\Gamma^2)=3$. Furthermore, the diagonal form of $x_1(I_1+I_2)+x_2\Gamma^2$ must be $I_{2,1}$.

A.3 Case $\operatorname{rk}\Gamma^1=2$ **and** $g\Gamma^1g^T=I_1-I_2$**.** Let us write $S(\theta)=\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}$, so that $g\in\mathrm{O}(I_1-I_2)$ if and only if $g=\begin{pmatrix} S(\theta) & v\\ 0 & a\end{pmatrix}$ with $v\in\mathbb{R}^2$ and $a\in\mathbb{R}-\{0\}$. Write $\Gamma^2=\begin{pmatrix} B & w\\ w^T & b\end{pmatrix}$ as before, and see that,

$$g\,\Gamma^2g^T=\begin{pmatrix} S(\theta)BS(\theta)+vw^TS(\theta)+S(\theta)wv^T+bvv^T & a(S(\theta)w+bv)\\ a(S(\theta)w+bv)^T & a^2b\end{pmatrix}.$$

Thus, in order to simultaneously diagonalize I_1-I_2, and Γ^2 it is necessary that $S(\theta)w+bv=0$.

A.3.1 Case $w=0$**.** We choose $g\in\mathrm{O}(I_1-I_2)$ having $v=0$ and find that,

$$[I_1-I_2,\Gamma^2,0]=\left[I_1-I_2,\begin{pmatrix} X & 0\\ 0 & a^2b\end{pmatrix},0\right]\qquad X=\operatorname{Diag}(d_1,d_2),K,L$$

where again, the stated possibilities for X follow from Prop. in §6.1 of [8].

A.3.2 Case $w\neq 0$ We subdivide the analysis depending on whether $b=0$ or $b\neq 0$.

3.2.1. $b\neq 0$**.** In this case we may choose $g\in\mathrm{O}(I_1-I_2)$ having $v=-\frac{1}{b}S(\theta)w$, to obtain

$$g\,\Gamma^2g^T=\begin{pmatrix} S(\theta)(B-\frac{1}{b}ww^T)S(\theta) & 0\\ 0 & a^2b\end{pmatrix}.$$

So that,

$$[I_1-I_2,\Gamma^2,0]=\left[I_1-I_2,\begin{pmatrix} X & 0\\ 0 & 1\end{pmatrix},0\right],\quad X=\operatorname{Diag}(d_1,d_2),K,L\,.$$

3.2.2. $b=0$**.** In this case we choose $g\in \mathrm{O}(I_1-I_2)$ having $v=0$, and get

$$g\Gamma^2 g^T = \begin{pmatrix} S(\theta)BS(\theta) & aS(\theta)w \\ a(S(\theta)w)^T & 0 \end{pmatrix}.$$

Whence,

$$[I_1-I_2,\Gamma^2,0] = \left[I_1-I_2, \begin{pmatrix} X & w \\ w^T & 0\end{pmatrix}, 0\right], \qquad X = \mathrm{Diag}(d_1,d_2), K, L\,.$$

One might further choose a in order to make $\|w\|=1$; but, since we are going to choose orbit representatives having Γ^1 with maximal rank, and in these cases we are going to find a $\widetilde{\Gamma}^1$ with $\mathrm{rk}\,\widetilde{\Gamma}^1=3$, there is no point in being more economical about the choice of w stating that $\|w\|=1$. Nevertheless, at some point it is useful to know that we might have chosen a unitary vector w in order to prove that we may find another triple in the orbit having $\mathrm{rk}\,\widetilde{\Gamma}^1=3$.

We may now conclude the following:

Lemma A.3.3. *Write Γ^2 be as before, and assume $a(I_1-I_2,\Gamma^2,0)=2$. Then,*

$$[I_1-I_2,\Gamma^2,0] = \begin{cases} \left[I_1-I_2, \begin{pmatrix} X & 0 \\ 0 & \varepsilon\end{pmatrix}, 0\right], & X=K,L; \quad \varepsilon=\pm 1, 0\,, \\ \left[I_1-I_2, \begin{pmatrix} X & w \\ w^T & 0\end{pmatrix}, 0\right], & X=\mathrm{Diag}(d_1,d_2),K,L\,;\ w\in\mathbb{R}^2-\{0\}\,. \end{cases}$$

Furthermore, we may find real scalars x_1 and x_2 such that $\mathrm{rk}(x_1(I_1-I_2)+x_2\Gamma^2)=3$ in all cases but the one with $\varepsilon=0$.

A.4 Case $\mathrm{rk}\,\Gamma^1=3$ with $g\Gamma^1g^T=I_{2,1}$. In this case, $g=\begin{pmatrix} a & u^T \\ v & A\end{pmatrix}\in \mathrm{O}(2,1)$ if and only if,

$$1=a^2+u^THu, \qquad 0 = AHu+av, \qquad H = AHA^T+vv^T\,.$$

In particular, we shall write $g(\theta):=\begin{pmatrix} \pm 1 & 0 \\ 0 & S(\theta)\end{pmatrix}\in \mathrm{O}(2,1)$. Write $\Gamma^2=\begin{pmatrix} b & w^T \\ w & B\end{pmatrix}$, and $\widetilde{\Gamma}^2=g\Gamma^2g^T$, to obtain the following blocks:

$$\begin{aligned}
\widetilde{\Gamma}^2_{11} &= a^2b+au^Tw+aw^Tu+u^TBu\,,\\
\widetilde{\Gamma}^2_{21} &= A(bHu+aw+a^{-1}Huw^Tu+Bu)\,,\\
\widetilde{\Gamma}^2_{12} &= \widetilde{\Gamma}^{2\ T}_{21}\,,\\
\widetilde{\Gamma}^2_{22} &= bH+a^{-1}A(abH+wu^TH+Huw^T+aB)A^T\,.
\end{aligned}$$

In order to diagonalize $\widetilde{\Gamma}^2$, say with $g\in\mathrm{O}(I_{2,1})$ having an invertible A in its lower right block, we must have

$$bHu+aw+a^{-1}Huw^Tu+Bu=0\,.$$

There are two complementary conditions under which we may approach this equation; namely $w_1^2<w_2^2$, and $w_1^2\geq w_2^2$, where $w=\begin{pmatrix} w_1 \\ w_2\end{pmatrix}$.

A.4.1 Case $w_2^2 > w_1^2$. Under this condition it is possible to find $g \in \mathrm{O}(I_{2,1})$ such that $\widetilde{\Gamma}^2 = g\Gamma^2 g^T = \begin{pmatrix} \widetilde{\Gamma}^2_{11} & 0 \\ 0 & \widetilde{\Gamma}^2_{22} \end{pmatrix}$. But then, a linear combination of the latter with $I_{2,1}$ let us assume that $\widetilde{\Gamma}^2_{11} = 0$. Finally, acting with $g(\theta)$'s, we may assume that:

$$\widetilde{\Gamma}^2 = g\Gamma^2 g^T = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \qquad X = \mathrm{Diag}(d_1, d_2), K, L\,. \tag{5}$$

A.4.2 Case $w_2^2 \leq w_1^2$. In this case we may act with a suitable $g(\theta) \in \mathrm{O}(I_{2,1})$ (*ie,* $u = v = 0$) to obtain $\widetilde{\Gamma}^2 = g\Gamma^2 g^T = \begin{pmatrix} b & \pm w^T A^T \\ \pm Aw & 2bH + ABA^T \end{pmatrix}$. Now, if $b \neq 0$ a suitable linear combination of the later with $I_{2,1}$ let us assume that:

$$\widetilde{\Gamma}^2 = g\Gamma^2 g^T = \begin{pmatrix} 0 & u^T \\ u & X \end{pmatrix} \qquad X = \mathrm{Diag}(d_1, d_2), K, L,\ u \in \mathbb{R}^2\,.$$

In either case we conclude the following:

Lemma A.4.3. *Write Γ^2 be as before, and assume $a(I_{2,1}, \Gamma^2, 0) = 2$. Then,*

$$[I_{2,1}, \Gamma^2, 0] = \left[I_{2,1}, \begin{pmatrix} 0 & u^T \\ u & X \end{pmatrix}, 0\right], \qquad X = \mathrm{Diag}(d_1, d_2), K, L,\ u \in \mathbb{R}^2\,.$$

A.4.4 Remark. The analysis of the case $a(\Gamma) = 2$ and $b(\Gamma) = 3$ follows easily from **A.1**-**A.4.3**.

A.5 Orbits with $\Gamma^1 = I_{2,1}$. We shall consider triples of the form $(I_{2,1}, \Gamma^2, \Gamma^3)$. Let us write $\Gamma^i = \begin{pmatrix} b_i & w_i^T \\ w_i & B_i \end{pmatrix}$, $i = 2, 3$ as before. Let $k = \dim \mathrm{Span}_{\mathbb{F}}\{w_2, w_3\}$. Clearly, $k = 0$ if and only if $\Gamma^i = \begin{pmatrix} b_i & 0 \\ 0 & B_i \end{pmatrix}$ $(i = 2, 3)$. If $k = 1$ we may write $w_3 = \lambda w_2$, and it is then clear that $\widetilde{\Gamma}^3 = \Gamma^3 - \lambda\Gamma^2$ is block-diagonal. If $k = 2$ we may find scalars λ_{ij} $(i, j = 1, 2)$, such that $\lambda_{i1} w_2 + \lambda_{i2} w_3 = \delta_i$ ($\delta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\delta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). Using the same λ_{ij}'s we have $\widetilde{\Gamma}^2 = \lambda_{11}\Gamma^2 + \lambda_{12}\Gamma^3$ and $\widetilde{\Gamma}^3 = \lambda_{21}\Gamma^2 + \lambda_{22}\Gamma^3$, and using our findings in **A.4.1** we see that for $\widetilde{\Gamma}^3$ there is some $g \in \mathrm{O}(2, 1)$ such that $g\widetilde{\Gamma}^3 g^T$ is again block diagonal. Therefore we conclude the following:

Lemma A.5.1. *For any orbit of the form $[I_{2,1}, \Gamma^2, \Gamma^3]$ we may always assume that $[I_{2,1}, \Gamma^2, \Gamma^3] = [I_{2,1}, \widetilde{\Gamma}^2, \widetilde{\Gamma}^3]$, where at least $\widetilde{\Gamma}^2$ is block diagonal. Moreover, if both $\widetilde{\Gamma}^2$, and $\widetilde{\Gamma}^3$ are block diagonal, then $a(I_{2,1}, \Gamma^2, \Gamma^3) \geq 2$.*

Proof. Let us assume that Γ^2 and $I_{2,1}$ cannot be simultaneously diagonalized. Then (*cf,* **A.4.1** above), there is some $g \in \mathrm{O}(I_{2,1})$, such that $g\Gamma^2 g^T = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ with $X = K, L$. Since Γ^3 is block diagonal, the linear combination $\Gamma^3 - \Gamma^3_{32}(g\Gamma^2 g^T)$ becomes diagonal. ∎

A.6 Case $a(\Gamma) = 1$ y $b(\Gamma) = 3$.

Lemma A.6.1. *Let* $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ *with* $b(\Gamma) = 3$. *Then,* $[\Gamma^1, \Gamma^2, \Gamma^3] = [I_1, I_2, \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right)]$ *if and only if for any choice of scalars* x_1, x_2, *and* x_3, $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) = 0$.

Proof. Since $\mathrm{rk}(\Gamma^i) \leq 2$ we may assume that $\mathrm{rk}(\Gamma^1) = 2 \geq \mathrm{rk}(\Gamma^2) \geq \mathrm{rk}(\Gamma^3)$. Now, if $a(\Gamma) = 3$, the linear independence of the Γ^i's yields $[\Gamma^1, \Gamma^2, \Gamma^3] = [I_1, I_2, I_3]$. Thus, we must have $a(\Gamma) \leq 2$. If there is some $S \in \mathrm{GL}_3$ such that $S\Gamma^1 S^T = I_1 + I_2$, our Remark **A.2.1** above shows that there are scalars x_i such that $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) \neq 0$. Therefore, the diagonal form of Γ^1 must be $I_1 - I_2$. From Lemma **A.3.3** above we conclude that $\Gamma^i = \left(\begin{smallmatrix} B_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ $(i = 2, 3)$, with symmetric 2×2 matrices B_i. Now, H, B_2 and B_3 become a basis for the symmetric 2×2 matrices, from which the statement now follows. [Note that if $\mathrm{rk}\,\Gamma^1 = 1$ the analysis may be reduced to the one just given by finding a suitable linear combination of Γ^1 and Γ^2 —which cannot be simultaneously diagonalized— that yields a new Γ^1 with rank ≥ 2]. ■

Lemma A.6.2. *Assume* $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ *is such that* $\mathrm{rk}(\Gamma^i) = 1$ *(*$i = 1, 2, 3$*), and* $b(\Gamma) = 3$. *Then* $a(\Gamma) = 3$, *and* $[\Gamma^1, \Gamma^2, \Gamma^3] = [I_1, I_2, I_3]$.

Proof. Suppose Γ^1 and Γ^2 cannot be simultaneously diagonalized. Then, Lemma **A.1.2** shows that $\mathrm{rk}(\Gamma^2) \geq 2$. The same argument applies to Γ^1 and Γ^3. In any case we may act on the triple with (I, S) $(S \in \mathrm{GL}_3)$, and $\mathrm{rk}(\Gamma^i) = \mathrm{rk}(S\,\Gamma^i S^T)$. ■

A.6.3 Remark. Note that under the assumption $b(\Gamma) = 3$ and $\det(x_1\Gamma^1 + x_2\Gamma^2 + x_3\Gamma^3) \neq 0$ for some choice of scalars x_i, we may further assume right from the start that $\Gamma^1 = I_{2,1}$ and that Γ^2 is block diagonal and cannot be simultaneously diagonalized with $I_{2,1}$. Therefore, we finally obtain the following:

Proposition A.6.4. *The* **only orbits** *having* $a(\Gamma) = 1$ *and* $b(\Gamma) = 3$ *are,*

$$\left[I_{2,1}, \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} 0 & w^T \\ w & \mathrm{Diag}(c,d) \end{pmatrix} \right]$$

where $X = K, L$, *and* $\mathbb{R}^2 \ni w \neq 0$.

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