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Reduced 1-Cohomology of Connected Locally Compact Groups and Applications

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Abstract. In this article we focus on the reduced 1-cohomology spaces of locally compact connected groups with coefficients in unitary representations. The vanishing of these spaces for every unitary irreducible representation characterizes Kazhdan's property (T). The main theorem states that for a connected locally compact group, there are only a finite number of unitary irreducible representations for which the reduced 1-cohomology does not vanish. Moreover, a description of these representations is given.
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1. Introduction

The vanishing of the reduced 1-cohomology spaces for every unitary irreducible representation characterizes Kazhdan's property (T) (see Y.Shalom [14]) for a compactly generated locally compact group (in particular a connected group). For connected solvable Lie groups, P. Delorme [5] showed that for every unitary irreducible representations of dimension at least 2 of a connected solvable Lie group, the reduced 1-cohomology vanishes and that there are only finitely many characters for which the reduced 1-cohomology is not zero.

As (non compact) solvable Lie groups do not have property (T) we can interpret this result by saying that the lack of property (T) of such groups is, from a cohomological point of view, concentrated in the 1-dimensional representations.

The main goal of this paper is to understand for connected Lie groups where the lack of property (T) is concentrated. Delorme's theorem provides the answer for connected solvable Lie groups. The main theorem that will be proven is the:

Theorem 6.3. *Let G be a connected locally compact group. Then, up to unitary equivalence, there are only finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*
Moreover, any such representation π is either trivial or factors through an irreducible unitary representation σ of a group H isomorphic to $PO(n,1), PU(m,1)$,

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or to a non-compact amenable non-nilpotent group H *such that* $\overline{H^1}(H,\sigma) \cong \overline{H^1}(G,\pi) \neq 0$.

This goal is achieved by enlarging the class of solvable groups in several steps:

1) In section 3, we focus on the class of amenable groups. We show:

Theorem 3.3 *Let* G *be a connected amenable Lie group. Up to unitary equivalence, there are finitely many irreducible unitary representations* π *of* G *with* $\overline{H^1}(G,\pi) \neq 0$. *Moreover all these representations are finite dimensional and their dimensions are less than the (real) dimension of the solvable radical of* G.

An interesting corollary of this fact is the vanishing of $\overline{H^1}(G, L^2(G))$ for these groups (Theorem 7.2). As this vanishing result is also true for discrete amenable groups (see [12]), we conjecture that $\overline{H^1}(G, L^2(G))$ is zero for every amenable locally compact group.

2) In section 4, a much larger class of groups is investigated: the groups having the Haagerup property. Recall that a locally compact group G has the Haagerup property if there exists a proper conditionally negative definite function on G. For such connected locally compact groups, we show that there are only finitely many irreducible representations which characterize the lack of property (T). However these representations are *not* finite dimensional in general.

3) In section 5, some known facts on the (reduced) 1-cohomology of group having the relative property (T) with respect to a closed normal subgroup are recalled.

4) Finally in section 6, the proof of theorem 6.3 is given.

In the theorem 6.3, the irreducible representations for which the reduced 1-cohomology space does not vanish (and hence the non-reduced 1-cohomology) are described algebraically.

The topological description of the irreducible representations π of a group G for which $H^1(G,\pi) \neq 0$ is given by the Vershik-Karpushev theorem (see [17] and [11]). Let us recall that the support of a representation π of a group G is the set of irreducible representations of G which are weakly contained in π and that the cortex of the group, $\mathrm{Cor}\,(G)$, is the set of all irreducible representation which are not separated from the trivial representation for the Fell-Jacobson topology on the dual space $\widehat{G}$ (see [11] for a nice presentation of this). The Vershik-Karpushev theorem states that if π is a unitary factorial representation of a second countable locally compact group G with $H^1(G,\pi) \neq 0$ then $\operatorname{supp}\pi \subset \mathrm{Cor}\,(G)$. This result can be interpreted by saying that the lack of property (T) is topologically concentrated in the cortex of the group.

In the last section, these vanishing results are applied to

- The description of the reduced 1-cohomology of a locally compact amenable group with coefficients in the regular representation;

- The study of smooth μ-harmonic Dirichlet finite functions on smooth manifolds which are homogeneous spaces of connected unimodular Lie groups. We show (Theorem 7.6) that if G is a unimodular connected Lie group acting transitively on a smooth connected non-compact manifold M with $\overline{H^1}(G, L^2(M)) = 0$, and if μ is a symmetric probability measure on G whose support is a compact generating set of G, then the only smooth Dirichlet-finite μ-harmonic functions on M are the constant functions.

In [12], the authors proved the analogous result in the case where the groups are discrete. In [1], G. Alexopoulos proved this kind of result for the bounded functions on discrete polycyclic groups.

2. 1-cohomology and reduced 1-cohomology

Let G be a locally compact σ-compact separable group and let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation of G.

Definition 2.1. 1) A continuous map $b : G \to \mathcal{H}_\pi$ is a 1-cocycle with respect to π if it satisfies the following relation:

$$b(gh) = b(g) + \pi(g)b(h) \ (*)$$

for all $g, h \in G$. The space of cocycles endowed with the topology of uniform convergence on compact sets of G is a Frechet space, denoted by $Z^1(G, \pi)$.

2) A cocycle b is a coboundary if there exists an element $\xi \in \mathcal{H}_\pi$ such that $b(g) = \pi(g)\xi - \xi$. The set of coboundaries is a subspace of $Z^1(G, \pi)$ denoted by $B^1(G, \pi)$. The closure of the coboundaries in $Z^1(G, \pi)$ will be denoted by $\overline{B^1}(G, \pi)$. An element of this space is called an almost coboundary.

3) The 1-cohomology of G with coefficients in π is the quotient space

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

4) The 1- *reduced*-cohomology of G with coefficients in π is the Hausdorff quotient space

$$\overline{H^1}(G, \pi) = Z^1(G, \pi)/\overline{B^1}(G, \pi).$$

We have a nice geometrical interpretation of these spaces in terms of affine isometric actions of the group G.

Definition 2.2. Let $\mathcal{H}$ be an affine Hilbert space. An affine isometric action of G on $\mathcal{H}$ is a strongly continuous group homomorphism $\alpha : G \to Is(\mathcal{H})$ to the group of affine isometries of $\mathcal{H}$.

The next lemma establishes a relationship between affine isometric actions, unitary representations and 1-cocycles.

Lemma 2.3. *Any affine isometric action $\alpha : G \to Is(\mathcal{H})$ can be written as $\alpha(g)v = \pi(g)v + b(g)$ $(v \in \mathcal{H})$ where π is an unitary representation of G on the underlying Hilbert space of $\mathcal{H}$ and $b : G \to \mathcal{H}$ is a 1-cocycle.The representation π is called the linear part of α and b is the translation part of α. Conversely, given π an unitary representation of G on a Hilbert space $\mathcal{H}$ and $b : G \to \mathcal{H}$ a 1-cocycle, we can define an affine isometric action by setting $\alpha(g)\xi = \pi(g)\xi + b(g)$, $\forall \xi \in \mathcal{H}_\pi$.*

For the proof, we refer to [10]. It is an easy exercise to show that, given an unitary representation π of G, the coboundaries $b \in B^1(G,\pi)$ correspond to affine isometric actions with linear part π which have fixed points. Moreover, almost coboundaries $b \in \overline{B^1}(G,\pi)$ correspond to those actions α which almost have fixed points in the sense that for every $\varepsilon > 0$ and for every compact subset K of G, there exists an element $\xi \in \mathcal{H}_\pi$ such that

$$\max_{k\in K}\|\alpha(k)\xi - \xi\| < \varepsilon$$

2.1. Some general facts on 1-cohomology.

For a given unitary representation π of a locally compact group G, one can ask if the associated 1-cohomology and reduced 1-cohomology coincide. The answer is given by A.Guichardet in [7]:

Proposition 2.4. *Let π be a unitary representation of G (σ-compact) without non zero invariant vectors. The following are equivalent:*

i) π does not almost have invariant vectors (i.e. there exists $\varepsilon > 0$, a compact subset K of G, such that $\max_{k\in K}\|\pi(k)\xi - \xi\| \geq \varepsilon\|\xi\|$) for all $\xi \in \mathcal{H}_\pi$;

ii) $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$;

iii) $H^1(G,\pi) = \overline{H^1}(G,\pi)$.

In the case where π has a non zero invariant vector, one can decompose π into an orthogonal direct sum of the form $\pi_0 \oplus 1$ where π_0 has no non zero invariant vectors and where 1 denotes the trivial action of G. As $H^1(G,1) = \overline{H^1}(G,1) = Z^1(G,1)$, one can compare the 1-cohomology with the reduced 1-cohomology spaces by using the following property [6]:

Lemma 2.5. *Let $\pi_1, ..., \pi_n$ be a finite set of unitary representations of a group G. Then*

$$H^1(G, \oplus_{i=1}^n \pi_i) = \oplus_{i=1}^n H^1(G,\pi_i)$$

One can show (see [7]) that this statement is no longer true in general for an infinite family of unitary representations. However, we have:

Lemma 2.6. *(see [2]) If π is a unitary representation of a locally compact σ-compact group G, then*

$$H^1(G,\pi) = 0 \Leftrightarrow H^1(G,\infty\cdot\pi) = 0$$

If we deal with reduced 1-cohomology these kind of properties behave quite nicely (see [7]):

Proposition 2.7. *Let (X,μ) be a measured space and $(\pi_x)_{x\in X}$ a measurable field of unitary representations of a locally compact group G. If $\overline{H^1}(G,\pi_x) = 0$ for μ-almost every $x \in X$, then*

$$\overline{H^1}(G, \int_X^\oplus \pi_x \, d\mu(x)) = 0.$$

2.2. Normal subgroups.

The aim of this section is to study rigidity phenomena of the following type: If the restriction of an affine action of a group G to a closed normal subgroup N admits a fixed point (resp. almost fixed points) what can be said about the existence of a G-fixed point (resp. almost fixed points)?

Lemma 2.8. *Let N be a closed normal subgroup of a locally compact group G and α an affine isometric action of G whose linear part has no non zero N-invariant vectors. If the restriction of α to N has a fixed point, then α has a fixed point.*

We can give a short geometrical proof if this fact:

Let α be an affine isometric action with linear part π,whose restriction to N has a fixed point. Let $\mathcal{H}^N$ be the set of $\alpha(N)$-fixed points. If $\xi, \eta \in \mathcal{H}^N$, then $\xi - \eta = \alpha(n)\xi - \alpha(n)\eta = \pi(n)(\xi - \eta)$. But we assume π not to have N-invariant non zero vectors; so we conclude that $\mathcal{H}^N$ is reduced to a single point. On the other hand, $\mathcal{H}^N$ is $\alpha(G)$-invariant by normality of N in G.

The preceding lemma can be also stated as : *Let N be a closed normal subgroup of G and π a unitary representation of G without non zero N-invariant vectors. Then the restriction map induced by restriction of cocycles from G to N, $Res : H^1(G, \pi) \to H^1(N, \pi)$ is injective.*

As an immediate consequence:

Corollary 2.9. *Let G be a locally compact group and π an irreducible unitary representation of G such that $\overline{H^1}(G, \pi) \neq 0$. If N is a closed normal subgroup of G having the property (T) then $\pi|_N = 1$.*

The analogous statement of Lemma 2.8 in the context of non-reduced cohomology is not true in general (see [12]). Under cocompactness condition on the normal subgroup, we can state:

Proposition 2.10. *Let G be a locally compact group and N a closed, normal, cocompact subgroup of G. Let π be a unitary representation of G. Then the restriction map $Res : \overline{H^1}(G, \pi) \to \overline{H^1}(N, \pi|_N)$ is injective.*

In particular if $\overline{H^1}(N, \pi|_N) = 0$ then $\overline{H^1}(G, \pi) = 0$.

Proof. By [9], there exists a Borel regular section $s : G/N \to G$ whose image is relatively compact. For all $x \in G/N$, and all $g \in G$, $gs(x)$, $s(gx)$ has the same image in G/N. Let us define a cocycle $\sigma : G/N \times G \to N$; $\sigma(x, g) = (s(gx)^{-1} gs(x))^{-1}$. So that $\sigma(x, g)$ is the unique element of N satisfying $gs(x)\sigma(x, g) \in s(G/N)$. Remark that $\sigma(G/N, K)$ is relatively compact whenever K is a compact subset of G.

Let α be an affine action of G such that $\alpha|_N$ almost has fixed points and let us show that it almost has fixed points. Let K be a compact subset of G, it is contained in a compact subset of the form $K_0\, \overline{s(G/N)}$, where K_0 is a compact subset of N. Let K_X be the compact subset (by normality of N) of N defined by:

$$K_X = \mathrm{Cl}_N\{s(x)^{-1} n s(x) \sigma(x, x_0^{-1}) \,|\, n \in K_0,\, x \in G/N,\, x_0 \in s(G/N)\}.$$

Then for $\varepsilon > 0$ fixed, there exists by assumption a point ξ in the Hilbert space of π such that

$$\sup_{n\in K_X} \| \alpha(n)\xi - \xi \| < \varepsilon.$$

Denote by dx the finite G-invariant normalized measure (for the action $g \cdot s(x) = gs(x)\sigma(x,g)$) induced by the Haar measure on G/N.

For $g_0 \in G$, there exists a unique $x_0 \in s(G/N)$ and a unique $n_0 \in N$ such that $g_0 = n_0 x_0$. For $g_0 \in K$, we have:

$$\begin{aligned}
& \| \alpha(g_0) \int_{G/N} \alpha(s(x))\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \alpha(n_0 x_0) \int_{G/N} \alpha(s(x))\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \alpha(n_0) \int_{G/N} \alpha(x_0 s(x)\sigma(x,x_0)\sigma(x,x_0)^{-1})\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \alpha(n_0) \int_{G/N} \alpha(x_0 \cdot s(x)\sigma(x,x_0)^{-1})\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \alpha(n_0) \int_{G/N} \alpha(s(x)\sigma(x_0^{-1}\cdot x,x_0)^{-1})\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \alpha(n_0) \int_{G/N} \alpha(s(x)\sigma(x,x_0^{-1}))\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \int_{G/N} \alpha(n_0 s(x)\sigma(x,x_0^{-1}))\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \\
= & \| \int_{G/N} \alpha(n_0 s(x)\sigma(x,x_0^{-1}))\xi - \alpha(s(x))\xi dx \| \\
\leq & \sup_{x\in G/N} \| \alpha(n_0 s(x)\sigma(x,x_0^{-1}))\xi - \alpha(s(x))\xi \| \\
= & \sup_{x\in G/N} \| \alpha(s(x)^{-1} n_0 s(x)\sigma(x,x_0^{-1}))\xi - \xi \| .
\end{aligned}$$

So,

$$\sup_{g_0\in K} \| \alpha(g_0) \int_{G/N} \alpha(s(x))\xi dx - \int_{G/N} \alpha(s(x))\xi dx \| \leq \sup_{n\in K_X} \| \alpha(n)\xi - \xi \| < \varepsilon. \quad \blacksquare$$

We recall now some results of Guichardet [7] which describe the relationship between the 1-cohomology (resp., reduced 1-cohomology) of a group G and the 1-cohomology (resp., reduced 1-cohomology) of a quotient by a closed normal subgroup, with value in a unitary representation of G which is trivial on this normal subgroup .

Theorem 2.11. *Let G be a locally compact group, N a closed normal subgroup of G and π a unitary representation of G such that $\pi|_N = 1$. Then:*

i) Let $A(G,N,\pi)$ be the image of the restriction map $Z^1(G,\pi) \to Z^1(N,1)$. We have the isomorphisms:

$$H^1(G,\pi) \cong H^1(G/N,\dot{\pi}) \oplus A(G,N,\pi);$$

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/N,\dot{\pi}) \oplus A(G,N,\pi).$$

Notice that $A(G,N,\pi)$ is contained in $Hom_G(N,\pi)$, the space of G-equivariant homomorphisms from N to the additive group $\mathcal{H}_\pi$.

ii) If G is the semi-direct product $N \rtimes H$ for some group H, then $A(G,N,\pi) = Hom_G(N,\pi)$.

Corollary 2.12. *Let K be a compact normal subgroup of a locally compact group G and let π a unitary representation of G which is trivial on K. Then:*

$$H^1(G,\pi) \cong H^1(G/K,\pi)$$

and

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/K,\pi).$$

Let K be the closed normal subgroup of $N/\overline{[N,N]}$ generated by the closure of the union of the compact subgroups of $N/\overline{[N,N]}$, and set $V = (N/\overline{[N,N]})/K$. The group G acts by conjugation on N and this give rise to an action of G on V. The latter factors through an action of G/N on V which will be denoted by σ. Every continuous morphism f from N to $\mathcal{H}_\pi$ factors through a continuous morphism $\tilde{f}$ from V to $\mathcal{H}_\pi$, and f belongs to $Hom_G(N,\pi)$ if and only if $\tilde{f}$ satisfies

$$\tilde{f}(\sigma(g)(v)) = \pi(g)(\tilde{f}(v))$$

for all $g \in G/N$ and all $v \in V$.

If moreover N is a connected Lie group, $N/\overline{[N,N]}$ can be identified to $\mathbb{R}^n \times \mathbb{T}^k$ for some n, k. Consequently $V = \mathbb{R}^n$, and in this case, σ is a real finite dimensional representation (non unitary in general) of G/N. Hence the following result from [7]:

Proposition 2.13. *Let N be a connected Lie group; $Hom_G(N,\pi)$ is isomorphic to the space of $\mathbb{R}$-linear maps from V to $\mathcal{H}_\pi$ which intertwine σ and π.*

If $(\sigma^{\mathbb{C}}, V^{\mathbb{C}})$ is the complexified representation obtained from (σ,V), the space $Hom_G(N,\pi)$ can be identified with the space of $\mathbb{C}$-linear maps from $V^{\mathbb{C}}$ to $\mathcal{H}_\pi$ which intertwine $\sigma^{\mathbb{C}}$ and π.

Corollary 2.14. *Let π be a unitary irreducible representation of a connected Lie group N; then $Hom_G(N,\pi)$ does not vanish if and only if π is equivalent to a subrepresentation of $\sigma^{\mathbb{C}}$. In particular there are only finitely many such representations and there are all of dimension at most $dim(\sigma) \leq dim(N/\overline{[N,N]})$.*

In the case where N is a central subgroup, we have [7]:

Lemma 2.15. *Let π a non trivial irreducible unitary representation of G and let C be a closed central subgroup of G. If $\overline{H^1}(G,\pi) \neq 0$, then $\pi|_C = 1$ and $\overline{H^1}(G,\pi) \cong \overline{H^1}(G/C,\dot{\pi})$.*

3. $\overline{H^1}(G,\pi)$ of connected amenable locally compact groups

In this section we will establish an analogue of Delorme's theorem (see introduction) for connected amenable locally compact groups. More precisely, we will show that the reduced 1-cohomology of such a group is zero for all irreducible unitary representation except a finite number of finite dimensional ones.

We first establish the result for a connected amenable Lie group:

Theorem 3.1. *Let G be a connected amenable Lie group. Up to unitary equivalence, there are finitely many irreducible unitary representations π of G with $\overline{H^1}(G,\pi) \neq 0$. Moreover all these representations are finite dimensional and their dimensions are less or equal than the (real) dimension of the solvable radical R of G.*

Proof. Let π be an irreducible unitary representation of G such that $\overline{H^1}(G,\pi) \neq 0$. Let us prove the theorem in three steps:

Step 1: The restriction $\pi|_R$ has a finite dimensional subrepresentation.

Indeed, assume by contradiction that this is not the case. Then, as R is a connected solvable Lie group, $\overline{H^1}(R,\pi|_R) = 0$ by Delorme's theorem and proposition 2.7. Proposition 2.10 implies $\overline{H^1}(G,\pi) = 0$, contradicting our assumption.

Step 2: π is finite dimensional.

Consider the Lévi decomposition $G{=}RS$ of G, where R is the radical and S is semisimple (hence compact by amenability of G). Let χ be a finite dimensional subrepresentation of $\pi|_R$. Then we have that $\pi|_R \otimes \overline{\chi}$ contains the trivial representation which implies that $Ind_R^G(\pi|_R \otimes \overline{\chi}) = \pi \otimes Ind_R^G\overline{\chi}$ contains $\lambda_{G/R} = Ind_R^G 1$. But the quasi-regular representation $\lambda_{G/R}$ contains the trivial representation by compactness of G/R. So π must be finite dimensional.

Step 3: There are only finitely many finite-dimensional irreducible representations of G with $\overline{H^1}(G,\pi) \neq 0$.

Let $\tilde{G}$ be the universal cover of G. If π is a unitary representation of G and if $\tilde{\pi}$ denotes the $\tilde{G}$-representation obtained by pulling π back , then $\overline{H^1}(G,\pi) \cong \overline{H^1}(\tilde{G},\tilde{\pi})$ (see [5]). So we can assume G to be simply connected. The Lvi decomposition of G is then a semi-direct product $R \rtimes S$. Let π be a finite dimensional irreducible unitary representation of G. By Lie's theorem, $\pi|_{[R,R]} = 1$ and because $[R,R]$ is a closed normal subgroup of G (see [8] Chap. XII Thm. 2.2), theorem 2.11 applies and gives

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/[R,R],\dot{\pi}) \oplus A(G,[R,R],\pi).$$

where $\dot{\pi}$ denotes the representation of the quotient group defined by $\pi = q \circ \dot{\pi}$ where q is the canonical projection of G on its quotient. By Corollary 2.14, $A(G,[R,R],\pi)$ is non zero only for finitely many representations π, of dimension at most $dim(R)$. So we will show that $\overline{H^1}(G/[R,R],\dot{\pi})$ is non zero only for finitely many irreducible unitary representations.By connectedness of R, $G/[R,R] = (\mathbb{R}^n \times \mathbb{T}^k) \rtimes S$ for some n,k. If $\dot{\pi}$ does not have non-zero $(\mathbb{R}^n \times \mathbb{T}^k)$-invariant vectors, then by proposition 2.9 and by the vanishing of the space $\overline{H^1}(\mathbb{R}^n \times \mathbb{T}^k,\sigma)$ for all unitary representations without non-zero invariant vectors (see [7]), we have $\overline{H^1}(G/[R,R],\dot{\pi}) = 0$.

If $\dot{\pi}$ has non-zero $(\mathbb{R}^n \times \mathbb{T}^k)$-invariant vectors, we get by irreducibility that $\dot{\pi}|_{(\mathbb{R}^n \times \mathbb{T}^k)} = 1$.
So by applying theorem 2.11 $i)$:

$$\overline{H^1}(G/[R,R],\dot{\pi}) \cong \overline{H^1}(S,\dot{\pi}) \oplus A(G/[R,R],(\mathbb{R}^n \times \mathbb{T}^k),\dot{\pi}).$$

But S is compact, so $\overline{H^1}(S,\dot{\pi}) = 0$ and we apply the Corollary 2.14 to conclude that the space $A(G/[R,R],(\mathbb{R}^n \times \mathbb{T}^k),\dot{\pi})$ is non zero for only finitely many irreducible finite dimensional unitary representations, whose dimensions are less than the (real) dimension of the radical of G. ∎

Example 3.2. Let $G = \mathbb{C}^n \rtimes U(n)$ be the rigid motion group of $\mathbb{C}^n$; and let π be the unitary irreducible representation of G in $\mathbb{C}^n$ given by

$$\pi(x,g) = g.$$

Define a cocycle in $Z^1(G,\pi)$ by setting $b(x,g) = x$. The corresponding affine action is the tautological one on the affine space underlying $\mathbb{C}^n$. This cocycle is not almost a coboundary, so $\overline{H^1}(G,\pi) \neq 0$.

This example shows that in the previous theorem the upper bound on the dimension of irreducible unitary representations with non vanishing reduced 1-cohomology, is optimal.

In the sequel, we will use several times the well-known theorem by Montgomery-Zippin (see [13]): *Let G be a connected locally compact group. Then for every neighborhood V of the neutral element there exists a normal compact subgroup K_V of G contained in V, such that G/K_V is a real Lie group.*

Theorem 3.3. *Let G a connected amenable locally compact group. The unitary irreducible representations with non vanishing reduced 1-cohomology are all finite dimensional and there are only finitely many such representations up to unitary equivalence.*

Proof. By Montgomery-Zippin's theorem, there exists a normal compact subgroup K of G such that G/K is a Lie group. If π is a unitary irreducible representation of G, then:

i) Either $\pi|_K$ has no non zero invariant vectors and then Lemma 2.8 implies that $H^1(G,\pi) = 0$ which implies $\overline{H^1}(G,\pi) = 0$.

ii) Or $\pi|_K$ has non zero invariant vectors, and by irreducibility, $\pi|_K = 1$. So by Corollary 2.12, $\overline{H^1}(G,\pi) = \overline{H^1}(G/K,\dot{\pi})$, and the previous theorem applies. ∎

Recently, Y. Shalom introduced the property (H_{FD}) for a locally compact group (see [15]):

A locally compact group G has the property (H_{FD}) if for all irreducible representation π, $\overline{H^1}(G,\pi) \neq 0$ implies that π is finite dimensional. He shows in particular that this property is a quasi-isometry invariant among the class of finitely generated amenable groups.

Corollary 3.4. *A connected locally compact amenable group has the property (H_{FD}).*

4. $\overline{H^1}(G,\pi)$ and the Haagerup property

In [3], the authors classify connected Lie groups having the Haagerup property. They show that such a group is necessarily locally isomorphic to a product $M \times SO(n_1,1)\times ...\times SO(n_k,1)\times SU(m_1,1)\times ...\times SU(m_l,1)$, where M is an amenable Lie group. By using Delorme's theorem [5] on the 1-cohomology of the groups $SO(n,1)$ and $SU(m,1)$, we will classify the irreducible unitary representations of a connected group having Haagerup property that give rise to non zero first reduced cohomology space. Delorme's theorem that we will need is the following:

Theorem 4.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$. Then there exists at least one irreducible unitary representation and at most two with non trivial 1-cohomology. Moreover, these representations are infinite dimensional.*

From this and the previous theorem, we deduce:

Theorem 4.2. *Let G be a connected locally compact group with the Haagerup property. There are finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*

Proof. As in the proof of Theorem 3.1, we can assume G to be simply connected. Because G has the Haagerup property, it is isomorphic to a product ([3], thm 4.0.1)

$$M\times \widetilde{SO(n_1,1)}\times ...\times \widetilde{SO(n_k,1)}\times \widetilde{SU(m_1,1)}\times ...\times \widetilde{SU(m_l,1)}.$$

where M is amenable.

Let us show the result by induction on the number of factors in the preceding direct product. If there is only one factor, then the result follows from theorem 3.1 and 4.1. Let us assume that there are n factors in the direct product decomposition of G and let π be an irreducible unitary representation of G.

If π has no non zero invariant vectors for each factor, then $\overline{H^1}(G,\pi)=0$ (see [14]). If π has a nonzero invariant vector by at least one factor, set N to be the product of those factors where π has non zero invariant vectors. By irreducibility, $\pi|_N=1$ so by theorem 2.11

$$\overline{H^1}(G,\pi)=\overline{H^1}(G/N,\dot{\pi})\oplus Hom_G(N,\pi).$$

then we conclude, by the induction assumption and Corollary 2.14. ■

5. $\overline{H^1}(G,\pi)$ and the relative property (T)

Now we shall study 1-cohomology and the reduced 1-cohomology with values in an irreducible unitary representation of a locally compact group G having a closed normal subgroup N such that the pair (G,N) has the relative property (T).

Proposition 5.1. *Let G be a locally compact and N a closed normal subgroup such that (G,N) has relative property (T). Let π be an irreducible unitary representation of G. We have the following alternative :*

i) either $\pi|_N$ does not have non zero invariant vectors, and then $H^1(G,\pi) = \overline{H^1}(G,\pi) = 0$;

ii) or $\pi|_N = 1$ and we have the isomorphisms $H^1(G,\pi) \cong H^1(G/N,\dot{\pi})$, $\overline{H^1}(G,\pi) \cong \overline{H^1}(G/N,\dot{\pi})$.

Proof. By definition of relative property (T), the restriction map $Res : H^1(G,\pi) \to H^1(N,\pi|_N)$ is identically zero. So if $\pi|_N$ does not have non zero invariant vectors, $H^1(G,\pi) = 0$, by Lemma 2.8. If $\pi|_N$ has non zero invariant vectors then by irreducibility, $\pi|_N = 1$, and we apply theorem 2.11 to get the isomorphisms:

$$H^1(G,\pi) \cong H^1(G/N,\pi) \oplus A(G,N,\pi),$$

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/N,\pi) \oplus A(G,N,\pi).$$

But by the relative property (T), the second summand is zero. ∎

6. $\overline{H^1}(G,\pi)$ of locally compact connected groups

Recently, Y. de Cornulier showed the following nice result [4]:

Theorem 6.1. *Let G be a non compact connected Lie group. Then either G has the Haagerup property, or there exists a closed non compact characteristic subgroup N such that the pair (G,N) has the relative property (T) and G/N has the Haagerup property.*

Remark 6.2. Theorem 6.1 improves on Theorem 4.0.1 of [3], where it was shown that, for a connected Lie group G, either G has the Haagerup property or there exists a closed non-compact subgroup H such that (G,H) has the relative property (T).

Theorem 6.3. *Let G be a connected locally compact group. Then, up to unitary equivalence, there are only finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*

Moreover, any such representation π is either trivial or factors through an irreducible unitary representation σ of a group H isomorphic to $PO(n,1), PU(m,1)$, or to a non-compact amenable non-nilpotent group H such that $\overline{H^1}(H,\sigma) \cong \overline{H^1}(G,\pi) \neq 0$.

Proof. By Montgomery-Zippin's theorem and Corollary 2.12, G can be assumed to be a real Lie group.

Step 1: There are only finitely many irreducible unitary representations π *with non vanishing* $\overline{H^1}(G,\pi)$.

If G is compact then the result is clear. Assume that G is non-compact. If G has the Haagerup property, we use Theorem 4.2. If it is not the case, by Theorem 6.1, there exists a non compact closed normal subgroup N such that (G,N) has the relative property (T) and G/N has the Haagerup property. By proposition 5.1, the irreducible unitary representations π of G that have non vanishing reduced-1 cohomology satisfy $\pi|_N = 1$ and then proposition 5.1 applies to give :

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/N,\dot{\pi}).$$

If the only irreducible representation having non trivial reduced 1-cohomology is the trivial representation there is nothing to prove. Let π be a non trivial unitary representation of G with $\overline{H^1}(G,\pi) \neq 0$.

Step 2: There exists a non compact closed connected subgroup N *such that* $G_N \doteq G/N$ *has the Haagerup property,* $\pi|_N = 1$, *and* $\overline{H^1}(G_N,\pi) \cong \overline{H^1}(G,\pi) \neq 0$.

Indeed if G has the Haagerup property, we end here. If not, by theorem 6.1, there exists a closed normal subgroup N_0 of G such that (G,N_0) has relative property (T) and G/N_0 has the Haagerup property. By proposition 5.1, $\pi|_{N_0} = 1$ and $\overline{H^1}(G/N_0,\dot{\pi}) \cong \overline{H^1}(G,\pi) \neq 0$.

Step 3: Factorization through an irreducible unitary representation σ *of a group* H *isomorphic either to* $PO(n,1), PU(m,1)$ *or to a non-compact amenable non-nilpotent group* H *such that* $\overline{H^1}(H,\sigma) \cong \overline{H^1}(G,\pi) \neq 0$.

Let $\widetilde{\pi}$ be the representation defined canonically on the universal cover $\widetilde{G_N}$ of G_N. By [5], $\overline{H^1}(\widetilde{G_N},\widetilde{\pi}) \cong \overline{H^1}(G_N,\pi) \neq 0$. Applying the classification theorem of [3], $\widetilde{G_N}$ is a product of (simply connected) groups $\widetilde{SO(n,1)}$, and/or $\widetilde{SU(n,1)}$ and/or amenable groups. By proposition 3.2 of [14], π is trivial on at least one factor. But then, by proposition 2.13 (applied to $\sigma^{\mathbb{C}} = 1$), and as $\widetilde{\pi}$ is not trivial, $\widetilde{\pi}$ is trivial on all factors except one, that we will denote by $\widetilde{H}$. Moreover we have $\overline{H^1}(G,\pi) \cong \overline{H^1}(\widetilde{H},\widetilde{\pi}) \neq 0$. However, $\widetilde{\pi}$ is trivial on the center $\mathcal{Z}(\widetilde{H})$ of $\widetilde{H}$. So if we denote by H the quotient $\widetilde{H}/\mathcal{Z}(\widetilde{H})$, we have that $\overline{H^1}(H,\widetilde{\pi}) \cong \overline{H^1}(\widetilde{H},\widetilde{\pi}) \neq 0$. Notice that as $\widetilde{\pi}$ is irreducible and non trivial, H cannot be nilpotent (because if it was the case the reduced 1-cohomology should vanish [7]) .

By construction, G maps onto H, π is trivial on the kernel of this surjection, and $\pi = \widetilde{\pi}$ on H. So $\overline{H^1}(G_N,\pi) \cong \overline{H^1}(H,\pi) \neq 0$ and by construction, H is isomorphic to either $PO(n,1)$ or $PU(n,1)$ or an amenable group. ∎

Remark 6.4. There is no analogue of the Theorem 6.3 for non-connected groups. To see this, consider the free group $G = \mathbb{F}_2$ on 2 generators. A.Guichardet [7] observed that $H^1(G,\pi) \neq 0$ for every unitary representation π of G. Now, if π is finite dimensional, we even have $\overline{H^1}(G,\pi) \neq 0$. In particular, for every character χ of G, $\overline{H^1}(G,\chi) \neq 0$, so we get a continuum of irreducible representations carrying reduced 1-cohomology.

7. Applications

7.1. $\overline{H^1}(G, L^2(G))$ and amenability.

In this section, we will focus on the reduced 1-cohomology with coefficient in the regular representation for amenable connected locally compact groups. It is shown in [12], that for a discrete amenable group Γ, $\overline{H^1}(\Gamma, l^2(\Gamma)) = 0$. In this section, the analogous result will be proved for a connected amenable locally compact group.

Lemma 7.1. *Let G be a locally compact group. If for every neighborhood V of the identity , there exists a normal compact subgroup K contained in V such that $\overline{H^1}(G/K, \lambda_{G/K}) = 0$, then $\overline{H^1}(G, \lambda_G) = 0$.*

Proof. Let $b \in Z^1(G, \lambda_G)$. For any compact normal subgroup K let us define a cocycle in $Z^1(G, L^2(G)^K)$ (where $L^2(G)^K$ is the space of (right) K-invariant vectors in $L^2(G)$) by:

$$(b^K(g))(h) = \int_K b(g)(hk)\, dk$$

(dk is the normalized Haar measure on K). So we have (ρ denotes the right regular representation):

$$\begin{aligned}
\| b^K(g) - b(g) \|_2^2 &= \int_G | b^K(g)(h) - b(g)(h) |^2 \, dh \\
&= \int_G | \int_K (b(g)(hk) - b(g)(h)) dk |^2 \, dh \\
&\leq \int_G \int_K | b(g)(hk) - b(g)(h) |^2 \, dk dh \\
&= \int_K \int_G | b(g)(hk) - b(g)(h) |^2 \, dh dk \\
&= \int_K \| \rho(k) b(g) - b(g) \|_2^2 \, dk.
\end{aligned}$$

Finally as ρ is strongly continuous at the neutral element, there exists for every $\varepsilon > 0$, every compact subset Q of G, a neighborhood V of e such that $\| \rho(k)b(g) - b(g) \|_2 \leq \varepsilon$, $\forall g \in Q$, $\forall k \in V$. We easily conclude by using the cohomological assumption. ■

Theorem 7.2. *Let G be a locally compact connected group. If G is amenable, then $\overline{H^1}(G, L^2(G)) = 0$.*

Proof. Let us recall that if N is a closed subgroup of G, then $\lambda_G|_N = [G : N] \cdot \lambda_N$ and so $\overline{H^1}(N, \lambda_G|_N) = 0 \Leftrightarrow \overline{H^1}(N, \lambda_N) = 0$ (see e.g. [12]). So by proposition 2.9, we can replace G by its connected component of 1; i.e. we can assume that G is connected and non compact.

By Montgomery-Zippin's theorem, for every neighborhood V of the identity in G, there exists a compact normal subgroup K_V, such that G/K_V is a Lie group. So G/K_V is an amenable connected Lie group. Since G/K_V is non-compact a finite set of finite dimensional representations cannot appear discretely in the direct integral decomposition into irreducible representations of the regular representation of G/K_V. So by Theorem 3.3, $\overline{H^1}(G/K_V, \lambda_{G/K_V}) = 0$ and by Lemma 7.1, $\overline{H^1}(G, \lambda_G)$ must vanish. ■

This result leads to the following conjecture:

Conjecture 7.3. Let G be an amenable separable locally compact group then $\overline{H^1}(G, L^2(G)) = 0$.

7.2. Application to harmonic analysis.

Let G be a connected unimodular Lie group and let (M, ν) a smooth non compact connected manifold on which G acts transitively by diffeomorphisms and preserving a σ-finite measure ν. If μ is a probability measure on G, we say that a smooth function f on M is μ-harmonic if $f(x) = \int_G f(q^{-1} \cdot x)\, d\mu(q)$ (where $\cdot$ denote the action of G on M).

Recall that if $(X_1, \ldots, X_n)$ is a Hrmander system of smooth G-invariants vector fields (i.e. a family of smooth vector fields such that the Lie algebra they generates is the whole tangent space at each point), the gradient of a function $f \in \mathcal{C}^\infty(M)$ is defined by $\nabla f = (X_1 f, \ldots, X_n f)$ and that $|\nabla f| = (\sum_{i=1}^{n} |X_i f|^2)^{\frac{1}{2}}$.

A right invariant Hrmander system always exists on G. Consequently as G acts transitively by diffeomorphisms, we obtain a G-invariant Hrmander system on M. Fix once and for all a G-invariant Hrmander system on M.

Finally, $f \in \mathcal{C}^\infty(M)$ is said to be Dirichlet finite, if $||\nabla f||_{L^2(M,\nu)} < \infty$. We will denote by π the action of G on $\mathcal{C}^\infty(M)$ defined by $\pi(g)f(x) = f(g^{-1} \cdot x)$.

With these definitions and notations, we will establish in this section a link between the existence of Dirichlet-finite harmonic functions on M and the reduced 1-cohomology of G with values in $L^2(M)$. Some preliminary technical lemmas (adapted from [16]) are needed:

Lemma 7.4. *Let M be a manifold, $(X_1, \ldots, X_n)$ a Hrmander system and $\gamma : [0, a] \to M$ a differentiable path on M tangent to the Hrmander system (i.e $\gamma'(t)$ is in the vector subspace generated by $(X_1, \ldots, X_n)$) with $\|\gamma'(t)\|_2 \leq 1$. For $f \in \mathcal{C}^\infty(M)$, we have the following inequality:*

$$|f(\gamma(a)) - f(\gamma(0))| \leq \int_0^a |\nabla f(\gamma(t))| dt.$$

Proof. For $t \in [0, a]$ we have:

$$\begin{aligned} |f(\gamma(a)) - f(\gamma(0))| &= |\int_0^a \frac{d}{dt} f(\gamma(t))\, dt| \\ &\leq \int_0^a |df_{\gamma(t)}(\gamma'(t))|\, dt. \end{aligned}$$

Moreover if we write $\gamma'(t) = \sum_{i=0}^{k} a_i(t)X_i(\gamma(t))$, we have as $df_{\gamma(t)}(X_i(\gamma(t))) = X_i f(\gamma(t))$, using the Cauchy-Schwartz inequality:

$$\begin{aligned} |df_{\gamma(t)}(\gamma'(t))| &= |\sum_{i=0}^{k} a_i(t)X_i f(\gamma(t))| \\ &\leq \|\gamma'(t)\||\nabla f(\gamma(t))| \\ &\leq |\nabla f(\gamma(t))| \end{aligned}$$

Hence the claimed inequality. ∎

Lemma 7.5. *Let f be a smooth Dirichlet finite function on M. Then for all $h \in G$, there exists $a = a(h) > 0$ such that*

$$\|\pi(h)f - f\|_{L^2(M,\nu)} \leq a \cdot ||\nabla f||_{L^2(M,\nu)}.$$

Proof. Let $h \in G$ and let $\gamma : [0, a] \to G$ be an absolutely continuous path such that $\gamma(0) = e$, $\gamma(a) = h$, and $\gamma'(t) = \sum_{i=1}^{n} a_i(t)X_i(\gamma(t))$ a.e. with $\sum_{i=1}^{n} a_i^2(t) \leq 1$ (such a path always exists, see [16] III.4). As the action is smooth and the Hrmander system is invariant, we apply the preceding lemma to the path $t \mapsto \gamma(t)^{-1} \cdot x$ and we get:

$$|f(h^{-1} \cdot x) - f(x)| \leq \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|dt.$$

So by Cauchy-Schwarz, $|f(h^{-1} \cdot x) - f(x)|^2 \leq a\int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|^2 dt$.

Therefore,

$$\begin{aligned} \int_M |f(h^{-1} \cdot x) - f(x)|^2 dx &\leq a\int_M \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|^2 dt d\nu(x) \\ &= a\int_0^a \int_M |\nabla f(\gamma(t)^{-1} \cdot x)|^2 d\nu(x)dt \\ &= a\int_0^a \int_M |\nabla f(x)|^2 d\nu(x)dt \\ &= a^2\|\nabla f\|_2^2. \end{aligned}$$

which proves the lemma. ∎

Here is the main theorem of this section:

Theorem 7.6. *Let G be a connected unimodular Lie group acting smoothly and transitively on a non-compact connected smooth manifold M endowed with a G-invariant (σ-finite) measure ν and let μ be a probability measure on G with compact symmetric support generating G. If $\overline{H^1}(G, L^2(M,\nu)) = 0$, then every Dirichlet-finite μ-harmonic smooth function on M is constant.*

Proof. Set $L^2(M) = L^2(M,\nu)$ and let $\mathcal{D}(M)$ be the following quotient space: $\{f \in \mathcal{C}^\infty(M) \mid ||\pi(g)f - f||_2 < \infty \, \forall g \in G\}/\mathbb{C}$. Consider the pre-Hilbert structure on $\mathcal{D}(M)$ given by $||f||^2_{\mathcal{D}(M)} = \int_G ||\pi(q)f - f||^2_{L^2(M)} d\mu(q)$. Notice that $\mathcal{D}(M)$ is Hausdorff because $||f||^2_{\mathcal{D}(M)} = 0$ iff $\pi(g)f = f$, $\forall g \in supp(\mu)$, which is equivalent to $\pi(g)f = f$, $\forall g \in G$ (because $supp(\mu)$ generates G) and which is also equivalent to the fact that f is constant (this follows from the transitivity of the action). Let i be the canonical embedding of $\mathcal{C}^\infty(M) \cap L^2(M)$ in $\mathcal{D}(M)$. For all $f \in \mathcal{D}(M)$, denote by $\theta(f)$ the algebraic cocycle given by $g \mapsto \pi(g)f - f$. This cocycle is weakly measurable, so by [7], it is continuous for the topology of uniform convergence on compact subsets (we use here the fact that G is separable).

By assumption $\theta(f)$ is almost a coboundary. As $\mathcal{C}^\infty(M) \cap L^2(M)$ is $|| \, ||_2-$dense in $L^2(M)$, there exists a sequence $(\xi_n)_{n\geq 1}$ in $\mathcal{C}^\infty(M) \cap L^2(M)$ such that $\theta(f)(g) = \lim_{n\to\infty} \pi(g)\xi_n - \xi_n$ uniformly on compact subsets of G. Hence $\int_G ||\pi(q)(f - \xi_n) - (f - \xi_n)||^2_{L^2(M)} d\mu(q) \overset{n\mapsto\infty}{\longrightarrow} 0$ since μ has compact support.

This shows that $i(\xi_n) \overset{n\mapsto\infty}{\longrightarrow} f$ in $\mathcal{D}(M)$. In other words, $i(\mathcal{C}^\infty(M) \cap L^2(M))$ is dense in $\mathcal{D}(M)$. So $i(\mathcal{C}^\infty(M) \cap L^2(M))^\perp = 0$, because $\mathcal{D}(M)$ is Hausdorff.

Let us compute this orthogonal complement :

$$
\begin{array}{cc}
 & f \in i(\mathcal{C}^\infty(M) \cap L^2(M))^\perp \\
\Leftrightarrow & \int_G \langle \rho(q)f - f \mid \rho(q)\xi - \xi \rangle_2 d\mu(q) = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow & \begin{array}{r} \int_G \langle \rho(q)f - f \mid \rho(q)\xi \rangle_2 d\mu(q) - \int_G \langle \rho(q)f - f \mid \xi \rangle_2 d\mu(q) = 0 \ , \\ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \end{array} \\
\Leftrightarrow & \begin{array}{r} \int_G \langle f - \rho(q^{-1})f \mid \xi \rangle_2 d\mu(q) - \int_G \langle \rho(q)f - f \mid \xi \rangle_2 d\mu(q) = 0 \ , \\ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \end{array} \\
\Leftrightarrow & -2\int_G \langle \rho(q)f - f \mid \xi \rangle_2 d\mu(q) = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) (\text{as } \mu \text{ is symmetric}) \\
\Leftrightarrow & \langle \int_G (\rho(q)f - f) d\mu(q) \mid \xi \rangle_2 = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow & \int_G (\rho(q)f - f) d\mu(q) = 0 \\
\Leftrightarrow & \int_G \rho(q) f d\mu(q) = f \\
\Leftrightarrow & \int_G f(q^{-1} \cdot x) d\mu(q) = f(x) \ , \ \forall x \in M
\end{array}
$$

So the orthogonal complement of $i(\mathcal{C}^\infty(M) \cap L^2(M))$ is nothing else than the space of μ-harmonic functions in $\mathcal{D}(M)$.

Now, let f be a smooth Dirichlet finite function. By the preceding lemma,

$$||\pi(g)f - f||_{L^2(M)} \leq a(g)||\nabla f||_{L^2(M)}, \ \forall g \in G.$$

So such a f is (modulo constant functions) in $\mathcal{D}(M)$. So if f is μ-harmonic and Dirichlet finite, then it is constant. ■

We get immediately the following corollary

Corollary 7.7. *Let G be a connected Lie group having property (T). If G acts smoothly and transitively on a non-compact connected smooth manifold M endowed with a G-invariant (σ-finite) measure ν and if μ is a probability measure on G with compact symmetric support generating G, then a Dirichlet-finite μ-harmonic smooth function on M is constant.*

In the case where G acts by translation on itself, we obtain immediately:

Corollary 7.8. *Let G be a connected unimodular Lie group such that $\overline{H^1}(G, L^2(G)) = 0$ and let μ be a probability measure on G with compact symmetric support generating G. Then a Dirichlet-finite μ-harmonic smooth function on G is constant.*

By Theorem 3.6, we also have

Corollary 7.9. *Let G be a amenable connected unimodular Lie group and let μ be a probability measure on G with compact symmetric support generating G. Then a finite Dirichlet μ-harmonic smooth function on G is constant.*

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