Alpha-determinant Cyclic Modules of $\mathfrak{gl}_n(\mathbb{C})$

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Abstract. The alpha-determinant unifies and interpolates the notion of the determinant and permanent. We determine the irreducible decomposition of the cyclic module of $\mathfrak{gl}_n(\mathbb{C})$ defined by the alpha-determinant. The degeneracy of the irreducible decomposition is determined by the content polynomial of a given partition.

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1. Introduction

Let $X = (x_{ij})_{1 \leq i,j \leq n}$ be a matrix with commutative variables $x_{ij}$. For a complex number $\alpha$, the $\alpha$-determinant of $X$ is defined by

$$\det_\alpha(X) = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu_n(\sigma)} \prod_{i=1}^{n} x_{i\sigma(i)},$$

where $\mathfrak{S}_n$ is the symmetric group of degree $n$ and $\nu_n(\sigma)$ stands for the number of cycles in the cycle decomposition of a permutation $\sigma \in \mathfrak{S}_n$. The $\alpha$-determinant is nothing but the permanent if $\alpha = 1$ and the (usual) determinant if $\alpha = -1$, and hence, it interpolates these two. It appears as a coefficient in the Taylor expansion of the power $\det(I - \alpha X)^{-1/\alpha}$ of the characteristic polynomial of $X$ and defines a generalization of the boson, poisson and fermion point processes, see [9, 10]. Also, its Pfaffian analogue has been developed in [8].

It is a natural question whether the $\alpha$-determinant can be interpreted as an invariant like the usual determinant (and also the $q$-determinant in quantum group theory). Denote by $\mathcal{P}(\text{Mat}_{n \times n})$ the ring of polynomials in variables $\{x_{ij}\}_{1 \leq i,j \leq n}$. Let $\{E_{ij}\}_{1 \leq i,j \leq n}$ be the natural basis of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. When $n = 2$...

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and \( \alpha \neq 0 \), consider the linear map \( \rho_2^{(\alpha)} \) of \( \mathfrak{gl}_2(\mathbb{C}) \) on \( \mathcal{P}(\text{Mat}_{n \times n}) \) determined by
\[
\rho_2^{(\alpha)}(E_{11}) = x_{11}\partial_{11} + x_{12}\partial_{12}, \quad \rho_2^{(\alpha)}(E_{12}) = \frac{1}{\sqrt{-\alpha}}(x_{11}\partial_{21} - \alpha x_{12}\partial_{22}),
\]
\[
\rho_2^{(\alpha)}(E_{21}) = \frac{1}{\sqrt{-\alpha}}(-\alpha x_{21}\partial_{11} + x_{22}\partial_{12}), \quad \rho_2^{(\alpha)}(E_{22}) = x_{21}\partial_{21} + x_{22}\partial_{22},
\]
where \( \partial_{ij} = \frac{\partial}{\partial x_{ij}} \). Then \( \rho_2^{(\alpha)} \) defines a representation of \( \mathfrak{gl}_2(\mathbb{C}) \) and
\[
\rho_2^{(\alpha)}(E_{ii})\det_\alpha(X) = \det_\alpha(X),
\]
\[
\rho_2^{(\alpha)}(E_{ij})\det_\alpha(X) = 0 \quad \text{for} \quad i \neq j. \quad \text{This is not, however, true for} \quad n \geq 3. \quad \text{Precisely,}
\]
\[
\text{although the map} \quad \rho_n^{(\alpha)} \text{given by} \quad \rho_n^{(\alpha)}(E_{ij}) = \sum_{k=1}^{n}[\beta^{[i]-k}-\beta^{[j]-k}]x_{ik}\partial_{jk}, \quad \text{where} \quad \beta = \sqrt{-\alpha},
\]
defines a representation of \( \mathfrak{gl}_n(\mathbb{C}) \) on \( \mathcal{P}(\text{Mat}_{n \times n}) \), \( \rho_n^{(\alpha)}(E_{ij})\det_\alpha(X) \neq 0 \), (i \neq j) in general. (One can actually show that \( \rho_n^{(\alpha)} \) is equivalent to the usual action of \( \mathfrak{gl}_n(\mathbb{C}) \) determined by \( \rho(E_{ij}) = \rho_n^{-1}(E_{ij}) = \sum_{k=1}^{n}x_{ik}\partial_{jk} \) on \( \mathcal{P}(\text{Mat}_{n \times n}) \) when \( \alpha \neq 0 \). Indeed, the map \( f(x_{ij}) \rightarrow f(\beta^{[i]-j}x_{ij}) \) is the intertwining operator from \( (\rho, \mathcal{P}(\text{Mat}_{n \times n})) \) to \( (\rho_n^{(\alpha)}, \mathcal{P}(\text{Mat}_{n \times n})). \)

Then, a question which subsequently arises is what the structure of the smallest invariant subspace of \( \mathcal{P}(\text{Mat}_{n \times n}) \) which contains \( \det_\alpha(X) \) is. Thus, the aim of the present paper is to investigate a cyclic module \( V_n^{(\alpha)} = U(\mathfrak{g})\det_\alpha(X) \), under the representation \( \rho \) on \( \mathcal{P}(\text{Mat}_{n \times n}) \) for each \( \alpha \in \mathbb{C} \). Clearly, \( V_n^{(-1)} \) is the one-dimensional determinant representation.

We adopt the notations for partitions used in [7] and for representations of \( \mathfrak{gl}_n(\mathbb{C}) \) used in [1] and [11]. A partition \( \lambda \) is a weakly decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of non-negative integers such that \( \lambda_j = 0 \) for sufficiently large \( j \). We usually identify a partition \( \lambda \) with the corresponding Young diagram. Write \( \lambda \vdash n \) if \( \sum_{j \geq 1} \lambda_j = n \) and denote by \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) the conjugate partition of \( \lambda \). Let \( E^\lambda \) denote the Schur module (or called the Weyl module) corresponding to \( \lambda \) and \( f^\lambda \) the number of standard tableaux of shape \( \lambda \).

The following is our main result, which describes the irreducible decomposition of \( V_n^{(\alpha)} \).

**Theorem 1.1.** For \( k = 1, 2, \ldots, n-1 \),
\[
V_n^{(\frac{k}{2})} \cong \bigoplus_{\lambda \vdash n, \lambda'_1 \leq k} (E^\lambda)^{\otimes f^\lambda} \quad \text{and} \quad V_n^{(-\frac{k}{2})} \cong \bigoplus_{\lambda \vdash n, \lambda'_1 \leq k} (E^\lambda)^{\otimes f^\lambda}. \quad (1)
\]

For \( \alpha \in \mathbb{C} \setminus \{ \pm 1, \pm \frac{1}{2}, \ldots, \pm \frac{1}{n-1} \} \),
\[
V_n^{(\alpha)} \cong (\mathbb{C}^n)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} (E^\lambda)^{\otimes f^\lambda}. \quad (2)
\]

**Example 1.2.** When \( n = 3 \) the irreducible decomposition of \( V_3^{(\alpha)} \) is given by
\[
V_3^{(\alpha)} \cong \begin{cases} 
E^{(3)} & \text{if} \; \alpha = 1, \\
E^{(3)} \oplus E^{(2,1)} \oplus E^{(2,1)} & \text{if} \; \alpha = \frac{1}{2}, \\
E^{(1,1,1)} & \text{if} \; \alpha = -1, \\
E^{(2,1)} \oplus E^{(2,1)} \oplus E^{(1,1,1)} & \text{if} \; \alpha = -\frac{1}{2}, \\
E^{(3)} \oplus E^{(2,1)} \oplus E^{(2,1)} \oplus E^{(1,1,1)} & \text{otherwise}.
\end{cases}
\]
Note that each Schur module possesses a canonical basis formed by $\alpha$-determinants (see Theorem 3.9). Although the $\alpha$-determinant is not an invariant, this fact implies that it has a rich symmetry.

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2. The $U(\mathfrak{gl}_n)$-module $V_n^{(\alpha)}$

Let $V_n^{(\alpha)} = \rho(U(\mathfrak{gl}_n))\det_\alpha(X)$. Put $[n] = \{1, 2, \ldots, n\}$ and

$$D^{(\alpha)}(i_1, i_2, \ldots, i_n) = \det_\alpha \begin{pmatrix} x_{i_11} & x_{i_12} & \ldots & x_{i_1n} \\ x_{i_21} & x_{i_22} & \ldots & x_{i_2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_n1} & x_{i_n2} & \ldots & x_{i nn} \end{pmatrix}$$

for any $i_1, \ldots, i_n \in [n]$. In particular, $\det_\alpha(X) = D^{(\alpha)}(1, 2, \ldots, n)$. We abbreviate $\rho(E_{ij})$ to $E_{ij}$ for simplicity. When there is no fear of confusion, we abbreviate $\nu_\alpha$ to $\nu$.

Lemma 2.1.

$$E_{pq} \cdot D^{(\alpha)}(i_1, \ldots, i_n) = \sum_{k=1}^n \delta_{ik,q} D^{(\alpha)}(i_1, \ldots, i_{k-1}, p, i_{k+1}, \ldots, i_n). \quad (3)$$

Proof. It is straightforward. In fact,

$$E_{pq} \cdot D^{(\alpha)}(i_1, \ldots, i_n) = \sum_{j=1}^n x_{pj} \frac{\partial}{\partial x_{qj}} \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} x_{i_1\sigma(1)} \cdots x_{i_n\sigma(n)}$$

$$= \sum_{j=1}^n \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} x_{pj} \delta_{ik,q} \delta_{\sigma(k),j} x_{i_1\sigma(1)} \cdots \widehat{x_{ik\sigma(k)}} \cdots x_{i_n\sigma(n)}$$

$$= \sum_{k=1}^n \delta_{ik,q} \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} x_{p\sigma(k)} x_{i_1\sigma(1)} \cdots \widehat{x_{ik\sigma(k)}} \cdots x_{i_n\sigma(n)}$$

$$= \sum_{k=1}^n \delta_{ik,q} D^{(\alpha)}(i_1, \ldots, i_{k-1}, p, i_{k+1}, \ldots, i_n),$$

where $\widehat{x_{kl}}$ stands for the omission of $x_{kl}$.

Example 2.2. We see that $E_{21} \cdot D^{(\alpha)}(4, 1, 2, 1) = D^{(\alpha)}(4, 2, 2, 1) + D^{(\alpha)}(4, 1, 2, 2)$, $E_{11} \cdot D^{(\alpha)}(4, 1, 2, 1) = 2D^{(\alpha)}(4, 1, 2, 1)$, and $E_{43} \cdot D^{(\alpha)}(4, 1, 2, 1) = 0$.

The symmetric group $\mathfrak{S}_n$ acts also on $V_n^{(\alpha)}$ from the right by $D^{(\alpha)}(i_1, \ldots, i_n) \cdot \sigma = D^{(\alpha)}(i_{\sigma(1)}, \ldots, i_{\sigma(n)})$. 


Lemma 2.3. The space $V_n^{(a)}$ is the complex vector space spanned by 
\[ \{ D^{(a)}(i_1, \ldots, i_n) \mid i_1, \ldots, i_n \in [n] \} \].

Proof. Since the vector space spanned by all $D^{(a)}(i_1, \ldots, i_n)$ contains $V_n^{(a)}$ by Lemma 2.1, we prove that all $D^{(a)}(i_1, \ldots, i_n)$ are contained in $V_n^{(a)}$. For $1 \leq p < q \leq n$, we have

\[
V_n^{(a)} \ni (E_{pq}E_{qp} - 1) \cdot D^{(a)}(1, 2, \ldots, n) = D^{(a)}(\tau(1), \ldots, \tau(n)),
\]

where $\tau$ is the transposition $(p, q)$ of $p$ and $q$. It follows that, for each $\sigma \in \mathfrak{S}_n$, $D^{(a)}(\sigma(1), \ldots, \sigma(n)) = Y \cdot D^{(a)}(1, \ldots, n)$ for some $Y = Y_\sigma \in U(\mathfrak{g})$. For any $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq n$, suppose there exists $X = X_{i_1 \ldots i_n} \in U(\mathfrak{g})$ such that $D^{(a)}(i_1, \ldots, i_n) = X \cdot D^{(a)}(1, \ldots, n)$. For any $j_1, \ldots, j_n \in [n]$, we have $D^{(a)}(j_1, \ldots, j_n) = D^{(a)}(i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ for some $\sigma \in \mathfrak{S}_n$ and $i_1 \leq \cdots \leq i_n$. Hence, since the action of $\mathfrak{gl}_n(\mathbb{C})$ and of $\mathfrak{S}_n$ commute, we see that

\[
D^{(a)}(j_1, \ldots, j_n) = D^{(a)}(i_1, \ldots, i_n) \cdot \sigma = (X \cdot D^{(a)}(1, \ldots, n)) \cdot \sigma = X \cdot (D^{(a)}(1, \ldots, n) \cdot \sigma) = XY \cdot D^{(a)}(1, \ldots, n)
\]

and $D^{(a)}(j_1, \ldots, j_n)$ is contained in $V_n^{(a)}$. Therefore it is sufficient to prove $\forall n \in V_n^{(a)}$ for $i_1 \leq \cdots \leq i_n$.

For any sequence $(i_1, \ldots, i_n)$ such that $i_k \leq k$ for any $k$, we have

\[
D^{(a)}(i_1, i_2, \ldots, i_n) = E_{i_1n} \cdots E_{i_21} \cdot D^{(a)}(1, 2, \ldots, n) \in V_n^{(a)}.
\]

In fact, by Lemma 2.1, we see that $E_{ikk} \cdot D^{(a)}(i_1, \ldots, i_{k-1}, k, k+1, \ldots, n) = D^{(a)}(i_1, \ldots, i_{k-1}, i_k, k+1, \ldots, n)$ because $i_1 \leq \cdots \leq i_{k-1} < k$ for any $1 \leq k \leq n$. Suppose there exists $k$ such that $i_j \leq j$ for any $j < k$ and $i_k > k$. We prove $D^{(a)}(i_1, \ldots, i_n) \in V_n^{(a)}$ for such sequences $(i_1, \ldots, i_n)$ by induction with respect to the lexicographic order. Since $i_1 \leq \cdots \leq i_{k-1} < k \leq i_k - 1 < i_{k+1} \leq \cdots \leq i_n$ and $D^{(a)}(i_1, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_n) \in V_n^{(a)}$ by the induction assumption, we have $D^{(a)}(i_1, \ldots, i_k, \ldots, i_n) = E_{i_ki_{k-1}} \cdot D^{(a)}(i_1, \ldots, i_k - 1, \ldots, i_n) \in V_n^{(a)}$ by Lemma 2.1. Hence we obtain our claim.

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ acts on the $n$-tensor product $(\mathbb{C}^n)^\otimes n = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ from the left by

\[
E_{pq}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \sum_{k=1}^{n} e_{i_1} \otimes \cdots \otimes E_{pq} e_{i_k} \otimes \cdots \otimes e_{i_n} = \sum_{k=1}^{n} \delta_{ik,q} e_{i_1} \otimes \cdots \otimes e_{p} \otimes \cdots \otimes e_{i_n},
\]

where $\{e_k\}_{k=1}^{n}$ is the natural basis of $\mathbb{C}^n$. From this fact together with Lemma 2.1 and Lemma 2.3, we have the

Proposition 2.4. Let $\Phi_n^{(a)}$ be the linear map from $(\mathbb{C}^n)^\otimes n$ to $V_n^{(a)}$ defined by

\[
\Phi_n^{(a)}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = D^{(a)}(i_1, \ldots, i_n)
\]

for each $i_1, \ldots, i_n \in [n]$. Then $\Phi_n^{(a)}$ is a $U(\mathfrak{g})$-module homomorphism. In particular, $V_n^{(a)}$ is isomorphic to a quotient module $(\mathbb{C}^n)^\otimes n / \text{Ker} \Phi_n^{(a)}$ of $(\mathbb{C}^n)^\otimes n$. 

...
Notice that, when $\alpha = 0$, the homomorphism $\Phi_n(0)(e_{i_1} \otimes \cdots \otimes e_{i_n}) = D(0)(i_1, \ldots, i_n) = x_{i_1} \cdots x_{i_n}$ is clearly bijective, and therefore $V_n(0) \cong (\mathbb{C}^n)^{\otimes n}$. Hence we have the irreducible decomposition of $V_n(0)$ as

$$V_n(0) \cong \bigoplus_{\lambda \vdash n} (E^\lambda)^{\oplus f_\lambda}. \quad (4)$$

The symmetric group $\mathfrak{S}_n$ acts on $(\mathbb{C}^n)^{\otimes n}$ from the right by $(e_{i_1} \otimes \cdots \otimes e_{i_n}) \cdot \sigma = e_{i_\sigma(1)} \otimes \cdots \otimes e_{i_\sigma(n)}$ for any $\sigma \in \mathfrak{S}_n$.

3. A formula for the number of cycles

A numbering of shape $\lambda \vdash n$ is a way of putting distinct elements in $[n]$ in each box of the Young diagram $\lambda$. Let $R(T)$ be the row group (or called the Young subgroup) of a numbering represented by a tableau $T$, i.e., permutations in $R(T)$ permutate the entries of each row among themselves. The column group $C(T)$ is also defined similarly.

Recall the Frobenius notation $(a_1, a_2, \ldots, a_d|b_1, b_2, \ldots, b_d)$ of a partition $\lambda$, where $a_i = \lambda_i - i \geq 0$ and $b_i = \lambda'_i - i \geq 0$ for $1 \leq i \leq d$. Then the content polynomial $f_\lambda(\alpha)$ ([7, I-1]) for the partition $\lambda$ is written as

$$f_\lambda(\alpha) = \prod_{i=1}^{d} \left\{ \prod_{j=1}^{a_i} (1 + j\alpha) \cdot \prod_{j=1}^{b_i} (1 - j\alpha) \right\}. \quad (5)$$

Note that $f_\lambda(\alpha)$ satisfies $f_\lambda(\alpha) = f_{\lambda'}(-\alpha)$. We have the following formula for the number $\nu = \nu_n$ of cycles.

**Proposition 3.1.** Let $T$ be a numbering of shape $\lambda \vdash n$. Then

$$\sum_{q \in C(T)} \text{sgn}(q) \sum_{p \in R(T)} \alpha^{n-\nu(pq\sigma)} \quad (6)$$

$$= \begin{cases} \text{sgn}(q_0)f_\lambda(\alpha) & \text{if } \sigma = q_0p_0 \text{ for some } q_0 \in C(T) \text{ and } p_0 \in R(T), \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The formula

$$\alpha^{n-\nu_n(\sigma)} = \sum_{\mu \vdash n} \frac{f_\mu}{n!} f_\mu(\alpha) \chi_\mu(\sigma) \quad (7)$$

is a specialization of the Frobenius character formula, see [7, I-7, Example 17]. Here $\chi^\mu$ is the irreducible character of $\mathfrak{S}_n$ corresponding to $\mu$. Moreover, for a numbering $T$ of shape $\lambda$ and a partition $\mu$, the well-known equation

$$\chi^\mu \cdot c_T = \delta_{\lambda,\mu} \frac{n!}{f_\mu} c_T \quad (8)$$

in the group algebra $\mathbb{C}\mathfrak{S}_n$ holds, which is obtained by Young. Here $c_T$ is the Young symmetrizer

$$c_T = \sum_{q \in C(T)} \text{sgn}(q) \sum_{p \in R(T)} qp.$$
Define $\phi_{\alpha} = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \sigma \in \mathbb{C}[S_n]$. It follows from (7) that

$$\phi_{\alpha} = \sum_{\mu \vdash n} \frac{f_{\mu}}{n!} f_{\mu}(\alpha) \chi_{\mu}.$$ Hence by (8) we have

$$\phi_{\alpha} \cdot c_T = \sum_{\mu \vdash n} \frac{f_{\mu}}{n!} f_{\mu}(\alpha) \delta_{\lambda,\mu} \frac{n!}{f_{\mu}} c_T = f_{\lambda}(\alpha) c_T.$$ 

In other words,

$$\sum_{\sigma \in S_n} \sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} \alpha^{n-\nu(pq\sigma)} \sigma = f_{\lambda}(\alpha) \sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} qp.$$ 

This gives our desired formula. \hfill \blacksquare

**Example 3.2.** When $T = \begin{array}{c|c|c} 1 & 2 & n \end{array}$ we have

$$\sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} \alpha^{3-\nu(pq\sigma)} = \begin{cases} (1 + \alpha)(1 - \alpha) & \text{for } \sigma = (1) \text{ or } (12), \\ -(1 + \alpha)(1 - \alpha) & \text{for } \sigma = (13) \text{ or } (123), \\ 0 & \text{for } \sigma = (23) \text{ or } (132). \end{cases}$$

**Example 3.3.** For $T = \begin{array}{c|c|c} 1 & 2 & \cdots & n \end{array}$ Proposition 3.1 says

$$\sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} = \prod_{j=1}^{n-1} (1 + j \alpha). \quad (9)$$

For a sequence $(i_1, \ldots, i_n) \in [n]^n$ and a numbering $T$, we define

$$v_T^{(\alpha)}(i_1, \ldots, i_n) = D^{(\alpha)}(i_1, \ldots, i_n) \cdot c_T$$

$$= \sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} D^{(\alpha)}(i_{qp(1)}, \ldots, i_{qp(n)}), \quad (10)$$

where $c_T = \sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} qp$ is the Young symmetrizer.

**Corollary 3.4.** Let $\lambda \vdash n$. For each sequence $(i_1, \ldots, i_n) \in [n]^n$ and numbering $T$ of shape $\lambda$, we have

$$v_T^{(\alpha)}(i_1, \ldots, i_n) = f_{\lambda}(\alpha) v_T^{(0)}(i_1, \ldots, i_n). \quad (11)$$

**Proof.** For a numbering $T$ of shape $\lambda$, denote by $W_T^{(\alpha)}$ the space spanned by $\left\{ v_T^{(\alpha)}(i_1, \ldots, i_n) \mid i_1, \ldots, i_n \in [n] \right\}$. Since the Schur module $E^\lambda$ is isomorphic to the image of the map $(\mathbb{C}^n)^{\otimes n} \to (\mathbb{C}^n)^{\otimes n}$ given by $c_T$ for any numbering $T$, it follows from Proposition 2.4 and (10) that $W_T^{(\alpha)}$ is isomorphic to $\{0\}$ or $E^\lambda$. If (11) is proved for a certain sequence $(j_1, \ldots, j_n)$ satisfying $v_T^{(0)}(j_1, \ldots, j_n) \neq 0$, (11) holds for any $(i_1, \ldots, i_n) \in [n]^n$ because $W_T^{(\alpha)}$ is the cyclic module $U(\mathfrak{g}) v_T^{(\alpha)}(j_1, \ldots, j_n)$ and the action of $\mathfrak{g}$ is independent of $\alpha$. 


We prove (11) for the case where \((i_1, i_2, \ldots, i_n) = (1, 2, \ldots, n)\). Then we have
\[
v_T^{(\alpha)}(1, \ldots, n) = \sum_{q \in C(T)} \text{sgn}(q) \sum_{p \in R(T)} \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} x_{q p \sigma(1)} x_{q p \sigma(2)} \cdots x_{q p \sigma(n)},
\]
By Proposition 3.1 we see that
\[
v_T^{(\alpha)}(1, \ldots, n) = f_\lambda(\alpha) \sum_{q_0 \in C(T)} \text{sgn}(q_0) \sum_{p_0 \in R(T)} x_{q_0 p_0(1)} x_{q_0 p_0(2)} \cdots x_{q_0 p_0(n)} = f_\lambda(\alpha)v_T^{(0)}(1, \ldots, n).
\]
If \(q_0 \neq q_0'\) or \(p_0 \neq p_0'\) then \(q_0 p_0 \neq q_0' p_0'\). Indeed, if \(q_0 p_0 = q_0' p_0'\) then \(C(T) \supseteq (q_0')^{-1} q_0 = p_0 (p_0')^{-1} \in R(T)\). But, since \(C(T) \cap R(T) = \{(1)\}\), we have \(q_0 = q_0'\) and \(p_0 = p_0'\). Hence
\[
v_T^{(0)}(1, \ldots, n) = \sum_{q_0 \in C(T)} \text{sgn}(q_0) \sum_{p_0 \in R(T)} x_{q_0 p_0(1)} x_{q_0 p_0(2)} \cdots x_{q_0 p_0(n)} \neq 0
\]
and so we have proved the corollary.

**Example 3.5.** For \(T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \), we have
\[
v_T^{(\alpha)}(1, 2, 1) = D^{(\alpha)}(1, 2, 1) \cdot (13 + (13) - (12) - (132)) = 2D^{(\alpha)}(1, 2, 1) - D^{(\alpha)}(2, 1, 1) - D^{(\alpha)}(1, 1, 2) = (1 + \alpha)(1 - \alpha)(2x_{11}x_{22}x_{13} - x_{21}x_{12}x_{13} - x_{11}x_{12}x_{23}).
\]

**Example 3.6.** For \(T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \), we have
\[
v_T^{(\alpha)}(1, 2, 4) = D^{(\alpha)}(1, 2, 4) + D^{(\alpha)}(1, 2, 4, 2) - 2D^{(\alpha)}(1, 4, 2, 2) + D^{(\alpha)}(2, 1, 2, 4) + D^{(\alpha)}(2, 1, 4, 2) - 2D^{(\alpha)}(2, 1, 2, 4) + D^{(\alpha)}(2, 2, 1, 4) + D^{(\alpha)}(2, 2, 4, 1) + D^{(\alpha)}(4, 1, 2, 2) + D^{(\alpha)}(4, 1, 2, 4) + D^{(\alpha)}(4, 2, 1, 2) + D^{(\alpha)}(4, 2, 1, 4) + D^{(\alpha)}(4, 2, 2, 1) = (1 + \alpha)(1 - \alpha)(x_{11}x_{22}x_{23}x_{44} + x_{11}x_{22}x_{43}x_{24} - 2x_{11}x_{42}x_{23}x_{24} - 2x_{11}x_{23}x_{24} + x_{21}x_{12}x_{13}x_{24} + x_{21}x_{12}x_{24} - 2x_{21}x_{22}x_{13}x_{24} - 2x_{21}x_{22}x_{43}x_{14} + x_{21}x_{42}x_{13}x_{24} + x_{21}x_{42}x_{23}x_{14} + x_{41}x_{12}x_{23}x_{24} + x_{41}x_{12}x_{23}x_{24} + x_{41}x_{12}x_{23}x_{14}).
\]
For a semi-standard tableau \(S\) and a standard tableau \(T\) of the same shape, we define the sequence \(i^{(S,T)} = (i_1^{(S,T)}, \ldots, i_n^{(S,T)})\) as follows. For each \(k\), we let \(B_k\) be the box numbered by \(k\) in \(T\) and denote by \(i_k^{(S,T)}\) the number in box \(B_k\) of \(S\). For example, for
\[
S = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 4 & 6 & \hline \end{array} \quad \text{and} \quad T = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 9 \\ \hline 7 & 8 & \hline \end{array} ,
\]
we have \(i^{(S,T)} = (1, 3, 2, 3, 2, 3, 4, 6, 4)\). Put \(v_T^{(\alpha)} = v_T^{(\alpha)}(i^{(S,T)})\).
Example 3.7. 

\[ v_{S,T}^{(a)} = 2D^{(a)}(1, 2, 1) - D^{(a)}(2, 1, 1) - D^{(a)}(1, 1, 2) \text{ for } (S, T) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right), \]

\[ v_{S,T}^{(a)} = D^{(a)}(1, 3, 2) - D^{(a)}(3, 1, 2) + D^{(a)}(2, 3, 1) - D^{(a)}(2, 1, 3) \text{ for } (S, T) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right), \]

\[ v_{S,T}^{(a)} = D^{(a)}(1, 2, 3) - D^{(a)}(2, 1, 3) + D^{(a)}(3, 2, 1) - D^{(a)}(3, 1, 2) \text{ for } (S, T) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right). \]

Example 3.8. For

\[ S = T = \left( \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \right) \]

we have \( v_{S,T}^{(a)} = \sum_{q \in [n]} \text{sgn}(q)D^{(a)}(q(1), \ldots, q(n)) = \prod_{j=1}^{n-1} (1 - j\alpha) \det(X). \)

Finally, we obtain the following theorem.

Theorem 3.9. Denote by \( W_{T}^{(a)} \) the image of the map \( V_{n}^{(a)} \rightarrow V_{n}^{(a)} \) given by \( c_{T} \) for each standard tableau \( T \) of shape \( \lambda \vdash n \). Then \( V_{n}^{(a)} = \bigoplus_{\lambda \vdash n} \bigoplus_{T} W_{T}^{(a)} \), where \( T \) run over standard tableaux, and

\[ W_{T}^{(a)} \cong f_{\lambda(\alpha)}E^{\lambda} = \begin{cases} \{0\} & \text{for } \alpha \in \{1, \frac{1}{2}, \ldots, \frac{1}{\lambda_{1}-1}, -1, -\frac{1}{2}, \ldots, -\frac{1}{\lambda_{1}-1}\}; \\ E^{\lambda} & \text{otherwise.} \end{cases} \]

When \( W_{T}^{(a)} \cong E^{\lambda} \), the \( v_{S,T}^{(a)} = f_{\lambda}(\alpha)v_{S,T}^{(0)} \), where \( S \) run over all semi-standard tableaux of shape \( \lambda \) with entries in \([n]\), form a basis of \( W_{T}^{(a)} \). Further, the vector \( v_{S,T}^{(a)} \) is the highest weight vector of \( W_{T}^{(a)} \) if all entries in the \( r \)-th row of \( S \) are \( r \), and \( v_{S,T}^{(a)} \) is the lowest weight vector if entries in each \( r \)-th column of \( S \) are given as \( n - \lambda_{r} + 1, \ldots, n - 1, n \) from the top.

Proof. By Proposition 2.4, it is clear that \( V_{n}^{(a)} = \bigoplus_{\lambda} \bigoplus_{T} W_{T}^{(a)} \) and each \( W_{T}^{(a)} \) is \( \{0\} \) or isomorphic to \( E^{\lambda} \). The space \( W_{T}^{(a)} \) is generated by \( v_{T}^{(a)}(i_{1}, \ldots, i_{n}) \), where \((i_{1}, \ldots, i_{n}) \in [n]^{n}\). Since \( W_{T}^{(0)} \cong E^{\lambda} \) by (4), it follows from Corollary 3.4 that \( W_{T}^{(a)} \cong E^{\lambda} \) unless \( f_{a}(\alpha) = 0 \). It is easy to see that \( f_{a}(\alpha) = 0 \) if and only if \( \alpha = 1/k \) for \( 1 \leq k \leq \lambda_{1} - 1 \) or \( \alpha = -1/k \) for \( 1 \leq k \leq \lambda_{1} - 1 \).

Suppose \( W_{T}^{(a)} \cong E^{\lambda} \). Elements \( \{v_{S,T}^{(0)} \mid S \text{ are semi-standard tableaux}\} \) are linearly independent. In fact, for any semi-standard tableau \( S_{0} \), the term \( D^{(0)}(i_{1}^{(s_{0},T)}, \ldots, i_{n}^{(s_{0},T)}) \) appears only in \( v_{S_{0},T}^{(0)} \) among all \( v_{S,T}^{(0)} \). Since the dimension of \( E^{\lambda} \) is equal to the number of semi-standard tableaux of shape \( \lambda \), the \( v_{S,T}^{(a)} = f_{\lambda(\alpha)}v_{S,T}^{(0)} \) form a basis of \( W_{T}^{(a)} \). It is immediate to check the last claim. ■
Theorem 3.9 says that \( \{ D^{(\alpha)}(i_1, \ldots, i_n) \mid i_1, \ldots, i_n \in [n] \} \) are linearly independent if \( \alpha \in \mathbb{C} \setminus \{ \pm 1/k \mid k = 1, \ldots, n-1 \} \). Theorem 1.1 follows from Theorem 3.9 immediately.

A trick of doubling the variables ([3], [11]) suggests the following corollary. In fact, since \( \dim \mathbb{C}(E^\lambda \otimes (E^\lambda)^*)_{\mathfrak{sl}_n(\mathbb{C})} = 1 \), we can express \( \det(X)^2 \) by \( \alpha \)-determinants except a finite number of \( \alpha \).

**Corollary 3.10.** Let \( \alpha \in \mathbb{C} \setminus \{ \frac{1}{k} \mid 1 \leq k \leq \frac{n-1}{2} \} \). Then there exists \( \lambda \vdash n \) such that \( f_\lambda(\alpha) \neq 0 \), which has the following property: for any standard tableau \( T \) of shape \( \lambda \) there exists a \( \mathfrak{sl}_n(\mathbb{C}) \)-intertwining operator \( A^{(\alpha)} : (W_T^{(\alpha)})^* \rightarrow W_T^{(\alpha)} \) satisfying \( A^{(\alpha)}((v_{ST}^{(\alpha)})^*) = v_{TT}^{(\alpha)} \) and

\[
\det(X)^2 = f_\lambda(\alpha)^{-2} \sum_S v_{ST}^{(\alpha)} \cdot A^{(\alpha)}((v_{ST}^{(\alpha)})^*).
\]  

(12)

Here the sum runs over all semi-standard tableaux \( S \) of shape \( \lambda \) and \((v_{ST}^{(\alpha)})^*\) are defined by \((v_{ST}^{(\alpha)})(v_{ST}^{(\alpha)})^* = \delta_{SS'}\). More precisely, one may take

\[
\lambda = (\underbrace{2, 2, \ldots, 2}_n) \quad \text{if \( n \) is even} \quad \text{or} \quad \lambda = (\underbrace{2, \ldots, 2, 1}_n) \quad \text{if \( n \) is odd},
\]

which satisfies the condition.

**Proof.** For \( \alpha \) and \( \lambda \) in the corollary, it is easy to see that \( f_\lambda(\alpha) \neq 0 \). Consider a standard tableau \( T \) of shape \( \lambda \). Then we see that \( W_T^{(\alpha)} = W_T^{(0)} \) and \((v_{ST}^{(\alpha)})^* = f_\lambda(\alpha)^{-1}(v_{ST}^{(0)})^*\). Suppose the corollary is true for \( \alpha = 0 \). Using the intertwining operator \( A^{(0)} \), we define \( A^{(\alpha)} \) by \( A^{(\alpha)} = f_\lambda(\alpha)^2 A^{(0)} \). Then we see that \( A^{(\alpha)}((v_{TT}^{(\alpha)})^*) = f_\lambda(\alpha)^2 A^{(0)}(f_\lambda(\alpha)^{-1}(v_{TT}^{(0)})^*) = v_{TT}^{(\alpha)} \) and

\[
\sum_S v_{ST}^{(\alpha)} \cdot A^{(\alpha)}((v_{ST}^{(\alpha)})^*) = f_\lambda(\alpha)^2 \sum_S v_{ST}^{(0)} \cdot A^{(0)}((v_{ST}^{(0)})^*) = f_\lambda(\alpha)^2 \det(X)^2.
\]

It is hence sufficient to prove the corollary for the case \( \alpha = 0 \).

In general, for a finite-dimensional irreducible \( U(\mathfrak{sl}_n) \)-module \( V \) and its dual module \( V^* \),

\[
\mathcal{J} = \sum_i v_i \otimes v_i^* \in V \otimes V^*
\]

defines an invariant of \( \mathfrak{sl}_n(\mathbb{C}) \), see [3]. Here \( v_i \) are a basis of \( V \) and \( v_i^* \) are the dual basis, i.e., \( v_i^*(v_j) = \delta_{ij} \). Let \( \lambda \) be a partition whose parts are 1 or 2. Then \( V = W_T^{(0)} \cong E^\lambda \) is self-dual, i.e., \( V^* \cong_{\mathfrak{sl}_n(\mathbb{C})} V \). Therefore there exists an intertwining operator \( A' \) from \( V^* \) to \( V \). Then the polynomial \( \sum_S v_{ST}^{(0)} \cdot A'((v_{ST}^{(0)})^*) \in \mathcal{P}(\text{Mat}_{n \times n}) \) of degree \( 2n \) determined by \( \mathcal{J} \) is an invariant of \( \mathfrak{sl}_n \) and hence

\[
\sum_S v_{ST}^{(0)} \cdot A'((v_{ST}^{(0)})^*) = c \det(X)^2
\]  

(13)

for some constant \( c \). Comparing the coefficients of \( x_{11}^2 \cdots x_{1m}^2 \) in both sides in (13), we have \( A'((v_{TT}^{(0)})^*) = cx_{11} \cdots x_{mn} = cv_{TT}^{(0)} \) and so \( c \neq 0 \). Hence \( A^{(0)} = c^{-1} A' \) is our desired operator. \( \blacksquare \)
Example 3.11. Let $T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The module $W_T^{(\alpha)}$ has a basis consisting of $v_+ = v^{(\alpha)}_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 2D^{(\alpha)}(1,1)$, $v = v^{(\alpha)}_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = D^{(\alpha)}(1,2) + D^{(\alpha)}(2,1)$, $v_- = v^{(\alpha)}_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 2D^{(\alpha)}(2,2)$ if $\alpha \neq -1$. The linear map $A$ determined by

$$A(v^+_-) = -\frac{1}{2}v^+_-,$$ $A(v^+) = v,$ $A(v^-) = -\frac{1}{2}v_-$

from $(W_T^{(\alpha)})^*$ to $W_T^{(\alpha)}$ defines an intertwining operator of $\mathfrak{sl}_2(\mathbb{C})$. Hence, by the corollary, we have

$$(1 + \alpha)^2 \det(X)^2 = v_+ \cdot A(v^+_-) + v \cdot A(v^+) + v_- \cdot A(v^+) = v^2 - v_+ \cdot v_-$$
$$= (D^{(\alpha)}(1,2) + D^{(\alpha)}(2,1))^2 - 4D^{(\alpha)}(1,1)D^{(\alpha)}(2,2).$$

4. Concluding remarks

4.1. Quantum analogue.

We give here a brief comment on a possible generalization of our theorems to the quantum group $U_q(\mathfrak{sl}_n)$ because it produces new interesting phenomena that have never appeared in the classical case.

Define the quantum $\alpha$-determinant by

$$\det_{\alpha,q}(X) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)}a^{-\nu(\sigma)}x_{\sigma(1)} \cdots x_{\sigma(n)}n,$$ (14)

where $\text{inv}(\sigma)$ is the inversion number of $\sigma$; $\text{inv}(\sigma) = \#\{(i,j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$. In particular, $\det_q(X) = \det_{1,q}(X)$ is the (usual) quantum determinant, see e.g. [4]. We define quantum analogue $v_{a,T}^{(\alpha,q)}$ of the element $v_{a,T}^{(\alpha,q)}$ by the $q$-Young symmetrizers studied by Gyoja [2]. For each standard tableau $\lambda$, denote by $W_T^{(\alpha,q)}$ the quantum analogue of $W_T^{(\alpha)}$ given by the $q$-Young symmetrizer. Let $\lambda$ be a partition of $n$ and let $T_1, \ldots, T_d$ be all standard tableaux of shape $\lambda$, where $d = f^\lambda$. Let $v_k^{(\alpha,q)} = v_{a,T_k}^{(\alpha,q)}$ be the highest weight vector of each $W_{T_k}^{(\alpha,q)}$. Then there exists a $d \times d$ matrix $F_\lambda(\alpha; q)$ such that

$$(v_1^{(\alpha,q)}, \ldots, v_d^{(\alpha,q)}) = (v_1^{(0,q)}, \ldots, v_d^{(0,q)})F_\lambda(\alpha; q).$$

In the classical case, as we have seen in Corollary 3.4, $F_\lambda(\alpha; 1)$ is the scalar matrix $f_\lambda(\alpha)I$. It is observed, however, $F_\lambda(\alpha; q)$ is not, in general, a scalar matrix, not even a diagonal matrix, see [6]. Therefore, it is necessary to find a new basis other than the one obtained by the $q$-Young symmetrizers in order to diagonalize $F_\lambda(\alpha; q)$. In particular, one notes that the $q$-Young symmetrizer does not provide a formula like (6) in Proposition 3.1 in the quantum group case.

4.2. Immanant.

Recall the immanant. For a partition $\lambda$ of $n$, the $\lambda$-immanant of $X$ is defined by

$$\text{Imm}_\lambda(X) = \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\prod_{i=1}^n x_{\sigma(i)}.$$
Then, if we use the formula (8), we find the cyclic module \( U(\mathfrak{gl}_n) \text{Imm}_\lambda(X) \) is decomposed as \( U(\mathfrak{gl}_n) \text{Imm}_\lambda(X) \cong (E^\lambda) \oplus f^\lambda \) as in the case of \( \alpha \)-determinants. Also, since the function \( \sigma \mapsto \nu_n(\sigma) \) is a class function, the \( \alpha \)-determinant is expanded by immanants:

\[
\det_\alpha(X) = \sum_{\lambda \vdash n} \frac{f^\lambda}{n!} f_\lambda(\alpha) \text{Imm}_\lambda(X)
\]

by (7). Combining these facts, we have \( V_n^{(\alpha)} \cong \bigoplus_{\lambda \vdash n} (E^\lambda) \oplus f^\lambda \). This agrees with Theorem 1.1. However, if we consider the quantum group case, this discussion cannot be applied, because the function \( \sigma \mapsto \text{inv}(\sigma) \) appeared in (14) is not a class function.

4.3. The case where \( \alpha = \infty \).

We consider the case \( \alpha = \infty \) and describe the irreducible decomposition of \( V_n^{(\infty)} \). Since \( \det_\alpha(X) \) is a polynomial of degree \( n - 1 \) in variable \( \alpha \), we can define a limit

\[
\det_\infty(X) = \lim_{|\alpha| \to \infty} \alpha^{1-n} \det_\alpha(X) = \sum_{\sigma \in S_n, \nu_n(\sigma) = 1} x_{\sigma(1)} \cdots x_{\sigma(n)}.
\]

For example,

\[
\det_\infty \begin{pmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{pmatrix} = x_{21}x_{32}x_{13} + x_{31}x_{12}x_{23}.
\]

Denote by \( V_n^{(\infty)} \) the cyclic module \( U(\mathfrak{g}) \det_\infty(X) \). Then we have the following irreducible decomposition of \( V_n^{(\infty)} \) as a corollary of Theorem 3.9.

**Corollary 4.1.**

\[
V_n^{(\infty)} \cong \bigoplus_{\lambda \vdash n} (E^\lambda) \oplus f^\lambda = \bigoplus_{k=1}^{n} \left( E^{(k,1^{n-k})} \right) \oplus \binom{n-1}{k-1},
\]

where \( \lambda \) run over all hook partitions of \( n \).

**Proof.** The degree of polynomial \( f_\lambda(\alpha) \in \mathbb{Z}[\alpha] \) is equal to \( n - d \), where \( d \) is the number of the main diagonal of the Young diagram \( \lambda \). Therefore \( \lim_{|\alpha| \to \infty} \alpha^{1-n} f_\lambda(\alpha) \) is zero unless \( d = 1 \), i.e., \( \lambda \) is a hook. For a hook \( \lambda = (k, 1^{n-k}) \), the number \( f^\lambda \) is given by the binomial coefficient \( \binom{n-1}{k-1} \). Hence, the claim follows from Theorem 3.9.

**Example 4.2.** When \( n = 5 \),

\[
V_5^{(\infty)} \cong E^{(5)} \oplus (E^{(4,1)})^\oplus 4 \oplus (E^{(3,1,1)})^\oplus 6 \oplus (E^{(2,1,1,1)})^\oplus 4 \oplus E^{(1,1,1,1,1,1)}.
\]
Remark 4.3. By (9), for each \( \alpha > 0 \), we can define a probability measure \( \mathcal{M}_n^{(\alpha)} \) on \( \mathfrak{S}_n \) by

\[
\mathcal{M}_n^{(\alpha)}(\sigma) = \frac{\alpha^{n-\nu_n(\sigma)}}{\prod_{j=1}^{n-1} (1 + j\alpha)} \quad \text{for each } \sigma \in \mathfrak{S}_n.
\]

This is called the Ewens measure in [5] but the definition is slightly different from ours. It is clear that \( \mathcal{M}_n^{(1)} \) is the uniform measure on \( \mathfrak{S}_n \) and \( \mathcal{M}_n^{(0)} = \lim_{\alpha \to 0^+} \mathcal{M}_n^{(\alpha)} \) is the Dirac measure at the identity. Also we see that

\[
\mathcal{M}_n^{(\infty)}(\sigma) = \lim_{\alpha \to +\infty} \mathcal{M}_n^{(\alpha)}(\sigma) = \begin{cases} 1/(n-1)! & \text{if } \nu_n(\sigma) = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Given a matrix \( X \) with non-negative entries \( x_{ij} \), define a random variable \( X_\sigma \) by \( X_\sigma = \prod_{i=1}^n x_{i\sigma(i)} \) on \( \mathfrak{S}_n \). Then for \( \alpha \in [0, +\infty] \) the \( \alpha \)-determinant of \( X \) is essentially the mean value of \( X_\sigma \) with respect to \( \mathcal{M}_n^{(\alpha)} \):

\[
det_\alpha(X) = \prod_{j=1}^{n-1} (1 + j\alpha) \sum_{\sigma \in \mathfrak{S}_n} X_\sigma \mathcal{M}_n^{(\alpha)}(\sigma) \quad \text{for } 0 \leq \alpha < +\infty,
\]

\[
det_\infty(X) = (n-1)! \sum_{\sigma \in \mathfrak{S}_n} X_\sigma \mathcal{M}_n^{(\infty)}(\sigma).
\]

References


