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Alpha-determinant Cyclic Modules of $\mathfrak{gl}_n(\mathbb{C})$

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Abstract. The alpha-determinant unifies and interpolates the notion of the determinant and permanent. We determine the irreducible decomposition of the cyclic module of $\mathfrak{gl}_n(\mathbb{C})$ defined by the alpha-determinant. The degeneracy of the irreducible decomposition is determined by the content polynomial of a given partition.
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1. Introduction

Let $X = (x_{ij})_{1\le i,j\le n}$ be a matrix with commutative variables x_{ij}. For a complex number α, the α-determinant of X is defined by

$$\det_\alpha(X) = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu_n(\sigma)} \prod_{i=1}^{n} x_{i\sigma(i)},$$

where $\mathfrak{S}_n$ is the symmetric group of degree n and $\nu_n(\sigma)$ stands for the number of cycles in the cycle decomposition of a permutation $\sigma \in \mathfrak{S}_n$. The α-determinant is nothing but the permanent if $\alpha = 1$ and the (usual) determinant if $\alpha = -1$, and hence, it interpolates these two. It appears as a coefficient in the Taylor expansion of the power $\det(I - \alpha X)^{-1/\alpha}$ of the characteristic polynomial of X and defines a generalization of the boson, poisson and fermion point processes, see [9, 10]. Also, its Pfaffian analogue has been developed in [8].

It is a natural question whether the α-determinant can be interpreted as an invariant like the usual determinant (and also the q-determinant in quantum group theory). Denote by $\mathcal{P}(\mathrm{Mat}_{n\times n})$ the ring of polynomials in variables $\{x_{ij}\}_{1\le i,j\le n}$. Let $\{E_{ij}\}_{1\le i,j\le n}$ be the natural basis of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. When $n = 2$

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and $\alpha \neq 0$, consider the linear map $\rho_2^{(\alpha)}$ of $\mathfrak{gl}_2(\mathbb{C})$ on $\mathcal{P}(\mathrm{Mat}_{n\times n})$ determined by

$$\rho_2^{(\alpha)}(E_{11}) = x_{11}\partial_{11} + x_{12}\partial_{12}, \qquad \rho_2^{(\alpha)}(E_{12}) = \frac{1}{\sqrt{-\alpha}}\left(x_{11}\partial_{21} - \alpha x_{12}\partial_{22}\right),$$
$$\rho_2^{(\alpha)}(E_{21}) = \frac{1}{\sqrt{-\alpha}}\left(-\alpha x_{21}\partial_{11} + x_{22}\partial_{12}\right), \quad \rho_2^{(\alpha)}(E_{22}) = x_{21}\partial_{21} + x_{22}\partial_{22},$$

where $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$. Then $\rho_2^{(\alpha)}$ defines a representation of $\mathfrak{gl}_2(\mathbb{C})$ and

$$\rho_2^{(\alpha)}(E_{ii})\mathrm{det}_\alpha(X) = \mathrm{det}_\alpha(X),$$

$\rho_2^{(\alpha)}(E_{ij})\mathrm{det}_\alpha(X) = 0$ for $i \neq j$. This is not, however, true for $n \geq 3$. Precisely, although the map $\rho_n^{(\alpha)}$ given by $\rho_n^{(\alpha)}(E_{ij}) = \sum_{k=1}^n \beta^{|i-k|-|j-k|} x_{ik}\partial_{jk}$, where $\beta = \sqrt{-\alpha}$, defines a representation of $\mathfrak{gl}_n(\mathbb{C})$ on $\mathcal{P}(\mathrm{Mat}_{n\times n})$, $\rho_n^{(\alpha)}(E_{ij})\mathrm{det}_\alpha(X) \neq 0$, $(i \neq j)$ in general. (One can actually show that $\rho_n^{(\alpha)}$ is equivalent to the usual action of $\mathfrak{gl}_n(\mathbb{C})$ determined by $\rho(E_{ij}) = \rho_n^{(-1)}(E_{ij}) = \sum_{k=1}^n x_{ik}\partial_{jk}$ on $\mathcal{P}(\mathrm{Mat}_{n\times n})$ when $\alpha \neq 0$. Indeed, the map $f(x_{ij}) \mapsto f(\beta^{|i-j|}x_{ij})$ is the intertwining operator from $(\rho, \mathcal{P}(\mathrm{Mat}_{n\times n}))$ to $(\rho_n^{(\alpha)}, \mathcal{P}(\mathrm{Mat}_{n\times n}))$.)

Then, a question which subsequently arises is what the structure of the smallest invariant subspace of $\mathcal{P}(\mathrm{Mat}_{n\times n})$ which contains $\mathrm{det}_\alpha(X)$ is. Thus, the aim of the present paper is to investigate a cyclic module $V_n^{(\alpha)} = U(\mathfrak{g})\mathrm{det}_\alpha(X)$, under the representation ρ on $\mathcal{P}(\mathrm{Mat}_{n\times n})$ for each $\alpha \in \mathbb{C}$. Clearly, $V_n^{(-1)}$ is the one-dimensional determinant representation.

We adopt the notations for partitions used in [7] and for representations of $\mathfrak{gl}_n(\mathbb{C})$ used in [1] and [11]. A partition λ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_j = 0$ for sufficiently large j. We usually identify a partition λ with the corresponding Young diagram. Write $\lambda \vdash n$ if $\sum_{j\geq 1}\lambda_j = n$ and denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ the conjugate partition of λ. Let E^λ denote the Schur module (or called the Weyl module) corresponding to λ and f^λ the number of standard tableaux of shape λ.

The following is our main result, which describes the irreducible decomposition of $V_n^{(\alpha)}$.

Theorem 1.1. *For $k = 1, 2, \dots, n-1$,*

$$V_n^{(\frac{1}{k})} \cong \bigoplus_{\substack{\lambda\vdash n,\\ \lambda'_1\leq k}} (E^\lambda)^{\oplus f^\lambda} \qquad \text{and} \qquad V_n^{(-\frac{1}{k})} \cong \bigoplus_{\substack{\lambda\vdash n,\\ \lambda_1\leq k}} (E^\lambda)^{\oplus f^\lambda}. \tag{1}$$

For $\alpha \in \mathbb{C} \setminus \{\pm 1, \pm\frac{1}{2}, \dots, \pm\frac{1}{n-1}\}$,

$$V_n^{(\alpha)} \cong (\mathbb{C}^n)^{\otimes n} \cong \bigoplus_{\lambda\vdash n} (E^\lambda)^{\oplus f^\lambda}. \tag{2}$$

Example 1.2. When $n = 3$ the irreducible decomposition of $V_n^{(\alpha)}$ is given by

$$V_3^{(\alpha)} \cong \begin{cases} E^{(3)} & \text{if } \alpha = 1, \\ E^{(3)} \oplus E^{(2,1)} \oplus E^{(2,1)} & \text{if } \alpha = \frac{1}{2}, \\ E^{(1,1,1)} & \text{if } \alpha = -1, \\ E^{(2,1)} \oplus E^{(2,1)} \oplus E^{(1,1,1)} & \text{if } \alpha = -\frac{1}{2}, \\ E^{(3)} \oplus E^{(2,1)} \oplus E^{(2,1)} \oplus E^{(1,1,1)} & \text{otherwise.} \end{cases}$$

Note that each Schur module possesses a canonial basis formed by α-determinants (see Theorem 3.9). Although the α-determinant is not an invariant, this fact implies that it has a rich symmetry.

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2. The $U(\mathfrak{gl}_n)$-module $V_n^{(\alpha)}$

Let $V_n^{(\alpha)} = \rho(U(\mathfrak{gl}_n))\mathrm{det}_\alpha(X)$. Put $[n] = \{1, 2, \ldots, n\}$ and

$$D^{(\alpha)}(i_1, i_2, \ldots, i_n) = \mathrm{det}_\alpha \begin{pmatrix} x_{i_1 1} & x_{i_1 2} & \ldots & x_{i_1 n} \\ x_{i_2 1} & x_{i_2 2} & \ldots & x_{i_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_n 1} & x_{i_n 2} & \ldots & x_{i_n n} \end{pmatrix}$$

for any $i_1, \ldots, i_n \in [n]$. In particular, $\mathrm{det}_\alpha(X) = D^{(\alpha)}(1, 2, \ldots, n)$. We abbreviate $\rho(E_{ij})$ to E_{ij} for simplicity. When there is no fear of confusion, we abbreviate ν_n to ν.

Lemma 2.1.

$$E_{pq} \cdot D^{(\alpha)}(i_1, \ldots, i_n) = \sum_{k=1}^{n} \delta_{i_k,q} D^{(\alpha)}(i_1, \ldots, i_{k-1}, p, i_{k+1}, \ldots, i_n). \tag{3}$$

Proof. It is straightforward. In fact,

$$\begin{aligned} E_{pq} \cdot D^{(\alpha)}(i_1, \ldots, i_n) &= \sum_{j=1}^{n} x_{pj} \frac{\partial}{\partial x_{qj}} \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu(\sigma)} x_{i_1\sigma(1)} \cdots x_{i_n\sigma(n)} \\ &= \sum_{j=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu(\sigma)} \sum_{k=1}^{n} x_{pj} \delta_{i_k,q} \delta_{\sigma(k),j} x_{i_1\sigma(1)} \cdots \widehat{x_{i_k\sigma(k)}} \cdots x_{i_n\sigma(n)} \\ &= \sum_{k=1}^{n} \delta_{i_k,q} \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu(\sigma)} x_{p\sigma(k)} x_{i_1\sigma(1)} \cdots \widehat{x_{i_k\sigma(k)}} \cdots x_{i_n\sigma(n)} \\ &= \sum_{k=1}^{n} \delta_{i_k,q} D^{(\alpha)}(i_1, \ldots, i_{k-1}, p, i_{k+1}, \ldots, i_n), \end{aligned}$$

where $\widehat{x_{kl}}$ stands for the omission of x_{kl}. ∎

Example 2.2. We see that $E_{21} \cdot D^{(\alpha)}(4,1,2,1) = D^{(\alpha)}(4,2,2,1) + D^{(\alpha)}(4,1,2,2)$, $E_{11} \cdot D^{(\alpha)}(4,1,2,1) = 2D^{(\alpha)}(4,1,2,1)$, and $E_{43} \cdot D^{(\alpha)}(4,1,2,1) = 0$.

The symmetric group $\mathfrak{S}_n$ acts also on $V_n^{(\alpha)}$ from the right by
$D^{(\alpha)}(i_1, \ldots, i_n) \cdot \sigma = D^{(\alpha)}(i_{\sigma(1)}, \ldots, i_{\sigma(n)})$.

Lemma 2.3. *The space $V_n^{(\alpha)}$ is the complex vector space spanned by* $\{D^{(\alpha)}(i_1,\dots,i_n) \mid i_1,\dots,i_n \in [n]\}$.

Proof. Since the vector space spanned by all $D^{(\alpha)}(i_1,\dots,i_n)$ contains $V_n^{(\alpha)}$ by Lemma 2.1, we prove that all $D^{(\alpha)}(i_1,\dots,i_n)$ are contained in $V_n^{(\alpha)}$. For $1 \le p < q \le n$, we have

$$V_n^{(\alpha)} \ni (E_{pq}E_{qp} - 1) \cdot D^{(\alpha)}(1,2,\dots,n) = D^{(\alpha)}(\tau(1),\dots,\tau(n)),$$

where τ is the transposition (p,q) of p and q. It follows that, for each $\sigma \in \mathfrak{S}_n$, $D^{(\alpha)}(\sigma(1),\dots,\sigma(n)) = Y \cdot D^{(\alpha)}(1,\dots,n)$ for some $Y = Y_\sigma \in U(\mathfrak{g})$. For any $1 \le i_1 \le i_2 \le \cdots \le i_n \le n$, suppose there exists $X = X_{i_1\dots i_n} \in U(\mathfrak{g})$ such that $D^{(\alpha)}(i_1,\dots,i_n) = X \cdot D^{(\alpha)}(1,\dots,n)$. For any $j_1,\dots,j_n \in [n]$, we have $D^{(\alpha)}(j_1,\dots,j_n) = D^{(\alpha)}(i_{\sigma(1)},\dots,i_{\sigma(n)})$ for some $\sigma \in \mathfrak{S}_n$ and $i_1 \le \cdots \le i_n$. Hence, since the action of $\mathfrak{gl}_n(\mathbb{C})$ and of $\mathfrak{S}_n$ commute, we see that

$$\begin{aligned} &D^{(\alpha)}(j_1,\dots,j_n) = D^{(\alpha)}(i_1,\dots,i_n) \cdot \sigma \\ &= \big(X \cdot D^{(\alpha)}(1,\dots,n)\big) \cdot \sigma = X \cdot \big(D^{(\alpha)}(1,\dots,n) \cdot \sigma\big) = XY \cdot D^{(\alpha)}(1,\dots,n) \end{aligned}$$

and $D^{(\alpha)}(j_1,\dots,j_n)$ is contained in $V_n^{(\alpha)}$. Therefore it is sufficient to prove $D^{(\alpha)}(i_1,\dots,i_n) \in V_n^{(\alpha)}$ for $i_1 \le \cdots \le i_n$.

For any sequence $(i_1,\dots,i_n)$ such that $i_k \le k$ for any k, we have

$$D^{(\alpha)}(i_1,i_2,\dots,i_n) = E_{i_n n} \cdots E_{i_2 2} E_{i_1 1} \cdot D^{(\alpha)}(1,2,\dots,n) \in V_n^{(\alpha)}.$$

In fact, by Lemma 2.1, we see that $E_{i_k k} \cdot D^{(\alpha)}(i_1,\dots,i_{k-1},k,k+1,\dots,n) = D^{(\alpha)}(i_1,\dots,i_{k-1},i_k,k+1,\dots,n)$ because $i_1 \le \cdots \le i_{k-1} < k$ for any $1 \le k \le n$. Suppose there exists k such that $i_j \le j$ for any $j < k$ and $i_k > k$. We prove $D^{(\alpha)}(i_1,\dots,i_n) \in V_n^{(\alpha)}$ for such sequences $(i_1,\dots,i_n)$ by induction with respect to the lexicographic order. Since $i_1 \le \cdots \le i_{k-1} < k \le i_k - 1 < i_{k+1} \le \cdots \le i_n$ and $D^{(\alpha)}(i_1,\dots,i_{k-1},i_k-1,i_{k+1},\dots,i_n) \in V_n^{(\alpha)}$ by the induction assumption, we have $D^{(\alpha)}(i_1,\dots,i_k,\dots,i_n) = E_{i_k,i_k-1} \cdot D^{(\alpha)}(i_1,\dots,i_k-1,\dots,i_n) \in V_n^{(\alpha)}$ by Lemma 2.1. Hence we obtain our claim. ■

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ acts on the n-tensor product $(\mathbb{C}^n)^{\otimes n} = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ from the left by

$$E_{pq} \cdot (\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n}) = \sum_{k=1}^{n} \boldsymbol{e}_{i_1} \otimes \cdots \otimes E_{pq}\boldsymbol{e}_{i_k} \otimes \cdots \otimes \boldsymbol{e}_{i_n} = \sum_{k=1}^{n} \delta_{i_k,q} \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_p \otimes \cdots \otimes \boldsymbol{e}_{i_n},$$

where $\{\boldsymbol{e}_k\}_{k=1}^n$ is the natural basis of $\mathbb{C}^n$. ¿From this fact together with Lemma 2.1 and Lemma 2.3, we have the

Proposition 2.4. *Let $\Phi_n^{(\alpha)}$ be the linear map from $(\mathbb{C}^n)^{\otimes n}$ to $V_n^{(\alpha)}$ defined by*

$$\Phi_n^{(\alpha)}(\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n}) = D^{(\alpha)}(i_1,\dots,i_n)$$

for each $i_1,\dots,i_n \in [n]$. Then $\Phi_n^{(\alpha)}$ is a $U(\mathfrak{g})$-module homomorphism. In particular, $V_n^{(\alpha)}$ is isomorphic to a quotient module $(\mathbb{C}^n)^{\otimes n}/\operatorname{Ker}\Phi_n^{(\alpha)}$ of $(\mathbb{C}^n)^{\otimes n}$.

Notice that, when $\alpha = 0$, the homomorphism $\Phi_n^{(0)}(\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n}) = D^{(0)}(i_1, \ldots, i_n) = x_{i_1 1} \cdots x_{i_n n}$ is clearly bijective, and therefore $V_n^{(0)} \cong (\mathbb{C}^n)^{\otimes n}$. Hence we have the irreducible decomposition of $V_n^{(0)}$ as

$$V_n^{(0)} \cong \bigoplus_{\lambda \vdash n} (E^\lambda)^{\oplus f^\lambda}. \tag{4}$$

The symmetric group $\mathfrak{S}_n$ acts on $(\mathbb{C}^n)^{\otimes n}$ from the right by $(\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n}) \cdot \sigma = \boldsymbol{e}_{i_{\sigma(1)}} \otimes \cdots \otimes \boldsymbol{e}_{i_{\sigma(n)}}$ for any $\sigma \in \mathfrak{S}_n$.

3. A formula for the number of cycles

A numbering of shape $\lambda \vdash n$ is a way of putting distinct elements in $[n]$ in each box of the Young diagram λ. Let $R(T)$ be the row group (or called the Young subgroup) of a numbering represented by a tableau T, i.e., permutations in $R(T)$ permutate the entries of each row among themselves. The column group $C(T)$ is also defined similarly.

Recall the Frobenius notation $(a_1, a_2, \ldots, a_d | b_1, b_2, \ldots, b_d)$ of a partition λ, where $a_i = \lambda_i - i \geq 0$ and $b_i = \lambda'_i - i \geq 0$ for $1 \leq i \leq d$. Then the content polynomial $f_\lambda(\alpha)$ ([7, I-1]) for the partition λ is written as

$$f_\lambda(\alpha) = \prod_{i=1}^{d} \left\{ \prod_{j=1}^{a_i} (1 + j\alpha) \cdot \prod_{j=1}^{b_i} (1 - j\alpha) \right\}. \tag{5}$$

Note that $f_\lambda(\alpha)$ satisfies $f_\lambda(\alpha) = f_{\lambda'}(-\alpha)$. We have the following formula for the number $\nu = \nu_n$ of cycles.

Proposition 3.1. *Let T be a numbering of shape $\lambda \vdash n$. Then*

$$\sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} \alpha^{n - \nu(pq\sigma)} \tag{6}$$
$$= \begin{cases} \operatorname{sgn}(q_0) f_\lambda(\alpha) & \text{if } \sigma = q_0 p_0 \text{ for some } q_0 \in C(T) \text{ and } p_0 \in R(T), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The formula

$$\alpha^{n - \nu_n(\sigma)} = \sum_{\mu \vdash n} \frac{f^\mu}{n!} f_\mu(\alpha) \chi^\mu(\sigma) \tag{7}$$

is a specialization of the Frobenius character formula, see [7, I-7, Example 17]. Here χ^μ is the irreducible character of $\mathfrak{S}_n$ corresponding to μ. Moreover, for a numbering T of shape λ and a partition μ, the well-known equation

$$\chi^\mu \cdot c_T = \delta_{\lambda,\mu} \frac{n!}{f^\mu} c_T \tag{8}$$

in the group algebra $\mathbb{C}\mathfrak{S}_n$ holds, which is obtained by Young. Here c_T is the Young symmetrizer

$$c_T = \sum_{q \in C(T)} \operatorname{sgn}(q) \sum_{p \in R(T)} qp.$$

Define $\phi_\alpha = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu(\sigma)}\sigma \in \mathbb{C}[\mathfrak{S}_n]$. It follows from (7) that $\phi_\alpha = \sum_{\mu\vdash n} \frac{f^\mu}{n!} f_\mu(\alpha)\chi^\mu$. Hence by (8) we have

$$\phi_\alpha \cdot c_T = \sum_{\mu\vdash n} \frac{f^\mu}{n!} f_\mu(\alpha)\delta_{\lambda,\mu}\frac{n!}{f^\mu} c_T = f_\lambda(\alpha)c_T.$$

In other words,

$$\sum_{\sigma\in\mathfrak{S}_n}\sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} \alpha^{n-\nu(pq\sigma)}\sigma = f_\lambda(\alpha) \sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} qp.$$

This gives our desired formula. ■

Example 3.2. When $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ we have

$$\sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} \alpha^{3-\nu(pq\sigma)} = \begin{cases} (1+\alpha)(1-\alpha) & \text{for } \sigma = (1) \text{ or } (12), \\ -(1+\alpha)(1-\alpha) & \text{for } \sigma = (13) \text{ or } (123), \\ 0 & \text{for } \sigma = (23) \text{ or } (132). \end{cases}$$

Example 3.3. For $T = \boxed{1\,|\,2\,|\cdot\cdot|\,n}$ Proposition 3.1 says

$$\sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu(\sigma)} = \prod_{j=1}^{n-1}(1+j\alpha). \tag{9}$$

For a sequence $(i_1,\dots,i_n)\in[n]^n$ and a numbering T, we define

$$\begin{aligned} v_T^{(\alpha)}(i_1,\dots,i_n) =& D^{(\alpha)}(i_1,\dots,i_n)\cdot c_T \\ =& \sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} D^{(\alpha)}(i_{qp(1)},\dots,i_{qp(n)}), \end{aligned} \tag{10}$$

where $c_T = \sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} qp$ is the Young symmetrizer.

Corollary 3.4. *Let $\lambda\vdash n$. For each sequence $(i_1,\dots,i_n)\in[n]^n$ and numbering T of shape λ, we have*

$$v_T^{(\alpha)}(i_1,\dots,i_n) = f_\lambda(\alpha)v_T^{(0)}(i_1,\dots,i_n). \tag{11}$$

Proof. For a numbering T of shape λ, denote by $W_T^{(\alpha)}$ the space spanned by $\{v_T^{(\alpha)}(i_1,\dots,i_n) \mid i_1,\dots,i_n\in[n]\}$. Since the Schur module E^λ is isomorphic to the image of the map $(\mathbb{C}^n)^{\otimes n}\to(\mathbb{C}^n)^{\otimes n}$ given by c_T for any numbering T, it follows from Proposition 2.4 and (10) that $W_T^{(\alpha)}$ is isomorphic to $\{0\}$ or E^λ. If (11) is proved for a certain sequence $(j_1,\dots,j_n)$ satisfying $v_T^{(0)}(j_1,\dots,j_n)\neq 0$, (11) holds for any $(i_1,\dots,i_n)\in[n]^n$ because $W_T^{(\alpha)}$ is the cyclic module $U(\mathfrak{g})v_T^{(\alpha)}(j_1,\dots,j_n)$ and the action of $\mathfrak{g}$ is independent of α.

We prove (11) for the case where $(i_1, i_2, \ldots, i_n) = (1, 2, \ldots, n)$. Then we have

$$\begin{aligned} v_T^{(\alpha)}(1,\ldots,n) &= \sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu(\sigma)} x_{qp\sigma(1),1}\cdots x_{qp\sigma(n),n} \\ &= \sum_{\sigma\in\mathfrak{S}_n} \left(\sum_{q\in C(T)} \operatorname{sgn}(q) \sum_{p\in R(T)} \alpha^{n-\nu(pq\sigma)} \right) x_{\sigma(1),1}\cdots x_{\sigma(n),n}. \end{aligned}$$

By Proposition 3.1 we see that

$$\begin{aligned} &v_T^{(\alpha)}(1,\ldots,n) \\ &= f_\lambda(\alpha) \sum_{q_0\in C(T)} \operatorname{sgn}(q_0) \sum_{p_0\in R(T)} x_{q_0p_0(1),1}\cdots x_{q_0p_0(n),n} = f_\lambda(\alpha) v_T^{(0)}(1,\ldots,n). \end{aligned}$$

If $q_0 \neq q_0'$ or $p_0 \neq p_0'$ then $q_0p_0 \neq q_0'p_0'$. Indeed, if $q_0p_0 = q_0'p_0'$ then $C(T) \ni (q_0')^{-1}q_0 = p_0(p_0')^{-1} \in R(T)$. But, since $C(T) \cap R(T) = \{(1)\}$, we have $q_0 = q_0'$ and $p_0 = p_0'$. Hence

$$v_T^{(0)}(1,\ldots,n) = \sum_{q_0\in C(T)} \operatorname{sgn}(q_0) \sum_{p_0\in R(T)} x_{q_0p_0(1),1}\cdots x_{q_0p_0(n),n} \neq 0$$

and so we have proved the corollary. ■

Example 3.5. For $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$, we have

$$\begin{aligned} v_T^{(\alpha)}(1,2,1) =& D^{(\alpha)}(1,2,1)\cdot((1)+(13)-(12)-(132)) \\ =& 2D^{(\alpha)}(1,2,1) - D^{(\alpha)}(2,1,1) - D^{(\alpha)}(1,1,2) \\ =& (1+\alpha)(1-\alpha)(2x_{11}x_{22}x_{13} - x_{21}x_{12}x_{13} - x_{11}x_{12}x_{23}). \end{aligned}$$

Example 3.6. For $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$, we have

$$\begin{aligned} &v_T^{(\alpha)}(1,2,2,4) \\ &= D^{(\alpha)}(1,2,2,4) + D^{(\alpha)}(1,2,4,2) - 2D^{(\alpha)}(1,4,2,2) + D^{(\alpha)}(2,1,2,4) \\ &+ D^{(\alpha)}(2,1,4,2) - 2D^{(\alpha)}(2,2,1,4) - 2D^{(\alpha)}(2,2,4,1) + D^{(\alpha)}(2,4,1,2) \\ &+ D^{(\alpha)}(2,4,2,1) - 2D^{(\alpha)}(4,1,2,2) + D^{(\alpha)}(4,2,1,2) + D^{(\alpha)}(4,2,2,1) \\ =&(1+\alpha)(1-\alpha)(x_{11}x_{22}x_{23}x_{44} + x_{11}x_{22}x_{43}x_{24} - 2x_{11}x_{42}x_{23}x_{24} + x_{21}x_{12}x_{23}x_{44} \\ &+ x_{21}x_{12}x_{43}x_{24} - 2x_{21}x_{22}x_{13}x_{44} - 2x_{21}x_{22}x_{43}x_{14} + x_{21}x_{42}x_{13}x_{24} \\ &+ x_{21}x_{42}x_{23}x_{14} - 2x_{41}x_{12}x_{23}x_{24} + x_{41}x_{22}x_{13}x_{24} + x_{41}x_{22}x_{23}x_{14}). \end{aligned}$$

For a semi-standard tableau S and a standard tableau T of the same shape, we define the sequence $\boldsymbol{i}^{(S,T)} = (i_1^{(S,T)}, \ldots, i_n^{(S,T)})$ as follows. For each k, we let B_k be the box numbered by k in T and denote by $i_k^{(S,T)}$ the number in box B_k of S. For example, for

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & \\ \cline{1-3} 4 & 6 & & \\ \cline{1-2}\end{array} \qquad \text{and} \qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 9 & \\ \cline{1-3} 7 & 8 & & \\ \cline{1-2}\end{array},$$

we have $\boldsymbol{i}^{(S,T)} = (1,3,2,3,2,3,4,6,4)$. Put $v_{S,T}^{(\alpha)} = v_T^{(\alpha)}(\boldsymbol{i}^{(S,T)})$.

Example 3.7.

$$
\begin{aligned}
v^{(\alpha)}_{S,T} =& 2D^{(\alpha)}(1,2,1) - D^{(\alpha)}(2,1,1) - D^{(\alpha)}(1,1,2) \quad \text{for } (S,T) = \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right), \\
v^{(\alpha)}_{S,T} =& D^{(\alpha)}(1,3,2) - D^{(\alpha)}(3,1,2) + D^{(\alpha)}(2,3,1) - D^{(\alpha)}(2,1,3) \\
& \qquad \text{for } (S,T) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right), \\
v^{(\alpha)}_{S,T} =& D^{(\alpha)}(1,2,3) - D^{(\alpha)}(2,1,3) + D^{(\alpha)}(3,2,1) - D^{(\alpha)}(3,1,2) \\
& \qquad \text{for } (S,T) = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right).
\end{aligned}
$$

Example 3.8. For

$$
S = T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}
$$

we have $v^{(\alpha)}_{S,T} = \sum_{q \in \mathfrak{S}_n} \operatorname{sgn}(q) D^{(\alpha)}(q(1), \ldots, q(n)) = \prod_{j=1}^{n-1}(1 - j\alpha)\det(X)$.

Finally, we obtain the following theorem.

Theorem 3.9. *Denote by $W^{(\alpha)}_T$ the image of the map $V^{(\alpha)}_n \to V^{(\alpha)}_n$ given by c_T for each standard tableau T of shape $\lambda \vdash n$. Then $V^{(\alpha)}_n = \bigoplus_{\lambda \vdash n} \bigoplus_T W^{(\alpha)}_T$, where T run over standard tableaux, and*

$$
W^{(\alpha)}_T \cong f_\lambda(\alpha) E^\lambda = \begin{cases} \{0\} & \text{for } \alpha \in \{1, \frac{1}{2}, \ldots, \frac{1}{\lambda'_1 - 1}, -1, -\frac{1}{2}, \ldots, -\frac{1}{\lambda_1 - 1}\}, \\ E^\lambda & \text{otherwise.} \end{cases}
$$

When $W^{(\alpha)}_T \cong E^\lambda$, the $v^{(\alpha)}_{S,T} = f_\lambda(\alpha) v^{(0)}_{S,T}$, where S run over all semi-standard tableaux of shape λ with entries in $[n]$, form a basis of $W^{(\alpha)}_T$. Further, the vector $v^{(\alpha)}_{S,T}$ is the highest weight vector of $W^{(\alpha)}_T$ if all entries in the r-th row of S are r, and $v^{(\alpha)}_{S,T}$ is the lowest weight vector if entries in each r-th column of S are given as $n - \lambda'_r + 1, \ldots, n-1, n$ from the top.

Proof. By Proposition 2.4, it is clear that $V^{(\alpha)}_n = \bigoplus_\lambda \bigoplus_T W^{(\alpha)}_T$ and each $W^{(\alpha)}_T$ is $\{0\}$ or isomorphic to E^λ. The space $W^{(\alpha)}_T$ is generated by $v^{(\alpha)}_T(i_1, \ldots, i_n)$, where $(i_1, \ldots, i_n) \in [n]^n$. Since $W^{(0)}_T \cong E^\lambda$ by (4), it follows from Corollary 3.4 that $W^{(\alpha)}_T \cong E^\lambda$ unless $f_\alpha(\lambda) = 0$. It is easy to see that $f_\alpha(\lambda) = 0$ if and only if $\alpha = 1/k$ for $1 \le k \le \lambda'_1 - 1$ or $\alpha = -1/k$ for $1 \le k \le \lambda_1 - 1$.

Suppose $W^{(\alpha)}_T \cong E^\lambda$. Elements $\{v^{(0)}_{S,T} \mid S \text{ are semi-standard tableaux}\}$ are linearly independent. In fact, for any semi-standard tableau S_0, the term $D^{(0)}(\boldsymbol{i}^{(S_0,T)}) = x_{i_1^{(S_0,T)},1} \cdots x_{i_n^{(S_0,T)},n}$ appears only in $v^{(0)}_{S_0,T}$ among all $v^{(0)}_{S,T}$. Since the dimension of E^λ is equal to the number of semi-standard tableaux of shape λ, the $v^{(\alpha)}_{S,T} = f_\lambda(\alpha) v^{(0)}_{S,T}$ form a basis of $W^{(\alpha)}_T$. It is immediate to check the last claim. ■

Theorem 3.9 says that $\{D^{(\alpha)}(i_1,\dots,i_n) \mid i_1,\dots,i_n \in [n]\}$ are linearly independent if $\alpha \in \mathbb{C}\setminus\{\pm 1/k \mid k=1,\dots,n-1\}$. Theorem 1.1 follows from Theorem 3.9 immediately.

A trick of doubling the variables ([3], [11]) suggests the following corollary. In fact, since $\dim_{\mathbb{C}}(E^\lambda \otimes (E^\lambda)^*)^{\mathfrak{sl}_n(\mathbb{C})} = 1$, we can express $\det(X)^2$ by α-determinants except a finite number of α.

Corollary 3.10. *Let $\alpha \in \mathbb{C}\setminus\{\frac{1}{k} \mid 1 \le k \le \frac{n-1}{2}\}$. Then there exists $\lambda \vdash n$ such that $f_\lambda(\alpha) \neq 0$, which has the following property; for any standard tableau T of shape λ there exists a $\mathfrak{sl}_n(\mathbb{C})$-intertwining operator $A^{(\alpha)} : (W_T^{(\alpha)})^* \to W_T^{(\alpha)}$ satisfying $A^{(\alpha)}((v_{T,T}^{(\alpha)})^*) = v_{T,T}^{(\alpha)}$ and*

$$\det(X)^2 = f_\lambda(\alpha)^{-2} \sum_S v_{S,T}^{(\alpha)} \cdot A^{(\alpha)}((v_{S,T}^{(\alpha)})^*). \tag{12}$$

Here the sum runs over all semi-standard tableaux S of shape λ and $(v_{S,T}^{(\alpha)})^*$ are defined by $(v_{S,T}^{(\alpha)})^*(v_{S',T}^{(\alpha)}) = \delta_{S,S'}$. More precisely, one may take

$$\lambda = (\overbrace{2,2,\dots,2}^{\frac{n}{2}}) \quad \text{if } n \text{ is even} \quad \text{or} \quad \lambda = (\overbrace{2,\dots,2}^{\frac{n-1}{2}},1) \quad \text{if } n \text{ is odd},$$

which satisfies the condition.

Proof. For α and λ in the corollary, it is easy to see that $f_\lambda(\alpha) \neq 0$. Consider a standard tableau T of shape λ. Then we see that $W_T^{(\alpha)} = W_T^{(0)}$ and $(v_{S,T}^{(\alpha)})^* = f_\lambda(\alpha)^{-1}(v_{S,T}^{(0)})^*$. Suppose the corollary is true for $\alpha = 0$. Using the intertwining operator $A^{(0)}$, we define $A^{(\alpha)}$ by $A^{(\alpha)} = f_\lambda(\alpha)^2 A^{(0)}$. Then we see that $A^{(\alpha)}((v_{T,T}^{(\alpha)})^*) = f_\lambda(\alpha)^2 A^{(0)}(f_\lambda(\alpha)^{-1}(v_{T,T}^{(0)})^*) = v_{T,T}^{(\alpha)}$ and

$$\sum_S v_{S,T}^{(\alpha)} \cdot A^{(\alpha)}((v_{S,T}^{(\alpha)})^*) = f_\lambda(\alpha)^2 \sum_S v_{S,T}^{(0)} \cdot A^{(0)}((v_{S,T}^{(0)})^*) = f_\lambda(\alpha)^2 \det(X)^2.$$

It is hence sufficient to prove the corollary for the case $\alpha = 0$.

In general, for a finite-dimensional irreducible $U(\mathfrak{sl}_n)$-module V and its dual module V^*,

$$\mathfrak{I} = \sum_i v_i \otimes v_i^* \in V \otimes V^*$$

defines an invariant of $\mathfrak{sl}_n(\mathbb{C})$, see [3]. Here v_i are a basis of V and v_i^* are the dual basis, i.e, $v_i^*(v_j) = \delta_{ij}$. Let λ be a partition whose parts are 1 or 2. Then $V = W_T^{(0)} \cong E^\lambda$ is self-dual, i.e., $V^* \cong_{\mathfrak{sl}_n(\mathbb{C})} V$. Therefore there exists an intertwining operator A' from V^* to V. Then the polynomial $\sum_S v_{S,T}^{(0)} \cdot A'((v_{S,T}^{(0)})^*) \in \mathcal{P}(\mathrm{Mat}_{n\times n})$ of degree $2n$ determined by $\mathfrak{I}$ is an invariant of $\mathfrak{sl}_n$ and hence

$$\sum_S v_{S,T}^{(0)} \cdot A'((v_{S,T}^{(0)})^*) = c \det(X)^2 \tag{13}$$

for some constant c. Comparing the coefficients of $x_{11}^2 \cdots x_{nn}^2$ in both sides in (13), we have $A'((v_{T,T}^{(0)})^*) = c x_{11} \cdots x_{nn} = c v_{T,T}^{(0)}$ and so $c \neq 0$. Hence $A^{(0)} = c^{-1} A'$ is our desired operator. ∎

Example 3.11. Let $T = \boxed{1}\boxed{2}$. The module $W_T^{(\alpha)}$ has a basis consisting of $v_+ = v^{(\alpha)}_{\boxed{1}\boxed{1},\boxed{1}\boxed{2}} = 2D^{(\alpha)}(1,1)$, $v = v^{(\alpha)}_{\boxed{1}\boxed{2},\boxed{1}\boxed{2}} = D^{(\alpha)}(1,2) + D^{(\alpha)}(2,1)$, $v_- = v^{(\alpha)}_{\boxed{2}\boxed{2},\boxed{1}\boxed{2}} = 2D^{(\alpha)}(2,2)$ if $\alpha \neq -1$. The linear map A determined by

$$A(v_+^*) = -\frac{1}{2}v_-, \quad A(v^*) = v, \quad A(v_-^*) = -\frac{1}{2}v_+$$

from $(W_T^{(\alpha)})^*$ to $W_T^{(\alpha)}$ defines an intertwining operator of $\mathfrak{sl}_2(\mathbb{C})$. Hence, by the corollary, we have

$$\begin{aligned}(1+\alpha)^2 \det(X)^2 =& v_+ \cdot A(v_+^*) + v \cdot A(v^*) + v_- \cdot A(v_-^*) = v^2 - v_+ \cdot v_- \\ =& (D^{(\alpha)}(1,2) + D^{(\alpha)}(2,1))^2 - 4D^{(\alpha)}(1,1)D^{(\alpha)}(2,2).\end{aligned}$$

4. Concluding remarks

4.1. Quantum analogue.

We give here a brief comment on a possible generalization of our theorems to the quantum group $U_q(\mathfrak{gl}_n)$ because it produces new interesting phenomena that have never appeared in the classical case.

Define the quantum α-determinant by

$$\det_{\alpha,q}(X) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} \alpha^{n-\nu(\sigma)} x_{\sigma(1)1} \cdots x_{\sigma(n)n}, \tag{14}$$

where $\mathrm{inv}(\sigma)$ is the inversion number of σ; $\mathrm{inv}(\sigma) = \#\{(i,j) \mid 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}$. In particular, $\det_q(X) = \det_{-1,q}(X)$ is the (usual) quantum determinant, see e.g. [4]. We define quantum analogue $v_{S,T}^{(\alpha,q)}$ of the element $v_{S,T}^{(\alpha)}$ by the q-Young symmetrizers studied by Gyoja [2]. For each standard tableau T, denote by $W_T^{(\alpha,q)}$ the quantum analogue of $W_T^{(\alpha)}$ given by the q-Young symmetrizer. Let λ be a partition of n and let $T_1, \dots, T_d$ be all standard tableaux of shape λ, where $d = f^\lambda$. Let $v_k^{(\alpha,q)} = v_{S_k,T_k}^{(\alpha,q)}$ be the highest weight vector of each $W_{T_k}^{(\alpha,q)}$. Then there exists a $d \times d$ matrix $F_\lambda(\alpha; q)$ such that

$$(v_1^{(\alpha,q)}, \dots, v_d^{(\alpha,q)}) = (v_1^{(0,q)}, \dots, v_d^{(0,q)}) F_\lambda(\alpha; q).$$

In the classical case, as we have seen in Corollary 3.4, $F_\lambda(\alpha; 1)$ is the scalar matrix $f_\lambda(\alpha)I$. It is observed, however, $F_\lambda(\alpha; q)$ is not, in general, a scalar matrix, not even a diagonal matrix, see [6]. Therefore, it is necessary to find a new basis other than the one obtained by the q-Young symmetrizers in order to diagonalize $F_\lambda(\alpha; q)$. In particular, one notes that the q-Young symmetrizer does not provide a formula like (6) in Proposition 3.1 in the quantum group case.

4.2. Immanant.

Recall the immanant. For a partition λ of n, the λ-immanant of X is defined by

$$\mathrm{Imm}_\lambda(X) = \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma) \prod_{i=1}^n x_{i\sigma(i)}.$$

Then, if we use the formula (8), we find the cyclic module $U(\mathfrak{gl}_n)\mathrm{Imm}_\lambda(X)$ is decomposed as $U(\mathfrak{gl}_n)\mathrm{Imm}_\lambda(X) \cong (E^\lambda)^{\oplus f^\lambda}$ as in the case of α-determinants. Also, since the function $\sigma \to \nu_n(\sigma)$ is a class function, the α-determinant is expanded by immanants;

$$\det_\alpha(X) = \sum_{\lambda \vdash n} \frac{f^\lambda}{n!} f_\lambda(\alpha) \mathrm{Imm}_\lambda(X)$$

by (7). Combining these facts, we have $V_n^{(\alpha)} \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} (E^\lambda)^{\oplus f^\lambda}$. This agrees with Theorem 1.1. However, if we consider the quantum group case, this discussion can not be applied, because the function $\sigma \mapsto \mathrm{inv}(\sigma)$ appeared in (14) is not a class function.

4.3. The case where $\alpha = \infty$.

We consider the case "$\alpha = \infty$" and describe the irreducible decomposition of $V_n^{(\infty)}$. Since $\det_\alpha(X)$ is a polynomial of degree $n-1$ in variable α, we can define a limit

$$\det_\infty(X) = \lim_{|\alpha| \to \infty} \alpha^{1-n} \det_\alpha(X) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \nu_n(\sigma) = 1}} x_{\sigma(1),1} \cdots x_{\sigma(n),n}. \tag{15}$$

For example,

$$\det_\infty \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = x_{21}x_{32}x_{13} + x_{31}x_{12}x_{23}.$$

Denote by $V_n^{(\infty)}$ the cyclic module $U(\mathfrak{g})\det_\infty(X)$. Then we have the following irreducible decomposition of $V_n^{(\infty)}$ as a corollary of Theorem 3.9.

Corollary 4.1.

$$V_n^{(\infty)} \cong \bigoplus_{\lambda:\mathrm{hook}} (E^\lambda)^{\oplus f^\lambda} = \bigoplus_{k=1}^{n} \left(E^{(k,1^{n-k})}\right)^{\oplus \binom{n-1}{k-1}},$$

where λ run over all hook partitions of n.

Proof. The degree of polynomial $f_\lambda(\alpha) \in \mathbb{Z}[\alpha]$ is equal to $n-d$, where d is the number of the main diagonal of the Young diagram λ. Therefore $\lim_{|\alpha|\to\infty} \alpha^{1-n} f_\lambda(\alpha)$ is zero unless $d = 1$, i.e., λ is a hook. For a hook $\lambda = (k, 1^{n-k})$, the number f^λ is given by the binomial coefficient $\binom{n-1}{k-1}$. Hence, the claim follows from Theorem 3.9. ∎

Example 4.2. When $n = 5$,

$$V_5^{(\infty)} \cong E^{(5)} \oplus \left(E^{(4,1)}\right)^{\oplus 4} \oplus \left(E^{(3,1,1)}\right)^{\oplus 6} \oplus \left(E^{(2,1,1,1)}\right)^{\oplus 4} \oplus E^{(1,1,1,1,1)}.$$

Remark 4.3. By (9), for each $\alpha > 0$, we can define a probability measure $\mathfrak{M}_n^{(\alpha)}$ on $\mathfrak{S}_n$ by

$$\mathfrak{M}_n^{(\alpha)}(\sigma) = \frac{\alpha^{n-\nu_n(\sigma)}}{\prod_{j=1}^{n-1}(1+j\alpha)} \qquad \text{for each } \sigma \in \mathfrak{S}_n.$$

This is called the Ewens measure in [5] but the definition is slightly different from ours. It is clear that $\mathfrak{M}_n^{(1)}$ is the uniform measure on $\mathfrak{S}_n$ and $\mathfrak{M}_n^{(0)} = \lim_{\alpha\to 0+} \mathfrak{M}_n^{(\alpha)}$ is the Dirac measure at the identity. Also we see that

$$\mathfrak{M}_n^{(\infty)}(\sigma) = \lim_{\alpha\to+\infty} \mathfrak{M}_n^{(\alpha)}(\sigma) = \begin{cases} 1/(n-1)! & \text{if } \nu_n(\sigma)=1, \\ 0 & \text{otherwise.} \end{cases}$$

Given a matrix X with non-negative entries x_{ij}, define a random variable X_σ by $X_\sigma = \prod_{i=1}^n x_{i\sigma(i)}$ on $\mathfrak{S}_n$. Then for $\alpha \in [0,+\infty]$ the α-determinant of X is essentially the mean value of X_σ with respect to $\mathfrak{M}_n^{(\alpha)}$:

$$\begin{aligned} \det_\alpha(X) =& \prod_{j=1}^{n-1}(1+j\alpha) \sum_{\sigma\in\mathfrak{S}_n} X_\sigma \mathfrak{M}_n^{(\alpha)}(\sigma) \qquad \text{for } 0 \le \alpha < +\infty, \\ \det_\infty(X) =& (n-1)! \sum_{\sigma\in\mathfrak{S}_n} X_\sigma \mathfrak{M}_n^{(\infty)}(\sigma). \end{aligned}$$

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