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Minimal polynomials and annihilators of generalized Verma modules of the scalar type

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Abstract. We construct a generator system of the annihilator of a generalized Verma module of a reductive Lie algebra induced from a character of a parabolic subalgebra as an analogue of the minimal polynomial of a matrix. *Mathematics Subject Classification*: 22E47, 16S30.
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1. Introduction

In the representation theory of a real reductive Lie group G the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of the complexification $\mathfrak{g}$ of the Lie algebra of G plays an important role. For example, any irreducible admissible representation τ of G realized in a subspace E of sections of a certain G-homogeneous vector bundle is a simultaneous eigenspace of $Z(\mathfrak{g})$ parameterized by the infinitesimal character of τ. The differential equations induced from $Z(\mathfrak{g})$ are often used to characterize the subspace E.

If the representation τ is small, we expect more differential equations corresponding to the primitive ideal I_τ, that is, the annihilator of τ in $U(\mathfrak{g})$. For the study of I_τ and these differential equations it is interesting and important to get a good generator system of I_τ.

Let $\mathfrak{p}_\Theta$ be a parabolic subalgebra containing a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and let λ be a character of $\mathfrak{p}_\Theta$. Then the generalized Verma module of the scalar type is by definition

$$M_\Theta(\lambda) = U(\mathfrak{g})/J_\Theta(\lambda) \quad \text{with } J_\Theta(\lambda) = \sum_{X\in\mathfrak{p}_\Theta} U(\mathfrak{g})\bigl(X-\lambda(X)\bigr). \tag{1}$$

In this paper we construct generator systems of the annihilator $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ of the generalized Verma module $M_\Theta(\lambda)$ in a unified way. If τ can be realized in a space E of sections of a line bundle over a generalized flag manifold, the annihilator of the corresponding generalized Verma module kills E.

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When $\mathfrak{g} = \mathfrak{gl}_n$, [20] and [21] construct such a generator system by *generalized Capelli operators* defined through *quantized elementary divisors.* This is a good generator system and in fact it is used there to characterize the image of the Poisson integrals on various boundaries of the symmetric space and also to define generalized hypergeometric functions. A similar generator system is studied by [18] for $\mathfrak{g} = \mathfrak{o}_n$ but it is difficult to construct the corresponding generator system in the case of other general reductive Lie groups. On the other hand, in [22] we give other generator systems as a quantization of *minimal polynomials* when $\mathfrak{g}$ is classical.

Associated to a faithful finite dimensional representation π of $\mathfrak{g}$ and a $\mathfrak{g}$-module M, [22] defines a minimal polynomial $q_{\pi,M}(x)$ as is quoted in Definition 2.3 and Definition 2.5. If $\mathfrak{g} = \mathfrak{gl}_n$ and π is a natural representation of $\mathfrak{g}$, $q_{\pi,M}(x)$ is characterized by the condition $q_{\pi,M}(F_\pi)M = 0$. Here $F_\pi = \Bigl(E_{ij}\Bigr)_{\substack{1\le i\le n\\ 1\le j\le n}}$ is the matrix whose (i,j)-component is the fundamental matrix unit E_{ij} and then F_π is identified with a square matrix with components in $\mathfrak{g} \subset U(\mathfrak{g})$. In this case $q_{\pi,M_\Theta(\lambda)}(x)$ is naturally regarded as a quantization of the minimal polynomial which corresponds to the conjugacy class of matrices given by a *classical limit* of $M_\Theta(\lambda)$. For example, if $\mathfrak{p}_\Theta$ is a maximal parabolic subalgebra of $\mathfrak{gl}_n$, the minimal polynomial $q_{\pi,M_\Theta(\lambda)}(x)$ is a polynomial of degree 2.

For general π and $\mathfrak{g}$, the matrix F_π is the image $\Bigl(p(E_{ij})\Bigr)$ of $\bigl(E_{ij}\bigr)$ under the contragredient map p of π and then F_π is a square matrix of the size $\dim\pi$ with components in $\mathfrak{g}$. For example, if π is the natural representation of $\mathfrak{o}_n$, then the (i,j)-component of F_π equals $\frac{1}{2}(E_{ij} - E_{ji})$.

In [22] we calculate the minimal polynomial $q_{\pi,M_\Theta(\lambda)}(x)$ for the natural representation π of each type of classical Lie algebra $\mathfrak{g}$ and by putting

$$I_{\pi,\Theta}(\lambda) = \sum_{i,j} U(\mathfrak{g}) q_{\pi,M_\Theta(\lambda)}(F_\pi)_{ij} + \sum_{\Delta\in Z(\mathfrak{g})\cap \operatorname{Ann} M_\Theta(\lambda)} U(\mathfrak{g})\Delta, \tag{2}$$

it is shown that

$$J_\Theta(\lambda) = I_{\pi,\Theta}(\lambda) + J(\lambda_\Theta) \quad \text{with } J(\lambda_\Theta) = \sum_{X\in\mathfrak{b}} U(\mathfrak{g})\bigl(X - \lambda(X)\bigr) \tag{3}$$

for a generic λ. This equality is essential because it shows that $q_{\pi,M_\Theta(\lambda)}(F_\pi)_{ij}$ give elements killing $M_\Theta(\lambda)$ which cannot be described by $Z(\mathfrak{g})$ and define differential equations characterizing the local sections of the corresponding line bundle of a generalized flag manifold. Moreover (3) assures that $I_{\pi,\Theta}(\lambda)$ equals $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ for a generic λ (Proposition 3.11).

In this paper, π may be any faithful irreducible finite dimensional representation of a reductive Lie algebra $\mathfrak{g}$. In Theorem 2.24 we calculate a polynomial $q_{\pi,\Theta}(x;\lambda)$ which is divisible by the the minimal polynomial $q_{\pi,M_\Theta(\lambda)}(x)$ and it is shown in Theorem 2.29 that the former polynomial equals the latter for a generic λ. If $\mathfrak{p}_\Theta = \mathfrak{b}$, this result gives the *characteristic polynomial* associated to π as is stated in Theorem 2.33, which is studied by [11]. We prove Theorem 2.24 in a similar way as in [22] but in a more generalized way and the proof is used to get the condition for (3). Another proof which is similar as is given in [11] is also possible and it is based on the decomposition of the tensor product of some finite dimensional representations of $\mathfrak{g}$ given by Proposition 2.27. The proof of Theorem 2.29 uses *infinitesimal Mackey's tensor product theorem* which is explained in Appendix A.

In §3 we examine (3) and obtain a sufficient condition for (3) by Theorem 3.21. Proposition 3.25 and Proposition 3.27 assure that a generic λ satisfies this condition if π is one of many proper representations including minuscule representations, adjoint representations, representations of multiplicity free, and representations with regular highest weights. In such cases the sufficient condition is satisfied if λ is not in the union of a certain finite number of complex hypersurfaces in the parameter space, which are defined by the difference of certain weights of the representation π. On the other hand, in Appendix B, we give counter examples for which our sufficient condition is never satisfied by any λ. In Proposition 3.3 we also study the element of $Z(\mathfrak{g})$ contained in $I_{\pi,\Theta}(\lambda)$.

A corresponding problem in the *classical limit* is to construct a generator system of the defining ideal of the coadjoint orbit of $\mathfrak{g}$ and in fact Theorem 3.28 is considered to be the classical limit of Corollary 3.22.

If π is smaller, the two-sided ideal $I_{\pi,\Theta}(\lambda)$ is better in general and therefore in §4 we give examples of the characteristic polynomials of some small π for every simple $\mathfrak{g}$ and describe some minimal polynomials, especially in each case where $\mathfrak{p}_\Theta$ is maximal. Note that the minimal polynomial is a divisor of the characteristic polynomial evaluated at the infinitesimal character. In Proposition 4.12 we present a two-sided ideal of $U(\mathfrak{g})$ for every $(\mathfrak{g}, \mathfrak{p}_\Theta)$ and examine the condition (3) for this ideal by applying Theorem 3.21. In particular, the condition is satisfied if the infinitesimal character of $M_\Theta(\lambda)$ is regular in the case when $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{o}_{2n+1}$, $\mathfrak{sp}_n$ or G_2. The condition is also satisfied if the infinitesimal character is in the positive Weyl chamber containing the infinitesimal characters of the Verma modules which have finite dimensional irreducible quotients.

Some applications of our results in this paper to the integral geometry will be found in [22, §5] and [23].

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2. Minimal Polynomials and Characteristic Polynomials

For an associative algebra $\mathfrak{A}$ and a positive integer N, we denote by $M(N, \mathfrak{A})$ the associative algebra of square matrices of size N with components in $\mathfrak{A}$. We use the standard notation $\mathfrak{gl}_n$, $\mathfrak{o}_n$ and $\mathfrak{sp}_n$ for classical Lie algebras over $\mathbb{C}$. The exceptional simple Lie algebra is denoted by its type E_6, E_7, E_8, F_4 or G_2.

The Lie algebra $\mathfrak{gl}_N$ is identified with $M(N, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^N)$ with the bracket $[X, Y] = XY - YX$. In general, if we fix a base $\{v_1, \ldots, v_N\}$ of an N-dimensional vector space V over $\mathbb{C}$, we naturally identify an element $X = (X_{ij})$ of $M(N, \mathbb{C})$ with an element of $\mathrm{End}(V)$ by $Xv_j = \sum_{i=1}^N X_{ij} v_i$. Let $E_{ij} = \left(\delta_{\mu i}\delta_{\nu j}\right)_{\substack{1 \le \mu \le N \\ 1 \le \nu \le N}} \in M(N, \mathbb{C})$ be the standard matrix units and put $E_{ij}^* = E_{ji}$. Note that the symmetric bilinear form

$$\langle X, Y\rangle = \operatorname{Trace} XY \quad \text{for} \quad X, Y \in \mathfrak{gl}_N \tag{4}$$

on $\mathfrak{gl}_N$ is non-degenerate and satisfies

$$\begin{aligned} \langle E_{ij}, E_{\mu\nu}\rangle = \langle E_{ij}, E^*_{\nu\mu}\rangle &= \delta_{i\nu}\delta_{j\mu}, \\ X &= \sum_{i,j}\langle X, E_{ji}\rangle E_{ij}, \\ \langle \operatorname{Ad}(g)X, \operatorname{Ad}(g)Y\rangle = \langle X, Y\rangle &\quad \text{for } X,\ Y \in \mathfrak{gl}_N \text{ and } g \in GL(N,\mathbb{C}). \end{aligned} \tag{5}$$

In general, for a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we denote by $U(\mathfrak{g})$ and $Z(\mathfrak{g})$ the *universal enveloping algebra* of $\mathfrak{g}$ and the *center* of $U(\mathfrak{g})$, respectively. Then we have the following lemma.

Lemma 2.1 ([22, Lemma 2.1]). *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $(\pi, \mathbb{C}^N)$ be a representation of $\mathfrak{g}$. Let p be a linear map of $\mathfrak{gl}_N$ to $U(\pi(\mathfrak{g}))$ satisfying*

$$p([X, Y]) = [X, p(Y)] \quad \textit{for } X \in \pi(\mathfrak{g}) \textit{ and } Y \in \mathfrak{gl}_N, \tag{6}$$

that is, $p \in \operatorname{Hom}_{\pi(\mathfrak{g})}\big(\mathfrak{gl}_N, U(\pi(\mathfrak{g}))\big)$.

Fix $q(x) \in \mathbb{C}[x]$ and put

$$\begin{cases} F = \Big(p(E_{ij})\Big)_{\substack{1\le i\le N\\ 1\le j\le N}} \in M\big(N, U(\pi(\mathfrak{g}))\big), \\ \Big(Q_{ij}\Big)_{\substack{1\le i\le N\\ 1\le j\le N}} = q(F) \in M\big(N, U(\pi(\mathfrak{g}))\big). \end{cases} \tag{7}$$

Then

$$\Big(p(\operatorname{Ad}(g)E_{ij})\Big)_{\substack{1\le i\le N\\ 1\le j\le N}} = {}^t g\, F\, {}^t g^{-1} \quad \textit{for} \ \ g \in GL(n,\mathbb{C}) \tag{8}$$

and

$$\begin{aligned} [X, Q_{ij}] &= \sum_{\mu=1}^{N} X_{\mu i}Q_{\mu j} - \sum_{\nu=1}^{N} X_{j\nu}Q_{i\nu} \\ &= \sum_{\mu=1}^{N}\langle X, E_{i\mu}\rangle Q_{\mu j} - \sum_{\nu=1}^{N} Q_{i\nu}\langle X, E_{\nu j}\rangle \quad \textit{for } X = \Big(X_{\mu\nu}\Big)_{\substack{1\le\mu\le N\\ 1\le\nu\le N}} \in \pi(\mathfrak{g}). \end{aligned} \tag{9}$$

Hence the linear map $\mathfrak{gl}_N \to U(\pi(\mathfrak{g}))$ defined by $E_{ij} \mapsto Q_{ij}$ is an element of $\operatorname{Hom}_{\pi(\mathfrak{g})}\big(\mathfrak{gl}_N, U(\pi(\mathfrak{g}))\big)$. In particular, $\sum_{i=1}^N Q_{ii} \in Z(\pi(\mathfrak{g}))$.

Remark 2.2. The referee suggested that we should give the reader the following conceptual explanation of Lemma 2.1: Since $(\mathfrak{gl}_N)^* \simeq M(N,\mathbb{C})^*$ is naturally identified with $M(N,\mathbb{C})$ via (4), the linear $\pi(\mathfrak{g})$-homomorphism $p : \mathfrak{gl}_N \to U(\pi(\mathfrak{g}))$ is considered as an element of $(\mathfrak{gl}_N)^* \otimes U(\pi(\mathfrak{g})) \simeq M(N,\mathbb{C}) \otimes U(\pi(\mathfrak{g})) \simeq M(N, U(\pi(\mathfrak{g})))$. By this identification, the image of p equals ${}^tF$ and hence (8) holds almost immediately. Furthermore, (9) is equivalent to the fact that $(M(N,\mathbb{C}) \otimes U(\pi(\mathfrak{g})))^{\pi(\mathfrak{g})}$ is a subalgebra of $M(N,\mathbb{C})\otimes U(\pi(\mathfrak{g})) \simeq M(N, U(\pi(\mathfrak{g})))$.

Now we introduce the minimal polynomial defined by [22], which will be studied in this section.

Definition 2.3 (characteristic polynomials and minimal polynomials)**.** Given a Lie algebra $\mathfrak{g}$, a faithful finite dimensional representation $(\pi, \mathbb{C}^N)$ and a $\mathfrak{g}$-homomorphism p of $\mathrm{End}(\mathbb{C}^N) \simeq \mathfrak{gl}_N$ to $U(\mathfrak{g})$. Here we identify $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ through π. Let $\hat{Z}(\mathfrak{g})$ denote the quotient field of $Z(\mathfrak{g})$. (Recall $Z(\mathfrak{g})$ is an integral domain.) Put $F = \bigl(p(E_{ij})\bigr) \in M(N, U(\mathfrak{g}))$. We say $q_F(x) \in \hat{Z}(\mathfrak{g})[x]$ is the *characteristic polynomial* of F if it is the monic polynomial with the minimal degree which satisfies

$$q_F(F) = 0$$

in $M\bigl(N, \hat{Z}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g})\bigr)$. Suppose moreover a $\mathfrak{g}$-module M is given. Then we say $q_{F,M}(x) \in \mathbb{C}[x]$ is the *minimal polynomial* of the pair (F, M) if it is the monic polynomial with the minimal degree which satisfies

$$q_{F,M}(F)M = 0.$$

Remark 2.4. The uniqueness of the characteristic (or minimal) polynomial is clear if it exists. Suppose $\mathfrak{g}$ is reductive. Then the characteristic polynomial actually exists by [22, Theorem 2.6]. The same theorem assures the existence of the minimal polynomial if M has a finite length or an infinitesimal character.

Definition 2.5. If the symmetric bilinear form (4) is non-degenerate on $\pi(\mathfrak{g})$, the orthogonal projection of $\mathfrak{gl}_N$ onto $\pi(\mathfrak{g})$ satisfies the assumption for p in Lemma 2.1, which we call the *canonical projection* of $\mathfrak{gl}_N$ to $\pi(\mathfrak{g}) \simeq \mathfrak{g}$. In this case we put $F_\pi = \bigl(p(E_{ij})\bigr)$. Then we call $q_{F_\pi}(x)$ (resp. $q_{F_\pi,M}(x)$) in Definition 2.3 the characteristic polynomial of π (resp. the minimal polynomial of the pair (π, M)) and denote it by $q_\pi(x)$ (resp. $q_{\pi,M}(x)$).

Remark 2.6. For a given involutive automorphism σ of $\mathfrak{gl}_N$, put

$$\mathfrak{g} = \{X \in \mathfrak{gl}_N;\ \sigma(X) = X\}$$

and let π be the inclusion map of $\mathfrak{g} \subset \mathfrak{gl}_N$. Then $p(X) = \frac{X+\sigma(X)}{2}$.

Hereafter in the general theory of minimal polynomials which we shall study, we restrict our attention to a fixed finite dimensional representation (π, V) of $\mathfrak{g}$ such that

$$\begin{cases} \mathfrak{g} \text{ is a reductive Lie algebra over } \mathbb{C}, \\ \pi \text{ is faithful and irreducible.} \end{cases} \tag{10}$$

Moreover we put $N = \dim V$ and identify V with $\mathbb{C}^N$ through some basis of V. The assumption of Definition 2.5 is then satisfied.

Remark 2.7. i) The dimension of the center of $\mathfrak{g}$ is at most one.
ii) Fix $g \in GL(V)$. If we replace (π, V) by (π^g, V) with $\pi^g(X) = \mathrm{Ad}(g)\pi(X)$ for $X \in \mathfrak{g}$ in Lemma 2.1, $F_\pi \in M(N, \mathfrak{g})$ is naturally changed into ${}^t g^{-1} F_\pi\, {}^t g$ under the fixed identification $V \simeq \mathbb{C}^N$. This is clear from Lemma 2.1 (cf. [22, Remark 2.7 ii)]). iii) Exceptionally the condition (10) will not be assumed in Definition 2.36 and Proposition 2.37.

Definition 2.8 (root system)**.** We fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and let $\Sigma(\mathfrak{g})$ be a root system for the pair $(\mathfrak{g},\mathfrak{a})$. We choose an order in $\Sigma(\mathfrak{g})$ and denote by $\Sigma(\mathfrak{g})^+$ and $\Psi(\mathfrak{g})$ the set of the positive roots and the fundamental system, respectively. For each root $\alpha\in\Sigma(\mathfrak{g})$ we fix a root vector $X_\alpha\in\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\bar{\mathfrak{n}}$ be the triangular decomposition of $\mathfrak{g}$ so that $\mathfrak{n}$ is spanned by X_α with $\alpha\in\Sigma(\mathfrak{g})^+$. We say $\mu\in\mathfrak{a}^*$ is *dominant* if and only if

$$2\frac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\notin\{-1,-2,\ldots\}\quad\text{for any }\alpha\in\Sigma(\mathfrak{g})^+.$$

Let us prepare some lemmas and definitions.

Lemma 2.9. *Let U be a k-dimensional subspace of $\mathfrak{gl}_N$ such that $\langle\ ,\ \rangle|_U$ is non-degenerate. Let p_U be the orthogonal projection of $\mathfrak{gl}_N$ to U and let $\{v_1,\ldots,v_k\}$ be a basis of U with $\langle v_2,v_j\rangle=0$ for $2\le j\le k$. Suppose that $u\in\mathfrak{gl}_N$ satisfies $\langle u,v_j\rangle=0$ for $2\le j\le k$. Then $p_U(u)=\frac{\langle u,v_1\rangle}{\langle v_1,v_2\rangle}v_2$.*

The proof of this lemma is easy and we omit it.

Lemma 2.10. *Choose a base $\{v_i;\,i=1,\ldots,N\}$ of V for the identification $V\simeq\mathbb{C}^N$ so that v_i are weight vectors with weights $\varpi_i\in\mathfrak{a}^*$, respectively. We identify $\mathfrak{g}$ with the subalgebra $\pi(\mathfrak{g})$ of $\mathfrak{gl}_N\simeq M(N,\mathbb{C})$ and put $\mathfrak{a}_N=\sum_{i=1}^N\mathbb{C}E_{ii}$. For $F_\pi=\Big(F_{ij}\Big)_{\substack{1\le i\le N\\1\le j\le N}}$ we have*

$$\begin{aligned}
&F_{ii}=\varpi_i=\sum_{j=1}^N\varpi_i(E_{jj})E_{jj},\\
&\operatorname{ad}(H)(F_{ij})=(\varpi_i-\varpi_j)(H)F_{ij}\ (\forall H\in\mathfrak{a}),\\
&\langle F_{ij},E_{\mu\nu}\rangle\neq0\textit{ with }i\neq j\textit{ implies }\varpi_i-\varpi_j=\varpi_\nu-\varpi_\mu\in\Sigma(\mathfrak{g}),\\
&\mathfrak{a}=\sum_{i=1}^N\mathbb{C}F_{ii}\subset\mathfrak{a}_N,\ \mathfrak{n}=\sum_{\varpi_i-\varpi_j\in\Sigma(\mathfrak{g})^+}\mathbb{C}F_{ij},\ \bar{\mathfrak{n}}=\sum_{\varpi_j-\varpi_i\in\Sigma(\mathfrak{g})^+}\mathbb{C}F_{ij}
\end{aligned}\tag{11}$$

under the identification $\mathfrak{a}^*\simeq\mathfrak{a}\subset\mathfrak{a}_N\simeq\mathfrak{a}_N^*$ by the bilinear form (4).

Proof. Note that $H\in\mathfrak{a}$ is identified with $\sum_{j=1}^N\varpi_j(H)E_{jj}\in\mathfrak{a}_N\subset\mathfrak{gl}_N$. Hence $\operatorname{ad}(H)(E_{ij})=(\varpi_i-\varpi_j)(H)E_{ij}$ and therefore $\operatorname{ad}(H)(F_{ij})=(\varpi_i-\varpi_j)(H)F_{ij}$. In particular we have $F_{ii}\in\mathfrak{a}$. Since

$$\langle H,F_{ii}\rangle=\langle H,E_{ii}\rangle=\langle\sum_{j=1}^N\varpi_j(H)E_{jj},E_{ii}\rangle=\varpi_i(H)\quad(\forall H\in\mathfrak{a}),$$

we get $F_{ii}=\varpi_i$.

For each root α, the condition $(X_\alpha)_{ij}=\langle X_\alpha,E_{ji}\rangle\neq0$ means $\varpi_i-\varpi_j=\alpha$. Hence if $i\neq j$ and $X\in\mathfrak{a}+\sum_{\alpha\in\Sigma(\mathfrak{g}),\,\alpha\neq\varpi_j-\varpi_i}\mathbb{C}X_\alpha$, then $\langle E_{ij},X\rangle=0$ and therefore $\langle F_{ij},X\rangle=0$. Hence $F_{ij}=0$ if $i\neq j$ and $\varpi_j-\varpi_i\notin\Sigma(\mathfrak{g})$. On the other hand, if $\varpi_j-\varpi_i\in\Sigma(\mathfrak{g})$, we can easily get $F_{ij}=CX_{\varpi_i-\varpi_j}$ for some $C\in\mathbb{C}$. Hence $\langle F_{ij},E_{\mu\nu}\rangle=0$ if $\varpi_i-\varpi_j\neq\varpi_\nu-\varpi_\mu$. ∎

Through the identification of $\mathfrak{a}^* \simeq \mathfrak{a} \subset \mathfrak{a}_N$ in the lemma, we introduce the symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{a}^*$. We note this bilinear form is real-valued and positive definite on $\sum_{\alpha\in\Psi(\mathfrak{g})} \mathbb{R}\alpha$.

Now we take a subset $\Theta \subset \Psi(\mathfrak{g})$ with $\Theta \neq \Psi(\mathfrak{g})$ and fix it.

Definition 2.11 (generalized Verma module). Put

$$\begin{aligned}
\mathfrak{a}_\Theta &= \{H \in \mathfrak{a};\ \alpha(H) = 0, \quad \forall\alpha \in \Theta\},\\
\mathfrak{g}_\Theta &= \{X \in \mathfrak{g};\ [X, H] = 0, \quad \forall H \in \mathfrak{a}_\Theta\},\\
\mathfrak{m}_\Theta &= \{X \in \mathfrak{g}_\Theta;\ \langle X, H\rangle = 0, \quad \forall H \in \mathfrak{a}_\Theta\},\\
\Sigma(\mathfrak{g})^- &= \{\alpha;\ -\alpha \in \Sigma(\mathfrak{g})^+\},\\
\Sigma(\mathfrak{g}_\Theta) &= \{\alpha \in \Sigma(\mathfrak{g});\ \alpha(H) = 0, \quad \forall H \in \mathfrak{a}_\Theta\},\\
\Sigma(\mathfrak{g}_\Theta)^+ &= \Sigma(\mathfrak{g}_\Theta) \cap \Sigma(\mathfrak{g})^+, \quad \Sigma(\mathfrak{g}_\Theta)^- = \{-\alpha;\ \alpha \in \Sigma(\mathfrak{g}_\Theta)^+\},\\
\mathfrak{n}_\Theta &= \sum_{\alpha\in\Sigma(\mathfrak{g})^+\setminus\Sigma(\mathfrak{g}_\Theta)} \mathbb{C}X_\alpha, \quad \bar{\mathfrak{n}}_\Theta = \sum_{\alpha\in\Sigma(\mathfrak{g})^-\setminus\Sigma(\mathfrak{g}_\Theta)} \mathbb{C}X_\alpha,\\
\mathfrak{b} &= \mathfrak{a} + \mathfrak{n}, \quad \mathfrak{p}_\Theta = \mathfrak{g}_\Theta + \mathfrak{n}_\Theta,\\
\rho &= \frac{1}{2}\sum_{\alpha\in\Sigma(\mathfrak{g})^+} \alpha, \quad \rho(\Theta) = \frac{1}{2}\sum_{\alpha\in\Sigma(\mathfrak{g}_\Theta)^+} \alpha, \quad \rho_\Theta = \rho - \rho(\Theta).
\end{aligned}$$

For $\Lambda \in \mathfrak{a}^*$ which satisfies $2\frac{\langle\Lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \{0, 1, 2, \ldots\}$ for $\alpha \in \Theta$, let $U_{(\Theta,\Lambda)}$ denote the finite dimensional irreducible $\mathfrak{g}_\Theta$-module with highest weight Λ. By the trivial action of $\mathfrak{n}_\Theta$, we consider $U_{(\Theta,\Lambda)}$ to be a $\mathfrak{p}_\Theta$-module. Put

$$M_{(\Theta,\Lambda)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} U_{(\Theta,\Lambda)}. \tag{12}$$

Then $M_{(\Theta,\Lambda)}$ is called a *generalized Verma module of the finite type.*

Remark 2.12. i) $\mathfrak{p}_\Theta$ is a parabolic subalgebra containing the Borel subalgebra $\mathfrak{b}$. $\mathfrak{p}_\Theta = \mathfrak{m}_\Theta + \mathfrak{a}_\Theta + \mathfrak{n}_\Theta$ gives its direct sum decomposition.
ii) Every finite dimensional irreducible $\mathfrak{p}_\Theta$-module is isomorphic to $U_{(\Theta,\Lambda)}$ with a suitable choice of Λ.
iii) $M_{(\emptyset,\Lambda)}$ is nothing but the Verma module for the highest weight $\Lambda \in \mathfrak{a}^*$.
iv) Let u_Λ be a highest weight vector of $U_{(\Theta,\Lambda)}$. Then $1 \otimes u_\Lambda$ is a highest weight vector of $M_{(\Theta,\Lambda)}$. Moreover $1 \otimes u_\Lambda$ generates $M_{(\Theta,\Lambda)}$ because

$$\begin{aligned}
M_{(\Theta,\Lambda)} &= U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} U_{(\Theta,\Lambda)} = U(\bar{\mathfrak{n}}_\Theta) \otimes_{\mathbb{C}} U(\mathfrak{p}_\Theta) \otimes_{U(\mathfrak{p}_\Theta)} U_{(\Theta,\Lambda)}\\
&= U(\bar{\mathfrak{n}}_\Theta) \otimes_{\mathbb{C}} U_{(\Theta,\Lambda)} = U(\bar{\mathfrak{n}}_\Theta) \otimes_{\mathbb{C}} U(\bar{\mathfrak{n}} \cap \mathfrak{g}_\Theta) u_\Lambda = U(\bar{\mathfrak{n}})(1 \otimes u_\Lambda).
\end{aligned}$$

Hence $M_{(\Theta,\Lambda)}$ is a highest weight module and is therefore a quotient of the Verma module $M_{(\emptyset,\Lambda)}$.
v) If $\langle\Lambda, \alpha\rangle = 0$ for each $\alpha \in \Theta$, then $\dim U_{(\Theta,\Lambda)} = 1$ and we have the character λ_Θ of $\mathfrak{p}_\Theta$ such that $Xu_\Lambda = \lambda_\Theta(X)u_\Lambda$ for $X \in \mathfrak{p}_\Theta$. Since

$$U(\mathfrak{g}) = U(\bar{\mathfrak{n}}_\Theta) \oplus \sum_{X\in\mathfrak{p}_\Theta} U(\mathfrak{g})\bigl(X - \lambda_\Theta(X)\bigr)$$

is a direct sum and $M_{(\Theta,\Lambda)} = U(\bar{\mathfrak{n}}_\Theta) \otimes_{\mathbb{C}} \mathbb{C}u_\Lambda$, we have the kernel of the surjective $U(\mathfrak{g})$-homomorphism $U(\mathfrak{g}) \to M_{(\Theta,\Lambda)}$ defined by $D \mapsto D(1 \otimes u_\Lambda)$ equals $\sum_{X\in\mathfrak{p}_\Theta} U(\mathfrak{g})\bigl(X - \lambda_\Theta(X)\bigr)$.

Definition 2.13 (generalized Verma module of the scalar type). For $\lambda \in \mathfrak{a}_\Theta^*$ define a character λ_Θ of $\mathfrak{p}_\Theta$ by $\lambda_\Theta(X+H) = \lambda(H)$ for $X \in \mathfrak{m}_\Theta + \mathfrak{n}_\Theta$ and $H \in \mathfrak{a}_\Theta$. Put

$$\begin{aligned} J_\Theta(\lambda) &= \sum_{X \in \mathfrak{p}_\Theta} U(\mathfrak{g})\bigl(X - \lambda_\Theta(X)\bigr),\\ J(\lambda_\Theta) &= \sum_{X \in \mathfrak{b}} U(\mathfrak{g})\bigl(X - \lambda_\Theta(X)\bigr),\\ M_\Theta(\lambda) &= U(\mathfrak{g})/J_\Theta(\lambda), \quad M(\lambda_\Theta) = U(\mathfrak{g})/J(\lambda_\Theta). \end{aligned} \tag{13}$$

Then $M_\Theta(\lambda)$ is isomorphic to $M_{(\Theta,\lambda_\Theta)}$, which is called a *generalized Verma module of the scalar type*. If $\Theta = \emptyset$, we denote $J_\emptyset(\lambda)$ and $M_\emptyset(\lambda)$ by $J(\lambda)$ and $M(\lambda)$, respectively.

Definition 2.14 (Weyl group). Let W denote the Weyl group of $\Sigma(\mathfrak{g})$, which is generated by the reflections $w_\alpha : \mathfrak{a}^* \ni \mu \mapsto \mu - 2\frac{\langle \mu,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha \in \mathfrak{a}^*$ with respect to $\alpha \in \Psi(\mathfrak{g})$. Put

$$\begin{aligned} W_\Theta &= \{w \in W;\ w\bigl(\Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\bigr) = \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\},\\ W(\Theta) &= \{w \in W;\ w\bigl(\Sigma(\mathfrak{g}_\Theta)^+\bigr) \subset \Sigma(\mathfrak{g})^+\}. \end{aligned} \tag{14}$$

Then each element $w \in W(\Theta)$ is a unique element with the smallest length in the right coset wW_Θ and the map $W(\Theta) \times W_\Theta \ni (w_1, w_2) \mapsto w_1 w_2 \in W$ is a bijection.

For $w \in W$ and $\mu \in \mathfrak{a}^*$, define

$$w.\mu = w(\mu + \rho) - \rho. \tag{15}$$

Here we note that W_Θ is generated by the reflections w_α with $\alpha \in \Theta$ and

$$\langle \rho_\Theta, \alpha\rangle = 0 \quad \text{for} \quad \alpha \in \Sigma(\mathfrak{g}_\Theta). \tag{16}$$

Definition 2.15 (infinitesimal character). Let $D \in U(\mathfrak{g})$. We denote by $D_\mathfrak{a}$ the element of $U(\mathfrak{a})$ which satisfies $D - D_\mathfrak{a} \in \bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$ and identify $D_\mathfrak{a} \in U(\mathfrak{a}) \simeq S(\mathfrak{a})$ with a polynomial function on $\mathfrak{a}^*$. Then $\Delta_\mathfrak{a}(\mu) = \Delta_\mathfrak{a}(w.\mu)$ for $\Delta \in Z(\mathfrak{g})$, $\mu \in \mathfrak{a}^*$, and $w \in W$.

Let $\mu \in \mathfrak{a}^*$. We say a $\mathfrak{g}$-module M has *infinitesimal character* μ if each $\Delta \in Z(\mathfrak{g})$ operates by the scalar $\Delta_\mathfrak{a}(\mu)$ in M. We say an infinitesimal character μ is regular if $\langle \mu + \rho, \alpha\rangle \neq 0$ for any $\alpha \in \Sigma(\mathfrak{g})$.

Remark 2.16. The generalized Verma module $M_{(\Theta,\Lambda)}$ in Definition 2.11 has infinitesimal character Λ. It is clear by Remark 2.12 iv).

Definition 2.17 (Casimir operator). Let $\{X_i;\ i = 1, \ldots, \omega\}$ be a basis of $\mathfrak{g}$. Then put

$$\Delta_\pi = \sum_{i=1}^{\omega} X_i X_i^*$$

with the dual basis $\{X_i^*\}$ of $\{X_i\}$ with respect to the symmetric bilinear form (4) under the identification $\mathfrak{g} \subset \mathfrak{gl}_N$ through π and call Δ_π the *Casimir operator* of $\mathfrak{g}$ for π.

Remark 2.18. As is well-known, $\Delta_\pi \in Z(\mathfrak{g})$ and Δ_π does not depend on the choice of $\{X_i\}$.

We may assume in Definition 2.17 that $\{X_1, \ldots, X_{\omega'}\}$ and $\{X_{\omega'+1}, \ldots, X_\omega\}$ be bases of $\mathfrak{g}_\Theta$ and $\bar{\mathfrak{n}}_\Theta + \mathfrak{n}_\Theta$, respectively. Then $X_i^* \in \mathfrak{g}_\Theta$ for $i = 1, \ldots, \omega'$ and

$$\Delta_\pi^\Theta = \sum_{i=1}^{\omega'} X_i X_i^* \tag{17}$$

is the Casimir operator of $\mathfrak{g}_\Theta$ for π.

Lemma 2.19. *Fix a basis $\{H_1, \ldots, H_r\}$ of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$.*
i) *Let $\{H_1^*, \ldots, H_r^*\}$ be the dual basis of $\{H_1, \ldots, H_r\}$. Put $H_\alpha = [X_\alpha, X_{-\alpha}]$. Then*

$$\begin{aligned}
\Delta_\pi &= \sum_{\alpha \in \Sigma(\mathfrak{g})} \frac{X_\alpha X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle} + \sum_{i=1}^r H_i H_i^* \\
&= \sum_{i=1}^r H_i H_i^* + \sum_{\alpha \in \Sigma(\mathfrak{g})^+} \left(\frac{2X_{-\alpha} X_\alpha}{\langle X_\alpha, X_{-\alpha}\rangle} + \frac{\alpha(H_\alpha) H_\alpha}{\langle H_\alpha, H_\alpha \rangle} \right) \\
&= \Delta_\pi^\Theta + \sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)} \left(\frac{2X_\alpha X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle} - \frac{\alpha(H_\alpha) H_\alpha}{\langle H_\alpha, H_\alpha \rangle} \right).
\end{aligned}$$

ii) *Let M be a highest weight module of $\mathfrak{g}$ with highest weight $\mu \in \mathfrak{a}^*$. Then $\Delta_\pi v = \langle \mu, \mu + 2\rho \rangle v$ for any $v \in M$.*
iii) *Let v be a weight vector of π belonging to an irreducible representation of $\mathfrak{g}_\Theta$ realized as a subrepresentation of $\pi|_{\mathfrak{g}_\Theta}$ and let ϖ denote the lowest weight of the irreducible subrepresentation. Then*

$$\begin{aligned}
\Delta_\pi v &= \langle \bar{\pi}, \bar{\pi} - 2\rho \rangle v, \\
\Delta_\pi^\Theta v &= \langle \varpi, \varpi - 2\rho(\Theta) \rangle v, \\
\sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)} \frac{X_\alpha X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle} v &= \frac{1}{2} \langle \bar{\pi} - \varpi, \bar{\pi} + \varpi - 2\rho \rangle v.
\end{aligned}$$

Here $\bar{\pi}$ denotes the lowest weight of π.
iv) *Fix $\beta \in \Sigma(\mathfrak{g})^+$ and put $\mathfrak{g}(\beta) = \mathbb{C}X_\beta + \mathbb{C}X_{-\beta} + \sum_{i=1}^r \mathbb{C}H_i$. Let v be a weight vector of π belonging to an irreducible representation of $\mathfrak{g}(\beta)$ realized as a subrepresentation of $\pi|_{\mathfrak{g}(\beta)}$ and let ϖ denote the lowest weight of the irreducible subrepresentation. Let $\varpi + \ell\beta$ be the weight of v. Then*

$$\frac{X_\beta X_{-\beta}}{\langle X_\beta, X_{-\beta}\rangle} v = -\Big(\ell \langle \varpi + \frac{\ell - 1}{2} \beta, \beta \rangle \Big) v. \tag{18}$$

v) *Suppose $\mathfrak{g}$ is simple. Let $\alpha_{\max}$ is the maximal root of $\Sigma(\mathfrak{g})^+$ and let $B(\ ,\ )$ be the Killing form of $\mathfrak{g}$. Then*

$$B(\alpha_{\max}, \alpha_{\max} + 2\rho) = 1.$$

Proof. i) Note that

$$\langle H_\alpha, H_\alpha\rangle = \langle H_\alpha, [X_\alpha, X_{-\alpha}]\rangle = \langle [H_\alpha, X_\alpha], X_{-\alpha}\rangle = \alpha(H_\alpha)\langle X_\alpha, X_{-\alpha}\rangle \tag{19}$$

Since the dual base of $\{X_\alpha,\ H_i;\ \alpha\in\Sigma(\mathfrak{g}),\ i=1,\dots,r\}$ equals $\{\frac{X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle},\ H_i^*;\ \alpha\in\Sigma(\mathfrak{g}),\ i=1,\dots,r\}$, the claim is clear.

ii) Let v_μ be a highest weight vector of M. Then

$$\begin{aligned}\Delta_\pi v_\mu &= \sum_{i=1}^r H_i H_i^* v_\mu + \sum_{\alpha\in\Sigma(\mathfrak{g})^+} \frac{\alpha(H_\alpha)H_\alpha}{\langle H_\alpha, H_\alpha\rangle} v_\mu\\ &= \sum_{i=1}^r \mu(H_i)\mu(H_i^*)v_\mu + \sum_{\alpha\in\Sigma(\mathfrak{g})^+}\frac{\alpha(H_\alpha)\mu(H_\alpha)}{\langle H_\alpha, H_\alpha\rangle}v_\mu.\end{aligned}$$

Hence $\Delta_\pi v_\mu = \langle \mu, \mu+2\rho\rangle v_\mu$ because H_α is a non-zero constant multiple of α with the identification $\mathfrak{a}^*\simeq\mathfrak{a}$ by $\langle\ ,\ \rangle$ and therefore $\Delta_\pi v = \langle\mu,\mu+2\rho\rangle v$ because M is generated by v_μ.

iii) Let $v_{\bar\pi}$ be a lowest weight vector of π. Then we have $\Delta_\pi v_{\bar\pi} = \langle \bar\pi, \bar\pi - 2\rho\rangle v_{\bar\pi}$ and therefore $\Delta_\pi v = \langle\bar\pi,\bar\pi-2\rho\rangle v$. Similarly we have $\Delta_\pi^\Theta v = \langle\varpi,\varpi-2\rho(\Theta)\rangle v$.

Let ϖ' be the weight of v. Then we have

$$\begin{aligned}\sum_{\alpha\in\Sigma(\mathfrak{g})^+\setminus\Sigma(\mathfrak{g}_\Theta)}\frac{X_\alpha X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle}v &= \frac12\Delta_\pi v - \frac12\Delta_\pi^\Theta v + \langle\varpi',\rho_\Theta\rangle v\\ &= \frac12\langle\bar\pi-\varpi,\bar\pi+\varpi-2\rho\rangle v.\end{aligned}$$

Here we note that $\langle\varpi',\rho_\Theta\rangle = \langle\varpi,\rho_\Theta\rangle$.

iv) By the same argument as above we have

$$\frac{2X_\beta X_{-\beta}}{\langle X_\beta, X_{-\beta}\rangle}v + \sum_{i=1}^r H_iH_i^* v - \frac{\beta(H_\beta)H_\beta}{\langle H_\beta, H_\beta\rangle}v = \langle\varpi,\varpi-\beta\rangle v.$$

Hence

$$\begin{aligned}\frac{2X_\beta X_{-\beta}}{\langle X_\beta, X_{-\beta}\rangle}v &= \langle\varpi,\varpi-\beta\rangle v - \langle\varpi+\ell\beta,\varpi+\ell\beta\rangle v + \langle\beta,\varpi+\ell\beta\rangle v\\ &= -\big(2\ell\langle\varpi,\beta\rangle + \ell(\ell-1)\langle\beta,\beta\rangle\big)v\end{aligned}$$

v) Suppose π is the adjoint representation of the simple Lie algebra $\mathfrak{g}$. Then for $H\in\mathfrak{a}$ we have

$$\begin{aligned}\langle\pi(\Delta_\pi)(H), H\rangle &= \sum_{\alpha\in\Sigma(\mathfrak{g})}\frac{\langle[X_\alpha,[X_{-\alpha},H]],H\rangle}{\langle X_\alpha, X_{-\alpha}\rangle}\\ &= \sum_{\alpha\in\Sigma(\mathfrak{g})}\frac{-\langle[X_{-\alpha},H],[X_\alpha,H]\rangle}{\langle X_\alpha,X_{-\alpha}\rangle}\\ &= \sum_{\alpha\in\Sigma(\mathfrak{g})}\alpha(H)^2\\ &= \langle H, H\rangle.\end{aligned}$$

Hence $\pi(\Delta_\pi)(H) = H$ and $B(\alpha_{\max},\alpha_{\max}+2\rho) = B(-\alpha_{\max},-\alpha_{\max}-2\rho) = 1$. ∎

Definition 2.20 (weights)**.** Let $\mathcal{W}(\pi)$ denote the set of the weights of the finite dimensional irreducible representation π of $\mathfrak{g}$. For $\varpi \in \mathcal{W}(\pi)$ define a real constant

$$D_\pi(\varpi) = \frac{1}{2}\langle \bar{\pi} - \varpi, \bar{\pi} + \varpi - 2\rho \rangle. \tag{20}$$

Here $\bar{\pi}$ is the lowest weight of π. Put $R_+ = \{\sum_{\alpha \in \Psi(\mathfrak{g})} m_\alpha \alpha;\ m_\alpha \in \{0, 1, 2, \ldots\}\}$. We define a partial order among the elements of $\mathcal{W}(\pi)$ so that $\varpi \le \varpi'$ if and only if $\varpi' - \varpi \in R_+$.

Moreover we put

$$\begin{aligned} \mathcal{W}_\Theta(\pi) &= \{\varpi \text{ are the highest weights of the irreducible components of } \pi|_{\mathfrak{g}_\Theta}\}, \\ \overline{\mathcal{W}}_\Theta(\pi) &= \{\varpi \text{ are the lowest weights of the irreducible components of } \pi|_{\mathfrak{g}_\Theta}\}, \\ \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta} &= \{\varpi|_{\mathfrak{a}_\Theta};\ \varpi \in \mathcal{W}(\pi)\}. \end{aligned}$$

Let μ and $\mu' \in \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}$. Then we define $\mu \le_\Theta \mu'$ if and only if $\mu' - \mu \in \{\sum_{\alpha \in \Psi(\mathfrak{g}) \setminus \Theta} m_\alpha \alpha|_{\mathfrak{a}_\Theta};\ m_\alpha \in \{0, 1, 2, \ldots\}\}$.

Remark 2.21. i) $\mathcal{W}_\emptyset(\pi) = \overline{\mathcal{W}}_\emptyset(\pi) = \mathcal{W}(\pi)$ and $\overline{\mathcal{W}}_\Theta(\pi) = -\mathcal{W}_\Theta(\pi^*)$. Here (π^*, V^*) denotes the contragredient representation of (π, V) defined by

$$(\pi^*(X)v^*)(v) = -v^*(\pi(X)v) \text{ for } X \in \mathfrak{g},\ v^* \in V^* \text{ and } v \in V. \tag{21}$$

ii) $\mathcal{W}(\pi)|_{\mathfrak{a}_\Theta} = \{\varpi|_{\mathfrak{a}_\Theta};\ \varpi \in \mathcal{W}_\Theta(\pi)\} = \{\varpi|_{\mathfrak{a}_\Theta};\ \varpi \in \overline{\mathcal{W}}_\Theta(\pi)\}$.
iii) Suppose ϖ and $\varpi' \in \mathcal{W}(\pi)$ and put $\varpi' - \varpi = \sum_{\alpha \in \Psi(\mathfrak{g})} m_\alpha \alpha$. Then $\varpi|_{\mathfrak{a}_\Theta} \le_\Theta \varpi'|_{\mathfrak{a}_\Theta}$ if and only if $m_\alpha \ge 0$ for any $\alpha \in \Psi(\mathfrak{g}) \setminus \Theta$. Hence $\bar{\pi}|_{\mathfrak{a}_\Theta}$ is the smallest element of $\mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}$. Note that $\varpi \le \varpi'$ if and only if $\varpi \le_\emptyset \varpi'$.

Lemma 2.22. *Let ϖ and $\varpi' \in \mathcal{W}(\pi)$.*
i) *If $\alpha = \varpi' - \varpi \in \Psi(\mathfrak{g})$, then $D_\pi(\varpi) - D_\pi(\varpi') = \langle \varpi, \varpi' - \varpi \rangle$.*
ii) *Suppose $\varpi' \in \overline{\mathcal{W}}_\Theta(\pi)$, $\varpi < \varpi'$ and $\varpi|_{\mathfrak{a}_\Theta} = \varpi'|_{\mathfrak{a}_\Theta}$. Then $D_\pi(\varpi) < D_\pi(\varpi')$.*

Proof. ii) Note that

$$D_\pi(\varpi) - D_\pi(\varpi') = \frac{1}{2}\langle \varpi' - \varpi, \varpi + \varpi' - 2\rho \rangle.$$

The assumption in ii) implies $\varpi' - \varpi = \sum_{\alpha \in \Theta} m_\alpha \alpha$ with $m_\alpha \ge 0$. Here at least one of m_α is positive. Hence $\langle \varpi' - \varpi, \rho \rangle > 0$. Since ϖ' are the lowest weights of irreducible representations of $\mathfrak{g}_\Theta$, $\langle \alpha, \varpi' \rangle \le 0$ for $\alpha \in \Theta$. Thus we have $\langle \sum_{\alpha \in \Theta} m_\alpha \alpha, 2\varpi' - \sum_{\alpha \in \Theta} m_\alpha \alpha - 2\rho \rangle < 0$.

i) Put $\alpha = \varpi' - \varpi$. Then

$$D_\pi(\varpi) - D_\pi(\varpi') - \langle \varpi, \varpi' - \varpi \rangle = -\frac{1}{2}\langle \alpha, 2\rho - \alpha \rangle,$$

which equals 0 if $\alpha \in \Psi(\mathfrak{g})$ because $w_\alpha(\Sigma(\mathfrak{g})^+ \setminus \{\alpha\}) = \Sigma(\mathfrak{g})^+ \setminus \{\alpha\}$. ∎

Now we give a key lemma which is used to calculate our minimal polynomial.

Lemma 2.23. *Fix an irreducible decomposition $\bigoplus_{i=1}^{\kappa}(\pi_i, V_i)$ of $(\pi|_{\mathfrak{g}_\Theta}, V)$ and a basis $\{v_{i,1}, \ldots, v_{i,m_i}\}$ of V_i so that $v_{i,j}$ are weight vectors for $\mathfrak{a}$. Let $\varpi_{i,j}$ and ϖ_i be the weight of $v_{i,j}$ and the lowest weight of the representation π_i, respectively.*

Suppose $\varpi_{i,j} = \varpi_{i',j'}$. Then for a positive integer k with $k \geq 2$ and complex numbers $\mu_1, \ldots, \mu_k$

$$\Bigl(\prod_{\nu=1}^{k}(F_\pi - \mu_\nu)\Bigr)_{(i',j')(i,j)} \equiv \Bigl(\prod_{\nu=1}^{k-1}(F_\pi - \mu_\nu)\Bigr)_{(i',j')(i,j)}\bigl(\varpi_i - \mu_k + D_\pi(\varpi_i)\bigr)$$
$$\mod U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta) + \sum_{\substack{(s,t),(s'',t'');\\ \varpi_s|_{\mathfrak{a}_\Theta} <_\Theta \varpi_i|_{\mathfrak{a}_\Theta}\\ \varpi_{s,t} = \varpi_{s'',t''}}} \mathbb{C}\,\Bigl(\prod_{\nu=1}^{k-1}(F_\pi - \mu_\nu)\Bigr)_{(s'',t'')(s,t)}.$$

Proof. Note that $\varpi_{i,j} \equiv \varpi_i \mod U(\mathfrak{g})\mathfrak{m}_\Theta$. It follows from Lemma 2.10 that

$$F_{(s,t)(i,j)} \equiv \delta_{si}\delta_{tj}\varpi_i \mod U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta)$$

if $\varpi_{s,t} - \varpi_{i,j} \notin \Sigma(\mathfrak{g})^- \setminus \Sigma(\mathfrak{g}_\Theta)$.

Put $F^\ell = \prod_{\nu=1}^{\ell}(F_\pi - \mu_\nu)$. Then Lemma 2.10 implies

$$\begin{aligned}
&F^k_{(i',j')(i,j)} - F^{k-1}_{(i',j')(i,j)}(\varpi_i - \mu_k)\\
&\equiv \sum_{\varpi_{s,t} - \varpi_{i,j} \in \Sigma(\mathfrak{g})^- \setminus \Sigma(\mathfrak{g}_\Theta)} [F^{k-1}_{(i',j')(s,t)}, F_{(s,t)(i,j)}] \mod U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta) + \bar{\mathfrak{n}}_\Theta U(\mathfrak{g})\\
&= \sum_{\substack{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\\ \varpi_{s,t} = \varpi_{i,j} - \alpha}} \frac{\langle E_{(s,t)(i,j)}, X_\alpha\rangle}{\langle X_\alpha, X_{-\alpha}\rangle}[F^{k-1}_{(i',j')(s,t)}, X_{-\alpha}]\\
&= \sum_{\substack{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\\ \varpi_{s,t} = \varpi_{i,j} - \alpha\\ \varpi_{s',t'} - \varpi_{s'',t''} = \alpha}} \frac{\langle E_{(s,t)(i,j)}, X_\alpha\rangle\langle E_{(s',t')(s'',t'')}, X_{-\alpha}\rangle}{\langle X_\alpha, X_{-\alpha}\rangle}\\
&\quad\cdot \bigl(\delta_{ss''}\delta_{tt''}F^{k-1}_{(i',j')(s',t')} - \delta_{i's'}\delta_{j't'}F^{k-1}_{(s'',t'')(s,t)}\bigr)\\
&= \frac{1}{2}\langle \bar{\pi} - \varpi_i, \bar{\pi} + \varpi_i - 2\rho\rangle F^{k-1}_{(i',j')(i,j)}\\
&\quad - \sum_{\substack{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\\ \varpi_{s,t} = \varpi_{s'',t''} = \varpi_{i,j} - \alpha}} \frac{\langle E_{(s,t)(i,j)}, X_\alpha\rangle\langle E_{(i',j')(s'',t'')}, X_{-\alpha}\rangle}{\langle X_\alpha, X_{-\alpha}\rangle} F^{k-1}_{(s'',t'')(s,t)}.
\end{aligned} \tag{22}$$

In the above the second equality follows from (11) and Lemma 2.9 with $U = \mathfrak{g}$. The third equality follows from Lemma 2.1 with

$$X_{-\alpha} = \sum_{\varpi_{s',t'} - \varpi_{s'',t''} = \alpha} \langle E_{(s',t')(s'',t'')}, X_{-\alpha}\rangle E_{(s'',t'')(s',t')}$$

which follows from the identification $\mathfrak{g} \subset \mathfrak{gl}_N$ together with the property of $\langle\ ,\ \rangle$.

Put $X^\vee = -{}^tX$ for $X \in M(N, \mathbb{C}) \simeq \mathfrak{gl}_N$. Let $\{v^*_{i,j}\}$ be the dual base of $\{v_{i,j}\}$ and consider the contragredient representation π^* of π. Then $\pi^*(X) = X^\vee$

for $X \in \mathfrak{g}$ with respect to these basis. Then $\langle X, Y\rangle = \langle X^\vee, Y^\vee\rangle$ for X, $Y \in \mathfrak{g}$ and

$$\sum_{\alpha\in\Sigma(\mathfrak{g})^+\setminus\Sigma(\mathfrak{g}_\Theta)} \frac{X^\vee_{-\alpha}X^\vee_\alpha}{\langle X^\vee_{-\alpha}, X^\vee_\alpha\rangle} v^*_{i,j} = \sum_{\substack{\alpha\in\Sigma(\mathfrak{g})^+\setminus\Sigma(\mathfrak{g}_\Theta)\\ \varpi_{s,t}=\varpi_{i,j}-\alpha\\ \varpi_{s',t'}=\varpi_{i,j}}} \frac{\langle E_{(s,t)(s',t')}, X^\vee_{-\alpha}\rangle\langle E_{(i,j)(s,t)}, X^\vee_\alpha\rangle}{\langle X^\vee_{-\alpha}, X^\vee_\alpha\rangle} v^*_{s',t'},$$

which is proved to be equal to $D_\pi(\varpi_i)v^*_{i,j}$ by Lemma 2.19 iii) because $(\bar\pi, \varpi_i, \rho)$ for π changes into $(-\bar\pi, -\varpi_i, -\rho)$ in the dual π^* with the reversed order of roots. This implies the last equality in (22).

Note that if $D \in \bar{\mathfrak{n}}_\Theta U(\mathfrak{g}) + U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta)$ satisfies $[H, D] = 0$ for all $H \in \mathfrak{a}_\Theta$, then $D \in U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta)$. Since the condition $\varpi_{i,j} - \varpi_{s,t} \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)$ implies $\varpi_s|_{\mathfrak{a}_\Theta} <_\Theta \varpi_i|_{\mathfrak{a}_\Theta}$, we have the lemma. ■

Theorem 2.24. *Retain the notation in* Definition 2.20. *For $\varpi \in \mathfrak{a}^*$ we identify $\varpi|_{\mathfrak{a}_\Theta}$ with a linear function on $\mathfrak{a}^*_\Theta$ by $\varpi|_{\mathfrak{a}_\Theta}(\lambda) = \langle\lambda_\Theta, \varpi\rangle$ for $\lambda \in \mathfrak{a}^*_\Theta$. Put*

$$\begin{aligned}\Omega_{\pi,\Theta} &= \{(\varpi|_{\mathfrak{a}_\Theta}, D_\pi(\varpi));\ \varpi \in \overline{\mathcal{W}}_\Theta(\pi)\},\\ q_{\pi,\Theta}(x;\lambda) &= \prod_{(\mu,C)\in\Omega_{\pi,\Theta}} \bigl(x - \mu(\lambda) - C\bigr).\end{aligned}\tag{23}$$

Then $q_{\pi,\Theta}(F_\pi;\lambda)M_\Theta(\lambda) = 0$ for any $\lambda \in \mathfrak{a}^*_\Theta$.

Proof. For any $D \in U(\mathfrak{g})$ there exists a unique constant $T(D) \in \mathbb{C}$ satisfying

$$T(D) \equiv D \mod \bar{\mathfrak{n}}U(\mathfrak{g}) + J_\Theta(\lambda)$$

because the dimension of the space $M_\Theta(\lambda)/\bar{\mathfrak{n}}M_\Theta(\lambda)$ equals 1. Notice that

$$J_\Theta(\lambda) = \sum_{H\in\mathfrak{a}_\Theta} U(\mathfrak{g})(H - \lambda(H)) + U(\mathfrak{g})(\mathfrak{m}_\Theta + \mathfrak{n}_\Theta).$$

Use the notation in Lemma 2.23. Since

$$\mathrm{ad}(H)q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)} = (\varpi_{i',j'} - \varpi_{i,j})(H)q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)} \quad \text{for} \quad H \in \mathfrak{a},$$

$T(q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)}) = 0$ if $\varpi_{i,j} \neq \varpi_{i',j'}$.

Next assume $\varpi_{i,j} = \varpi_{i',j'}$ and put

$$\begin{aligned}\Omega_{\pi,\Theta,i} &= \{(\mu, C) \in \Omega_{\pi,\Theta};\ \mu \le_\Theta \varpi_i|_{\mathfrak{a}_\Theta}\},\\ q_{\pi,\Theta,i}(x;\lambda) &= \prod_{(\mu,C)\in\Omega_{\pi,\Theta,i}} \bigl(x - \mu(\lambda) - C\bigr).\end{aligned}$$

Then $q(F_\pi)_{(i',j')(i,j)} \in J_\Theta(\lambda)$ for any $q(x) \in \mathbb{C}[x]$ which is a multiple of $q_{\pi,\Theta,i}(x;\lambda)$. It is proved by the induction on $\varpi_i|_{\mathfrak{a}_\Theta}$ with the partial order $\le_\Theta$. Take $i_0 \in \{1, \ldots, \kappa\}$ so that $\varpi_{i_0} = \bar\pi$. If $i = i_0$ then Lemma 2.10 and Lemma 2.23 with $D_\pi(\varpi_i) = D_\pi(\bar\pi) = 0$ imply our claim. If $i \neq i_0$ then $\bar\pi|_{\mathfrak{a}_\Theta} <_\Theta \varpi_i|_{\mathfrak{a}_\Theta}$ and therefore $\deg_x q_{\pi,\Theta,i}(x;\lambda) \ge 2$. Hence we can use Lemma 2.23 again to prove our claim inductively.

Thus we get the condition

$$T(q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)}) = 0 \text{ for any } (i,j) \text{ and } (i',j'). \tag{24}$$

Let $\mathbf{V}(\lambda)$ denote the $\mathbf{C}$-subspace of $U(\mathfrak{g})$ spanned by $q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)}$. Then $\mathbf{V}(\lambda)$ is $\mathrm{ad}(\mathfrak{g})$-stable by Lemma 2.1. The $\mathfrak{g}$-module

$$M_\lambda = \mathbf{V}(\lambda)M_\Theta(\lambda)$$

is contained in $\bar{\mathfrak{n}}M_\Theta(\lambda)$ because putting $u_\lambda = 1 \bmod J_\Theta(\lambda)$,

$$M_\lambda = \mathbf{V}(\lambda)U(\bar{\mathfrak{n}})u_\lambda = U(\bar{\mathfrak{n}})\mathbf{V}(\lambda)u_\lambda \subset U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}U(\mathfrak{g})u_\lambda = \bar{\mathfrak{n}}M_\Theta(\lambda).$$

On the other hand, since $M_\Theta(\lambda)$ is irreducible if λ belongs to a suitable open subset of $\mathfrak{a}_\Theta^*$, $M_\lambda = \{0\}$ in the open set. If we fix a base $\{Y_1,\ldots,Y_m\}$ of $\bar{\mathfrak{n}}_\Theta$, we have the unique expression

$$q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)} \equiv \sum_\nu Q_\nu(\lambda)Y_1^{\nu_1}\cdots Y_m^{\nu_m} \mod J_\Theta(\lambda)$$

with polynomial functions $Q_\nu(\lambda)$. All these $Q_\nu(\lambda)$ vanish on the open set and therefore they are identically zero and we have $\mathbf{V}(\lambda)\subset J_\Theta(\lambda)$ for any λ. We have then for any λ

$$M_\lambda = \mathbf{V}(\lambda)U(\mathfrak{g})u_\lambda = U(\mathfrak{g})\mathbf{V}(\lambda)u_\lambda = \{0\}. \qquad \blacksquare$$

Theorem 2.24 is one of our central results since $q_{\pi,\Theta}(x;\lambda) = q_{\pi,M_\Theta(\lambda)}(x)$ for a generic $\lambda\in\mathfrak{a}_\Theta^*$. Before showing this minimality, which will be done in Theorem 2.29, we mention the possibility of other approaches to Theorem 2.24. In fact we have three different proofs. The first one given above has the importance that the calculation in the proof is also used in §3. to study the properties of the two-sided ideal of $U(\mathfrak{g})$ generated by $q_{\pi,\Theta}(F_\pi;\lambda)_{ij}$. The second one comes from a straight expansion of the method in [10] and [11] to construct characteristic polynomials. In the following we first discuss it. The third one is based on *infinitesimal Mackey's tensor product theorem* which we explain in Appendix A. With this method we shall get the sufficient condition for the minimality of $q_{\pi,\Theta}(x;\lambda)$ (Theorem 2.29) and slightly strengthen the result of Theorem 2.24 (Theorem 2.31).

Definition 2.25. Let (π^*,V^*) be the contragredient representation of (π,V) and $\{v_1^*,\ldots,v_N^*\}$ the dual base of the base $\{v_1,\ldots,v_N\}$ of V. For a $\mathfrak{g}$-module M define the homomorphism

$$h_{(\pi,M)} : M(N,U(\mathfrak{g})) \to \mathrm{End}\,(M\otimes V^*)$$

of associative algebras by

$$\big(h_{(\pi,M)}(Q)\big)\Big(\sum_{j=1}^N u_j\otimes v_j^*\Big) = \sum_{i=1}^N\sum_{j=1}^N \big(Q_{ij}u_j\big)\otimes v_i^* \tag{25}$$

for $u_j\in M$ and $Q=\big(Q_{ij}\big)\in M(N,U(\mathfrak{g}))$. Then $QM=0$, namely, $Q_{ij}\in \mathrm{Ann}(M)$ for any i,j if and only if $h_{(\pi,M)}(Q)=0$.

The following lemma is considered in [10] and [11].

Lemma 2.26. *Let M be a $\mathfrak{g}$-module. For an element $\sum_{j=1}^N u_j \otimes v_j^*$ of $M \otimes V^*$ with $u_j \in M$, we have*

$$2h_{(\pi,M)}(F_\pi)\Big(\sum_{j=1}^N u_j \otimes v_j^*\Big) = \sum_{j=1}^N \Delta_\pi(u_j) \otimes v_j^* + \sum_{j=1}^N u_j \otimes \Delta_\pi(v_j^*) - \Delta_\pi\Big(\sum_{j=1}^N u_j \otimes v_j^*\Big).$$

In particular $h_{(\pi,M)}(F_\pi) \in \operatorname{End}_{\mathfrak{g}}(M \otimes V^*)$.

Proof. Let $\{X_1, \ldots, X_\omega\}$ be a base of $\mathfrak{g}$ and let $\{X_1^*, \ldots, X_\omega^*\}$ be its dual base with respect to $\langle\ ,\ \rangle$. Then

$$\begin{aligned}
&\sum_{j=1}^N \Delta_\pi(u_j) \otimes v_j^* + \sum_{j=1}^N u_j \otimes \Delta_\pi(v_j^*) - \Delta_\pi\Big(\sum_{j=1}^N u_j \otimes v_j^*\Big)\\
&\quad = -\sum_{j=1}^N \sum_{\nu=1}^\omega X_\nu^* u_j \otimes X_\nu v_j^* - \sum_{j=1}^N \sum_{\nu=1}^\omega X_\nu u_j \otimes X_\nu^* v_j^*\\
&\quad = \sum_{j=1}^N \sum_{\nu=1}^\omega \Big(X_\nu^* u_j \otimes \sum_{i=1}^N \langle X_\nu, E_{ij}\rangle v_i^* + X_\nu u_j \otimes \sum_{i=1}^N \langle X_\nu^*, E_{ij}\rangle v_i^*\Big)\\
&\quad = 2\sum_{i=1}^N \sum_{j=1}^N (p(E_{ij})u_j) \otimes v_i^*.
\end{aligned}$$

Here we use the fact that $Xv_j^* = -\sum_{i=1}^N \langle X, E_{ij}\rangle v_i^*$ for $X \in \mathfrak{g}$ because $Xv_j = \sum_{i=1}^N \langle X, E_{ji}\rangle v_i$. ■

Now we examine the tensor product $M \otimes V^*$ in the preceding lemma when M is realized as a finite dimensional quotient of a generalized Verma module $M_\Theta(\lambda)$.

Proposition 2.27 (a character identity for a tensor product). *Put*

$$\chi_\Lambda = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\Lambda+\rho)}}{\prod_{\alpha \in \Sigma(\mathfrak{g})^+}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})}$$

for $\Lambda \in \mathfrak{a}^*$. If $\langle \Lambda, \alpha\rangle = 0$ for any $\alpha \in \Theta$, then

$$\chi_{\pi^*}\chi_\Lambda = \sum_{\varpi \in \mathcal{W}(\pi^*)} m_{\pi^*,\Theta}(\varpi)\chi_{\Lambda+\varpi} \tag{26}$$

by denoting

$$m_{\pi^*,\Theta}(\varpi) = \dim\{v^* \in V^*;\ Hv^* = \varpi(H)v^*\ (\forall H \in \mathfrak{a}),\ \ Xv^* = 0\ (\forall X \in \mathfrak{g}_\Theta \cap \mathfrak{n})\}.$$

Here χ_{π^*} is the character of the representation (π^*, V^*) and for $\mu \in \mathfrak{a}^*$, e^μ denotes the function on $\mathfrak{a}$ which takes the value $e^{\mu(H)}$ at $H \in \mathfrak{a}$.

Proof. It is sufficient to prove (26) under the condition that $\langle\Lambda,\alpha\rangle$ is a sufficiently large real number for any $\alpha\in\Psi(\mathfrak g)\setminus\Theta$ because both hand sides of (26) are holomorphic with respect to $\Lambda\in\mathfrak a^*$. Put

$$\begin{aligned}
\mathfrak a_0^* &= \{\mu\in\mathfrak a^*;\ \langle\mu,\alpha\rangle\in\mathbb R\quad \bigl(\forall\alpha\in\Sigma(\mathfrak g)\bigr)\},\\
\mathfrak a_+^* &= \{\mu\in\mathfrak a_0^*;\ \langle\mu,\alpha\rangle\ge 0\quad \bigl(\forall\alpha\in\Sigma(\mathfrak g)^+\bigr)\},\\
\chi_\Lambda^+ &= \frac{\sum_{w\in W_\Theta}\operatorname{sgn}(w)e^{w(\Lambda+\rho)}}{\prod_{\alpha\in\Sigma(\mathfrak g)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})},\\
\bar\chi_\varpi &= \frac{\sum_{w'\in W_\Theta}\operatorname{sgn}(w')e^{w'(\varpi+\rho(\Theta))}}{\prod_{\alpha\in\Sigma(\mathfrak g_\Theta)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})}.
\end{aligned}$$

Then $\chi_{\pi^*}=\sum_{\varpi\in\mathcal W(\pi^*)}m_{\pi^*,\Theta}(\varpi)\bar\chi_\varpi$ by Weyl's character formula and if $\varpi\in\mathcal W(\pi^*)$ satisfies $m_{\pi^*,\Theta}(\varpi)>0$, then $\Lambda+\varpi\in\mathfrak a_+^*$ and

$$\begin{aligned}
\bar\chi_\varpi\chi_\Lambda^+\prod_{\alpha\in\Sigma(\mathfrak g)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}) &= \frac{\sum_{w\in W_\Theta}\operatorname{sgn}(w)e^{w(\varpi+\rho(\Theta))}}{\prod_{\alpha\in\Sigma(\mathfrak g_\Theta)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})}e^{\Lambda+\rho_\Theta}\sum_{w'\in W_\Theta}\operatorname{sgn}(w')e^{w'\rho(\Theta)}\\
&= \sum_{w\in W_\Theta}\operatorname{sgn}(w)e^{w(\Lambda+\varpi+\rho)}\\
&\equiv e^{\Lambda+\varpi+\rho}\mod\sum_{\mu\in\mathfrak a_0^*\setminus\mathfrak a_+^*}\mathbb Ze^\mu.
\end{aligned}$$

For any $w\in W\setminus W_\Theta$ there exists $\alpha\in\Sigma(\mathfrak g)^-\setminus\Sigma(\mathfrak g_\Theta)$ with $w\alpha\in\Sigma(\mathfrak g)^+$ and then the value $-\langle w(\Lambda+\rho),w\alpha\rangle=-\langle(\Lambda+\rho),\alpha\rangle$ is sufficiently large and therefore

$$\bar\chi_\varpi(\chi_\Lambda-\chi_\Lambda^+)\prod_{\alpha\in\Sigma(\mathfrak g)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})\in\sum_{\mu\in\mathfrak a_0^*\setminus\mathfrak a_+^*}\mathbb Ze^\mu.$$

Hence

$$\chi_{\pi^*}\chi_\Lambda\prod_{\alpha\in\Sigma(\mathfrak g)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})\equiv\sum_{\varpi\in\mathcal W_\Theta(\pi^*)}m_{\pi^*,\Theta}(\varpi)e^{\Lambda+\varpi+\rho}\mod\sum_{\mu\in\mathfrak a_0^*\setminus\mathfrak a_+^*}\mathbb Ze^\mu$$

and we have the proposition because $\chi_{\pi^*}\chi_\Lambda\prod_{\alpha\in\Sigma(\mathfrak g)^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})$ is an odd function under W. ■

Lemma 2.28 (eigenvalue). *Let (π_Λ,V_Λ) be an irreducible finite dimensional representation of $\mathfrak g$ with highest weight Λ. Suppose $\langle\Lambda,\alpha\rangle=0$ for $\alpha\in\Theta$ and $\langle\Lambda+\varpi,\alpha\rangle\ge 0$ for $\varpi\in\mathcal W_\Theta(\pi^*)$ and $\alpha\in\Psi(\mathfrak g)\setminus\Theta$. Then the set of the eigenvalues of $h_{\pi,V_\Lambda}(F_\pi)\in\operatorname{End}(V_\Lambda\otimes V^*)$ without counting their multiplicities equals*

$$\begin{aligned}
&\{-\langle\Lambda,\varpi\rangle+\frac12\langle\pi^*-\varpi,\pi^*+\varpi+2\rho\rangle;\ \varpi\in\mathcal W_\Theta(\pi^*)\}\\
&=\{\langle\Lambda,\varpi\rangle+\frac12\langle\bar\pi-\varpi,\bar\pi+\varpi-2\rho\rangle;\ \varpi\in\overline{\mathcal W}_\Theta(\pi)\}.
\end{aligned}$$

Here we identify π^* with the highest weight of (π^*,V^*).

Proof. The assumption of the lemma and Proposition 2.27 imply

$$\pi^* \otimes \pi_\Lambda = \sum_{\varpi \in \mathcal{W}_\Theta(\pi^*)} m_{\pi^*,\Theta}(\varpi)\pi_{\Lambda+\varpi}$$

and hence by Lemma 2.19 ii) and Lemma 2.26 the eigenvalues of $2h_{\pi,V_\Lambda}(F_\pi)$ are

$$\langle\Lambda,\Lambda+2\rho\rangle+\langle\pi^*,\pi^*+2\rho\rangle-\langle\Lambda+\varpi,\Lambda+\varpi+2\rho\rangle = -2\langle\Lambda,\varpi\rangle+\langle\pi^*-\varpi,\pi^*+\varpi+2\rho\rangle$$

with $\varpi \in \mathcal{W}_\Theta(\pi^*)$. Since $\overline{\mathcal{W}}_\Theta(\pi) = -\mathcal{W}_\Theta(\pi^*)$, we have the lemma. ■

Proof of Theorem 2.24 (the 2nd version)**.** This proof differs from the previous one in how to deduce the condition (24). The rests of two proofs are the same.

Note that for fixed (i,j) and (i',j') the value $T(q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)})$ depends algebraically on the parameter $\lambda \in \mathfrak{a}_\Theta^*$. Since the set

$$S = \{\lambda \in \mathfrak{a}_\Theta^*;\ \langle\lambda_\Theta+\varpi,\alpha\rangle \in \{0,1,2,\ldots\} \text{ for } \varpi \in \mathcal{W}_\Theta(\pi^*)\cup\{0\} \text{ and } \alpha \in \Psi(\mathfrak{g})\setminus\Theta\}$$

is Zariski dense in $\mathfrak{a}_\Theta^*$, we have only to show (24) for $\lambda \in S$. In this case we have from Lemma 2.28 and the definition of $q_{\pi,\Theta}(x;\lambda)$,

$$h_{\pi,V_{\lambda_\Theta}}(q_{\pi,\Theta}(F_\pi;\lambda)) = q_{\pi,\Theta}(h_{\pi,V_{\lambda_\Theta}}(F_\pi);\lambda) = 0.$$

Hence $q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)} \in \operatorname{Ann}(V_{\lambda_\Theta})$ for any (i,j) and (i',j'). On the other hand, if we take a highest weight vector v_λ of V_{λ_Θ}, we get

$$q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)}v_\lambda \in T(q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)})v_\lambda + \bar{\mathfrak{n}}V_{\lambda_\Theta}$$

and therefore $T(q_{\pi,\Theta}(F_\pi;\lambda)_{(i',j')(i,j)}) = 0$. ■

Theorem 2.29 (minimality)**.** *Let* $\lambda \in \mathfrak{a}_\Theta^*$.
i) *The set of the roots of* $q_{\pi,M_\Theta(\lambda)}(x)$ *equals* $\{\langle\lambda_\Theta,\varpi\rangle + D_\pi(\varpi);\ \varpi \in \overline{\mathcal{W}}_\Theta(\pi)\}$.
ii) *If each root of* $q_{\pi,\Theta}(x;\lambda)$ *is simple, then* $q_{\pi,\Theta}(x;\lambda) = q_{\pi,M_\Theta(\lambda)}(x)$. *Hence we call* $q_{\pi,\Theta}(x;\lambda)$ *the global minimal polynomial of the pair* $(\pi, M_\Theta(\lambda))$.

Proof. i) Fix an irreducible decomposition $\bigoplus_{i=1}^{\kappa} U_i$ of the $\mathfrak{g}_\Theta$-module $V^*|_{\mathfrak{g}_\Theta}$. Let $\varpi_i \in \mathfrak{a}^*$ be the highest weight of U_i. With a suitable change of indices we may assume $\varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_j|_{\mathfrak{a}_\Theta}$ implies $i > j$. Then putting $V_i = \bigoplus_{\nu=1}^{i} U_\nu$ we get a $\mathfrak{p}_\Theta$-stable filtration

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\kappa = V^*|_{\mathfrak{p}_\Theta}.$$

Note that $V_i/V_{i-1} \simeq U_i$ is an irreducible $\mathfrak{p}_\Theta$-module on which $\mathfrak{n}_\Theta$ acts trivially.

Recall $M_\Theta(\lambda) \simeq M_{(\Theta,\lambda_\Theta)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} U_{(\Theta,\lambda_\Theta)}$ and $\dim U_{(\Theta,\lambda_\Theta)} = 1$. Hence writing $\mathbb{C}_\lambda$ instead of $U_{(\Theta,\lambda_\Theta)}$ we get by Theorem A1 of Appendix A

$$M_\Theta(\lambda) \otimes V^* = \left(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} \mathbb{C}_\lambda\right) \otimes V^* \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} (\mathbb{C}_\lambda \otimes V^*|_{\mathfrak{p}_\Theta}).$$

Since $\mathbb{C}_\lambda \otimes_{\mathbb{C}} \cdot$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} \cdot = U(\bar{\mathfrak{n}}_\Theta) \otimes_{\mathbb{C}} \cdot$ are exact functors, putting $M_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} (\mathbb{C}_\lambda \otimes V_i)$ we get a $\mathfrak{g}$-stable filtration

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\kappa = M_\Theta(\lambda) \otimes V^*$$

with

$$M_i/M_{i-1} \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} (\mathbb{C}_\lambda \otimes U_i) = M_{(\Theta,\lambda_\Theta+\varpi_i)}. \tag{27}$$

Now as a subalgebra of $\operatorname{End}(M_\Theta(\lambda) \otimes V^*)$ we take

$$A = \{D;\ DM_i \subset M_i \text{ for } i = 1, \ldots, \kappa\}.$$

Then by Lemma 2.26 and Lemma 2.19 ii) we have $h_{(\pi,M_\Theta(\lambda))}(q(F_\pi)) \in A$ for any polynomial $q(x) \in \mathbb{C}[x]$. Let $\eta_i : A \to \operatorname{End}(M_i/M_{i-1}) \simeq \operatorname{End}\big(M_{(\Theta,\lambda_\Theta+\varpi_i)}\big)$ be a natural algebra homomorphism. Then using Lemma 2.26 and Lemma 2.19 ii) again we get

$$\begin{aligned}
&\eta_i\big(h_{(\pi,M_\Theta(\lambda))}(F_\pi)\big)\\
&\quad= \frac{1}{2}\langle\lambda_\Theta, \lambda_\Theta + 2\rho\rangle + \frac{1}{2}\langle-\bar{\pi}, -\bar{\pi} + 2\rho\rangle - \frac{1}{2}\langle\lambda_\Theta + \varpi_i, \lambda_\Theta + \varpi_i + 2\rho\rangle\\
&\quad= \langle\lambda_\Theta, -\varpi_i\rangle + D_\pi(-\varpi_i).
\end{aligned} \tag{28}$$

and therefore

$$\begin{aligned}
q_{\pi,M_\Theta(\lambda)}\left(\langle\lambda_\Theta, -\varpi_i\rangle + D_\pi(-\varpi_i)\right) &= q_{\pi,M_\Theta(\lambda)}\left(\eta_i\left(h_{(\pi,M_\Theta(\lambda))}(F_\pi)\right)\right)\\
&= \eta_i\left(h_{(\pi,M_\Theta(\lambda))}\left(q_{\pi,M_\Theta(\lambda)}(F_\pi)\right)\right)\\
&= 0.
\end{aligned}$$

Since $\{\varpi_i\} = \mathcal{W}_\Theta(\pi^*) = -\overline{\mathcal{W}}_\Theta(\pi)$ we can conclude $\langle\lambda_\Theta, \varpi\rangle + D_\pi(\varpi)$ is a root of the minimal polynomial for each $\varpi \in \overline{\mathcal{W}}_\Theta(\pi)$. Conversely Theorem 2.24 assures any other roots do not exist.

ii) The claim immediately follows from i) and the definition of $q_{\pi,\Theta}(x;\lambda)$. ∎

Remark 2.30. In general it may happen for a certain λ that $q_{\pi,\Theta}(x;\lambda) \neq q_{\pi,M_\Theta(\lambda)}(x)$. Such example is shown in [22] when $\mathfrak{g}$ is $\mathfrak{o}_{2n}$ and λ is invariant under an outer automorphism of $\mathfrak{g}$, which is related to the following theorem. It gives more precise information on our minimal polynomials.

Theorem 2.31. *Let $\lambda \in \mathfrak{a}_\Theta^*$. Let $\overline{\mathcal{W}}_\Theta(\pi) = \overline{\mathcal{W}}_\lambda^1 \sqcup \overline{\mathcal{W}}_\lambda^2 \cdots \sqcup \overline{\mathcal{W}}_\lambda^{m_\lambda}$ be a division of $\overline{\mathcal{W}}_\Theta(\pi)$ into non-empty subsets $\overline{\mathcal{W}}_\lambda^\ell$ such that the relation $\lambda_\Theta - \varpi \in \{w.(\lambda_\Theta - \varpi');\ w \in W\}$ holds for $\varpi, \varpi' \in \overline{\mathcal{W}}_\Theta(\pi)$ if and only if $\varpi, \varpi' \in \overline{\mathcal{W}}_\lambda^\ell$ for some ℓ. For each ℓ we denote by κ_ℓ the maximal length of sequences $\{\varpi, \varpi', \ldots, \varpi''\}$ of weights in $\overline{\mathcal{W}}_\lambda^\ell$ such that the restriction of each weight to $\mathfrak{a}_\Theta$ gives both strictly and linearly ordered sequences:*

$$\varpi|_{\mathfrak{a}_\Theta} <_\Theta \varpi'|_{\mathfrak{a}_\Theta} <_\Theta \cdots <_\Theta \varpi''|_{\mathfrak{a}_\Theta}.$$

i) *$\langle\lambda_\Theta, \varpi\rangle + D_\pi(\varpi) = \langle\lambda_\Theta, \varpi'\rangle + D_\pi(\varpi')$ if $\varpi, \varpi' \in \overline{\mathcal{W}}_\lambda^\ell$ for some ℓ.*
ii) *Let $q(x) \in \mathbb{C}[x]$ and suppose for each $\ell = 1, \ldots, m_\lambda$, $q(x)$ is a multiple of $(x - \langle\lambda_\Theta, \varpi\rangle - D_\pi(\varpi))^{\kappa_\ell}$ with $\varpi \in \overline{\mathcal{W}}_\lambda^\ell$. Then $q(F_\pi)M_\Theta(\lambda) = 0$.*

Proof. i) By the W-invariance of $\langle\ ,\ \rangle$ and the assumption, we have

$$\langle \lambda_\Theta + \rho - \varpi, \lambda_\Theta + \rho - \varpi \rangle = \langle \lambda_\Theta + \rho - \varpi', \lambda_\Theta + \rho - \varpi' \rangle,$$

which implies the claim.

ii) Use the notation in the proof of Theorem 2.29. Let M be a $\mathfrak{g}$-module and $\mu \in \mathfrak{a}^*$. We say that a non-zero vector v in M is a *generalized weight vector for the generalized infinitesimal character* μ if for any $\Delta \in Z(\mathfrak{g})$ there exists a positive integer k such that $(\Delta - \Delta_{\mathfrak{a}}(\mu))^k v = 0$. We denote by $(M)_{(\mu)}$ the submodule of M spanned by the generalized weight vectors for the generalized infinitesimal character μ. Note that $(M)_{(\mu)} = (M)_{(\mu')}$ if and only if $\mu = w.\mu'$ for some $w \in W$. By virtue of (27) and Remark 2.16, $M_\Theta(\lambda) \otimes V^*$ is uniquely decomposed as a direct sum of submodules in $\{(M_\Theta(\lambda) \otimes V^*)_{(\lambda_\Theta + \varpi_\nu)};\ \nu = 1, \ldots, i\}$.

For $i = 1, \ldots, \kappa$ using a $\mathfrak{p}_\Theta$-module

$$V_{[i]} = U_i \oplus \bigoplus_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} U_\nu \subset V_i,$$

define

$$M_{[i]} = U(\mathfrak{g}) \otimes_{\mathfrak{p}_\Theta} (\mathbb{C}_\lambda \otimes V_{[i]}) = U(\bar{\mathfrak{n}}_\Theta) \otimes \mathbb{C}_\lambda \otimes V_{[i]}.$$

It is naturally considered as a $\mathfrak{g}$-submodule of $M_i = U(\bar{\mathfrak{n}}_\Theta) \otimes \mathbb{C}_\lambda \otimes V_i$. If we define the surjective homomorphism

$$\tau_{[i]} : M_{[i]} \hookrightarrow M_i \to M_i/M_{i-1} \simeq M_{(\Theta, \lambda_\Theta + \varpi_i)},$$

then

$$\operatorname{Ker} \tau_{[i]} = \sum_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} M_{[\nu]}. \tag{29}$$

Since $M_{(\Theta, \lambda_\Theta + \varpi_i)}$ has infinitesimal character $\lambda_\Theta + \varpi_i$ we get

$$M_{[i]} = (M_{[i]})_{(\lambda_\Theta + \varpi_i)} + \sum_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} M_{[\nu]}.$$

Therefore we get inductively

$$M_{[i]} = (M_{[i]})_{(\lambda_\Theta + \varpi_i)} + \sum_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} (M_{[\nu]})_{(\lambda_\Theta + \varpi_\nu)}. \tag{30}$$

Notice that the $\mathfrak{g}$-homomorphism $h_{(\pi, M_\Theta(\lambda))}(F_\pi)$ leaves any $\mathfrak{g}$-submodule of $M_\Theta(\lambda) \otimes V^*$ stable. Then from (28) and (29)

$$\begin{aligned}
\Bigl(h_{(\pi, M_\Theta(\lambda))}(F_\pi) - \langle \lambda_\Theta, -\varpi_i \rangle - D_\pi(-\varpi_i)\Bigr)&(M_{[i]})_{(\lambda_\Theta + \varpi_i)} \\
&\subset \Biggl(\sum_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} M_{[\nu]}\Biggr)_{(\lambda_\Theta + \varpi_i)} \\
&= \Biggl(\sum_{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}} (M_{[\nu]})_{(\lambda_\Theta + \varpi_\nu)}\Biggr)_{(\lambda_\Theta + \varpi_i)} \\
&= \sum_{\substack{\nu;\, \varpi_i|_{\mathfrak{a}_\Theta} <_\Theta \varpi_\nu|_{\mathfrak{a}_\Theta}, \\ \lambda_\Theta + \varpi_\nu \in \{w.(\lambda_\Theta + \varpi_i);\, w \in W\}}} (M_{[\nu]})_{(\lambda_\Theta + \varpi_\nu)}.
\end{aligned}$$

By the relation $\{\varpi_i\} = \mathcal{W}_\Theta(\pi^*) = -\overline{\mathcal{W}}_\Theta(\pi)$ and the assumption of ii) we get inductively

$$h_{(\pi,M_\Theta(\lambda))}(q(F_\pi))(M_{[i]})_{(\lambda_\Theta+\varpi_i)} = q(h_{(\pi,M_\Theta(\lambda))}(F_\pi))(M_{[i]})_{(\lambda_\Theta+\varpi_i)} = \{0\}$$

for $i = 1,\dots,\kappa$. Now our claim is clear because by (30) we have

$$M_\Theta(\lambda)\otimes V^* = \sum_{i=1}^{\kappa} M_{[i]} = \sum_{i=1}^{\kappa}(M_{[i]})_{(\lambda_\Theta+\varpi_i)}. \qquad \blacksquare$$

Corollary 2.32. *Let τ be an involutive automorphism of $\mathfrak{g}$ which corresponds to an automorphism of the Dynkin diagram of $\mathfrak{g}$. Then $\tau(\mathfrak{a}) = \mathfrak{a}$ and $\tau(\mathfrak{n}) = \mathfrak{n}$. Furthermore we suppose $\tau(\mathfrak{p}_\Theta) = \mathfrak{p}_\Theta$, or equivalently, $\tau(\mathfrak{a}_\Theta) = \mathfrak{a}_\Theta$. For $\varpi \in \mathfrak{a}^*$ we identify $\varpi|_{(\mathfrak{a}_\Theta)^\tau}$ as a linear function on $(\mathfrak{a}_\Theta^*)^\tau$ by $\varpi|_{(\mathfrak{a}_\Theta)^\tau}(\lambda) = \langle\lambda_\Theta,\varpi\rangle$ for $\lambda\in(\mathfrak{a}_\Theta^*)^\tau$. Put*

$$\begin{aligned}\Omega_{\pi,\Theta,\tau} &= \{\bigl(\varpi|_{(\mathfrak{a}_\Theta)^\tau}, D_\pi(\varpi)\bigr);\ \varpi\in\overline{\mathcal{W}}_\Theta(\pi)\},\\ q_{\pi,\Theta,\tau}(x;\lambda) &= \prod_{(\mu,C)\in\Omega_{\pi,\Theta,\tau}}\bigl(x-\mu(\lambda)-C\bigr).\end{aligned}$$

Then for $\lambda\in(a_\Theta^*)^\tau$ we have the following.
i) $q_{\pi,\Theta,\tau}(F_\pi;\lambda)M_\Theta(\lambda) = 0$.
ii) *If each root of $q_{\pi,\Theta,\tau}(x;\lambda)$ is simple, then $q_{\pi,\Theta,\tau}(x;\lambda) = q_{\pi,M_\Theta(\lambda)}(x)$.*

Proof. We naturally identify ρ_Θ with an element in $(\mathfrak{a}_\Theta^*)^\tau$. For a given pair of weights $\varpi,\varpi'\in\overline{\mathcal{W}}_\Theta(\pi)$ with $\varpi|_{\mathfrak{a}_\Theta} <_\Theta \varpi'|_{\mathfrak{a}_\Theta}$, choose the non-negative integers $\{m_\alpha;\ \alpha\in\Psi(\mathfrak{g})\setminus\Theta\}$ so that $\varpi'|_{\mathfrak{a}_\Theta} - \varpi|_{\mathfrak{a}_\Theta} = \sum_{\alpha\in\Psi(\mathfrak{g})\setminus\Theta} m_\alpha\alpha|_{\mathfrak{a}_\Theta}$. Then $\varpi'|_{\mathfrak{a}_\Theta}(\rho_\Theta) - \varpi|_{\mathfrak{a}_\Theta}(\rho_\Theta) = \sum_{\alpha\in\Psi(\mathfrak{g})\setminus\Theta} m_\alpha\langle\alpha,\rho_\Theta\rangle > 0$. It simply shows

$$\bigl(\varpi|_{(\mathfrak{a}_\Theta)^\tau}, D_\pi(\varpi)\bigr) \neq \bigl(\varpi'|_{(\mathfrak{a}_\Theta)^\tau}, D_\pi(\varpi')\bigr).$$

Hence from Theorem 2.31 we get i). Now ii) is clear from Theorem 2.29. $\blacksquare$

We will shift $\mathfrak{a}^*$ by ρ so that the action $w.\mu = w(\mu+\rho)-\rho$ for $\mu\in\mathfrak{a}^*$ and $w\in W$ changes into the natural action of W and then we can give the characteristic polynomial as a special case of the global minimal polynomials. The result itself is not new and it has already been studied in [11].

Theorem 2.33 (Cayley-Hamilton [11]). *The characteristic polynomial $q_\pi(x)$ of π is given by*

$$q_\pi(x) = \prod_{\varpi\in\mathcal{W}(\pi)}\Bigl(x-\varpi-\frac{\langle\pi,\pi+2\rho\rangle-\langle\varpi,\varpi\rangle}{2}\Bigr) \tag{31}$$

under the identification $\mathbb{C}[x]\otimes S(\mathfrak{a}^*)^W \simeq \mathbb{C}[x]\otimes S(\mathfrak{a})^W \simeq Z(\mathfrak{g})[x]$ by the symmetric bilinear form $\langle\ ,\ \rangle$ and the Harish-Chandra isomorphism:

$$\begin{aligned}&Z(\mathfrak{g})\simeq U(\mathfrak{a})^W;\ \Delta\mapsto\Upsilon(\Delta),\\ &\qquad\Upsilon(\Delta)(\mu) = \Delta_{\mathfrak{a}}(\mu-\rho)\ \text{for}\ \mu\in\mathfrak{a}^*.\end{aligned}$$

Here π is identified with its highest weight. In particular $q_\pi(x)\in Z(\mathfrak{g})[x]$.

Proof. Note that $\langle \pi, \pi + 2\rho \rangle = \langle \bar{\pi}, \bar{\pi} - 2\rho \rangle$. Let $\tilde{q}_\pi(x)$ be the element of $Z(\mathfrak{g})[x]$ identified with the right-hand side of (31). Put $\mathbf{V} = \sum_{i,j} \mathbb{C}\tilde{q}_\pi(F_\pi)_{ij}$ and $\mathbf{V}_\mathfrak{a} = \{D_\mathfrak{a};\, D \in \mathbf{V}\}$. Then Theorem 2.24 with $\Theta = \emptyset$ shows $Q(\mu) = 0$ for any $\mu \in \mathfrak{a}^*$ and $Q \in \mathbf{V}_\mathfrak{a}$, which implies $\mathbf{V}_\mathfrak{a} = \{0\}$. Since $\mathbf{V}$ is $\mathrm{ad}(\mathfrak{g})$-stable, we have $\mathbf{V} = \{0\}$ as is shown in [19, Lemma 2.12]. Since the minimality of $\tilde{q}_\pi(x)$ follows from Theorem 2.29, we get $q_\pi(x) = \tilde{q}_\pi(x)$. ■

Corollary 2.34. i) *Let $\mathfrak{g}$ be a simple Lie algebra. Then the characteristic polynomial of the adjoint representation of $\mathfrak{g}$ is given by*

$$q_{\alpha_{\max}}(x) = \prod_{\alpha \in \Sigma(\mathfrak{g}) \cup \{0\}} \Bigl(x - \alpha - \frac{1 - B(\alpha, \alpha)}{2}\Bigr).$$

Here $B(\ ,\ )$ denotes the Killing form of $\mathfrak{g}$.

ii) *Suppose that the representation π is minuscule, that is, $\mathcal{W}(\pi)$ is a single W-orbit. Then*

$$q_\pi(x) = \prod_{\varpi \in \mathcal{W}(\pi)} \bigl(x - \varpi - \langle \pi, \rho \rangle\bigr).$$

Proof. This is a direct consequence of Theorem 2.33 and Lemma 2.19 v). ■

Corollary 2.35. *Put $q_\pi(x) = x^m + \Delta_1 x^{m-1} + \cdots + \Delta_{m-1}x + \Delta_m$ with $\Delta_j \in Z(\mathfrak{g})$ and define*

$$\tilde{F}_\pi = -F_\pi^{m-1} - \Delta_1 F_\pi^{m-2} - \cdots - \Delta_{m-1} I_N.$$

Then

$$F_\pi \tilde{F}_\pi = \tilde{F}_\pi F_\pi = \Delta_m I_N = \prod_{\varpi \in \mathcal{W}(\pi)} \Bigl(-\varpi - \frac{\langle \pi, \pi + 2\rho \rangle - \langle \varpi, \varpi \rangle}{2}\Bigr) I_N,$$

In particular, F_π is invertible in $M\bigl(N, \hat{Z}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g})\bigr)$ with the quotient field $\hat{Z}(\mathfrak{g})$ of $Z(\mathfrak{g})$.

In the next definition and the subsequent proposition, we do not assume (10). Namely, $\mathfrak{g}$ is a general reductive Lie algebra and (π, V) denotes a finite dimensional irreducible representation which is not necessarily faithful. Moreover we use the symbol $\langle\ ,\ \rangle$ for the symmetric bilinear form on $\mathfrak{a}^*$ defined by the restriction of the Killing form of $\mathfrak{g}$.

Definition 2.36 (dominant minuscule weight)**.** We say a weight $\pi_{\min}$ of π is *dominant* and *minuscule* if

$$\langle \pi_{\min}, \alpha \rangle \geq 0 \quad \text{for all } \alpha \in \Sigma(\mathfrak{g})^+$$

and

$$\langle \pi_{\min}, \pi_{\min} \rangle \leq \langle \varpi, \varpi \rangle \quad \text{for all } \varpi \in \mathcal{W}(\pi).$$

If the highest weight of π is dominant and minuscule, then (π, V) is called a *minuscule representation.*

Proposition 2.37. *Put $\Psi(\mathfrak{g}) = \{\alpha_1, \ldots, \alpha_r\}$ and define $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ for $\alpha \in \Sigma(\mathfrak{g})$. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{g}$. Let $\pi_{\min}$ be a dominant minuscule weight of π.*
i) *If the highest weight of π is in the root lattice, then $\pi_{\min} = 0$.*
ii) *$\pi_{\min}$ is uniquely determined by π. Moreover if (π', V') is a finite dimensional irreducible representation of $\mathfrak{g}$ such that the difference of the highest weight of π' and that of π is in the root lattice of $\Sigma(\mathfrak{g})$, then $\pi_{\min} = \pi'_{\min}$.*
iii) *$\varpi \in \mathcal{W}(\pi)$ is a dominant minuscule weight if and only if*

$$\langle\varpi, \alpha^\vee\rangle \in \{0, 1\} \quad \text{for all } \alpha \in \Sigma(\mathfrak{g})^+. \tag{32}$$

iv) *If π is a minuscule representation, then $\mathcal{W}(\pi) = W\pi_{\min}$.*
v) *Suppose $\mathfrak{g}$ is simple. Let $\Sigma(\mathfrak{g})^\vee := \{\alpha^\vee; \alpha \in \Sigma(\mathfrak{g})\}$ be the dual root system of $\Sigma(\mathfrak{g})$. Let β be the maximal root of $\Sigma(\mathfrak{g})^\vee$ and put $\beta = \sum_{i=1}^r n_i\alpha_i^\vee$. Define the fundamental weights Λ_i by $\langle\Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$. Then π is a minuscule representation if and only if its highest weight is 0 or Λ_i with $n_i = 1$.*

Proof. For $\alpha \in \Sigma(\mathfrak{g})$ we denote by $\mathfrak{g}^\alpha$ the Lie algebra generated by the root vectors corresponding to α and $-\alpha$. Note that $\mathfrak{g}^\alpha$ is isomorphic to $\mathfrak{sl}_2$.

i) Suppose the highest weight of π is in the root lattice. Put $\varpi = \sum_{i=1}^r m_i(\varpi)\alpha_i$ for $\varpi \in \mathcal{W}(\pi)$. Note that $m_i(\varpi)$ are integers. Let $\varpi_0 \in W(\pi)$ such that $m_i(\varpi_0) \geq 0$ and $\sum_{i=1}^r m_i(\varpi_0) \leq \sum_{i=1}^r m_i(\varpi)$ for $\varpi \in W(\pi)$ satisfying $m_i(\varpi) \geq 0$ for $i = 1, \ldots, r$. The existence of ϖ_0 is clear because $m_i(\pi) \geq 0$ for $i = 1, \ldots, r$. Suppose $\varpi_0 \neq 0$. Since $0 < \langle\varpi_0, \varpi_0\rangle = \sum_{i=1}^r m_i(\varpi_0)\langle\varpi_0, \alpha_i\rangle$, there exists an index k such that $\langle\varpi_0, \alpha_k\rangle > 0$ and $m_k(\varpi_0) > 0$. Hence $\varpi_0 - \alpha_k \in \mathcal{W}(\pi)$ by the representation $\pi|_{\mathfrak{g}^{\alpha_k}}$, which contradicts the assumption for ϖ_0. Thus $0 = \varpi_0 \in \mathcal{W}(\pi)$ and $\pi_{\min} = 0$.

ii) – iv) Suppose the existence of $\alpha \in \Sigma(\mathfrak{g})^+$ with $\langle\pi_{\min}, \alpha^\vee\rangle > 1$. Then it follows from the representation $\pi|_{\mathfrak{g}^\alpha}$ that $\pi_{\min} - \alpha \in \mathcal{W}(\pi)$ and $\langle\pi_{\min}, \pi_{\min}\rangle - \langle\pi_{\min} - \alpha, \pi_{\min} - \alpha\rangle = 2\langle\pi_{\min}, \alpha\rangle - \langle\alpha, \alpha\rangle > 0$, which contradicts the assumption of $\pi_{\min}$. Thus we have (32) for $\varpi = \pi_{\min}$.

Suppose π is an irreducible representation of $\mathfrak{g}$ with the highest weight ϖ satisfying (32). Suppose $\mathcal{W}(\pi) \neq W\varpi$. Then there exist $\mu \in W\varpi$ and $\mu' \in \mathcal{W}(\pi)$ such that $\mu' \notin W\varpi$ with $\alpha := \mu - \mu' \in \Sigma(\mathfrak{g})$. By the W-invariance we may assume $\mu = \varpi$ and therefore $\mu' = \varpi - \alpha$ with $\alpha \in \Sigma(\mathfrak{g})^+$. Then by the representation $\pi_{\mathfrak{g}^\alpha}$ together with the condition (32) we have $\langle\varpi, \alpha^\vee\rangle = 1$ and $\mu' = w_\alpha\varpi$, which is a contradiction. Thus we have iv).

Let ϖ and ϖ' be the elements of $\mathfrak{a}^*$ satisfying the condition (32). Then $\varpi'' := \varpi - \varpi'$ satisfies $\langle\varpi'', \alpha^\vee\rangle \in \{-1, 0, 1\}$ for $\alpha \in \Sigma(\mathfrak{g})$. Suppose that ϖ'' is in the root lattice. Let $\varpi_0 \in W\varpi''$ such that $\langle\varpi_0, \alpha\rangle \geq 0$ for $\alpha \in \Sigma(\mathfrak{g})^+$. Since ϖ_0 also satisfies (32), the finite dimensional irreducible representation π_0 with the highest weight ϖ_0 is minuscule by the argument above. Since ϖ_0 is in the root lattice, $\varpi_0 = 0$ by i) and hence $\varpi = \varpi'$. Thus we obtain ii) and iii).

v) Let $\alpha \in \Sigma(\mathfrak{g})^+$. If we denote $\alpha^\vee = \sum_{i=1}^r n_i(\alpha)\alpha_i^\vee$, then $n_i(\alpha) \leq n_i$ for $i = 1, \ldots, r$. Hence the claim is clear. ∎

Remark 2.38. Equivalent contents of Proposition 2.37 are found in exercises of [3], Ch. VI.

Restore the previous setting (10) on $\mathfrak{g}$ and (π, V).

Proposition 2.39. i) *Let* V_ϖ *denote the weight space of* V *with weight* $\varpi \in \mathcal{W}(\pi)$*. Define the projection map* $\bar{p}_\Theta : \mathcal{W}(\pi) \to \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}$ *by* $\bar{p}_\Theta(\varpi) = \varpi|_{\mathfrak{a}_\Theta}$ *and put* $V(\Lambda) = \sum_{\varpi\in\bar{p}_\Theta^{-1}(\Lambda)} V_\varpi$ *for* $\Lambda \in \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}$*. Then*

$$V = \bigoplus_{\Lambda\in\mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}} V(\Lambda) \tag{33}$$

is a direct sum decomposition of the $\mathfrak{g}_\Theta$*-module* V*.*

Let $V(\Lambda) = V(\Lambda)_1 \oplus \cdots \oplus V(\Lambda)_{k_\Lambda}$ *be a decomposition into irreducible* $\mathfrak{g}_\Theta$*-modules. We denote by* ϖ_Λ *the dominant minuscule weight of* $(\pi|_{\mathfrak{g}_\Theta}, V(\Lambda)_1)$*. Then*

$$V_{\varpi_\Lambda} = \bigoplus_{i=1}^{k_\Lambda} V_{\varpi_\Lambda} \cap V(\Lambda)_i \qquad \textit{with } \dim V_{\varpi_\Lambda} \cap V(\Lambda)_i > 0. \tag{34}$$

In particular, $V(\Lambda)$ *is an irreducible* $\mathfrak{g}_\Theta$*-module if* $\dim V_{\varpi_\Lambda} = 1$*.*
ii) *Put* $\Psi(\mathfrak{g}) = \{\alpha_1, \cdots, \alpha_r\}$ *and put* $\Psi(\mathfrak{g}) \setminus \Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$*, define the map*

$$\begin{array}{rccc} p_\Theta : & \Sigma(\mathfrak{g}) & \to & \mathbb{Z}^k \\ & \alpha = \sum m_i\alpha_i & \mapsto & (m_{i_1}, \ldots, m_{i_k}) \end{array}$$

and put

$$L_\Theta = \{0\} \cup \{p_\Theta(\alpha);\, \alpha \in \Sigma(\mathfrak{g})\},$$
$$V(\mathbf{m}) = \begin{cases} \displaystyle\sum_{\alpha\in p_\Theta^{-1}(\mathbf{m})} \mathbb{C}X_\alpha & \textit{if } \mathbf{m} \neq 0, \\ \displaystyle\mathfrak{a} + \sum_{\alpha\in p_\Theta^{-1}(\mathbf{m})} \mathbb{C}X_\alpha & \textit{if } \mathbf{m} = 0 \end{cases}$$

for $\mathbf{m} \in L_\Theta$*. Then*

$$\mathfrak{g} = \bigoplus_{\mathbf{m}\in L_\Theta} V(\mathbf{m}) \tag{35}$$

is a decomposition of the $\mathfrak{g}_\Theta$*-module* $\mathfrak{g}$*. If* $\mathbf{m} \neq 0$*, then* $V(\mathbf{m})$ *is an irreducible* $\mathfrak{g}_\Theta$*-module. On the other hand,* $V(0) = \mathfrak{g}_\Theta$ *is isomorphic to the adjoint representation of* $\mathfrak{g}_\Theta = \mathfrak{a}_\Theta \oplus \mathfrak{m}_\Theta$*. Let* $\Theta = \Theta_1 \sqcup \Theta_2 \sqcup \cdots \sqcup \Theta_\ell$ *be the division of* Θ *into the connected parts of vertexes in the Dynkin diagram of* $\Psi(\mathfrak{g})$*. Then* $\mathfrak{m}_\Theta = \mathfrak{m}_{\Theta_1} \oplus \mathfrak{m}_{\Theta_2} \oplus \cdots \oplus \mathfrak{m}_{\Theta_\ell}$ *gives a decomposition into irreducible* $\mathfrak{g}_\Theta$*-modules.*
iii) *Suppose that the representation* (π, V) *is minuscule. Put* $W^\pi = \{w \in W;\, w\pi = \pi\}$*. Here we identify* π *with its highest weight. Let* $\{w_1, \ldots, w_k\}$ *be a representative system of* $W^\pi\backslash W/W_\Theta$ *such that* $w_i \in W(\Theta)$*. Then with the notation in* i)

$$V = \bigoplus_{i=1}^{k} V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta}) \tag{36}$$

gives a decomposition into irreducible $\mathfrak{g}_\Theta$*-modules. Moreover the* $\mathfrak{g}_\Theta$*-submodule* $V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta})$ *has highest weight* $w_i^{-1}\pi$*.*

Proof. i) Since $\alpha|_{\mathfrak{a}_\Theta} = 0$ for $\alpha \in \Theta$, (33) is a decomposition into $\mathfrak{g}_\Theta$-modules. Then Proposition 2.37 ii) implies that ϖ_Λ is the minuscule weight for any $(\pi|_{\mathfrak{g}_\Theta}, V(\Lambda)_i)$ and therefore the other statements in i) are clear.

ii) Note that $\alpha_{i_k}|_{\mathfrak{a}_\Theta}, \ldots, \alpha_{i_1}|_{\mathfrak{a}_\Theta}$ are linearly independent and $\mathfrak{g}_\Theta = V(0)$. Then the statements in ii) follows from i).

iii) From i) each $V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta})$ is an irreducible $\mathfrak{g}_\Theta$-module and

$$V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta}) \supset \sum\{V_{w^{-1}\pi};\ w \in W^\pi w_i W_\Theta\}.$$

Since $w_i \in W(\Theta)$ we have $w_i^{-1}\pi + \alpha \notin \mathcal{W}(\pi)$ for $\alpha \in \Sigma(\mathfrak{g}_\Theta)^+$. It shows the highest weight of $V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta})$ is $w_i^{-1}\pi$. Since $w_i^{-1}\pi \neq w_j^{-1}\pi$ if $i \neq j$ we have (36). ∎

We give the minimal polynomials for some representations in the following proposition as a corollary of Lemma 2.19 v) and Proposition 2.39.

Proposition 2.40. *Retain the notation in* Theorem 2.24 *and* Proposition 2.39.
i) (multiplicity free representation) *Suppose* $\dim V_\varpi = 1$ *for any* $\varpi \in \mathcal{W}(\pi)$. *Let* $\bar\Lambda$ *be the lowest weight of* $(\pi|_{\mathfrak{g}_\Theta}, V(\Lambda))$ *for* $\Lambda \in \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}$. *Then*

$$\begin{aligned} q_{\pi,\Theta}(x;\lambda) &= \prod_{\Lambda \in \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}} \Big(x - \langle \lambda_\Theta, \bar\Lambda\rangle - \frac{1}{2}\langle \bar\pi - \bar\Lambda, \bar\pi + \bar\Lambda - 2\rho\rangle\Big) \\ &= \prod_{\Lambda \in \mathcal{W}(\pi)|_{\mathfrak{a}_\Theta}} \Big(x - \langle \lambda_\Theta + \rho, \bar\Lambda\rangle + \langle \bar\pi, \rho\rangle - \frac{\langle \bar\pi, \bar\pi\rangle - \langle \bar\Lambda, \bar\Lambda\rangle}{2}\Big). \end{aligned} \tag{37}$$

ii) (adjoint representation) *Suppose* $\mathfrak{g}$ *is simple and* $\Theta \neq \emptyset$. *Let* $\Theta = \Theta_1 \sqcup \cdots \sqcup \Theta_\ell$ *be the division in* Proposition 2.39 ii). *Let* $\alpha^i_{\max}$ *denote the maximal root of the simple Lie algebra* $\mathfrak{m}_{\Theta_i}$ *for* $i = 1, \ldots, \ell$. *Put*

$$\Omega_\Theta = \{B(\alpha^1_{\max}, \alpha^1_{\max} + 2\rho(\Theta_1)), \ldots, B(\alpha^\ell_{\max}, \alpha^\ell_{\max} + 2\rho(\Theta_\ell))\}.$$

Let $\alpha_{\mathbf{m}}$ *be the smallest root in* $p_\Theta^{-1}(\mathbf{m})$ *for* $\mathbf{m} \in L_\Theta \setminus \{0\}$ *under the order in* Definition 2.20. *Then for the adjoint representation of* $\mathfrak{g}$,

$$\begin{aligned} q_{\alpha_{\max},\Theta}(x;\lambda) = \Big(x - \frac{1}{2}\Big) &\prod_{C \in \Omega_\Theta} \Big(x - \frac{1 - C}{2}\Big) \\ &\cdot \prod_{\mathbf{m} \in L_\Theta \setminus \{0\}} \Big(x - B(\lambda_\Theta + \rho, \alpha_{\mathbf{m}}) - \frac{1 - B(\alpha_{\mathbf{m}}, \alpha_{\mathbf{m}})}{2}\Big). \end{aligned} \tag{38}$$

iii) (minuscule representation) *Suppose* (π, V) *is minuscule. Then with* $w_1, \ldots, w_k$ *in* Proposition 2.39 iii),

$$q_{\pi,\Theta}(x;\lambda) = \prod_{i=1}^{k} \Big(x - \langle w_i\big(\lambda_\Theta + \rho_\Theta - \rho(\Theta)\big) + \rho, \pi\rangle\Big). \tag{39}$$

Proof. It is easy to get i) and ii).

iii) Let $\bar{w}_\Theta$ denote the longest element in W_Θ. Then the $\mathfrak{g}_\Theta$-module $V(w_i^{-1}\pi|_{\mathfrak{a}_\Theta})$ has lowest weight $\bar{w}_\Theta w_i^{-1}\pi$. The claim follows from the next calculation:

$$\begin{aligned}\langle\lambda_\Theta, \bar{w}_\Theta w_i^{-1}\pi\rangle &+ \frac{1}{2}\langle\bar{\pi} - \bar{w}_\Theta w_i^{-1}\pi, \bar{\pi} + \bar{w}_\Theta w_i^{-1}\pi - 2\rho\rangle \\ &= \langle\lambda_\Theta + \rho, \bar{w}_\Theta w_i^{-1}\pi\rangle + \langle\rho, \pi\rangle \\ &= \langle w_i\bar{w}_\Theta(\lambda_\Theta + \rho) + \rho, \pi\rangle = \langle w_i\big(\lambda_\Theta + \rho_\Theta - \rho(\Theta)\big) + \rho, \pi\rangle. \qquad \blacksquare\end{aligned}$$

3. Two-sided ideals

Our main concern in this paper is the following two-sided ideal.

Definition 3.1 (gap). Let $\lambda \in \mathfrak{a}_\Theta^*$. If a two-sided ideal $I_\Theta(\lambda)$ of $U(\mathfrak{g})$ satisfies

$$J_\Theta(\lambda) = I_\Theta(\lambda) + J(\lambda_\Theta), \tag{40}$$

then we say that $I_\Theta(\lambda)$ *describes the gap* between the generalized Verma module $M_\Theta(\lambda)$ and the Verma module $M(\lambda_\Theta)$.

It is clear that there exists a two-sided ideal $I_\Theta(\lambda)$ satisfying (40) if and only if

$$J_\Theta(\lambda) = \mathrm{Ann}\big(M_\Theta(\lambda)\big) + J(\lambda_\Theta). \tag{41}$$

This condition depends on λ but such an ideal exists and is essentially unique for a generic λ (cf. Proposition 3.11, Theorem 3.12, Remark 4.14). The main purpose in this paper is to construct a good generator system of the ideal from a minimal polynomial.

Definition 3.2 (two-sided ideal). Using the global minimal polynomial defined in the last section, we define a two-sided ideal of $U(\mathfrak{g})$:

$$I_{\pi,\Theta}(\lambda) = \sum_{i,j} U(\mathfrak{g})q_{\pi,\Theta}(F_\pi;\lambda)_{ij} + \sum_{\Delta\in Z(\mathfrak{g})} U(\mathfrak{g})\big(\Delta - \Delta_{\mathfrak{a}}(\lambda_\Theta)\big). \tag{42}$$

From Theorem 2.24 and Remark 2.16 this ideal satisfies

$$I_{\pi,\Theta}(\lambda) \subset J_\Theta(\lambda). \tag{43}$$

In this section we will examine the condition so that

$$J_\Theta(\lambda) = I_{\pi,\Theta}(\lambda) + J(\lambda_\Theta). \tag{44}$$

Proposition 3.3 (invariant differential operators). *For $\Delta \in Z(\mathfrak{g})$ and a non-negative integer k we denote by $\Delta_{\mathfrak{a}}^{(k)}$ the homogeneous part of $\Delta_{\mathfrak{a}}$ with degree k and put*

$$T_\pi^{(k)} = \sum_{\varpi\in\mathcal{W}(\pi)} m_\pi(\varpi)\varpi^k. \tag{45}$$

Here $m_\pi(\varpi)$ is the multiplicity of the weight ϖ of π and we use the identification $\varpi \in \mathfrak{a}^* \simeq \mathfrak{a} \subset U(\mathfrak{a})$. Let $\{\Delta_1, \ldots, \Delta_r\}$ be a system of generators of $Z(\mathfrak{g})$ as an algebra over $\mathbb{C}$ and let d_i be the degree of $(\Delta_i)_{\mathfrak{a}}$ for $i = 1, \ldots, r$. We assume *that* $(\Delta_1)_{\mathfrak a}^{(d_1)},\dots,(\Delta_r)_{\mathfrak a}^{(d_r)}$ *are algebraically independent. Suppose a subset* A *of* $\{1,\dots,r\}$ *satisfies*

$$\begin{cases} d_k \ge \deg_x q_{\pi,\Theta}(x,\lambda) & \text{if } k\in\{1,\dots,r\}\setminus A,\\ \mathbb{C}[(\Delta_1)_{\mathfrak a}^{(d_1)},\dots,(\Delta_r)_{\mathfrak a}^{(d_r)}] = \mathbb{C}[(\Delta_i)_{\mathfrak a}^{(d_i)}, T_\pi^{(d_k)}; i\in A,\ k\in\{1,\dots,r\}\setminus A]. \end{cases} \tag{46}$$

Then

$$I_{\pi,\Theta}(\lambda) = \sum_{i,j} U(\mathfrak g) q_{\pi,\Theta}(F_\pi;\lambda)_{ij} + \sum_{i\in A} U(\mathfrak g)\bigl(\Delta_i - (\Delta_i)_{\mathfrak a}(\lambda_\Theta)\bigr). \tag{47}$$

Proof. Note that $\sum_{i,j} U(\mathfrak g) q_{\pi,\Theta}(F_\pi;\lambda)_{ij} \ni \operatorname{Trace}\bigl(F_\pi^\nu q_{\pi,\Theta}(F_\pi;\lambda)\bigr)$ if $\nu\ge 0$. On the other hand, since $\operatorname{Trace}\bigl(F_\pi^{\ell_k} q_{\pi,\Theta}(F_\pi;\lambda)\bigr)_{\mathfrak a}^{(d_k)} = T_\pi^{(d_k)}$ by Lemma 2.23 with $\Theta=\emptyset$ if the integer $\ell_k = d_k - \deg_x\bigl(q_{\pi,\Theta}(F_\pi;\lambda)\bigr)$ is non-negative, the assumption implies that for $k\notin A$, Δ_k may be replaced by $\operatorname{Trace}\bigl(F_\pi^{\ell_k} q_{\pi,\Theta}(F_\pi;\lambda)\bigr)$, which implies the proposition. ■

Lemma 3.4. *Let* $\mathbf V$ *be an* $\operatorname{ad}(\mathfrak g)$*-stable subspace of* $U(\mathfrak g)$ *and let* $\mathbf V = \bigoplus_\varpi \mathbf V_\varpi$ *be the decomposition of* $\mathbf V$ *into the weight spaces* $\mathbf V_\varpi$ *with weight* $\varpi\in\mathfrak a^*$. *Suppose* $D_{\mathfrak a}(\lambda_\Theta)=0$ *for* $D\in\mathbf V_0$. *Then the following three conditions are equivalent.*

i) $J_\Theta(\lambda)\subset U(\mathfrak g)\mathbf V + J(\lambda_\Theta)$.

ii) *For any* $\alpha\in\Theta$ *there exists* $D\in\mathbf V_{-\alpha}$ *such that* $D - X_{-\alpha}\in J(\lambda_\Theta)$.

iii) *For any* $\alpha\in\Theta$ *there exists* $D\in\mathbf V_0$ *such that* $D_{\mathfrak a}(\lambda_\Theta-\alpha)\ne 0$.

Proof. Let $U(\mathfrak g)=\bigoplus_\varpi U(\mathfrak g)_\varpi$ be the decomposition of $U(\mathfrak g)$ into the weight spaces $U(\mathfrak g)_\varpi$ with weight $\varpi\in\mathfrak a^*$. Let $\mu\in\mathfrak a^*$. Since $U(\mathfrak g)=U(\bar{\mathfrak n})\oplus J(\mu)$, to $D\in U(\mathfrak g)$, there corresponds a unique $D^\mu\in U(\bar{\mathfrak n})$ such that $D-D^\mu\in J(\mu)$. Here we note that $D\in U(\mathfrak g)_\varpi$ implies $D^\mu\in U(\bar{\mathfrak n})_\varpi$ and that $D^\mu = D_{\mathfrak a}(\mu)\in\mathbb C$ whenever $D\in U(\mathfrak g)_0$.

Put $\mathbf V^\mu = \{D^\mu;\ D\in\mathbf V\}$. Since $\operatorname{ad}(X)\mathbf V\subset\mathbf V$ for $X\in\mathfrak b$, we have $PD\in\mathbf V + J(\mu)$ and therefore $(PD)^\mu\in\mathbf V^\mu$ for every $P\in U(\mathfrak b)$ and $D\in\mathbf V$. Owing to $U(\mathfrak g)=U(\bar{\mathfrak n})\otimes U(\mathfrak b)$, we have

$$\{D^\mu; D\in U(\mathfrak g)\mathbf V\} = U(\bar{\mathfrak n})\mathbf V^\mu. \tag{48}$$

Note that

$$\mathbf V^\mu = \bigoplus\{(\mathbf V_\varpi)^\mu; \varpi = -\sum_{\gamma\in\Psi(\mathfrak g)} n_\gamma\gamma \text{ for some non-negative integers } n_\gamma\}. \tag{49}$$

Suppose i). Let $\alpha\in\Theta$. Since $X_{-\alpha}\in J_\Theta(\lambda)\setminus J(\lambda_\Theta)$, there exists $D\in U(\mathfrak g)\mathbf V$ with $D^{\lambda_\Theta}=X_{-\alpha}$. On the other hand, we can deduce $\bigl(U(\bar{\mathfrak n})\mathbf V^{\lambda_\Theta}\bigr)_{-\alpha} = (\mathbf V_{-\alpha})^{\lambda_\Theta}$ from (49) because the assumption of the lemma assures $(\mathbf V_0)^{\lambda_\Theta}=0$. Hence from (48) we may assume $D\in\mathbf V_{-\alpha}$. Thus we have ii).

It is clear that ii) implies i) because $J_\Theta(\lambda) = J(\lambda_\Theta) + \sum_{\alpha\in\Theta} U(\mathfrak g) X_{-\alpha}$.

Let $\alpha\in\Theta$. Since $\operatorname{ad}(H)X_{-\alpha} = -\alpha(H)X_{-\alpha}$ for $H\in\mathfrak a$, we have

$$H_1\cdots H_k X_{-\alpha} = X_{-\alpha}(H_1-\alpha(H_1))\cdots(H_k-\alpha(H_k)) \text{ for } H_1,\dots,H_k\in\mathfrak a.$$

We also have $X_\gamma X_{-\alpha} \in J(\lambda_\Theta)$ for $\gamma \in \Sigma(\mathfrak{g})^+$ because $\lambda_\Theta([X_\alpha, X_{-\alpha}]) = 0$ and $[X_\gamma, X_{-\alpha}] \in \mathfrak{n}$ if $\gamma \neq \alpha$. Hence for any $D \in U(\mathfrak{g})_0$,

$$(\mathrm{ad}(X_{-\alpha})D)^{\lambda_\Theta} = [X_{-\alpha}, D_\mathfrak{a}]^{\lambda_\Theta} = \big(D_\mathfrak{a}(\lambda_\Theta) - D_\mathfrak{a}(\lambda_\Theta - \alpha)\big)X_{-\alpha}. \tag{50}$$

Now it is clear that iii) implies ii).

Conversely suppose ii). Let $\alpha \in \Theta$. Since $\mathbf{V}_{-\alpha} = \mathrm{ad}(X_{-\alpha})\mathbf{V}_0$, there exists $D \in \mathbf{V}_0$ with $(\mathrm{ad}(X_{-\alpha})D)^{\lambda_\Theta} = X_{-\alpha}$ and we have iii) from (50). ■

Remark 3.5. In the above lemma $\lambda_\Theta - \alpha = w_\alpha.\lambda_\Theta$ for $\alpha \in \Theta$ because $\langle \lambda_\Theta, \alpha \rangle = 0$.

By the *Duflo theorem* ([6]), $\mathrm{Ann}\big(M(\mu)\big) = \sum_{\Delta \in Z(\mathfrak{g})} U(\mathfrak{g})\big(\Delta - \Delta_\mathfrak{a}(\mu)\big)$ for any $\mu \in \mathfrak{a}^*$. Then, by the following theorem, each $\mathrm{Ann}\big(M(\mu)\big)$ has the same $\mathrm{ad}(\mathfrak{g})$-module structure.

Theorem 3.6 (the Kostant theorem [14])**.** *There exists an* $\mathrm{ad}(\mathfrak{g})$*-submodule* $\mathcal{H}$ *of* $U(\mathfrak{g})$ *such that* $U(\mathfrak{g})$ *is naturally isomorphic to* $Z(\mathfrak{g}) \otimes \mathcal{H}$ *by the multiplication. For any finite dimensional* $\mathfrak{g}$*-module* $\mathbf{V}$*,* $\dim \mathrm{Hom}_\mathfrak{g}(\mathbf{V}, \mathcal{H}) = \dim \mathbf{V}_0$*.*

Similarly on the annihilators of generalized Verma modules we have

Proposition 3.7. *Suppose* $\lambda_\Theta + \rho$ *is dominant. Then for any finite dimensional* $\mathfrak{g}$*-module* $\mathbf{V}$ *and* $\mathcal{H}$ *in* Theorem 3.6*,*

$$\begin{aligned}\dim \mathrm{Hom}_\mathfrak{g}\big(\mathbf{V}, \mathrm{Ann}\big(M_\Theta(\lambda)\big)/\,\mathrm{Ann}\big(M(\lambda_\Theta)\big)\big)& \\ = \dim \mathrm{Hom}_\mathfrak{g}\big(\mathbf{V}, \mathcal{H} \cap \mathrm{Ann}\big(M_\Theta(\lambda)\big)\big) &= \dim \mathbf{V}_0 - \dim \mathbf{V}^{\mathfrak{g}_\Theta}\end{aligned}$$

where $\mathbf{V}^{\mathfrak{g}_\Theta} = \{v \in \mathbf{V};\ Xv = 0\ (\forall X \in \mathfrak{g}_\Theta)\}$*.*

Before proving the proposition, we accumulate some necessary facts from [2], [1] and [13].

Definition 3.8 (category $\mathcal{O}$ [2])**.** Let $\mathcal{O}$ be the abelian category consisting of the $\mathfrak{g}$-modules which are finitely generated, $\mathfrak{a}^*$-diagonalizable and $U(\mathfrak{n})$-finite. All subquotients of Verma modules are objects of $\mathcal{O}$. For $\mu \in \mathfrak{a}^*$ we denote by $L(\mu)$ the unique irreducible quotient of the Verma module $M(\mu)$. There exists a unique indecomposable projective object $P(\mu) \in \mathcal{O}$ such that $\mathrm{Hom}_\mathfrak{g}(P(\mu), L(\mu)) \neq 0$.

Proposition 3.9 ([2], [1])**.** i) *If* $\mu + \rho$ *is dominant, then* $P(\mu) = M(\mu)$ *and*

$$\dim \mathrm{Hom}_\mathfrak{g}(M(\mu), M(\mu')) = \begin{cases} 1 & \text{if } \mu' = \mu, \\ 0 & \text{if } \mu' \neq \mu. \end{cases}$$

ii) *For any* $\mu, \mu' \in \mathfrak{a}^*$

$$\dim \mathrm{Hom}_\mathfrak{g}(P(\mu), L(\mu')) = \begin{cases} 1 & \text{if } \mu' = \mu, \\ 0 & \text{if } \mu' \neq \mu. \end{cases}$$

iii) *For any finite dimensional* $\mathfrak{g}$*-module* $\mathbf{V}$ *and* $\mu \in \mathfrak{a}^*$*,* $\mathbf{V} \otimes P(\mu)$ *is a projective object in* $\mathcal{O}$*.*

Proposition 3.10 ([1], [13]). *Suppose $\mu \in \mathfrak{a}^*$ and $\mu+\rho$ is dominant. Then the map*

$$\{I \subset U(\mathfrak{g});\ \textit{two-sided ideal},\ I \supset \operatorname{Ann}\bigl(M(\mu)\bigr)\} \to \{M \subset M(\mu);\ \textit{submodule}\} \tag{51}$$

defined by $I \mapsto IM(\mu)$ is injective and hence $\operatorname{Ann}\bigl(M(\mu)/IM(\mu)\bigr) = I$ for any two-sided ideal I with $I \supset \operatorname{Ann}\bigl(M(\mu)\bigr)$. The image of the map (51) *consists of the submodules which are isomorphic to quotients of direct sums of $P(\mu')$ with*

$$2\frac{\langle \mu'+\rho, \beta\rangle}{\langle \beta,\beta\rangle} \in \{0,-1,-2,\ldots\} \textit{ for any } \beta \in \Sigma(\mathfrak{g})^+ \textit{ such that } \langle \mu+\rho,\beta\rangle = 0. \tag{52}$$

Proof of Proposition 3.7. We first show the map

$$\operatorname{Hom}_{\mathfrak{g}}(\mathbf{V},\mathcal{H}) \ni \varphi \mapsto \Phi \in \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M(\lambda_\Theta)) \tag{53}$$

defined by $\Phi(v\otimes u) = \varphi(v)u$ is a linear isomorphism. Since $U(\mathfrak{g}) = \mathcal{H} \oplus \operatorname{Ann}\bigl(M(\lambda_\Theta)\bigr)$ the map is injective. To show the surjectivity we calculate the dimensions of both spaces. By Theorem 3.6 $\dim \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V},\mathcal{H}) = \dim \mathbf{V}_0$. On the other hand, note that

$$\operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M(\lambda_\Theta)) \simeq \operatorname{Hom}_{\mathfrak{g}}(M(\lambda_\Theta), M(\lambda_\Theta)\otimes \mathbf{V}^*)$$

and there exist a sequence $\{\mu_1,\ldots,\mu_\ell\} \subset \mathfrak{a}^*$ and a $\mathfrak{g}$-stable filtration

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M(\lambda_\Theta)\otimes \mathbf{V}^*$$

such that $M_i/M_{i-1} \simeq M(\mu_i)$ for $i = 1,\ldots,\ell$. Here the number of appearances of λ_Θ in the sequence $\{\mu_1,\ldots,\mu_\ell\}$ equals $\dim \mathbf{V}_0^* = \dim \mathbf{V}_0$ (cf. the proof of Theorem 2.29). Since $\lambda_\Theta + \rho$ is dominant, it follows from Proposition 3.9 i) that $\dim \operatorname{Hom}_{\mathfrak{g}}(M(\lambda_\Theta), M(\lambda_\Theta)\otimes\mathbf{V}^*) = \dim \mathbf{V}_0$. Thus (53) is isomorphism.

Secondly, consider the exact sequence

$$0 \to J_\Theta(\lambda)/J(\lambda_\Theta) \to M(\lambda_\Theta) \to M_\Theta(\lambda) \to 0.$$

It is clear that under the isomorphism (53) the subspace

$$\operatorname{Hom}_{\mathfrak{g}}\bigl(\mathbf{V}, \mathcal{H}\cap \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)\bigr) \subset \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V},\mathcal{H})$$

corresponds to the subspace

$$\operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), J_\Theta(\lambda)/J(\lambda_\Theta)) \subset \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M(\lambda_\Theta)).$$

Let us calculate the dimension of the latter space. By Proposition 3.9 i) and iii), $\mathbf{V}\otimes M(\lambda_\Theta)$ is projective and therefore

$$\begin{aligned}&\dim \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), J_\Theta(\lambda)/J(\lambda_\Theta))\\ &\qquad= \dim \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M(\lambda_\Theta)) - \dim \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M_\Theta(\lambda)).\end{aligned}$$

Here we know

$$\operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}\otimes M(\lambda_\Theta), M_\Theta(\lambda)) \simeq \operatorname{Hom}_{\mathfrak{g}}(M(\lambda_\Theta), M_\Theta(\lambda)\otimes \mathbf{V}^*)$$

and there exist a sequence $\{\mu_1,\ldots,\mu_{\ell'}\} \subset \mathfrak{a}^*$ and a $\mathfrak{g}$-stable filtration

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{\ell'} = M_\Theta(\lambda)\otimes \mathbf{V}^*$$

such that $M_i/M_{i-1} \simeq M_{(\Theta,\mu_i)}$ for $i = 1,\ldots,\ell'$. The number of appearances of λ_Θ in the sequence $\{\mu_1,\ldots,\mu_{\ell'}\}$ equals $\dim(\mathbf{V}^*)^{\mathfrak{g}_\Theta} = \dim \mathbf{V}^{\mathfrak{g}_\Theta}$ (cf. the proof of Theorem 2.29). Since the generalized Verma module $M_{(\Theta,\mu_i)}$ is a quotient of $M(\mu_i)$, Proposition 3.9 i) implies $\dim \operatorname{Hom}_{\mathfrak{g}}(M(\lambda_\Theta), M_\Theta(\lambda)\otimes\mathbf{V}^*) = \dim \mathbf{V}^{\mathfrak{g}_\Theta}$. Thus the proposition is proved. ∎

Proposition 3.11 (Harish-Chandra homomorphism). *Let I be a two-sided ideal of $U(\mathfrak{g})$. Put $\mathcal{V}(I) = \{\mu \in \mathfrak{a};\ D_{\mathfrak{a}}(\mu) = 0\ (\forall D \in I)\}$.*
i) *Fix $\alpha \in \Psi(\mathfrak{g})$. If $\mu \in \mathcal{V}(I)$ and*

$$2\frac{\langle \mu + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \{1, 2, 3, \ldots\}, \tag{54}$$

then $w_\alpha.\mu \in \mathcal{V}(I)$.
ii) *Suppose $\lambda \in \mathfrak{a}_\Theta^*$ and*

$$J_\Theta(\lambda) = I + J(\lambda_\Theta). \tag{55}$$

Then $w.\lambda_\Theta \notin \mathcal{V}(I)$ for $w \in W_\Theta \setminus \{e\}$.
iii) *In addition to the assumption of* ii), *suppose $\lambda_\Theta + \rho$ is dominant and*

$$I \supset \operatorname{Ann}\bigl(M(\lambda_\Theta)\bigr). \tag{56}$$

Then $I = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ and

$$\mathcal{V}(I) = \{w.\lambda_\Theta;\ w \in W(\Theta)\}. \tag{57}$$

Proof. i) Note that $\mu \in \mathcal{V}(I)$ if and only if $I \subset \operatorname{Ann}\bigl(L(\mu)\bigr)$. It is known by [12] that $\operatorname{Ann}\bigl(L(\mu)\bigr) \subset \operatorname{Ann}\bigl(L(w_\alpha.\mu)\bigr)$ if (54) holds, which implies i).

ii) Since $I \subset \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr) \subset \operatorname{Ann}\bigl(L(\lambda_\Theta)\bigr)$ we have $\lambda_\Theta \in \mathcal{V}(I)$. Put $W' = \{w \in W_\Theta \setminus \{e\};\ w.\lambda_\Theta \in \mathcal{V}(I)\}$. Then, by Lemma 3.4 with $\mathbf{V} = I$, $w_\alpha \notin W'$ for any $\alpha \in \Theta$. Suppose $W' \neq \emptyset$. Let w' be an element of W' with the minimal length. Then there exists $\alpha \in \Theta$ such that the length of $w'' = w_\alpha w'$ is smaller than that of w'. Then $w'' \neq e$ and

$$2\frac{\langle w'.\lambda_\Theta + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\langle w'\rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} < 0.$$

Hence by i), we have $w''.\mu \in \mathcal{V}(I)$, which is a contradiction.

iii) It immediately follows from Proposition 3.10 that $I = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$. Since $\operatorname{Ann}\bigl(M(\lambda_\Theta)\bigr) = \sum_{\Delta \in Z(\mathfrak{g})} U(\mathfrak{g})\bigl(\Delta - \Delta_{\mathfrak{a}}(\lambda_\Theta)\bigr)$, $\mathcal{V}(I) \subset \{w.\lambda_\Theta;\ w \in W\}$. Let $w = w(\Theta)w_\Theta \in W$ with $w(\Theta) \in W(\Theta)$ and $w_\Theta \in W_\Theta$. Suppose $w(\Theta) \neq e$. Then there exists $\alpha \in \Psi(\mathfrak{g})$ such that the length of $w_\alpha w(\Theta)$ is less than that of $w(\Theta)$. For this root α we have $w_\alpha w(\Theta) \in W(\Theta)$ and $w(\Theta)^{-1}\alpha, w_\Theta^{-1}w(\Theta)^{-1}\alpha \in \Sigma(\mathfrak{g})^- \setminus \Sigma(\mathfrak{g}_\Theta)$. The assumption thereby implies

$$2\frac{\langle w.\lambda_\Theta + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \{1, 2, 3, \ldots\}.$$

Hence $(w_\alpha w).\lambda_\Theta \in \mathcal{V}(I)$ provided that $w.\lambda_\Theta \in \mathcal{V}(I)$, which assures

$$\mathcal{V}(I) \cap \bigl\{\bigl(W(\Theta)w_\Theta\bigr).\lambda_\Theta;\ w_\Theta \in W_\Theta \setminus \{e\}\bigr\} = \emptyset \tag{58}$$

by ii) and the induction on the length of $w(\Theta)$. Similarly we can show that $\mathcal{V}(I) \supset \{w.\lambda_\Theta;\ w \in W(\Theta)\}$ if

$$2\frac{\langle \lambda_\Theta + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \{1, 2, 3, \ldots\} \quad \bigl(\forall \alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)\bigr). \tag{59}$$

Let us remove the condition (59) by use of Proposition 3.7. Since $U(\mathfrak{g}) = \mathcal{H} \oplus \operatorname{Ann}\bigl(M(\lambda_\Theta)\bigr)$, we have only to show for each finite dimensional $\mathfrak{g}$-module $\mathbf{V}$

$$\bigl(\varphi(v)\bigr)_{\mathfrak{a}}(w.\lambda_\Theta) = 0 \quad \Bigl(\forall\varphi \in \operatorname{Hom}_{\mathfrak{g}}\bigl(\mathbf{V}, \mathcal{H} \cap \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)\bigr), \forall v \in \mathbf{V}, \forall w \in W(\Theta)\Bigr). \tag{60}$$

For $D \in U(\mathfrak{g})$ we denote by D^λ a unique element of $U(\bar{\mathfrak{n}}_\Theta)$ such that $D - D^\lambda \in J_\Theta(\lambda)$. Then $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}, \mathcal{H})$ belongs to $\operatorname{Hom}_{\mathfrak{g}}\bigl(\mathbf{V}, \mathcal{H} \cap \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)\bigr)$ if and only if $\varphi(v)^\lambda = 0$ for $v \in V$. Let $k = \dim \mathbf{V}_0$ and take $\varphi_1, \ldots, \varphi_k \in \operatorname{Hom}_{\mathfrak{g}}(\mathbf{V}, \mathcal{H})$ so that they constitute a basis. Note that for $v \in \mathbf{V}$ and $i = 1, \ldots, k$, $\varphi_i(v)^\lambda$ are $U(\bar{\mathfrak{n}}_\Theta)$-valued polynomials in λ. Let $\ell = k - \dim \mathbf{V}^{\mathfrak{g}_\Theta}$. Then by Proposition 3.7 there exist an open neighborhood $S \subset \mathfrak{a}_\Theta^*$ of the point in question and complex-valued rational functions $a_{ij}(\lambda)$ on S such that

$$a_{1j}(\lambda)\varphi_1 + a_{2j}(\lambda)\varphi_2 + \cdots + a_{kj}(\lambda)\varphi_k \quad (j = 1, \ldots, \ell)$$

form a basis of $\operatorname{Hom}_{\mathfrak{g}}\bigl(\mathbf{V}, \mathcal{H} \cap \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)\bigr)$ for any $\lambda \in S$. Since generic $\lambda \in S$ satisfy (59), (60) holds for any $\lambda \in S$. ■

On the existence of a two-sided ideal $I_\Theta(\lambda)$ satisfying (40), we have

Theorem 3.12. *Suppose $\lambda_\Theta + \rho$ is dominant. Then the following four conditions are equivalent.*

i) $J_\Theta(\lambda) = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr) + J(\lambda_\Theta)$.
ii) *If $\beta \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)$ satisfies $\langle \lambda_\Theta + \rho, \beta\rangle = 0$, then $\langle \beta, \alpha\rangle = 0$ for all $\alpha \in \Theta$.*
iii) $W(\Theta).\lambda_\Theta \cap W_\Theta.\lambda_\Theta = \{\lambda_\Theta\}$.
iv) *If $w_\Theta \in W_\Theta$ satisfies $\bigl(W(\Theta)w_\Theta\bigr).\lambda_\Theta \cap W(\Theta).\lambda_\Theta \neq \emptyset$, then $w_\Theta = e$.*

In particular, if $\lambda_\Theta + \rho$ is regular, these conditions are satisfied.

Proof. iv) $\Rightarrow$ iii) is obvious.

iii) $\Rightarrow$ ii). Suppose there exist $\beta \in \Sigma(\mathfrak{g})^+ \setminus \Sigma(\mathfrak{g}_\Theta)$ and $\alpha \in \Theta$ such that $\langle \lambda_\Theta + \rho, \beta\rangle = 0$ and $\langle \beta, \alpha\rangle \neq 0$. For $\gamma \in \Sigma(\mathfrak{g}_\Theta)^+$ we have

$$2\frac{\langle \lambda_\Theta + \rho, w_\beta\gamma\rangle}{\langle w_\beta\gamma, w_\beta\gamma\rangle} = 2\frac{\langle \lambda_\Theta + \rho, \gamma\rangle}{\langle \gamma, \gamma\rangle} = 2\frac{\langle \rho, \gamma\rangle}{\langle \gamma, \gamma\rangle} \in \{1, 2, \ldots\},$$

which shows $\langle \beta, \gamma\rangle \leq 0$ and $w_\beta \in W(\Theta)$. In particular $\langle \beta, \alpha\rangle < 0$ and hence $w_\alpha w_\beta \in W(\Theta)$. Now we get $(w_\alpha w_\beta).\lambda_\Theta = w_\alpha.\lambda_\Theta$, a contradiction.

ii) $\Rightarrow$ i). For each $\alpha \in \Theta$ we define the $\mathfrak{g}$-homomorphism $M(\lambda_\Theta - \alpha) \to M(\lambda_\Theta)$ by $D \bmod J(\lambda_\Theta - \alpha) \mapsto DX_{-\alpha} \bmod J(\lambda_\Theta)$. This is an injection and therefore we identify its image with $M(\lambda_\Theta - \alpha)$. Note that

$$\sum_{\alpha\in\Theta} M(\lambda_\Theta - \alpha) = \Bigl(J(\lambda_\Theta) + \sum_{\alpha\in\Theta} U(\mathfrak{g})X_{-\alpha}\Bigr)/J(\lambda_\Theta) = J_\Theta(\lambda)/J(\lambda_\Theta)$$

and we have a surjection $P(\lambda_\Theta - \alpha) \to M(\lambda_\Theta - \alpha)$ by Proposition 3.9 ii). Moreover it is clear that the condition (52) with $(\mu, \mu') = (\lambda_\Theta, \lambda_\Theta - \alpha)$ holds for each $\alpha \in \Theta$. Hence by Proposition 3.10 we have a two-sided ideal I containing $\operatorname{Ann}\bigl(M(\lambda_\Theta)\bigr)$ such that $IM(\lambda_\Theta) = J_\Theta(\lambda)/J(\lambda_\Theta)$. Then $I = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ and $J_\Theta(\lambda) = I + J(\lambda_\Theta)$.

i) $\Rightarrow$ iv) follows from (57) and (58). ■

Remark 3.13. Through $I_{\pi,\Theta}$, we will get in §4. many sufficient conditions for (41), which are effective even if $\lambda_\Theta + \rho$ is not dominant.

Definition 3.14 (extremal low weight)**.** For a simple root $\alpha \in \Psi(\mathfrak{g})$, we call a minimal element of $\{\varpi \in \mathcal{W}(\pi);\ \langle \varpi, \alpha \rangle \neq 0\}$ under the order $\leq$ in Definition 2.20 *an extremal low weight of π with respect to α.*

Since π is a faithful representation, $\pi(X_{-\alpha})$ is not zero and therefore an extremal low weight ϖ_α with respect to α always exists but it may not be unique. The main purpose in this section is to calculate the function

$$\mathfrak{a}_\Theta^* \ni \lambda \mapsto \Big(q_{\pi,\Theta}(F_\pi; \lambda)_{\varpi_\alpha \varpi_\alpha} \Big)_{\mathfrak{a}} (\lambda_\Theta - \alpha) \tag{61}$$

on $\mathfrak{a}_\Theta^*$. If for any $\alpha \in \Theta$ there exists ϖ_α such that the value of the corresponding function (61) does not vanish, Lemma 3.4 assures (44).

Lemma 3.15. *Fix $\alpha \in \Psi(\mathfrak{g})$ and let ϖ_α be an extremal low weight of π with respect to α. For $\lambda = \sum_{\beta \in \Psi(\mathfrak{g})} m_\beta \beta \in \mathfrak{a}^*$ put $|\lambda| = \sum_{\beta \in \Psi(\mathfrak{g})} m_\beta$. Then there exists $\{\gamma_1, \ldots, \gamma_K\} \subset \Psi(\mathfrak{g})$ with $\gamma_K = \alpha$ such that the following* (63)*–*(69) *hold by denoting*

$$\varpi_i = \varpi_\alpha - \sum_{i \leq \nu < K} \gamma_\nu. \tag{62}$$

$$K = |\varpi_\alpha - \bar{\pi}| + 1 \textit{ and } \varpi_1 = \bar{\pi}, \tag{63}$$

$$\langle \varpi_i, \gamma_i \rangle < 0 \textit{ for } i = 1, \ldots, K, \tag{64}$$

$$\langle \varpi_i, \gamma_j \rangle = 0 \textit{ if } 1 \leq i < j \leq K, \tag{65}$$

$$\langle \gamma_i, \gamma_j \rangle \neq 0 \textit{ if and only if } |i - j| \leq 1, \tag{66}$$

$$\{\varpi_1, \ldots, \varpi_{K-1}\} = \{\varpi' \in \mathcal{W}(\pi);\ \varpi' < \varpi_\alpha\}, \tag{67}$$

$$\varpi_i \textit{ is an extremal low weight of } \pi \textit{ with respect to } \gamma_i \textit{ for } i = 1, \ldots, K, \tag{68}$$

$$\textit{the multiplicity of the weight space of the weight } \varpi_i \textit{ equals } 1. \tag{69}$$

The sequence $\gamma_1, \ldots, \gamma_K$ is unique by the condition $\varpi_1, \ldots, \varpi_K \in \mathcal{W}(\pi)$. The part of the partially ordered set of the weights of π which are smaller or equal to ϖ_α is as follows:

$$\varpi_1 = \bar{\pi} \xrightarrow{\gamma_1} \varpi_2 \xrightarrow{\gamma_2} \varpi_3 \xrightarrow{\gamma_3} \cdots\cdots \xrightarrow{\gamma_{K-1}} \varpi_K = \varpi_\alpha \xrightarrow{\gamma_K = \alpha} \tag{70}$$

Proof. Let $\gamma_1, \ldots, \gamma_K$ be a sequence of $\Psi(\mathfrak{g})$ satisfying (63), $\gamma_K = \alpha$, and $\varpi_1, \ldots, \varpi_K \in \mathcal{W}(\pi)$ under the notation (62). The existence of such a sequence is clear. We shall prove by the induction on K that such a sequence is unique and that it satisfies (64)–(68).

By the minimality of ϖ_α we have $\langle \varpi_i, \alpha \rangle = 0$ for $i = 1, \ldots, K-1$. Hence $\langle \gamma_i, \alpha \rangle = \langle \varpi_{i+1} - \varpi_i, \alpha \rangle = 0$ for $i = 1, \ldots, K-2$ and $\langle \gamma_{K-1}, \alpha \rangle = \langle \varpi_\alpha - \varpi_{K-1}, \alpha \rangle = \langle \varpi_\alpha, \alpha \rangle < 0$. Thus we get $\gamma_i \neq \alpha$ for $i = 1, \ldots, K-1$. Moreover $\varpi_\alpha - \gamma_i \notin \mathcal{W}(\pi)$ for $i = 1, \ldots, K-2$ because $\langle \varpi_\alpha - \gamma_i, \alpha \rangle = \langle \varpi_\alpha, \alpha \rangle \neq 0$ and ϖ_α is minimal. This means $\{\varpi' \in \mathcal{W}(\pi);\ \varpi' < \varpi_\alpha\} = \{\varpi_{K-1}\} \cup \{\varpi' \in \mathcal{W}(\pi);\ \varpi' < \varpi_{K-1}\}$.

Suppose $\langle \varpi_{K-1}, \gamma_{K-1} \rangle \geq 0$. Then $\varpi_{K-1} - \gamma_{K-1} \in \mathcal{W}(\pi)$ because $\varpi_{K-1} + \gamma_{K-1} = \varpi_\alpha \in \mathcal{W}(\pi)$. Hence $\langle \varpi_{K-1} - \gamma_{K-1}, \alpha \rangle = -\langle \gamma_{K-1}, \alpha \rangle > 0$, which contradicts with the minimality of ϖ_α. Thus we get $\langle \varpi_{K-1}, \gamma_{K-1} \rangle < 0$.

Suppose ϖ_{K-1} is not an extremal low weight with respect to γ_{K-1}. Then there exists an extremal low weight ϖ' with respect to γ_{K-1} such that $\varpi' < \varpi_{K-1}$. Then $\mathcal{W}(\pi) \ni \varpi' + \gamma_{K-1} < \varpi_\alpha$ and $\langle \varpi' + \gamma_{K-1}, \alpha \rangle = \langle \gamma_{K-1}, \alpha \rangle = 0$ by the minimality of ϖ_α. It is a contradiction. Hence ϖ_{K-1} is an extremal low weight with respect to γ_{K-1}.

Now by the induction hypothesis we obtain the uniqueness and (64)–(68). Note that (69) follows from the uniqueness and the following lemma because $V = U(\mathfrak{n})v_{\bar{\pi}}$ with a lowest weight vector $v_{\bar{\pi}}$ of π. ■

Lemma 3.16. *$U(\mathfrak{n})$ is generated by $\{X_\gamma;\, \gamma \in \Psi(\mathfrak{g})\}$ as a subalgebra of $U(\mathfrak{g})$.*

Proof. Let U denote the algebra generated by $\{X_\gamma;\, \gamma \in \Psi(\mathfrak{g})\}$. It is sufficient to show that $X_\beta \in U$ for $\beta \in \Sigma(\mathfrak{g})^+$, which is proved by the induction on $|\beta|$ as follows. If $|\beta| > 1$, there exists $\gamma \in \Psi(\mathfrak{g})$ such that $\beta' = \beta - \gamma \in \Sigma(\mathfrak{g})^+$. Then $X_\beta = C(X_\gamma X_{\beta'} - X_{\beta'} X_\gamma)$ with a constant $C \in \mathbb{C}$. Hence the condition $X_\gamma,\, X_{\beta'} \in U$ implies $X_\beta \in U$. ■

Remark 3.17. By virtue of (66) the Dynkin diagram of the system $\{\gamma_1, \ldots, \gamma_K\}$ in Lemma 3.15 is of type A_K or B_K or C_K or F_4 or G_2 where γ_1 and γ_K correspond to the end points of the diagram. Note that

$$\langle \bar{\pi}, \gamma_1 \rangle < 0 \text{ and } \langle \bar{\pi}, \gamma_i \rangle = 0 \text{ for } i = 2, \ldots, K. \tag{71}$$

Conversely if a subsystem $\{\gamma_1, \ldots, \gamma_K\} \subset \Psi(\mathfrak{g})$ satisfies (66) and (71) then $\bar{\pi} + \gamma_1 + \cdots + \gamma_{K-1}$ is an extremal low weight with respect to γ_K. Hence we have at most three different extremal low weights of π with respect to a fixed $\alpha \in \Psi(\mathfrak{g})$.

The next lemma is studied in [22, Lemma 3.5]. It gives the solutions for the recursive equations which play key roles in the calculation of (61).

Lemma 3.18. *For $k = 0, 1, \ldots$ and $\ell = 1, 2, \ldots$, define the polynomial $f(k, \ell)$ in the variables $s_1, \ldots, s_{\ell-1}, \mu_1, \mu_2, \ldots$ recursively by*

$$f(k,\ell) = \begin{cases} 1 & \textit{if } \ k = 0, \\ f(k-1,\ell)(\mu_\ell - \mu_k) + \sum_{\nu=1}^{\ell-1} s_\nu f(k-1,\nu) & \textit{if } \ k \geq 1. \end{cases} \tag{72}$$

Moreover for $k = 1, 2, \ldots$ and $\ell = 1, 2, \ldots$, define the polynomial $g(k, \ell)$ in the variables $t, s_1, \ldots, s_{\ell-1}, \mu_1, \mu_2, \ldots$ recursively by

$$g(k,\ell) = \begin{cases} 1 & \textit{if } \ k = 1, \\ g(k-1,\ell)(t - \mu_k) + f(k-1,\ell) & \textit{if } \ k > 1. \end{cases} \tag{73}$$

Then the following (74)–(76) hold.

$$f(k,\ell) = 0 \quad \textit{for } \ k \geq \ell, \tag{74}$$

$$f(\ell-1,\ell) = \prod_{\nu=1}^{\ell-1} (\mu_\ell - \mu_\nu + s_\nu), \tag{75}$$

$$g(k,\ell) = \prod_{\nu=1}^{\ell-1} (t - \mu_\nu + s_\nu) \prod_{\nu=\ell+1}^{k} (t - \mu_\nu) \quad \textit{for } \ k \geq \ell. \tag{76}$$

Now recall (22) with $\Theta = \emptyset$. Let $F^k_{ii} \in U(\mathfrak{a})$ be the element in (22) corresponding to the weight ϖ_i for $i = 1, \ldots, K$ under the notation in Lemma 3.15. Then Lemma 3.15 and Lemma 2.19 iv) with $\ell = 1$, $\beta = \varpi_i - \varpi_\nu \in \Sigma(\mathfrak{g})^+$ $(1 \le \nu < i)$ and $\varpi = \varpi_\nu$ show that (22) is reduced to

$$F^k_{ii} - F^{k-1}_{ii}\big(\varpi_i - \mu_k + D_\pi(\varpi_i)\big) \equiv \sum_{1\le\nu<i} \langle \varpi_\nu, \varpi_i - \varpi_\nu\rangle F^{k-1}_{\nu\nu} \mod U(\mathfrak{g})\mathfrak{n}. \tag{77}$$

Since $\langle \varpi_i, \lambda_\Theta\rangle = \langle \varpi_i, \lambda_\Theta - \alpha\rangle$ for $i = 1, \ldots, K-1$, (77) inductively implies

$$\big(F^k_{ii}\big)_{\mathfrak{a}}(\lambda_\Theta) = \big(F^k_{ii}\big)_{\mathfrak{a}}(\lambda_\Theta - \alpha) \quad \text{for } i = 1, \ldots, K-1 \text{ and } k = 0, 1, \ldots. \tag{78}$$

From (65) we have

$$\langle \varpi_\nu, \varpi_i - \varpi_\nu\rangle = \langle \varpi_\nu, \gamma_\nu + \cdots + \gamma_{i-1}\rangle = \langle \varpi_\nu, \gamma_\nu\rangle$$

and hence

$$\begin{aligned} F^k_{i+1i+1} - F^k_{ii} &\equiv F^{k-1}_{i+1i+1}\big(\varpi_{i+1} - \mu_k + D_\pi(\varpi_{i+1})\big) \\ &\qquad + F^{k-1}_{ii}\langle \varpi_i, \varpi_{i+1} - \varpi_i\rangle - F^{k-1}_{ii}\big(\varpi_i - \mu_k + D_\pi(\varpi_i)\big) \mod U(\mathfrak{g})\mathfrak{n} \\ &= (F^{k-1}_{i+1i+1} - F^{k-1}_{ii})\big(\varpi_{i+1} - \mu_k + D_\pi(\varpi_{i+1})\big) + F^{k-1}_{ii}\gamma_i. \end{aligned}$$

The last equality above follows from Lemma 2.22 i) with $\varpi = \varpi_i$ and $\varpi' = \varpi_{i+1}$ because $\gamma_i = \varpi_{i+1} - \varpi_i \in \Psi(\mathfrak{g})$. Hence by the induction on k we have

$$F^k_{i+1i+1} \equiv F^k_{ii} \mod U(\mathfrak{g})\mathfrak{n} + U(\mathfrak{g})\gamma_i.$$

Now consider general $\Theta \subset \Psi(\mathfrak{g})$. Define integers n_0, $n_1, \ldots, n_L$ with $n_0 = 0 < n_1 < \cdots < n_L = K$ such that

$$\{n_1, \ldots, n_{L-1}\} = \{\nu \in \{1, \ldots, K-1\}; \gamma_\nu \notin \Theta\}.$$

If $n_{\ell-1} < \nu < n_\ell$, then $\gamma_\nu \in \Theta$, which implies $\langle \gamma_\nu, \lambda_\Theta\rangle = 0$ and hence

$$\big(F^k_{\nu+1\nu+1}\big)_{\mathfrak{a}}(\lambda_\Theta) = \big(F^k_{\nu\nu}\big)_{\mathfrak{a}}(\lambda_\Theta).$$

We note that $\varpi_{n_0+1}|_{\mathfrak{a}_\Theta} <_\Theta \varpi_{n_1+1}|_{\mathfrak{a}_\Theta} <_\Theta \cdots <_\Theta \varpi_{n_{L-1}+1}|_{\mathfrak{a}_\Theta}$ and

$$\{\varpi_{n_0+1}, \ldots, \varpi_{n_{L-1}+1}\} = \{\varpi' \in \overline{\mathcal{W}}_\Theta(\pi);\, \varpi' \le \varpi_\alpha\}.$$

Put $\mu_\ell = \langle \varpi_{n_{\ell-1}+1}, \lambda_\Theta\rangle + D_\pi(\varpi_{n_{\ell-1}+1})$ for $\ell = 1, \ldots, L$. Since $\prod_{\ell=1}^{L}(x - \mu_\ell)$ is a divisor of $q_{\pi,\Theta}(x;\lambda)$, we can take μ_ℓ for $\ell = L+1, L+2, \ldots, L' = \deg_x q_{\pi,\Theta}(x;\lambda)$ so that $q_{\pi,\Theta}(x;\lambda) = \prod_{\ell=1}^{L'}(x - \mu_\ell)$.

For $k = 0, 1, \ldots, L'$ and $\ell = 1, 2, \ldots, L$ we define

$$f(k,\ell) = \big(F^k_{n_{\ell-1}+1,n_{\ell-1}+1}\big)_{\mathfrak{a}}(\lambda_\Theta) = \cdots = \big(F^k_{n_\ell,n_\ell}\big)_{\mathfrak{a}}(\lambda_\Theta).$$

Then putting

$$s_\ell = \sum_{n_{\ell-1}<\nu\le n_\ell} \langle \varpi_\nu, \gamma_\nu\rangle,$$

we have from (77) with $i = n_{\ell-1}+1$

$$f(k,\ell) = f(k-1,\ell)(\mu_\ell - \mu_k) + \sum_{j=1}^{\ell-1} s_j f(k-1,j).$$

From (78) and (77) with $i = n_L = K$ we also have

$$\begin{aligned}\big(F^k_{KK}\big)_{\mathfrak a}(\lambda_\Theta - \alpha) &= \big(F^{k-1}_{KK}\big)_{\mathfrak a}(\lambda_\Theta-\alpha)\big(\langle\varpi_\alpha, \lambda_\Theta - \alpha\rangle + D_\pi(\varpi_\alpha) - \mu_k\big)\\ &\quad + \sum_{j=1}^{L-1} s_j f(k-1,j) + \left(\sum_{\nu=n_{L-1}+1}^{K-1}\langle\varpi_\nu,\gamma_\nu\rangle\right) f(k-1,L).\end{aligned}$$

Hence by Lemma 2.22 i)

$$\begin{aligned}&\frac{f(k,L) - \big(F^k_{KK}\big)_{\mathfrak a}(\lambda_\Theta-\alpha)}{\langle\varpi_\alpha,\alpha\rangle}\\ &\qquad = \frac{f(k-1,L) - \big(F^{k-1}_{KK}\big)_{\mathfrak a}(\lambda_\Theta-\alpha)}{\langle\varpi_\alpha,\alpha\rangle}\big(\langle\varpi_\alpha,\lambda_\Theta-\alpha\rangle + D_\pi(\varpi_\alpha) - \mu_k\big)\\ &\qquad\quad + f(k-1,L).\end{aligned}$$

Now applying Lemma 3.18 to

$$g(k,L) = \frac{f(k,L) - \big(F^k_{KK}\big)_{\mathfrak a}(\lambda_\Theta-\alpha)}{\langle\varpi_\alpha,\alpha\rangle}$$

with $t = \langle\varpi_\alpha,\lambda_\Theta-\alpha\rangle + D_\pi(\varpi_\alpha)$, we obtain

$$\begin{aligned}&\Big(q_{\pi,\Theta}(F_\pi;\lambda)_{\varpi_\alpha\varpi_\alpha}\Big)_{\mathfrak a}(\lambda_\Theta-\alpha)\\ &\qquad = \Big(F^{L'}_{KK}\Big)_{\mathfrak a}(\lambda_\Theta-\alpha)\\ &\qquad = -\langle\varpi_\alpha,\alpha\rangle\prod_{\ell=1}^{L-1}\Big(\langle\varpi_\alpha,\lambda_\Theta-\alpha\rangle + D_\pi(\varpi_\alpha) - \mu_\ell + s_\ell\Big)\\ &\qquad\qquad \cdot\prod_{\ell=L+1}^{L'}\Big(\langle\varpi_\alpha,\lambda_\Theta-\alpha\rangle + D_\pi(\varpi_\alpha) - \mu_\ell\Big)\\ &\qquad = -\langle\varpi_\alpha,\alpha\rangle\prod_{\ell=1}^{L-1}\Big(\langle\varpi'_\alpha - \varpi_{n_\ell},\lambda_\Theta\rangle + D_\pi(\varpi'_\alpha) - D_\pi(\varpi_{n_\ell+1})\Big)\\ &\qquad\qquad \cdot\prod_{(\mu,C)\in\Omega_{\pi,\Theta}\setminus\Omega^{\varpi_\alpha}_{\pi,\Theta}}\Big(\langle\varpi'_\alpha-\mu,\lambda_\Theta\rangle + D_\pi(\varpi'_\alpha) - C\Big).\end{aligned}$$

Here we put $\varpi'_\alpha = \varpi_\alpha + \alpha \in \mathcal W(\pi)$ and

$$\Omega^{\varpi_0}_{\pi,\Theta} = \{(\varpi|_{\mathfrak a_\Theta}, D_\pi(\varpi));\ \varpi\in\overline{\mathcal W}_\Theta(\pi),\ \varpi\le\varpi_0\} \tag{79}$$

for $\varpi_0\in\mathcal W(\pi)$. To deduce the last equality, we have used

$$\mu_\ell - s_\ell = \langle\varpi_{n_\ell},\lambda_\Theta\rangle + D_\pi(\varpi_{n_\ell+1}) \qquad \text{if } 1\le\ell\le L-1.$$

Definition 3.19. Suppose $\alpha \in \Theta$ and ϖ_α is an extremal low weight of π with respect to α. Put $\varpi'_\alpha = \varpi_\alpha + \alpha \in \mathcal{W}(\pi)$ and

$$\{\varpi_1, \ldots, \varpi_K\} = \{\varpi \in \overline{\mathcal{W}}(\pi);\ \varpi \le \varpi_\alpha\}$$

with $\varpi_1 < \varpi_2 < \cdots < \varpi_K$ and define $n_0 = 0 < n_1 < \cdots < n_L < K$ so that

$$\{\varpi_{n_0+1}, \ldots, \varpi_{n_L+1}\} = \{\varpi \in \overline{\mathcal{W}}_\Theta(\pi);\ \varpi \le \varpi_\alpha\}.$$

Under the notation in Definition 2.20 and (79), define

$$\begin{aligned} r_{\alpha,\varpi_\alpha}(\lambda) = &\prod_{(\mu,C)\in\Omega_{\pi,\Theta}\setminus\Omega_{\pi,\Theta}^{\varpi_\alpha}} \Bigl(\langle\lambda_\Theta, \varpi'_\alpha - \mu\rangle + D_\pi(\varpi'_\alpha) - C\Bigr) \\ &\cdot \prod_{i=1}^{L}\Bigl(\langle\lambda_\Theta, \varpi_\alpha - \varpi_{n_i}\rangle - \langle\alpha, \varpi_\alpha\rangle + D_\pi(\varpi_\alpha) - D_\pi(\varpi_{n_i+1})\Bigr). \end{aligned} \tag{80}$$

If there is no extremal low weights with respect to α other than ϖ_α, we use the simple symbol $r_\alpha(\lambda)$ for $r_{\alpha,\varpi_\alpha}(\lambda)$.

Remark 3.20. In the above definition we have the following.
i) If the lowest weight $\bar\pi$ is an extremal low weight of π with respect to α, then $L = 0$.
ii) The second factor

$$\prod_{i=1}^{L}\Bigl(\langle\lambda_\Theta, \varpi_\alpha - \varpi_{n_i}\rangle - \langle\alpha, \varpi_\alpha\rangle + D_\pi(\varpi_\alpha) - D_\pi(\varpi_{n_i+1})\Bigr)$$

is not identically zero because $\varpi_{n_i}|_{\mathfrak{a}_\Theta} <_\Theta \varpi_{n_i+1}|_{\mathfrak{a}_\Theta} \le_\Theta \varpi_\alpha|_{\mathfrak{a}_\Theta}$.
iii) For ϖ and $\varpi' \in \mathcal{W}(\pi)$

$$\langle\lambda_\Theta, \varpi - \varpi'\rangle + D_\pi(\varpi) - D_\pi(\varpi') = \langle\lambda_\Theta + \rho, \varpi - \varpi'\rangle + \frac{\langle\varpi', \varpi'\rangle - \langle\varpi, \varpi\rangle}{2}. \tag{81}$$

iv) Put $\gamma_\nu = \varpi_{\nu+1} - \varpi_\nu$ for $\nu = 1, \ldots, K-1$ and $\gamma_K = \alpha$. If

$$-2\frac{\langle\varpi_\nu, \gamma_\nu\rangle}{\langle\gamma_\nu, \gamma_\nu\rangle}\ \Bigl(= -2\frac{\langle\gamma_{\nu-1}, \gamma_\nu\rangle}{\langle\gamma_\nu, \gamma_\nu\rangle} \text{ if } \nu > 1\Bigr) = 1, \tag{82}$$

then $\langle\varpi_\nu, \varpi_\nu\rangle = \langle\varpi_{\nu+1}, \varpi_{\nu+1}\rangle$.
v) Suppose $\frac{2\langle\bar\pi,\gamma_1\rangle}{\langle\gamma_1,\gamma_1\rangle} = -1$ and the Dynkin diagram of the system $\{\gamma_1, \ldots, \gamma_K\}$ is of type A_K or of type B_K with short root γ_K or of type G_2 with short root γ_2. Then it follows from Lemma 3.15 and Lemma 2.22 i) that

$$\begin{aligned} &\langle\lambda_\Theta, \varpi_\alpha - \varpi_{n_i}\rangle - \langle\alpha, \varpi_\alpha\rangle + D_\pi(\varpi_\alpha) - D_\pi(\varpi_{n_i+1}) \\ &\qquad = \langle\lambda_\Theta, \varpi_\alpha - \varpi_{n_i}\rangle + D_\pi(\varpi_\alpha) - D_\pi(\varpi_{n_i}) \\ &\qquad\quad = \langle\lambda_\Theta + \rho, \varpi_\alpha - \varpi_{n_i}\rangle = \langle\lambda_\Theta + \rho, \gamma_{n_i} + \cdots + \gamma_{K-1}\rangle \end{aligned} \tag{83}$$

for $i = 1, \ldots, L$.

Theorem 3.21 (gap). *Let ϖ_α be an extremal low weight with respect to $\alpha \in \Theta$. Then*

$$X_{-\alpha} \in I_{\pi,\Theta}(\lambda) + J(\lambda_\Theta) \quad \text{if } r_{\alpha,\varpi_\alpha}(\lambda) \neq 0.$$

If for all $\alpha \in \Theta$ there exists an extremal low weight ϖ_α with respect to α such that $r_{\alpha,\varpi_\alpha}(\lambda) \neq 0$, then

$$J_\Theta(\lambda) = I_{\pi,\Theta}(\lambda) + J(\lambda_\Theta).$$

By Proposition 3.11 iii) we have the following corollary.

Corollary 3.22 (annihilator). *If $\lambda_\Theta + \rho$ is dominant and if for all $\alpha \in \Theta$ there exists an extremal low weight ϖ_α with respect to α such that $r_{\alpha,\varpi_\alpha}(\lambda) \neq 0$, then $I_{\pi,\Theta}(\lambda) = \operatorname{Ann}\big(M_\Theta(\lambda)\big)$.*

Remark 3.23. It does not always hold that for each $\alpha \in \Theta$ there exists an extremal low weight ϖ_α with respect to α such that the function $r_{\alpha,\varpi_\alpha}(\lambda)$ is not identically zero. In fact we construct counter examples in Appendix B. However this condition is valid for many π as we see below.

Recall the notation in Proposition 2.39.

Lemma 3.24. *Suppose ϖ_α is an extremal low weight with respect to $\alpha \in \Theta$. The function $r_{\alpha,\varpi_\alpha}(\lambda)$ is not identically zero if the space*

$$V(\varpi_\alpha|_{\mathfrak{a}_\Theta}) = \sum_{\varpi \in \mathcal{W}(\pi);\, \varpi|_{\mathfrak{a}_\Theta} = \varpi_\alpha|_{\mathfrak{a}_\Theta}} V_\varpi$$

is irreducible as a $\mathfrak{g}_\Theta$-module.

Proof. In this case we have $\mu|_{\mathfrak{a}_\Theta} \neq \varpi'_\alpha|_{\mathfrak{a}_\Theta}$ for $(\mu, C) \in \Omega_{\pi,\Theta} \setminus \Omega^{\varpi_\alpha}_{\pi,\Theta}$ and the first factor of (80) is not identically zero. ∎

Proposition 3.25. *Use the notation in* Lemma 3.15 *and suppose $\gamma_K = \alpha \in \Theta$. The function $r_{\alpha,\varpi_\alpha}(\lambda)$ is not identically zero if either one of the following conditions is satisfied.*

i) $\{\gamma_1, \ldots, \gamma_K\} \subset \Theta$.

ii) *The connected component of the Dynkin diagram of Θ containing α is orthogonal to $\bar\pi$. $\Theta \setminus \{\gamma_1, \ldots, \gamma_K\}$ is orthogonal to $\{\gamma_1, \ldots, \gamma_{K-1}\}$. Moreover the Dynkin diagram of the system $\{\gamma_1, \ldots, \gamma_{K-1}\}$ is of type A_{K-1}.*

Proof. i) Since $\varpi_\alpha|_{\mathfrak{a}_\Theta} = \bar\pi|_{\mathfrak{a}_\Theta}$ and $V(\bar\pi|_{\mathfrak{a}_\Theta})$ is an irreducible $\mathfrak{g}_\Theta$-module, the claim follows from Lemma 3.24.

ii) Suppose $\varpi \in \overline{\mathcal{W}}_\Theta(\pi)$ satisfies $\varpi|_{\mathfrak{a}_\Theta} = \varpi_\alpha|_{\mathfrak{a}_\Theta}$. Then we can write

$$\varpi = \bar\pi + \sum_{i=1}^{K} m_i \gamma_i + \sum_{\beta \in \Theta \setminus \{\gamma_1, \ldots, \gamma_K\}} n_\beta \beta$$

with non-negative integers m_i and n_β. Put

$$\Theta' = \{\gamma_i;\, m_i > 0\},$$

$$\Theta'' = \{\beta;\, n_\beta > 0\},$$

and define

$$V' = \sum\{V_{\varpi'};\, \varpi' \in \bar{\pi} \; + \sum_{\beta\in\Theta'\cup\Theta''} \mathbb{Z}\,\beta\}.$$

Since V' is an irreducible $\mathfrak{g}_{\Theta'\cup\Theta''}$-module with lowest weight $\bar{\pi}$ and $\{0\} \subsetneq V_\varpi \subset V'$, each connected component of the Dynkin diagram of the system $\Theta' \cup \Theta''$ is not orthogonal to $\bar{\pi}$.

Suppose $\gamma_K \in \Theta'$. Then the condition ii) implies $\Theta' = \{\gamma_1, \ldots, \gamma_K\}$ and therefore $\varpi'_\alpha = \varpi_\alpha + \alpha \le \varpi$. However it is clear $\dim V_{\varpi'_\alpha} = 1$ and $\varpi'_\alpha \notin \overline{\mathcal{W}}_\Theta(\pi)$. Thus we have $\varpi'_\alpha < \varpi$. In this case, by Lemma 2.22 ii), we have $D(\varpi'_\alpha) < D(\varpi)$.

Suppose $\gamma_K \notin \Theta'$. Then Θ' is orthogonal to Θ'' and hence we have the direct sum decomposition

$$\mathfrak{g}_{\Theta'\cup\Theta''} = \mathfrak{a}_{\Theta'\cup\Theta''} \oplus \mathfrak{m}_{\Theta'} \oplus \mathfrak{m}_{\Theta''}.$$

Since ϖ is the lowest weight of a $\mathfrak{m}_{\Theta''}$-submodule of V', which is an irreducible $\mathfrak{m}_{\Theta'} \oplus \mathfrak{m}_{\Theta''}$-module, Θ'' must be empty. On the other hand, we see $\Theta' = \{\gamma_1, \ldots, \gamma_{K'}\}$ with $K' < K$. Now we can find each weight ϖ' of the $\mathfrak{g}_{\Theta'}$-module V' is in the form

$$\varpi' = \bar{\pi} \; + \sum_{i=1}^{K'} m'_i \gamma_i \qquad \text{with } -2\frac{\langle\bar{\pi}, \gamma_1\rangle}{\langle\gamma_1, \gamma_1\rangle} \ge m'_1 \ge m'_2 \ge \cdots \ge m'_{K'} \ge 0$$

and its multiplicity is one (cf. Example 4.2 ii)). Fix $v \in V_\varpi \setminus \{0\}$. Take $i = 1, \ldots, K'$ so that $m_i > m_{i+1}$. Then $X_{-\gamma_i} v \ne 0$ and therefore $\gamma_i \notin \Theta$. Since $\varpi|_{\mathfrak{a}_\Theta} = \varpi_\alpha|_{\mathfrak{a}_\Theta}$, we conclude $i = K'$ and $m_{K'} = 1$. It shows

$$\varpi = \gamma_1 + \cdots + \gamma_{K'} \le \varpi_\alpha.$$

Thus we have proved the function (80) is not identically zero. ∎

Remark 3.26. The condition i) of the proposition is satisfied if the lowest weight $\bar{\pi}$ (or equivalently, the highest weight π) of (π, V) is regular.

Proposition 3.27. i) (multiplicity free representation) *Suppose* $\dim V_\varpi = 1$ *for any* $\varpi \in \mathcal{W}(\pi)$. *Then for any extremal low weight* ϖ_α *with respect to* $\alpha \in \Theta$, *the function* $r_{\alpha,\varpi_\alpha}(\lambda)$ *is not identically zero.*
ii) (adjoint representation) *Suppose* $\mathfrak{g}$ *is simple and* π *is the adjoint representation of* $\mathfrak{g}$. *Suppose* $\alpha \in \Theta$. *If the Dynkin diagram of* $\Psi(\mathfrak{g})$ *is of type* A_r, *then we have just two extremal low weights* ϖ_α *with respect to* α. *If the diagram is not of type* A_r, *then we have a unique* ϖ_α. *In either case, there is at least one* ϖ_α *such that* $r_{\alpha,\varpi_\alpha}(\lambda)$ *is not identically zero.*
iii) (minuscule representation) *Suppose* (π, V) *is minuscule. Then for any* $\alpha \in \Theta$ *there is a unique extremal low weight* ϖ_α *with respect to* α. *Moreover the function* $r_\alpha(\lambda)$ *is not identically zero.*

Proof. i) Thanks to Proposition 2.39 i), $V(\varpi_\alpha|_{\mathfrak{a}_\Theta})$ is an irreducible $\mathfrak{g}_\Theta$-module. Hence our claim follows from Lemma 3.24.

ii) The lowest weight of the adjoint representation is $-\alpha_{\max}$. Hence by Remark 3.17 we can determine the number of extremal low weights from the completed Dynkin diagram of each type, which is shown in §4..

Note that $\mathcal{W}(\pi) = \Sigma(\mathfrak{g}) \cup \{0\}$. Suppose $\varpi_\alpha \notin \Sigma(\mathfrak{g}_\Theta)$. Then Proposition 2.39 ii) assures the irreducibility of $V(\varpi_\alpha|_{\mathfrak{a}_\Theta})$. Hence $r_{\alpha,\varpi_\alpha}(\lambda)$ is not identically zero.

Suppose $\varpi_\alpha \in \Sigma(\mathfrak{g}_\Theta)$. Take $\{\gamma_1, \ldots, \gamma_K\} \subset \Psi(\mathfrak{g})$ as in Lemma 3.15 and put

$$\varpi_i = -\alpha_{\max} + \gamma_1 + \cdots + \gamma_{i-1} \qquad \text{for } i = 1, \ldots, K.$$

Let Θ_1 denote the connected component of the Dynkin diagram of Θ containing $\gamma_K = \alpha$. Then we can find an integer $K' \in \{1, \ldots, K-1\}$ such that $\{\gamma_1, \ldots, \gamma_{K'}\} \subset \Psi(\mathfrak{g}) \setminus \Theta_1$ and $\{\gamma_{K'+1}, \ldots, \gamma_K\} \subset \Theta_1$. Then it follows from Lemma 3.15 that the root vectors X_{ϖ_i} for $i = 1, \ldots, K'$ are lowest weight vectors of $\pi|_{\mathfrak{m}_{\Theta_1}}$. These lowest weight vectors generate the irreducible $\mathfrak{m}_{\Theta_1}$-submodules belonging to the same equivalence class because $\{\gamma_1, \ldots, \gamma_{K'-1}\}$ is orthogonal to Θ_1. On the other hand, we have $\varpi_{K'+1} \in \overline{\mathcal{W}}_\Theta(\pi)$. Then it follows from Proposition 2.39 ii) that $\varpi_{K'+1} \in \Sigma(\mathfrak{g}_{\Theta_1})^-$. Since $\varpi_{K'} - \varpi_{K'+1} = -\gamma_{K'} \in \Sigma(\mathfrak{g})$, $[X_{-\varpi_{K'+1}}, X_{\varpi_{K'}}] \neq 0$. It shows the equivalence class above is not the class of the trivial representation. Hence Θ_1 is not orthogonal to $\bar{\pi} = \varpi_1$. Now we can take another extremal low weight ϖ'_α with respect to α which satisfies the condition i) of Proposition 3.25.

iii) Since a minuscule representation is of multiplicity free, we have only to show the uniqueness of ϖ_α. Let $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ be the decomposition into simple Lie algebras. Then $\pi|_{[\mathfrak{g},\mathfrak{g}]}$ is a tensor product of faithful minuscule representations of $\mathfrak{g}_i$ for $i = 1, \ldots, m$. Hence, from Proposition 2.37 v), each connected component of the Dynkin diagram of $\Psi(\mathfrak{g})$, which corresponds to some $\Psi(\mathfrak{g}_i)$, has just one root γ which is not orthogonal to $\bar{\pi}$. Now the uniqueness follows from Remark 3.17. ∎

We conclude this section with a discussion of the commutative case. Consider $F_\pi = \Big(F_{ij}\Big)_{\substack{1 \le i \le N \\ 1 \le j \le N}}$ as an element of $M(N, S(\mathfrak{g}))$. Then we have

Theorem 3.28 (coadjoint orbit). *Put*

$$\begin{aligned} \bar{\Omega}_{\pi,\Theta} &= \{\varpi|_{\mathfrak{a}_\Theta};\, \varpi \in \overline{\mathcal{W}}_\Theta(\pi)\} \\ \bar{q}_{\pi,\Theta}(x;\lambda) &= \prod_{\mu \in \bar{\Omega}_{\pi,\Theta}} \big(x - \mu(\lambda)\big), \\ \bar{r}_\Theta(\lambda) &= \prod_{\mu,\,\mu' \in \bar{\Omega}_{\pi,\Theta},\ \mu \neq \mu'} \big(\mu(\lambda) - \mu'(\lambda)\big). \end{aligned} \tag{84}$$

Then if $\bar{r}_\Theta(\lambda) \neq 0$,

$$\sum_{i,j} S(\mathfrak{g})\bar{q}_{\pi,\Theta}(F_\pi;\lambda)_{ij} + \sum_{f \in I(\mathfrak{g})} S(\mathfrak{g})\big(f - f(\lambda_\Theta)\big) = \{f \in S(\mathfrak{g});\, f|_{\mathrm{Ad}(G)\lambda_\Theta} = 0\}.$$

Here $I(\mathfrak{g})$ *is the space of the* $\mathrm{ad}(\mathfrak{g})$*-invariant elements in the symmetric algebra* $S(\mathfrak{g})$ *of* $\mathfrak{g}$ *and* G *a connected complex Lie group with Lie algebra* $\mathfrak{g}$.

Proof. Let $\{v_i;\ i=1,\dots,N\}$ be a base of V such that each v_i is a weight vector with weight ϖ_i. Then

$$d\bar{q}_{\pi,\Theta}(F_\pi;\lambda)_{ij}|_{\lambda_\Theta}=\begin{cases}0 & \text{if } \langle\varpi_i-\varpi_j,\lambda_\Theta\rangle\neq 0,\\ \prod_{\mu\in\bar{\Omega}_{\pi,\Theta}\setminus\{\varpi_i|_{\mathfrak{a}_\Theta}\}}\big(\langle\varpi_i,\lambda_\Theta\rangle-\mu(\lambda)\big)dF_{ij} & \text{if } \langle\varpi_i-\varpi_j,\lambda_\Theta\rangle=0.\end{cases}$$

For $\alpha\in\Sigma(\mathfrak{g})\setminus\Sigma(\mathfrak{g}_\Theta)$ there exists a pair of weights of π whose difference equals α and therefore $\bar{r}_\Theta(\lambda)\neq 0$ implies $\langle\alpha,\lambda_\Theta\rangle\neq 0$, which assures that the centralizer of λ_Θ in $\mathfrak{g}$ equals $\mathfrak{g}_\Theta$. Since

$$\mathfrak{g}_\Theta=\sum_{i=j\text{ or }\varpi_i-\varpi_j\text{ is a root of }\mathfrak{g}_\Theta}\mathbb{C}F_{ij}$$

and $[H,F_{ij}]=(\varpi_i-\varpi_j)(H)F_{ij}$ for $H\in\mathfrak{a}$, we can prove the theorem as in the same way as in the proof of [22, Theorem 4.11]. ∎

Remark 3.29. There is a natural projection $\bar{p}_{\pi,\Theta}:\Omega_{\pi,\Theta}\to\bar{\Omega}_{\pi,\Theta}$. We say that $\mu\in\bar{\Omega}_{\pi,\Theta}$ is *ramified in the quantization* of $\bar{q}_{\pi,\Theta}$ to $q_{\pi,\Theta}$ if $\bar{p}_{\pi,\Theta}^{-1}(\mu)$ is not a single element.

If π is of multiplicity free, then there is no ramified element in $\bar{\Omega}_{\pi,\Theta}$ (cf. Proposition 2.39 i)). In this case, consider $\mathfrak{g}$ as an abelian Lie algebra acting on $S(\mathfrak{g})$ by the multiplication and define the $\mathfrak{g}$-module

$$M_\Theta^0(\lambda)=S(\mathfrak{g})/\sum_{X\in\mathfrak{p}_\Theta}S(\mathfrak{g})(X-\lambda_\Theta(X)).$$

Then taking a "classical limit" as in [22], we can prove $\bar{q}_{\pi,\Theta}(F_\pi;\lambda)M_\Theta^0(\lambda)=0$. Moreover if $\bar{r}_\Theta(\lambda)\neq 0$, the polynomial $\bar{q}_{\pi,\Theta}(x;\lambda)$ is minimal in the obvious sense.

4. Examples

In this section we give the explicit form of the characteristic polynomials of some small dimensional representations π of classical and exceptional Lie algebras $\mathfrak{g}$. (As in the previous sections, we always assume that $\mathfrak{g}$ and π satisfy (10).) In some special cases we also calculate the global minimal polynomials. Note that if $q_\pi(x)=\prod_{1\le i\le m}(x-\varpi_i-C_i)$ with suitable $\varpi_i\in\mathfrak{a}^*$ and $C_i\in\mathbb{C}$ is the characteristic polynomial, then the global minimal polynomial $q_{\pi,\Theta}(x,\lambda)$ for a given Θ equals $\prod_{i\in I}(x-\langle\varpi_i,\lambda_\Theta+\rho\rangle-C_i)$ with a certain subset I of $\{1,\dots,m\}$.

It is clear that if the dimension of π is small, then the degree of $q_{\pi,\Theta}(x,\lambda)$ is small, which means the corresponding ideal $I_{\pi,\Theta}(\lambda)$ is generated by elements with small degrees. In such a case, for an extremal low weight ϖ_α of π with respect to $\alpha\in\Theta$, the degree of the polynomial $r_{\alpha,\varpi_\alpha}(\lambda)$ defined by (80) is also small and hence the assumptions on λ of Theorem 3.21 and Corollary 3.22 become very weak.

Lemma 4.1 (bilinear form). *Let $(\ ,\ )$ be a symmetric bilinear form on $\mathfrak{a}^*$ and let $\mathfrak{a}^*=\mathfrak{a}_1^*\oplus\mathfrak{a}_2^*$ be a direct sum of linear subspaces with $(\mathfrak{a}_1^*,\mathfrak{a}_2^*)=\langle\mathfrak{a}_1^*,\mathfrak{a}_2^*\rangle=0$. If there exists $C\in\mathbb{C}\setminus\{0\}$ such that*

$$(\mu,\mu')=C\langle\mu,\mu'\rangle\quad(\forall\mu,\ \mu'\in\mathfrak{a}_1^*),$$

then

$$C = \sum_{\varpi \in \mathcal{W}(\pi)} m_\pi(\varpi) \frac{(\alpha, \varpi)^2}{(\alpha, \alpha)} \qquad \text{for } \alpha \in \mathfrak{a}_1^* \text{ such that } (\alpha, \alpha) \neq 0.$$

Here $m_\pi(\varpi)$ denotes the multiplicity of the weight $\varpi \in \mathcal{W}(\pi)$.

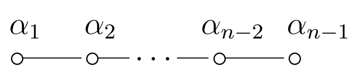
Proof. Let $H_\alpha \in \mathfrak{a}$ correspond to α by the bilinear form $\langle\ ,\ \rangle$. Then we have

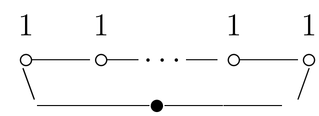
$$\begin{aligned} C(\alpha, \alpha) &= C^2 \langle \alpha, \alpha \rangle = C^2 \operatorname{Trace} \pi(H_\alpha)^2 \\ &= C^2 \sum_{\varpi \in \mathcal{W}(\pi)} m_\pi(\varpi) \left(\langle \alpha, \varpi \rangle\right)^2 = \sum_{\varpi \in \mathcal{W}(\pi)} m_\pi(\varpi)(\alpha, \varpi)^2. \end{aligned} \qquad \blacksquare$$

In the following examples $\varepsilon_1, \varepsilon_2, \ldots$ constitute a base of a vector space with symmetric bilinear form $(\ ,\ )$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. We consider $\mathfrak{a}^*$ a subspace of this space where $\varepsilon_1 - \varepsilon_2$ etc. are suitable elements in $\Psi(\mathfrak{g})$ (cf. [4]).

C_π equals the constant C in the above lemma for $\mathfrak{a}_1 = \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$. C'_π is the similar constant in the case when $\mathfrak{a}_1$ is the center of $\mathfrak{g}$. Then we can calculate $\langle\ ,\ \rangle$ under the base $\{\varepsilon_1, \varepsilon_2, \ldots\}$ by the above lemma.

Example 4.2 (A_{n-1})**.**





$\Psi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\}$
$\rho = \sum_{\nu=1}^{n} (\frac{n-1}{2} - (\nu - 1))\varepsilon_\nu = \sum_{\nu=1}^{n-1} \frac{\nu(n-\nu)}{2} \alpha_\nu$

i) $\mathfrak{g} = \mathfrak{gl}_n$

$\pi = \varpi_k := \varepsilon_1 + \cdots + \varepsilon_k = \bigwedge^k \varpi_1$ (minuscule, $k = 1, \ldots, n-1$)
$\dim \varpi_k = \binom{n}{k}$
$(\varpi_k, \rho) = \frac{k(n-k)}{2}$
$\mathcal{W}(\varpi_k) = \{\varepsilon_{\nu_1} + \cdots + \varepsilon_{\nu_k};\ 1 \leq \nu_1 < \cdots < \nu_k \leq n\}$
$C_{\varpi_k} = \frac{1}{2} \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} (\varepsilon_{\nu_1} + \cdots + \varepsilon_{\nu_k}, \varepsilon_1 - \varepsilon_2)^2 = \binom{n-2}{k-1}$
$C'_{\varpi_k} = \frac{1}{n} \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} (\varepsilon_{\nu_1} + \cdots + \varepsilon_{\nu_k}, \varepsilon_1 + \cdots + \varepsilon_n)^2 = k\binom{n-1}{k-1}$
$\langle \varepsilon_i, \varepsilon_j \rangle = \frac{(n-k)!(k-1)!}{n!} \left(\frac{n-1}{n-k}(n\delta_{ij} - 1) + \frac{1}{k} \right)$
$q_{\varpi_k}(x) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(x - (\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}) - \frac{k!(n-k)!}{2(n-2)!} \right)$

ii) $\mathfrak{g} = \mathfrak{gl}_n$

$V = V_m := \{$homogeneous polynomials of $(x_1, \ldots, x_n)$ with degree $m\}$
$\pi = m\varepsilon_1$ (multiplicity free, $m = 1, 2, \cdots$)
$\mathcal{W}(m\varepsilon_1) = \{m_1\varepsilon_1 + \cdots + m_n\varepsilon_n;\ m_1 + \cdots + m_n = m,\ m_j \in \mathbb{Z}_{\geq 0}\}$
$\dim m\varepsilon_1 = {}_nH_m = \binom{n+m-1}{m} = \frac{(n+m-1)!}{m!(n-1)!}$
$C_{m\varepsilon_1} = \frac{1}{2} \sum_{m_1 + \cdots + m_n = m} (m_1\varepsilon_1 + \cdots + m_n\varepsilon_n, \varepsilon_1 - \varepsilon_2)^2$
$= \frac{1}{2} \sum_{k=0}^{m} \sum_{m_1=0}^{k} (k - 2m_1)^2 {}_{n-2}H_{m-k}.$
$= \frac{1}{3!} \sum_{k=0}^{m} k(k+1)(k+2) \frac{(m+n-k-3)!}{(n-3)!(m-k)!}$
$= \frac{1}{3!(n-3)!} \sum_{k=0}^{m} k(k+1)(k+2)(m+n-(k+3)) \cdots (m+n-(k+n-1))$

$= \cdots = \frac{(m+n)!}{(n+1)!(m-1)!}$

$C'_{m\varepsilon_1} = \frac{1}{n}\sum_{m_1+\cdots+m_n=m}(m_1\varepsilon_1+\cdots+m_n\varepsilon_n, \varepsilon_1+\cdots+\varepsilon_n)^2 = \frac{m^2}{n}{}_nH_m$

$= \frac{(n+m-1)!}{(m-1)!n!}m = \frac{m(m+1)\cdots(m+n-1)}{n!}m$

$q_{m\varepsilon_1}(x) = \prod_{\substack{m_1+\cdots+m_n=m\\ m_i\in\mathbb{Z}_{\geq 0}}}\bigl(x - \sum_{i=1}^n m_i\varepsilon_i - \frac{m(m+n-1)-\sum_{i=1}^n m_i^2}{2C_{m\varepsilon_1}}\bigr)$

iii) $\mathfrak{g} = \mathfrak{sl}_n$

$\pi = \varpi_1 + \varpi_{n-1} = \varepsilon_1 - \varepsilon_n$ (adjoint)

$\dim(\varpi_1 + \varpi_n) = n^2 - 1$

$C_{\varpi_1+\varpi_{n-1}} = 2n$

$(\varpi_1 + \varpi_{n-1}, \rho) = n - 1$

$q_{\varpi_1+\varpi_{n-1}}(x) = (x - \frac{1}{2})\prod_{1\leq i<j\leq n}\bigl((x - \frac{n-1}{2n})^2 - (\varepsilon_i - \varepsilon_j)^2\bigr)$

In [22] we choose $\Psi' = \{\alpha'_1 = \varepsilon_2 - \varepsilon_1, \ldots, \alpha'_{n-1} = \varepsilon_n - \varepsilon_{n-1}\}$ as a fundamental system of $\mathfrak{gl}_n$ and then $\bar\pi = \varpi_1$ is the lowest weight of the natural representation π of $\mathfrak{gl}_n$. For a strictly increasing sequence

$$n_0 = 0 < n_1 < \cdots < n_L = n \tag{85}$$

we put $n'_j = n_j - n_{j-1}$ and $\Theta = \bigcup_{k=1}^L \bigcup_{n_{k-1}<\nu<n_k}\{\alpha'_\nu\}$ and study the minimal polynomial $q_{\pi,\Theta}(x;\lambda)$ in [22] for $\lambda = \bigl(\lambda_k\bigr) \in \mathbb{C}^L \simeq \mathfrak{a}_\Theta^*$. Define $\rho' = -\rho$ and put

$$\bar\lambda_1\varepsilon_1 + \cdots + \bar\lambda_n\varepsilon_n = \rho' + \sum_{k=1}^L \lambda_k\Bigl(\sum_{n_{k-1}<\nu\leq n_k}\varepsilon_\nu\Bigr). \tag{86}$$

The partially ordered set of the weights of π is as follows

$$\varepsilon_1 \xrightarrow{\alpha'_1} \varepsilon_2 \xrightarrow{\alpha'_2} \cdots\cdots \xrightarrow{\alpha'_{n_k-1}} \varepsilon_{n_k} \xrightarrow{\alpha'_{n_k}} \varepsilon_{n_k+1} \xrightarrow{\alpha'_{n_k+1}} \cdots\cdots \xrightarrow{\alpha'_{n-1}} \varepsilon_n.$$

Then $\overline{\mathcal{W}}_\Theta(\pi) = \{\varepsilon_{n_0+1}, \ldots, \varepsilon_{n_{L-1}+1}\}$ and Theorem 2.24 says

$$\begin{aligned} q_{\pi,\Theta}(x,\lambda) &= \prod_{k=1}^L\bigl(x - \lambda_k - \frac{1}{2}(\varepsilon_1 - \varepsilon_{n_{k-1}+1}, \varepsilon_1 + \varepsilon_{n_{k-1}+1} - 2\rho')\bigr) \\ &= \prod_{k=1}^L\bigl(x - \lambda_k - n_{k-1}\bigr) \end{aligned}$$

and it follows from Remark 3.20 that

$$r_{\alpha'_i}(\lambda) = \prod_{\nu=k+1}^L \bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\bigr)\prod_{\nu=1}^{k-1}\bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\bigr)$$

in Definition 3.19 if $n_{k-1} < i < n_k$. This result coincides with [22, Theorem 4.4]. Note that if λ satisfies the condition:

$$\langle\lambda + \rho', \beta\rangle = 0 \text{ with } \beta \in \Sigma(\mathfrak{g}) \ \Rightarrow\ \forall\alpha' \in \Theta\ \langle\beta, \alpha'\rangle = 0, \tag{87}$$

then $r_{\alpha'}(\lambda) \neq 0$ for each $\alpha' \in \Theta$.

Let π_{ϖ_k} be the minuscule representation ϖ_k in i) and we here adopt the fundamental system Ψ' as above. The decomposition

$$\pi_{\varpi_k}|_{\mathfrak{g}_\Theta} = \bigoplus_{\substack{k_1+\cdots+k_L=k \\ 0\le k_j\le n'_j\ (j=1,\dots,L)}} \pi_{k_1,\dots,k_L} \tag{88}$$

is a direct consequence of Proposition 2.39 i). Here $\pi_{k_1,\dots,k_L}$ denotes the irreducible representation of $\mathfrak{g}_\Theta$ with lowest weight $\sum_{j=1}^L(\varepsilon_{n_{j-1}+1}+\cdots+\varepsilon_{n_{j-1}+k_j})$. Then by Proposition 2.40 i) we have

$$\begin{aligned} q_{\pi_{\varpi_k},\Theta}(x;\lambda) &= \prod_{\substack{k_1+\cdots+k_L=k \\ 0\le k_j\le n'_j\ (j=1,\dots,L)}} \Big(x - \sum_{i=1}^n\sum_{j=1}^L\sum_{\nu=1}^{k_j}\bar\lambda_i\langle\varepsilon_i,\varepsilon_{n_{j-1}+\nu}\rangle - \frac{k!(n-k)!}{2(n-2)!}\Big) \\ &= \prod_{\substack{k_1+\cdots+k_L=k \\ 0\le k_j\le n'_j\ (j=1,\dots,L)}} \Big(x - C''_{\varpi_k}(n-1)\sum_{j=1}^L k_j\big(\lambda_j+n_{j-1}+\frac{k_j-k}{2}\big) \\ &\qquad + C''_{\varpi_k}(k-1)\sum_{j=1}^L n'_j\big(\lambda_j+n_{j-1}+\frac{n'_j-n}{2}\big)\Big) \end{aligned}$$

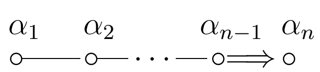
with $C''_{\varpi_k} = \frac{(n-k-1)!(k-1)!}{(n-1)!}$. To deduce the final form we have used the relation $\sum_{j=1}^L n'_j n_{j-1} = \frac{n^2-\sum_{j=1}^L n'_j}{2}$.

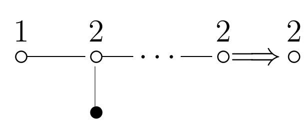
Remark 4.3 (A_{n-1})**.** Put $\mathfrak{g}'_\Theta = [\mathfrak{g}_\Theta,\mathfrak{g}_\Theta]$. Then the irreducible decomposition of $\pi_{\varpi_k}|_{\mathfrak{g}'_\Theta}$ is not of multiplicity free if and only if there exist an integer K and subsets I and J of $\{1,\dots,L\}$ such that

$$K = \sum_{i\in I} n'_i = \sum_{j\in J} n'_j \le k,\ \ K\le n-k \text{ and } I\ne J.$$

This is clear from (88) because $\pi_{k_1,\dots,k_L}|_{\mathfrak{g}'_\Theta} = \pi_{k'_1,\dots,k'_L}|_{\mathfrak{g}'_\Theta}$ if and only if $k_i=k'_i$ or $(k_i,k'_i)=(0,n'_i)$ or $(n'_i,0)$ for $i=1,\dots,L$.

Example 4.4 (B_n)**.** $\mathfrak{g}=\mathfrak{o}_{2n+1}$





$n\ge 3$

$\Psi = \{\alpha_1=\varepsilon_1-\varepsilon_2,\dots,\alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\ \alpha_n=\varepsilon_n\}$
$\rho = \sum_{\nu=1}^n(n-\nu+\frac12)\varepsilon_\nu = \sum_{\nu=1}^n\frac{\nu(2n-\nu)}{2}\alpha_\nu$

i) $\pi=\varpi_1:=\varepsilon_1$ (multiplicity free)
$\dim\varpi_1 = 2n+1$
$(\varpi_1,\rho) = n-\frac12$
$C_{\varpi_1} = \sum(\pm\varepsilon_\nu,\varepsilon_1)^2+(0,\varepsilon_1)^2 = 2$
$q_{\varpi_1}(x) = (x-\frac n2)\prod_{i=1}^n\big((x-\frac{2n-1}{4})^2-\varepsilon_i^2\big)$

ii) $\pi=\varpi_n:=\frac12(\varepsilon_1+\cdots+\varepsilon_n)$ (minuscule)

$\dim \varpi_n = 2^n$
$(\varpi_n, \rho) = \frac{(2n-1)+(2n-3)+\cdots+1}{4} = \frac{n^2}{4}$
$C_{\varpi_n} = \sum(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n, \varepsilon_1)^2 = 2^n$
$q_{\varpi_n}(x) = \prod_{c_1=\pm1,\cdots,c_n=\pm1}\bigl(x - \frac{1}{2}(c_1\varepsilon_1 + \cdots + c_n\varepsilon_n) - \frac{n^2}{2^{n+2}}\bigr)$

iii) $\pi = \varpi_2 := \varepsilon_1 + \varepsilon_2$ (adjoint) $\cdots$ ϖ_2 is not a fundamental weight if $n = 2$.
$\dim \varpi_2 = n(2n+1)$
$C_{\varpi_2} = 4n - 2$
$(\varpi_2, \rho) = 2n - 2$
$\varepsilon_1 = \varpi_2 - \alpha_2 - \cdots - \alpha_n$
$q_{\varpi_2}(x) = (x - \frac{1}{2})\prod_{1\le i<j\le n}\bigl((x - \frac{n-1}{2n-1})^2 - (\varepsilon_i - \varepsilon_j)^2\bigr)\bigl((x - \frac{n-1}{2n-1})^2$
$-(\varepsilon_i + \varepsilon_j)^2\bigr)\prod_{i=1}^n\bigl((x - \frac{4n-3}{8n-4})^2 - \varepsilon_i^2\bigr)$

Choose $\Psi' = \{\alpha'_1 = \varepsilon_2 - \varepsilon_1, \ldots, \alpha'_{n-1} = \varepsilon_n - \varepsilon_{n-1}, \alpha'_n = -\varepsilon_n\}$ as a fundamental system. Then the partially ordered set of the weights of the natural representation π of $\mathfrak{o}_{2n+1}$ is shown by

$$\varepsilon_1 \xrightarrow{\alpha'_1} \varepsilon_2 \xrightarrow{\alpha'_2} \cdots\cdots \xrightarrow{\alpha'_{n_k-1}} \varepsilon_{n_k} \xrightarrow{\alpha'_{n_k}} \varepsilon_{n_k+1} \xrightarrow{\alpha'_{n_k+1}} \cdots\cdots \xrightarrow{\alpha'_{n-1}} \varepsilon_n \xrightarrow{\alpha'_n} 0$$
$$\xrightarrow{\alpha'_n} -\varepsilon_n \xrightarrow{\alpha'_{n-1}} \cdots\cdots \xrightarrow{\alpha'_{n_k+1}} -\varepsilon_{n_k+1} \xrightarrow{\alpha'_{n_k}} -\varepsilon_{n_k} \xrightarrow{\alpha'_{n_k-1}} \cdots\cdots \xrightarrow{\alpha'_1} -\varepsilon_1.$$

Here we use the same notation as in (85) and (86). Put $\Theta = \bigcup_{k=1}^L \bigcup_{n_{k-1}<\nu<n_k}\{\alpha'_\nu\}$ and $\bar\Theta = \Theta \cup \{\alpha'_n\}$. Then

$$\overline{\mathcal{W}}_{\bar\Theta}(\pi) = \{\varepsilon_{n_0+1}, \ldots, \varepsilon_{n_{L-1}+1}, -\varepsilon_{n_{L-1}}, \ldots, -\varepsilon_{n_1}\},$$
$$\overline{\mathcal{W}}_{\Theta}(\pi) = \overline{\mathcal{W}}_{\bar\Theta}(\pi) \cup \{0,\, -\varepsilon_n\}.$$

Hence by Theorem 2.24

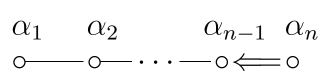
$$\begin{aligned}
q_{\pi,\Theta}(x;\lambda) &= \Bigl(x - \frac{1}{4}(\varepsilon_1, \varepsilon_1 - 2\rho')\Bigr)\\
&\quad\cdot\prod_{j=1}^L\Bigl(x - \frac{1}{2}\lambda_j - \frac{1}{4}(\varepsilon_1 - \varepsilon_{n_{j-1}+1}, \varepsilon_1 + \varepsilon_{n_{j-1}+1} - 2\rho')\Bigr)\\
&\quad\cdot\prod_{j=1}^L\Bigl(x + \frac{1}{2}\lambda_j - \frac{1}{4}(\varepsilon_1 + \varepsilon_{n_j}, \varepsilon_1 - \varepsilon_{n_j} - 2\rho')\Bigr)\\
&= \Bigl(x - \frac{n}{2}\Bigr)\prod_{j=1}^L\Bigl(x - \frac{\lambda_j}{2} - \frac{n_{j-1}}{2}\Bigr)\Bigl(x + \frac{\lambda_j}{2} - \frac{2n - n_j}{2}\Bigr),\\
q_{\pi,\bar\Theta}(x;\lambda) &= \Bigl(x - \frac{1}{4}(\varepsilon_1 - \varepsilon_{n_{L-1}+1}, \varepsilon_1 + \varepsilon_{n_{L-1}+1} - 2\rho')\Bigr)\\
&\quad\cdot\prod_{j=1}^{L-1}\Bigl(x - \frac{1}{2}\lambda_j - \frac{1}{4}(\varepsilon_1 - \varepsilon_{n_{j-1}+1}, \varepsilon_1 + \varepsilon_{n_{j-1}+1} - 2\rho')\Bigr)\\
&\quad\cdot\prod_{j=1}^{L-1}\Bigl(x + \frac{1}{2}\lambda_j - \frac{1}{4}(\varepsilon_1 + \varepsilon_{n_j}, \varepsilon_1 - \varepsilon_{n_j} - 2\rho')\Bigr)\\
&= \Bigl(x - \frac{n_{L-1}}{2}\Bigr)\prod_{j=1}^{L-1}\Bigl(x - \frac{\lambda_j}{2} - \frac{n_{j-1}}{2}\Bigr)\Bigl(x + \frac{\lambda_j}{2} - \frac{2n - n_j}{2}\Bigr).
\end{aligned}$$

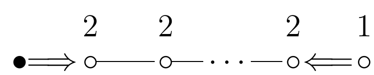
Moreover if $n_{k-1} < i < n_k$,

$$\begin{aligned}
2^{2L} r_{\alpha'_i,\Theta}(\lambda) &= \prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\Bigr) \prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\Bigr) \\
&\quad \cdot \Bigl(\bar\lambda_{i+1} - \frac{1}{2}\Bigr) \prod_{\nu=1}^{L}\bigl(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}\Bigr) \\
&= \frac{1}{2}\prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\Bigr) \prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\Bigr) \\
&\quad \cdot \Bigl(\bar\lambda_i + \bar\lambda_{i+1}\Bigr) \prod_{\nu=1}^{L}\bigl(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}\Bigr), \\
2^{2L-2} r_{\alpha'_i,\bar\Theta}(\lambda) &= \prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\Bigr) \prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\Bigr) \prod_{\nu=1}^{L-1}\bigl(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}\Bigr), \\
2^{2L-2} r_{\alpha'_n,\bar\Theta}(\lambda) &= \prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_n - \bar\lambda_{n_\nu}\Bigr) \prod_{\nu=1}^{L-1}\Bigl(\frac{1}{2} + \bar\lambda_{n_\nu}\Bigr) = (-1)^{L-1}\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_n - \bar\lambda_{n_\nu}\Bigr)^2.
\end{aligned}$$

Here we denote $r_\alpha(\lambda)$ corresponding to Θ and $\bar\Theta$ by $r_{\alpha,\Theta}(\lambda)$ and $r_{\alpha,\bar\Theta}(\lambda)$, respectively. Note that $r_{\alpha',\bar\Theta}(\lambda) \neq 0$ for $\alpha' \in \bar\Theta$ under the condition (87) for $\bar\Theta$. Moreover suppose $\lambda + \rho'$ is dominant. Then $\bar\lambda_i + \bar\lambda_{i+1} = 2\bar\lambda_{i+1} - 1 = -2\frac{\langle\lambda+\rho',-\varepsilon_{i+1}\rangle}{\langle -\varepsilon_{i+1},-\varepsilon_{i+1}\rangle} - 1 \neq 0$ and hence $r_{\alpha',\Theta}(\lambda) \neq 0$ for $\alpha' \in \Theta$ under the condition (87).

Example 4.5 (C_n). $\mathfrak{g} = \mathfrak{sp}_n$





$n \geq 2$

$\Psi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = 2\varepsilon_n\}$
$\rho = \sum_{\nu=1}^{n}(n-\nu+1)\varepsilon_\nu = \sum_{\nu=1}^{n-1}\frac{\nu(2n-\nu+1)}{2}\alpha_\nu + \frac{n(n+1)}{4}\alpha_n$

i) $\pi = \varpi_1 := \varepsilon_1$ (minuscule)
$\dim \varpi_1 = 2n$
$C_{\varpi_1} = \sum(\pm\varepsilon_\nu, \varepsilon_1)^2 = 2$
$(\varpi_1, \rho) = n$
$q_{\varpi_1}(x) = \prod_{i=1}^{n}\bigl((x-\frac{n}{2})^2 - \varepsilon_i^2\bigr)$

ii) $\pi = 2\varpi_1 = 2\varepsilon_1$ (adjoint)
$\dim 2\varpi_1 = n(2n+1)$
$C_{2\varpi_1} = 4(n+1)$
$(2\varpi_1, \rho) = 2n$
$q_{2\varpi_1}(x) = (x-\frac{1}{2})\prod_{i=1}^{n}\bigl((x-\frac{n}{2n+2})^2 - 2\varepsilon_i^2\bigr)\prod_{1\leq i<j\leq n}\bigl((x-\frac{2n+1}{4n+4})^2$
$\qquad -(\varepsilon_i - \varepsilon_j)^2\bigr)\bigl((x-\frac{2n+1}{4n+4})^2 - (\varepsilon_i+\varepsilon_j)^2\bigr)$

Choose $\Psi' = \{\alpha'_1 = \varepsilon_2 - \varepsilon_1, \ldots, \alpha'_{n-1} = \varepsilon_n - \varepsilon_{n-1},\ \alpha'_n = -2\varepsilon_n\}$ as a fundamental system. The partially ordered set of the weights of the natural representation π of $\mathfrak{sp}_n$ is shown by

$$\varepsilon_1 \xrightarrow{\alpha'_1} \varepsilon_2 \xrightarrow{\alpha'_2} \cdots\cdots \xrightarrow{\alpha'_{n_k-1}} \varepsilon_{n_k} \xrightarrow{\alpha'_{n_k}} \varepsilon_{n_k+1} \xrightarrow{\alpha'_{n_k+1}} \cdots\cdots \xrightarrow{\alpha'_{n-1}} \varepsilon_n$$

$$\xrightarrow{\alpha'_n} -\varepsilon_n \xrightarrow{\alpha'_{n-1}} \cdots\cdots \xrightarrow{\alpha'_{n_k+1}} -\varepsilon_{n_k+1} \xrightarrow{\alpha'_{n_k}} -\varepsilon_{n_k} \xrightarrow{\alpha'_{n_k-1}} \cdots\cdots \xrightarrow{\alpha'_1} -\varepsilon_1.$$

Under the same notation as in the previous example, we have

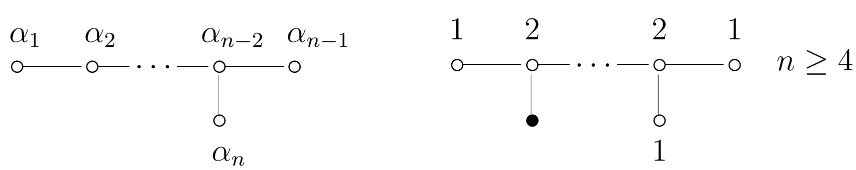
$$\begin{aligned}
\overline{\mathcal{W}}_{\bar\Theta}(\pi) &= \{\varepsilon_{n_0+1}, \dots, \varepsilon_{n_{L-1}+1}, -\varepsilon_{n_{L-1}}, \dots, -\varepsilon_{n_1}\},\\
\overline{\mathcal{W}}_{\Theta}(\pi) &= \overline{\mathcal{W}}_{\bar\Theta}(\pi) \cup \{-\varepsilon_n\}.
\end{aligned}$$

If $n_{k-1} < i < n_k$, it follows from Theorem 2.24, and Remark 3.20 that

$$\begin{aligned}
q_{\pi,\bar\Theta}(x;\lambda) &= \prod_{j=1}^{L}\Bigl(x - \frac{\lambda_j}{2} - \frac{n_{j-1}}{2}\Bigr)\prod_{j=1}^{L-1}\Bigl(x + \frac{\lambda_j}{2} - \frac{2n-n_j+1}{2}\Bigr),\\
q_{\pi,\Theta}(x;\lambda) &= \prod_{j=1}^{L}\Bigl(x - \frac{\lambda_j}{2} - \frac{n_{j-1}}{2}\Bigr)\Bigl(x + \frac{\lambda_j}{2} - \frac{2n-n_j+1}{2}\Bigr),\\
2^{2L-1}r_{\alpha'_i,\Theta}(\lambda) &= \prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\Bigr)\prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\Bigr)\prod_{\nu=1}^{L}\Bigl(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}\Bigr),\\
2^{2L-2}r_{\alpha'_i,\bar\Theta}(\lambda) &= \prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i - \bar\lambda_{n_\nu}\Bigr)\prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}\Bigr)\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}\Bigr),\\
2^{2L-2}r_{\alpha'_n,\bar\Theta}(\lambda) &= \prod_{\nu=1}^{L-1}\bar\lambda_{n_\nu}\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_n - \bar\lambda_{n_\nu}\Bigr).
\end{aligned}$$

If the condition (87) holds, then we have $r_{\alpha',\Theta}(\lambda) \neq 0$ and $r_{\alpha',\bar\Theta}(\lambda) \neq 0$ for $\alpha' \in \Theta$. Moreover suppose $\langle\lambda, \alpha'_n\rangle = 0$ and $\lambda + \rho'$ is dominant. In this case $\bar\lambda_n = -1$ and $\bar\lambda_{n_\nu} = -2\frac{\langle\lambda+\rho', \varepsilon_n - \varepsilon_{n_\nu}\rangle}{\langle\varepsilon_n - \varepsilon_{n_\nu}, \varepsilon_n - \varepsilon_{n_\nu}\rangle} - 1 \neq 0$. Hence $r_{\alpha'_n,\bar\Theta}(\lambda) \neq 0$ under the condition (87) for $\bar\Theta$.

Example 4.6 (D_n). $\mathfrak{g} = \mathfrak{o}_{2n}$



$$\Psi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$$
$$\rho = \textstyle\sum_{\nu=1}^{n}(n-\nu)\varepsilon_\nu = \sum_{\nu=1}^{n-2}\frac{\nu(2n-\nu-1)}{2}\alpha_\nu + \frac{n(n-1)}{4}(\alpha_{n-1} + \alpha_n)$$

i) $\pi = \varpi_1 := \varepsilon_1$ (minuscule)
$\dim\varpi_1 = 2n$
$C_{\varpi_1} = \sum(\pm\varepsilon_\nu, \varepsilon_1)^2 = 2$
$(\varpi_1, \rho) = n - 1$
$q_{\varpi_1}(x) = \prod_{i=1}^{n}\bigl((x - \frac{n-1}{2})^2 - \varepsilon_i^2\bigr)$

ii) $\pi = \begin{cases}\varpi_{n-1} := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n) & \text{(minuscule)}\\ \varpi_n := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n) & \text{(minuscule)}\end{cases}$
$\dim\varpi_{n-1} = \dim\varpi_n = 2^{n-1}$
$C_{\varpi_{n-1}} = C_{\varpi_n} = \sum(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n, \varepsilon_1)^2 = 2^{n-1}$

$(\varpi_{n-1},\rho)=(\varpi_n,\rho)=\frac{n(n-1)}{4}$.
$q_{\varpi_{n-1}}(x)=\prod_{\substack{c_1=\pm1,\dots,c_n=\pm1\\ c_1\cdots c_n=-1}}\bigl(x-\frac12(c_1\varepsilon_1+\cdots+c_n\varepsilon_n)-\frac{n(n-1)}{2^{n+1}}\bigr)$
$q_{\varpi_n}(x)=\prod_{\substack{c_1=\pm1,\dots,c_n=\pm1\\ c_1\cdots c_n=1}}\bigl(x-\frac12(c_1\varepsilon_1+\cdots+c_n\varepsilon_n)-\frac{n(n-1)}{2^{n+1}}\bigr)$

iii) $\pi=\varpi_2:=\varepsilon_1+\varepsilon_2$ (adjoint)
$\dim\varpi_2=n(2n-1)$
$C_{\varepsilon_1+\varepsilon_2}=4(n-1)$
$(\varpi_2,\rho)=2n-3$
$q_{\varpi_2}(x)=(x-\frac12)\prod_{1\le i<j\le n}\bigl((x-\frac{2n-3}{4n-4})^2-(\varepsilon_i-\varepsilon_j)^2\bigr)\bigl((x-\frac{2n-3}{4n-4})^2-(\varepsilon_i+\varepsilon_j)^2\bigr)$

Note that the coefficient of $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ in the polynomial $\sum_{\substack{c_1=\pm1,\dots,c_n=\pm1\\ c_1\cdots c_n=1}}(c_1\varepsilon_1+\cdots+c_n\varepsilon_n)^n$ of $(\varepsilon_1,\dots,\varepsilon_n)$ does not vanish. Hence

$$Z(\mathfrak g)=\mathbb C[\operatorname{Trace}F_{\varpi_1}^2,\operatorname{Trace}F_{\varpi_1}^4,\dots,\operatorname{Trace}F_{\varpi_1}^{2(n-1)},\operatorname{Trace}F_{\varpi_n}^n]. \tag{89}$$

Choose $\Psi'=\{\alpha'_1=\varepsilon_2-\varepsilon_1,\dots,\alpha'_{n-1}=\varepsilon_n-\varepsilon_{n-1},\ \alpha'_n=-\varepsilon_n-\varepsilon_{n-1}\}$ as a fundamental system. Then the partially ordered set of the weights of the natural representation π of $\mathfrak o_{2n}$ is shown by

$$\begin{array}{ccccccccccc}
\varepsilon_1\xrightarrow{\alpha'_1}\varepsilon_2\xrightarrow{\alpha'_2}\cdots\cdots\xrightarrow{\alpha'_{n-2}} & \varepsilon_{n-1} & \xrightarrow{\alpha'_{n-1}} & \varepsilon_n & & \\
& \downarrow{\scriptstyle\alpha'_n} & & \downarrow{\scriptstyle\alpha'_n} & & \\
& -\varepsilon_n & \xrightarrow{\alpha'_{n-1}} & -\varepsilon_{n-1} & \xrightarrow{\alpha'_{n-2}}\cdots\cdots\xrightarrow{\alpha'_2}-\varepsilon_2\xrightarrow{\alpha'_1}-\varepsilon_1.
\end{array}$$

Use the notation as in (85) and (86). Put $\Theta=\bigcup_{k=1}^{L}\bigcup_{n_{k-1}<\nu<n_k}\{\alpha'_\nu\}$. If $\alpha'_{n-1}\in\Theta$, we also put $\bar\Theta=\Theta\cup\{\alpha'_n\}$.

Then

$$\pi|_{\mathfrak g_\Theta}=\bigoplus_{j=0}^{L-1}\pi_{\varepsilon_{n_j+1}}\oplus\bigoplus_{j=1}^{L}\pi_{-\varepsilon_{n_j}},\qquad \pi|_{\mathfrak g_{\bar\Theta}}=\bigoplus_{j=0}^{L-1}\pi_{\varepsilon_{n_j+1}}\oplus\bigoplus_{j=1}^{L-1}\pi_{-\varepsilon_{n_j}}.$$

Here π_ε denotes the irreducible representation of $\mathfrak g_\Theta$ or $\mathfrak g_{\bar\Theta}$ with lowest weight ε. Hence if $n_{k-1}<i<n_k$,

$$\begin{aligned}
q_{\pi,\Theta}(x;\lambda)&=\prod_{j=1}^{L}\left(x-\frac{\lambda_j}{2}-\frac{n_{j-1}}{2}\right)\left(x+\frac{\lambda_j}{2}-\frac{2n-n_j-1}{2}\right),\\
q_{\pi,\bar\Theta}(x;\lambda)&=\prod_{j=1}^{L}\left(x-\frac{\lambda_j}{2}-\frac{n_{j-1}}{2}\right)\prod_{j=1}^{L-1}\left(x+\frac{\lambda_j}{2}-\frac{2n-n_j-1}{2}\right),\\
2^{2L-1}r_{\alpha'_i,\Theta}(\lambda)&=\prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i-\bar\lambda_{n_\nu}\Bigr)\prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1}-\bar\lambda_{n_{\nu-1}+1}\Bigr)\prod_{\nu=1}^{L}\Bigl(\bar\lambda_{i+1}+\bar\lambda_{n_\nu}\Bigr),\\
2^{2L-2}r_{\alpha'_i,\bar\Theta}(\lambda)&=\prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i-\bar\lambda_{n_\nu}\Bigr)\prod_{\nu=k+1}^{L}\Bigl(\bar\lambda_{i+1}-\bar\lambda_{n_{\nu-1}+1}\Bigr)\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_{i+1}+\bar\lambda_{n_\nu}\Bigr),\\
2^{2L-2}r_{\alpha'_n,\bar\Theta}(\lambda)&=(-1)^{L-1}\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_n-\bar\lambda_{n_\nu}\Bigr)\Bigl(\bar\lambda_{n-1}-\bar\lambda_{n_\nu}\Bigr).
\end{aligned}$$

If $\langle\lambda,\alpha'_n\rangle = 0$ and $i+1 = n_k < n$, then $\bar\lambda_{i+1}+\bar\lambda_{n_k} = 2(\bar\lambda_{i+1}+\bar\lambda_n)$. Hence $r_{\alpha',\bar\Theta}(\lambda)\neq 0$ for $\alpha'\in\bar\Theta$ under the condition (87) for $\bar\Theta$.

Now suppose $\alpha'_{n-1}\notin\Theta$. Then $n_{L-1} = n-1$. If $\lambda_L = 0$, then $q'_{\pi,\Theta}(F_\pi;\lambda)M_\Theta(\lambda) = 0$ by Corollary 2.32 with

$$q'_{\pi,\Theta}(x;\lambda) = \Bigl(x-\frac{\lambda_L}{2}-\frac{n-1}{2}\Bigr)\prod_{j=1}^{L-1}\Bigl(x-\frac{\lambda_j}{2}-\frac{n_{j-1}}{2}\Bigr)\Bigl(x+\frac{\lambda_j}{2}-\frac{2n-n_j-1}{2}\Bigr).$$

The analogue of $r_{\alpha'_i,\Theta}(\lambda)$ in this case is

$$r'_{\alpha'_i,\Theta}(\lambda) = 2^{2-2L}\bar\lambda_{i+1}\prod_{\nu=1}^{k-1}\Bigl(\bar\lambda_i-\bar\lambda_{n_\nu}\Bigr)\prod_{\nu=k+1}^{L-1}\Bigl(\bar\lambda_{i+1}-\bar\lambda_{n_{\nu-1}+1}\Bigr)\prod_{\nu=1}^{L-1}\Bigl(\bar\lambda_{i+1}+\bar\lambda_{n_\nu}\Bigr).$$

If $i+1 = n_k$ then $\bar\lambda_{i+1}+\bar\lambda_{n_k} = 2(\bar\lambda_{i+1}+\bar\lambda_n)$. Hence $r'_{\alpha',\bar\Theta}(\lambda)\neq 0$ for $\alpha'\in\Theta$ under the condition (87).

Let $\pi_{\varpi_{n-1}}$ be the half spin representation ϖ_{n-1} in ii) and we here use the fundamental system Ψ' defined above.

$$\pi_{\varpi_{n-1}}|_{\mathfrak g_\Theta} = \bigoplus_{(k_1,\ldots,k_L)\in\mathbf K_\Theta}\pi_{k_1,\ldots,k_L},\quad \pi_{\varpi_{n-1}}|_{\mathfrak g_{\bar\Theta}} = \bigoplus_{(k_1,\ldots,k_L)\in\mathbf K_{\bar\Theta}}\pi_{k_1,\ldots,k_L},$$

where

$$\begin{aligned}\mathbf K_\Theta &= \{(k_1,\ldots,k_L)\in\mathbb Z^L;\ 0\le k_j\le n'_j\ (j=1,\ldots,L),\\ &\qquad\qquad\qquad n-k_1-\cdots-k_L\equiv 1 \mod 2\},\\ \mathbf K_{\bar\Theta} &= \{(k_1,\ldots,k_L)\in\mathbf K_\Theta;\ k_L\ge n'_L-1\}\qquad (\text{Note } \alpha'_{n-1}\in\Theta \text{ and } n'_L>1)\end{aligned}$$

and $\pi_{k_1,\ldots,k_L}$ is the irreducible representation of $\mathfrak g_\Theta$ or $\mathfrak g_{\bar\Theta}$ with lowest weight

$$\sum_{j=1}^{L}\frac12(\varepsilon_{n_{j-1}+1}+\cdots+\varepsilon_{n_{j-1}+k_j}-\varepsilon_{n_{j-1}+k_j+1}-\cdots-\varepsilon_{n_j}).$$

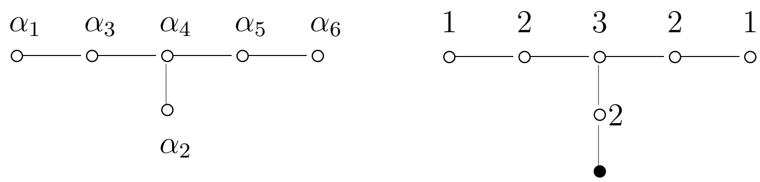
Then for $\Theta' = \Theta$ or $\bar\Theta$

$$\begin{aligned}q_{\pi_{\varpi_{n-1}},\Theta'}(x;\lambda) = \prod_{(k_1,\ldots,k_L)\in\mathbf K_{\Theta'}}&\Bigl(x-\frac{n(n-1)}{2^{n+1}}\\ &-\frac{1}{2^n}\sum_{j=1}^{L}(\bar\lambda_{n_{j-1}+1}+\cdots+\bar\lambda_{n_{j-1}+k_j}-\bar\lambda_{n_{j-1}+k_j+1}-\cdots-\bar\lambda_{n_j})\Bigr).\end{aligned}$$

If $n'_L > 1$, then

$$r_{\alpha'_{n-1},\Theta'}(\lambda) = \prod_{\substack{(k_1,\ldots,k_L)\in\mathbf{K}_{\Theta'}\\ (k_1,\ldots,k_L)\neq(n'_1,\ldots,n'_{L-1},n'_L-1)}} 2^{1-n} \cdot \Bigl(\sum_{j=1}^{L}(\bar\lambda_{n_{j-1}+k_j+1}+\cdots+\bar\lambda_{n_j-1}+\bar\lambda_{n_j})-\bar\lambda_{n-1}\Bigr).$$

Example 4.7 (E_6).



$\Psi = \{\alpha_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_8) - \frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7),\ \alpha_2 = \varepsilon_1+\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_1,\ \alpha_4 = \varepsilon_3-\varepsilon_2,\ \alpha_5 = \varepsilon_4-\varepsilon_3,\ \alpha_6 = \varepsilon_5-\varepsilon_4\}$

$\rho = \varepsilon_2+2\varepsilon_3+3\varepsilon_4+4\varepsilon_5+4(\varepsilon_8-\varepsilon_7-\varepsilon_6) = 8\alpha_1+11\alpha_2+15\alpha_3+21\alpha_4+15\alpha_5+8\alpha_6$

i) $\pi = \begin{cases} \varpi_1 := \frac{2}{3}(\varepsilon_8-\varepsilon_7-\varepsilon_6) & \text{(minuscule)}\\ \varpi_6 := \frac{1}{3}(\varepsilon_8-\varepsilon_7-\varepsilon_6)+\varepsilon_5 & \text{(minuscule)}\end{cases}$

$\dim\varpi_1 = \dim\varpi_6 = 27$

$C_{\varpi_1} = C_{\varpi_6} = 6$ (see below)

$(\varpi_1,\rho) = (\varpi_6,\rho) = 8$

$q_{\varpi_i}(x) = \prod_{\varpi\in W_{E_6}\varpi_i}(x-\varpi-\frac{4}{3})$ for $i = 1$ and 6.

ii) $\pi = \varpi_2 := \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$ (adjoint)

$\dim\varpi_2 = 78$

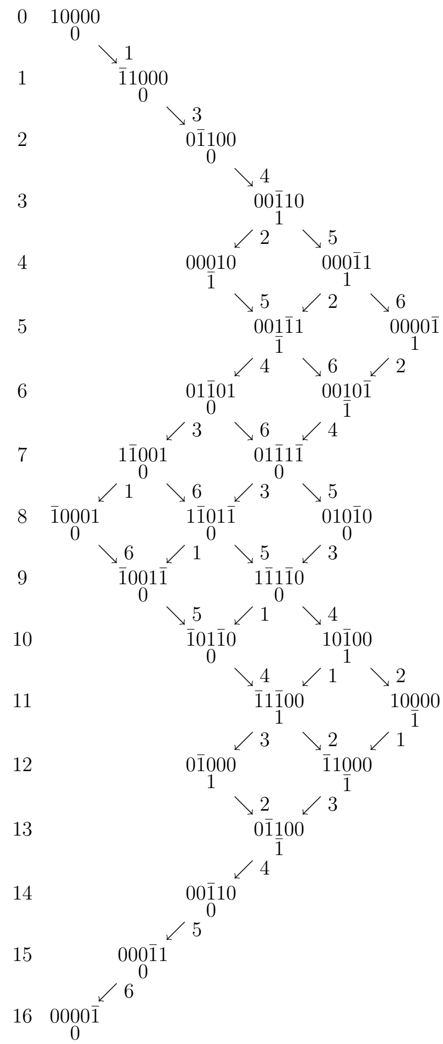
$C_{\varpi_2} = 24$

$(\varpi_2,\rho) = 11$

$q_{\varpi_2} = (x-\frac{1}{2})\prod_{\alpha\in\Sigma(E_6)}(x-\alpha-\frac{11}{24})$

Expressing a weight by the linear combination of the fundamental weights ϖ_j, we indicate the weight by the symbol arranging the coefficients in the corresponding position of the Dynkin diagram. For example, $\varpi = \sum_{j=1}^{6} m_j\varpi_j$ is indicated by the symbol $\underset{m_2}{m_1m_3m_4m_5m_6}$. Moreover for a positive integer m we will sometimes write $\bar m$ in place of $-m$.

Let π be the minuscule representation ϖ_1 in i). Then the partially ordered set of the weights of π is shown by the following. Here the number j beside an arrow represents $-\alpha_j$.



The type A_5 corresponding to $\{\alpha_1, \alpha_3, \ldots, \alpha_6\}$ is contained in type E_6. The highest weights of the restriction $(E_6, \pi)|_{A_5}$ are $\varpi_1 = \underset{0}{10000}$, $\varpi_5 - \varpi_2 = w_2 w_4 w_3 w_1 \varpi_1 = \underset{\bar{1}}{00010}$ and $\varpi_1 - \varpi_2 = w_2 w_4 w_5 w_6 w_3 w_4 w_5 (\varpi_5 - \varpi_2) = \underset{\bar{1}}{10000}$. Here we put $w_j = w_{\alpha_j}$. Hence $(E_6, \pi)|_{A_5} = 2(A_5, \varpi_1) + (A_5, \varpi_4)$ and $C_{\varpi_1} = C_{\varpi_6} = 2\binom{5-1}{1-1} + \binom{5-1}{2-1} = 6$.

Now use the fundamental system $\Psi' = \{\alpha'_1 = -\alpha_1, \ldots, \alpha'_6 = -\alpha_6\}$. Then the lowest weight $\bar{\pi}$ of π equals ϖ_1. Putting $\Theta_i = \Psi' \setminus \{\alpha'_i\}$, we have

$$\overline{\mathcal{W}}_{\Theta_1}(\pi) = \left\{\underset{0}{10000},\ \underset{0}{\bar{1}1000},\ \underset{0}{\bar{1}0001}\right\},$$
$$\overline{\mathcal{W}}_{\Theta_2}(\pi) = \left\{\underset{0}{10000},\ \underset{\bar{1}}{00010},\ \underset{\bar{1}}{10000}\right\},$$
$$\overline{\mathcal{W}}_{\Theta_3}(\pi) = \left\{\underset{0}{10000},\ \underset{0}{0\bar{1}100},\ \underset{0}{1\bar{1}001},\ \underset{1}{0\bar{1}000}\right\},$$
$$\overline{\mathcal{W}}_{\Theta_4}(\pi) = \left\{\underset{0}{10000},\ \underset{1}{00\bar{1}10},\ \underset{0}{01\bar{1}01},\ \underset{1}{10\bar{1}00},\ \underset{0}{00\bar{1}10}\right\},$$
$$\overline{\mathcal{W}}_{\Theta_5}(\pi) = \left\{\underset{0}{10000},\ \underset{1}{000\bar{1}1},\ \underset{0}{010\bar{1}0},\ \underset{0}{000\bar{1}1}\right\},$$
$$\overline{\mathcal{W}}_{\Theta_6}(\pi) = \left\{\underset{0}{10000},\ \underset{1}{0000\bar{1}},\ \underset{0}{0000\bar{1}}\right\}.$$

If we identify $\mathfrak{a}^*_{\Theta_i}$ with $\mathbb{C}$ by $\lambda_{\Theta_i} = \lambda\varpi_i$ and put $\bar\pi - \Lambda = \sum_j m^j_\Lambda \alpha_j$ for $\Lambda \in \mathcal{W}(\pi)$, then Proposition 2.40 i) implies

$$q_{\pi,\Theta_i}(x;\lambda) = \prod_{\Lambda\in\overline{\mathcal{W}}_{\Theta_i}(\pi)} \Bigl(x - (\langle\bar\pi,\varpi_i\rangle - m^i_\Lambda\langle\alpha_i,\varpi_i\rangle)\lambda - \sum_j m^j_\Lambda\langle\alpha_j,\rho\rangle\Bigr). \tag{90}$$

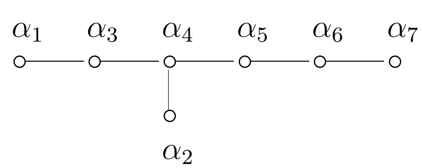
Since $\langle\alpha_j,\varpi_j\rangle = \langle\alpha_j,\rho\rangle = \frac{1}{2}\langle\alpha_j,\alpha_j\rangle = \frac{1}{6}$ and

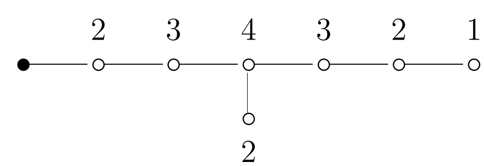
$$\langle\varpi_1,\varpi_1\rangle = \frac{2}{9}, \qquad \langle\varpi_1,\varpi_2\rangle = \frac{1}{6}, \qquad \langle\varpi_1,\varpi_3\rangle = \frac{5}{18},$$
$$\langle\varpi_1,\varpi_4\rangle = \frac{1}{3}, \qquad \langle\varpi_1,\varpi_5\rangle = \frac{2}{9}, \qquad \langle\varpi_1,\varpi_6\rangle = \frac{1}{9},$$

we get

$$\begin{aligned}
q_{\pi,\Theta_1}(x;\lambda) &= \Bigl(x - \frac{2}{9}\lambda\Bigr)\Bigl(x - \frac{1}{18}\lambda - \frac{1}{6}\Bigr)\Bigl(x + \frac{1}{9}\lambda - \frac{4}{3}\Bigr),\\
q_{\pi,\Theta_2}(x;\lambda) &= \Bigl(x - \frac{1}{6}\lambda\Bigr)\Bigl(x - \frac{2}{3}\Bigr)\Bigl(x + \frac{1}{6}\lambda - \frac{11}{6}\Bigr),\\
q_{\pi,\Theta_3}(x;\lambda) &= \Bigl(x - \frac{18}{5}\lambda\Bigr)\Bigl(x - \frac{1}{9}\lambda - \frac{1}{3}\Bigr)\Bigl(x + \frac{1}{18}\lambda - \frac{7}{6}\Bigr)\Bigl(x + \frac{2}{9}\lambda - 2\Bigr),\\
q_{\pi,\Theta_4}(x;\lambda) &= \Bigl(x - \frac{1}{3}\lambda\Bigr)\Bigl(x - \frac{1}{6}\lambda - \frac{1}{2}\Bigr)\Bigl(x - 1\Bigr)\Bigl(x + \frac{1}{6}\lambda - \frac{5}{3}\Bigr)\Bigl(x + \frac{1}{3}\lambda - \frac{7}{3}\Bigr),\\
q_{\pi,\Theta_5}(x;\lambda) &= \Bigl(x - \frac{2}{9}\lambda\Bigr)\Bigl(x - \frac{1}{18}\lambda - \frac{2}{3}\Bigr)\Bigl(x + \frac{1}{9}\lambda - \frac{4}{3}\Bigr)\Bigl(x + \frac{5}{18}\lambda - \frac{5}{2}\Bigr),\\
q_{\pi,\Theta_6}(x;\lambda) &= \Bigl(x - \frac{1}{9}\lambda\Bigr)\Bigl(x + \frac{1}{18}\lambda - \frac{5}{6}\Bigr)\Bigl(x + \frac{2}{9}\lambda - \frac{8}{3}\Bigr).
\end{aligned}$$

Example 4.8 (E_7)**.**





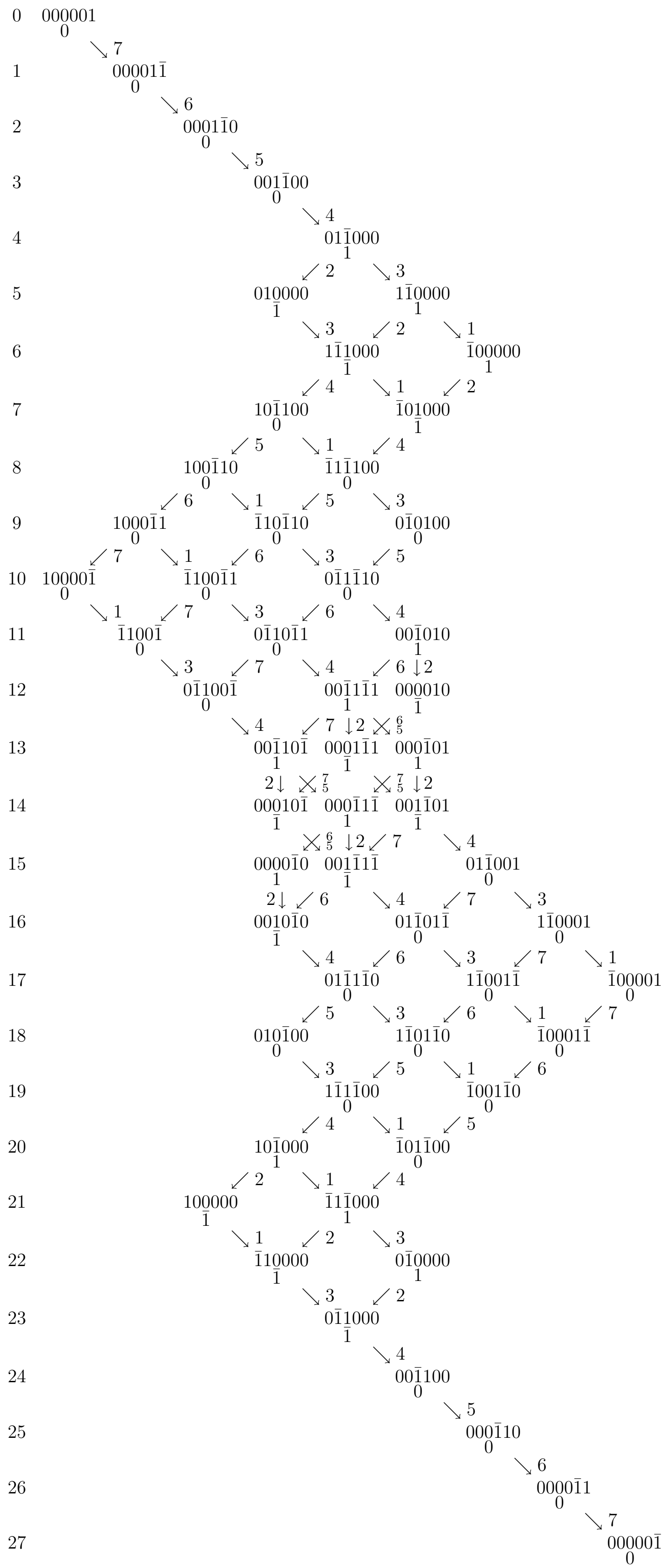
$\Psi = \{\alpha_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_8) - \frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7),\ \alpha_2 = \varepsilon_1+\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_1,\ \alpha_4 = \varepsilon_3-\varepsilon_2,\ \alpha_5 = \varepsilon_4-\varepsilon_3,\ \alpha_6 = \varepsilon_5-\varepsilon_4,\ \alpha_7 = \varepsilon_6-\varepsilon_5\}$

$\rho = \varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 5\varepsilon_6 - \frac{17}{2}\varepsilon_7 + \frac{17}{2}\varepsilon_8 = 17\alpha_1 + \frac{49}{2}\alpha_2 + 33\alpha_3 + 48\alpha_4 + \frac{75}{2}\alpha_5 + 26\alpha_6 + \frac{27}{2}\alpha_7$

i) $\pi = \varpi_7 := \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7)$ (minuscule)
$\dim\varpi_7 = 56$.
$C_{\varpi_7} = 12$ (see below)
$(\varpi_7,\rho) = \frac{27}{2}$
$q_{\varpi_7}(x) = \prod_{\varpi\in W_{E_7}\varpi_7}\bigl(x - \varpi - \frac{9}{8}\bigr)$

ii) $\pi = \varpi_1 := \varepsilon_8 - \varepsilon_2$ (adjoint)
$\dim\varpi_1 = 133$
$C_{\varpi_1} = 36$
$(\varpi_1,\rho) = 17$
$q_{\varpi_1}(x) = (x - \frac{1}{2})\prod_{\alpha\in\Sigma(E_7)}\bigl(x - \alpha - \frac{17}{36}\bigr)$

Let π be the minuscule representation ϖ_7 in i). Then the diagram of the partially ordered set of the weights of π is as follows.



Here we use the similar notation as in Example 4.7.

The type A_6 corresponding to $\{\alpha_1, \alpha_3, \ldots, \alpha_7\}$ is contained in type E_7. The highest weights of the restriction $(E_7, \pi)|_{A_6}$ are $\varpi_7 = 00\underset{0}{0}001$, $\varpi_3 - \varpi_2 = w_2w_4w_5w_6w_7\varpi_7 = 01\underset{\bar{1}}{0}000$, $\varpi_6 - \varpi_2 = w_2w_4w_3w_1w_5w_4w_3(\varpi_3 - \varpi_2) = 00\underset{\bar{1}}{0}010$ and $\varpi_1 - \varpi_2 = w_2w_4w_5w_6w_7w_3w_4w_5w_6(\varpi_6 - \varpi_2) = 10\underset{\bar{1}}{0}000$. Therefore $(E_7, \pi)|_{A_6} = (A_6, \varpi_6) + (A_6, \varpi_2) + (A_6, \varpi_5) + (A_6, \varpi_1)$ and $C_{\varpi_7} = \binom{6-1}{6-1} + \binom{6-1}{2-1} + \binom{6-1}{5-1} + \binom{6-1}{1-1} = 12$.

Now use $\Psi' = -\Psi$ and put $\Theta_i = \Psi' \setminus \{\alpha_i'\}$. Then

$$
\begin{aligned}
\overline{\mathcal{W}}_{\Theta_1}(\pi) &= \bigl\{00\underset{0}{0}001,\ \bar{1}0\underset{1}{0}00,\ \bar{1}0\underset{0}{0}001\bigr\},\\
\overline{\mathcal{W}}_{\Theta_2}(\pi) &= \bigl\{00\underset{0}{0}001,\ 01\underset{\bar{1}}{0}000,\ 00\underset{\bar{1}}{0}010,\ 10\underset{\bar{1}}{0}000\bigr\},\\
\overline{\mathcal{W}}_{\Theta_3}(\pi) &= \bigl\{00\underset{0}{0}001,\ 0\bar{1}\underset{1}{0}000,\ 0\bar{1}\underset{0}{0}100,\ 1\bar{1}\underset{0}{0}001,\ 0\bar{1}\underset{1}{0}000\bigr\},\\
\overline{\mathcal{W}}_{\Theta_4}(\pi) &= \bigl\{00\underset{0}{0}001,\ 01\underset{1}{\bar{1}}000,\ 10\underset{0}{\bar{1}}100,\ 00\underset{1}{\bar{1}}010,\ 01\underset{0}{\bar{1}}001,\ 10\underset{1}{\bar{1}}000,\ 00\underset{0}{\bar{1}}100\bigr\},\\
\overline{\mathcal{W}}_{\Theta_5}(\pi) &= \bigl\{00\underset{0}{0}001,\ 00\underset{0}{1}\bar{1}00,\ 10\underset{0}{0}\bar{1}10,\ 00\underset{1}{0}\bar{1}01,\ 01\underset{0}{0}\bar{1}00,\ 00\underset{0}{0}\bar{1}10\bigr\},\\
\overline{\mathcal{W}}_{\Theta_6}(\pi) &= \bigl\{00\underset{0}{0}001,\ 00\underset{0}{0}1\bar{1}0,\ 10\underset{0}{0}0\bar{1}1,\ 00\underset{1}{0}0\bar{1}0,\ 00\underset{0}{0}0\bar{1}1\bigr\},\\
\overline{\mathcal{W}}_{\Theta_7}(\pi) &= \bigl\{00\underset{0}{0}001,\ 00\underset{0}{0}01\bar{1},\ \bar{1}0\underset{0}{0}00\bar{1},\ 00\underset{0}{0}001\bigr\}.
\end{aligned}
$$

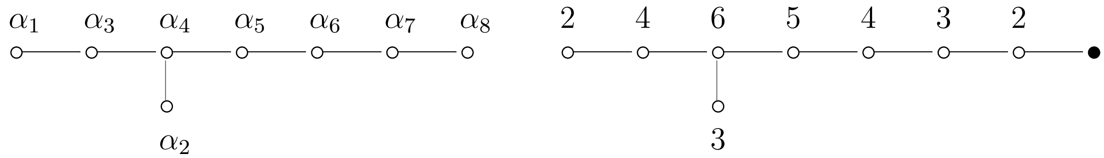
From (90) with $\langle \alpha_i, \varpi_i \rangle = \langle \alpha_i, \rho \rangle = \frac{1}{2}\langle \alpha_i, \alpha_i \rangle = \frac{1}{12}$ and

$$
\begin{aligned}
&\langle \varpi_7, \varpi_1 \rangle = \frac{1}{12}, \qquad \langle \varpi_7, \varpi_2 \rangle = \frac{1}{8}, \qquad \langle \varpi_7, \varpi_3 \rangle = \frac{1}{6}, \qquad \langle \varpi_7, \varpi_4 \rangle = \frac{1}{4},\\
&\langle \varpi_7, \varpi_5 \rangle = \frac{5}{24}, \qquad \langle \varpi_7, \varpi_6 \rangle = \frac{1}{6}, \qquad \langle \varpi_7, \varpi_7 \rangle = \frac{1}{8},
\end{aligned}
$$

we have

$$
\begin{aligned}
q_{\pi,\Theta_1}(x;\lambda) &= \Bigl(x - \frac{1}{12}\lambda\Bigr)\Bigl(x - \frac{1}{2}\Bigr)\Bigl(x + \frac{1}{12}\lambda - \frac{17}{12}\Bigr),\\
q_{\pi,\Theta_2}(x;\lambda) &= \Bigl(x - \frac{1}{8}\lambda\Bigr)\Bigl(x - \frac{1}{24}\lambda - \frac{5}{12}\Bigr)\Bigl(x + \frac{1}{24}\lambda - 1\Bigr)\Bigl(x + \frac{1}{8}\lambda - \frac{7}{4}\Bigr),\\
q_{\pi,\Theta_3}(x;\lambda) &= \Bigl(x - \frac{1}{6}\lambda\Bigr)\Bigl(x - \frac{1}{12}\lambda - \frac{5}{12}\Bigr)\Bigl(x - \frac{3}{4}\Bigr)\Bigl(x + \frac{1}{12}\lambda - \frac{4}{3}\Bigr)\Bigl(x + \frac{1}{6}\lambda - \frac{11}{6}\Bigr),\\
q_{\pi,\Theta_4}(x;\lambda) &= \Bigl(x - \frac{1}{4}\lambda\Bigr)\Bigl(x - \frac{1}{6}\lambda - \frac{1}{3}\Bigr)\Bigl(x - \frac{1}{12}\lambda - \frac{7}{12}\Bigr)\Bigl(x - \frac{11}{12}\Bigr)\\
&\qquad \cdot \Bigl(x + \frac{1}{12}\lambda - \frac{5}{4}\Bigr)\Bigl(x + \frac{1}{6}\lambda - \frac{5}{3}\Bigr)\Bigl(x + \frac{1}{4}\lambda - 2\Bigr),\\
q_{\pi,\Theta_5}(x;\lambda) &= \Bigl(x - \frac{5}{24}\lambda\Bigr)\Bigl(x - \frac{1}{8}\lambda - \frac{1}{4}\Bigr)\Bigl(x - \frac{1}{24}\lambda - \frac{2}{3}\Bigr)\Bigl(x + \frac{1}{24}\lambda - \frac{13}{12}\Bigr)\\
&\qquad \cdot \Bigl(x + \frac{1}{8}\lambda - \frac{3}{2}\Bigr)\Bigl(x + \frac{5}{24}\lambda - \frac{25}{12}\Bigr),\\
q_{\pi,\Theta_6}(x;\lambda) &= \Bigl(x - \frac{1}{6}\lambda\Bigr)\Bigl(x - \frac{1}{12}\lambda - \frac{1}{6}\Bigr)\Bigl(x - \frac{3}{4}\Bigr)\Bigl(x + \frac{1}{12}\lambda - \frac{5}{4}\Bigr)\Bigl(x + \frac{1}{6}\lambda - \frac{13}{6}\Bigr),\\
q_{\pi,\Theta_7}(x;\lambda) &= \Bigl(x - \frac{1}{8}\lambda\Bigr)\Bigl(x - \frac{1}{24}\lambda - \frac{1}{12}\Bigr)\Bigl(x + \frac{1}{24}\lambda - \frac{5}{6}\Bigr)\Bigl(x + \frac{1}{8}\lambda - \frac{9}{4}\Bigr).
\end{aligned}
$$

Example 4.9 (E_8).



$$\Psi = \{\alpha_1 = \tfrac{1}{2}(\varepsilon_1+\varepsilon_8) - \tfrac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7),\ \alpha_2 = \varepsilon_1+\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_1,\ \alpha_4 = \varepsilon_3-\varepsilon_2,\ \alpha_5 = \varepsilon_4-\varepsilon_3,\ \alpha_6 = \varepsilon_5-\varepsilon_4,\ \alpha_7 = \varepsilon_6-\varepsilon_5,\ \alpha_8 = \varepsilon_7-\varepsilon_6\}$$

$$\rho = \varepsilon_2+2\varepsilon_3+3\varepsilon_4+4\varepsilon_5+5\varepsilon_6+6\varepsilon_7+23\varepsilon_8 = 46\alpha_1+68\alpha_2+91\alpha_3+135\alpha_4 +110\alpha_5+84\alpha_6+57\alpha_7+29\alpha_8$$

i) $\pi = \alpha_{\max} := \varepsilon_7+\varepsilon_8$ (adjoint)

$\dim\alpha_{\max} = 248$ $(m_{\alpha_{\max}}(0) = 8)$

$C_{\alpha_{\max}} = 60$

$(\alpha_{\max}, \rho) = 29$

$q_{\alpha_{\max}}(x) = (x-\frac{1}{2})\prod_{\alpha\in\Sigma(E_8)}(x-\alpha-\frac{29}{60})$

Let π be the adjoint representation $\alpha_{\max}$ and $\alpha_{\max} = \sum_{i=1}^8 n_i\alpha_i$, that is, $n_1 = 2$, $n_2 = 3, \ldots$. Put $\Theta_i = \Psi\setminus\{\alpha_i\}$ for $i = 1,\ldots,8$. The irreducible decomposition of $\mathfrak{g}$ as a $\mathfrak{g}_{\Theta_i}$-module is given by Proposition 2.39 ii). In this case L_{Θ_i} in the proposition equals $\{-n_i, -n_i+1, \ldots, n_i\}$. Suppose $\mathbf{m}\in L_{\Theta_i}\setminus\{0\}$. Then $V(\mathbf{m})$ is a minuscule representation since E_8 is simply-laced. Let ϖ_i $(j = 1,\ldots,8)$ be the fundamental weights. If we write the lowest weight and the highest weight of $V(\mathbf{m})$ by $\alpha_{\mathbf{m}} = \sum_{j=1}^8 c_j\varpi_j$ and $\alpha'_{\mathbf{m}} = \sum_{j=1}^8 c'_j\varpi_j$ respectively, we clearly have

$$c_i = \begin{cases} 1 & \text{if } \mathbf{m}\neq 1, -n_i,\\ 2 & \text{if } \mathbf{m} = 1,\end{cases} \qquad c'_i = \begin{cases} -1 & \text{if } \mathbf{m}\neq -1, n_i,\\ -2 & \text{if } \mathbf{m} = -1,\end{cases}$$

and $\alpha_{\mathbf{m}} = -\alpha'_{-\mathbf{m}}$. Since we know the highest weights and the lowest weights of minuscule representations of $\mathfrak{g}_{\Theta_i}$ by the previous examples, starting with $\alpha_{\max} = \varpi_8 = 00\underset{0}{0}0001$, we can determine $\alpha_{\mathbf{m}}$ and $\alpha'_{\mathbf{m}}$ for $\mathbf{m}\in L_{\Theta_i}\setminus\{0\}$ step by step. For example, suppose $i = 4$. Then $L_{\Theta_4} = \{-6, -5, \ldots, 6\}$ and we have

$$V(6): \begin{cases} 00\underset{0}{0}0001 & \text{h.w.}\\ 00\underset{0}{1}\bar{1}000 & \text{l.w.}\end{cases} \quad\to\quad 00\underset{0}{1}\bar{1}000 - \alpha_4 = 01\underset{1}{\bar{1}}0000 \text{ is a weight of } V(5)$$

$$V(5): \begin{cases} 01\underset{1}{\bar{1}}0000 & \text{h.w.}\\ \bar{1}0\underset{\bar{1}}{1}0000 & \text{l.w.}\end{cases} \quad\to\quad \bar{1}0\underset{\bar{1}}{1}0000 - \alpha_4 = \bar{1}1\underset{0}{\bar{1}}1000 \text{ is a weight of } V(4)$$

$$V(4): \begin{cases} 10\underset{0}{\bar{1}}1000 & \text{h.w.}\\ \bar{1}1\underset{0}{\bar{1}}1000 & \\ 0\bar{1}\underset{0}{1}000\bar{1} & \text{l.w.}\end{cases} \quad\to\quad 0\bar{1}\underset{0}{1}000\bar{1} - \alpha_4 = 00\underset{1}{\bar{1}}100\bar{1} \text{ is a weight of } V(3)$$

$$V(3): \begin{cases} 00\underset{1}{\bar{1}}0100 & \text{h.w.}\\ 00\underset{1}{\bar{1}}100\bar{1} & \\ 00\underset{\bar{1}}{1}00\bar{1}0 & \text{l.w.}\end{cases} \quad\to\quad 00\underset{\bar{1}}{1}00\bar{1}0 - \alpha_4 = 01\underset{0}{\bar{1}}10\bar{1}0 \text{ is a weight of } V(2)$$

$$V(2):\begin{cases} 01\underset{0}{\bar{1}}0010 & \text{h.w.} \\ 01\underset{0}{\bar{1}}10\bar{1}0 & \\ \bar{1}0\underset{0}{1}0\bar{1}00 & \text{l.w.} \end{cases} \qquad \rightarrow \qquad \bar{1}0\underset{0}{1}0\bar{1}00-\alpha_4=\bar{1}1\underset{1}{\bar{1}}1\bar{1}00 \text{ is a weight of } V(1)$$

$$V(1):\begin{cases} 10\underset{1}{\bar{1}}0001 & \text{h.w.} \\ \bar{1}1\underset{1}{\bar{1}}1\bar{1}00 & \\ 0\bar{1}\underset{\bar{1}}{2}\bar{1}000 & \text{l.w.} \end{cases}$$

On the other hand, the non-trivial irreducible subrepresentations of $V(0)$ correspond to the connected parts of Dynkin diagram of Θ_i. If $\sum_{j=1}^{8} c_j\varpi_j$ is a lowest weight of such subrepresentations, then $c_i=1$. Hence, if $i=4$, the lowest weights of the non-trivial irreducible subrepresentations of $V(0)$ are

$$\bar{1}\bar{1}\underset{0}{1}0000, \qquad 00\underset{\bar{2}}{1}0000, \qquad 00\underset{0}{1}\bar{1}00\bar{1}.$$

Thus we get

$$\begin{aligned}\overline{\mathcal{W}}_{\Theta_4}(\pi)=\{&00\underset{0}{1}\bar{1}000,\bar{1}0\underset{\bar{1}}{1}0000,0\bar{1}\underset{0}{1}000\bar{1},00\underset{\bar{1}}{1}00\bar{1}0,\bar{1}0\underset{0}{1}0\bar{1}00,0\bar{1}\underset{\bar{1}}{2}\bar{1}000\}\\ &\cup\{0\}\cup\{\bar{1}\bar{1}\underset{0}{1}0000,00\underset{\bar{2}}{1}0000,00\underset{0}{1}\bar{1}00\bar{1}\}\\ &\cup\{-10\underset{1}{\bar{1}}0001,-01\underset{0}{\bar{1}}0010,-00\underset{1}{\bar{1}}0100,-10\underset{0}{\bar{1}}1000,-01\underset{1}{\bar{1}}0000,-00\underset{0}{0}0001\}.\end{aligned}$$

Put $\lambda_{\Theta_i}=\lambda\varpi_i$. Then, by (38), we have

$$\begin{aligned}q_{\pi,\Theta_4}(x;\lambda)=&\Big(x-\frac{1}{2}\Big)\Big(x-\frac{9}{20}\Big)\Big(x-\frac{7}{15}\Big)\Big(x-\frac{5}{12}\Big)\Big(x-\frac{1}{10}\lambda-\frac{9}{10}\Big)\\ &\cdot\Big(x-\frac{1}{12}\lambda-\frac{5}{6}\Big)\Big(x-\frac{1}{15}\lambda-\frac{11}{15}\Big)\Big(x-\frac{1}{20}\lambda-\frac{13}{20}\Big)\Big(x-\frac{1}{30}\lambda-\frac{17}{30}\Big)\\ &\cdot\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\Big(x+\frac{1}{60}\lambda-\frac{7}{20}\Big)\Big(x+\frac{1}{30}\lambda-\frac{4}{15}\Big)\Big(x+\frac{1}{20}\lambda-\frac{1}{5}\Big)\\ &\cdot\Big(x+\frac{1}{15}\lambda-\frac{2}{15}\Big)\Big(x+\frac{1}{12}\lambda-\frac{1}{12}\Big)\Big(x+\frac{1}{10}\lambda\Big).\end{aligned}$$

Similarly we get

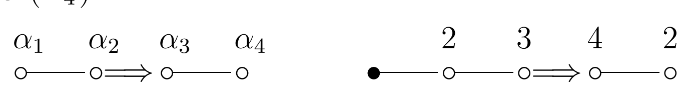
$$\begin{aligned}\overline{\mathcal{W}}_{\Theta_1}(\pi)&=\{10\underset{0}{0}000\bar{1},2\bar{1}\underset{0}{0}0000\}\cup\{0\}\cup\{10\underset{0}{0}00\bar{1}0\}\cup\{-\bar{1}0\underset{1}{0}0000,-00\underset{0}{0}0001\},\\ \overline{\mathcal{W}}_{\Theta_2}(\pi)&=\{\bar{1}0\underset{1}{0}0000,00\underset{1}{0}00\bar{1}0,00\underset{2}{\bar{1}}0000\}\cup\{0\}\cup\{\bar{1}0\underset{1}{0}000\bar{1}\}\\ &\qquad\cup\{-00\underset{\bar{1}}{0}0100,-01\underset{\bar{1}}{0}0000,-00\underset{0}{0}0001\},\\ \overline{\mathcal{W}}_{\Theta_3}(\pi)&=\{01\underset{\bar{1}}{0}0000,\bar{1}1\underset{0}{0}000\bar{1},01\underset{0}{0}\bar{1}00,\bar{1}2\underset{0}{\bar{1}}0000\}\cup\{0\}\cup\{\bar{2}1\underset{0}{0}0000,01\underset{\bar{1}}{0}000\bar{1}\}\\ &\qquad\cup\{-1\bar{1}\underset{0}{0}0010,-0\bar{1}\underset{0}{0}1000,-1\bar{1}\underset{1}{0}0000,-00\underset{0}{0}0001\},\\ \overline{\mathcal{W}}_{\Theta_5}(\pi)&=\{00\underset{0}{0}1\bar{1}00,0\bar{1}\underset{0}{0}1000,00\underset{\bar{1}}{0}100\bar{1},\bar{1}0\underset{0}{0}10\bar{1}0,00\underset{0}{\bar{1}}2\bar{1}00\}\\ &\qquad\cup\{0\}\cup\{\bar{1}0\underset{\bar{1}}{0}1000,00\underset{0}{0}1\bar{1}0\bar{1}\}\end{aligned}$$

$$
\begin{aligned}
&\cup\{-01\underset{0}{0}\bar{1}001,\ -00\underset{1}{0}\bar{1}010,\ -10\underset{0}{0}\bar{1}100,\ -00\underset{0}{1}\bar{1}000,\ -00\underset{0}{0}0001\},\\
\overline{\mathcal{W}}_{\Theta_6}(\pi) &= \{00\underset{0}{0}01\bar{1}0,\ 00\underset{\bar{1}}{0}0100,\ \bar{1}0\underset{0}{0}010\bar{1},\ 00\underset{0}{0}\bar{1}2\bar{1}0\}\cup\{0\}\cup\{0\bar{1}\underset{0}{0}0100,\ 00\underset{0}{0}01\bar{1}\bar{1}\}\\
&\cup\{-00\underset{1}{0}0\bar{1}01,\ -10\underset{0}{0}0\bar{1}10,\ -00\underset{0}{0}1\bar{1}00,\ -00\underset{0}{0}0001\},\\
\overline{\mathcal{W}}_{\Theta_7}(\pi) &= \{00\underset{0}{0}001\bar{1},\ \bar{1}0\underset{0}{0}0010,\ 00\underset{0}{0}0\bar{1}2\bar{1}\}\cup\{0\}\cup\{00\underset{\bar{1}}{0}0010,\ 00\underset{0}{0}001\bar{2}\}\\
&\cup\{-10\underset{0}{0}00\bar{1}1,\ -00\underset{0}{0}01\bar{1}0,\ -00\underset{0}{0}0001\},\\
\overline{\mathcal{W}}_{\Theta_8}(\pi) &= \{00\underset{0}{0}0001,\ 00\underset{0}{0}00\bar{1}2\}\cup\{0\}\cup\{\bar{1}0\underset{0}{0}0001\}\cup\{-00\underset{0}{0}001\bar{1},\ -00\underset{0}{0}0001\},
\end{aligned}
$$

and

$$
\begin{aligned}
q_{\pi,\Theta_1}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{3}{10}\Big)\Big(x-\frac{1}{30}\lambda-\frac{23}{30}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\\
&\quad\cdot\Big(x+\frac{1}{60}\lambda-\frac{7}{60}\Big)\Big(x+\frac{1}{30}\lambda\Big),\\
q_{\pi,\Theta_2}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{11}{30}\Big)\Big(x-\frac{1}{20}\lambda-\frac{17}{20}\Big)\Big(x-\frac{1}{30}\lambda-\frac{2}{3}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\\
&\quad\cdot\Big(x+\frac{1}{60}\lambda-\frac{13}{60}\Big)\Big(x+\frac{1}{30}\lambda-\frac{1}{10}\Big)\Big(x+\frac{1}{20}\lambda\Big),\\
q_{\pi,\Theta_3}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{7}{15}\Big)\Big(x-\frac{23}{60}\Big)\\
&\quad\cdot\Big(x-\frac{1}{15}\lambda-\frac{13}{15}\Big)\Big(x-\frac{1}{20}\lambda-\frac{3}{4}\Big)\Big(x-\frac{1}{30}\lambda-\frac{3}{5}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\\
&\quad\cdot\Big(x+\frac{1}{60}\lambda-\frac{17}{60}\Big)\Big(x+\frac{1}{30}\lambda-\frac{1}{6}\Big)\Big(x+\frac{1}{20}\lambda-\frac{1}{10}\Big)\Big(x+\frac{1}{15}\lambda\Big),\\
q_{\pi,\Theta_5}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{5}{12}\Big)\Big(x-\frac{13}{30}\Big)\Big(x-\frac{1}{12}\lambda-\frac{11}{12}\Big)\Big(x-\frac{1}{15}\lambda-\frac{4}{5}\Big)\\
&\quad\cdot\Big(x-\frac{1}{20}\lambda-\frac{7}{10}\Big)\Big(x-\frac{1}{30}\lambda-\frac{3}{5}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\Big(x+\frac{1}{60}\lambda-\frac{19}{60}\Big)\\
&\quad\cdot\Big(x+\frac{1}{30}\lambda-\frac{7}{30}\Big)\Big(x+\frac{1}{20}\lambda-\frac{3}{20}\Big)\Big(x+\frac{1}{15}\lambda-\frac{1}{15}\Big)\Big(x+\frac{1}{12}\lambda\Big),\\
q_{\pi,\Theta_6}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{11}{30}\Big)\Big(x-\frac{9}{20}\Big)\\
&\quad\cdot\Big(x-\frac{1}{15}\lambda-\frac{14}{15}\Big)\Big(x-\frac{1}{20}\lambda-\frac{3}{4}\Big)\Big(x-\frac{1}{30}\lambda-\frac{19}{30}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\\
&\quad\cdot\Big(x+\frac{1}{60}\lambda-\frac{4}{15}\Big)\Big(x+\frac{1}{30}\lambda-\frac{1}{6}\Big)\Big(x+\frac{1}{20}\lambda-\frac{1}{20}\Big)\Big(x+\frac{1}{15}\lambda\Big),\\
q_{\pi,\Theta_7}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{3}{10}\Big)\Big(x-\frac{7}{15}\Big)\Big(x-\frac{1}{20}\lambda-\frac{19}{20}\Big)\Big(x-\frac{1}{30}\lambda-\frac{2}{3}\Big)\\
&\quad\cdot\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\Big(x+\frac{1}{60}\lambda-\frac{11}{60}\Big)\Big(x+\frac{1}{30}\lambda-\frac{1}{30}\Big)\Big(x+\frac{1}{20}\lambda\Big),\\
q_{\pi,\Theta_8}(x;\lambda) &= \Big(x-\frac{1}{2}\Big)\Big(x-\frac{1}{5}\Big)\Big(x-\frac{1}{30}\lambda-\frac{29}{30}\Big)\Big(x-\frac{1}{60}\lambda-\frac{1}{2}\Big)\\
&\quad\cdot\Big(x+\frac{1}{60}\lambda-\frac{1}{60}\Big)\Big(x+\frac{1}{30}\lambda\Big).
\end{aligned}
$$

Example 4.10 (F_4).



$$\Psi = \{\alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = \varepsilon_4,\ \alpha_4 = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$$
$$\rho = \tfrac{11}{2}\varepsilon_1 + \tfrac{5}{2}\varepsilon_2 + \tfrac{3}{2}\varepsilon_3 + \tfrac{1}{2}\varepsilon_4 = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4$$

i) $\pi = \varpi_4 := \varepsilon_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ (dominant short root)

$$\dim \varpi_4 = 26 \ \ (m_{\varpi_4}(0) = 2)$$
$$C_{\varpi_4} = \textstyle\sum_{\nu=1}^{4}(\pm\varepsilon_\nu, \varepsilon_1)^2 + \frac{1}{4}\sum(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4, \varepsilon_1)^2 = 2 + \frac{16}{4} = 6$$
$$(\varpi_4, \rho) = \tfrac{11}{2}$$
$$q_{\varpi_4}(x) = \bigl(x - 1\bigr) \prod_{\substack{\alpha\in\Sigma(F_4)\\ |\beta|<|\alpha_{\max}|}} \bigl(x - \beta - \tfrac{11}{12}\bigr)$$

ii) $\pi = \varpi_1 := \varepsilon_1 + \varepsilon_2$ (adjoint)

$$\dim \varpi_1 = 52$$
$$C_{\varpi_1} = 18$$
$$(\varpi_1, \rho) = 8$$
$$q_{\varpi_1}(x) = (x - \tfrac{1}{2}) \prod_{\substack{\alpha\in\Sigma(F_4)\\ |\alpha|=|\alpha_{\max}|}} \bigl(x - \alpha - \tfrac{4}{9}\bigr) \prod_{\substack{\beta\in\Sigma(F_4)\\ |\beta|<|\alpha_{\max}|}} \bigl(x - \beta - \tfrac{17}{36}\bigr)$$

Let π be the representation ϖ_4 in i). Then the diagram of the partially ordered set of the weights of π is as follows. Here the weight $00\rangle 00$ is the only weight with the multiplicity 2 and hence indicated by $[00\rangle 00]$.

$$\begin{tikzcd}[row sep=small, column sep=tiny]
0 & 00\rangle 01 \arrow[dr, "4"] & & & & & \\
1 & & 00\rangle 1\bar{1} \arrow[dr, "3"] & & & & \\
2 & & & 01\rangle \bar{1}0 \arrow[dr, "2"] & & & \\
3 & & & & 1\bar{1}\rangle 10 \arrow[dl, "1"'] \arrow[dr, "3"] & & \\
4 & & & \bar{1}0\rangle 10 \arrow[dr, "3"] & & 10\rangle \bar{1}1 \arrow[dl, "1"'] \arrow[dr, "4"] & \\
5 & & & & \bar{1}1\rangle \bar{1}1 \arrow[dl, "2"'] \arrow[dr, "4"] & & 10\rangle 0\bar{1} \arrow[dl, "1"'] \\
6 & & & 0\bar{1}\rangle 11 \arrow[dl, "3"'] \arrow[dr, "4"] & & \bar{1}1\rangle 0\bar{1} \arrow[dl, "2"'] & \\
7 & & 00\rangle \bar{1}2 \arrow[dr, "4"] & & 0\bar{1}\rangle 2\bar{1} \arrow[dl, "3"'] & & \\
8 & & & {[00\rangle 00]} \arrow[dl, "4"'] \arrow[dr, "3"] & & & \\
9 & & 00\rangle 1\bar{2} \arrow[dr, "3"] & & 01\rangle \bar{2}1 \arrow[dl, "4"'] \arrow[dr, "2"] & & \\
10 & & & 01\rangle \bar{1}\bar{1} \arrow[dr, "2"] & & 1\bar{1}\rangle 01 \arrow[dl, "4"'] \arrow[dr, "1"] & \\
11 & & & & 1\bar{1}\rangle 1\bar{1} \arrow[dl, "3"'] \arrow[dr, "1"] & & \bar{1}0\rangle 01 \arrow[dl, "4"'] \\
12 & & & 10\rangle \bar{1}0 \arrow[dr, "1"] & & \bar{1}0\rangle 1\bar{1} \arrow[dl, "3"'] & \\
13 & & & & \bar{1}1\rangle \bar{1}0 \arrow[dl, "2"'] & & \\
14 & & & 0\bar{1}\rangle 10 \arrow[dl, "3"'] & & & \\
15 & & 00\rangle \bar{1}1 \arrow[dl, "4"'] & & & & \\
16 & 00\rangle 0\bar{1} & & & & &
\end{tikzcd}$$

Now use $\Psi' = \{\alpha'_1 = -\alpha_1, \ldots, \alpha'_4 = -\alpha_4\}$ and put $\Theta_i = \Psi' \setminus \{\alpha'_i\}$. Then we have

$$\overline{\mathcal{W}}_{\Theta_1}(\pi) = \bigl\{00\rangle 01,\ \bar{1}0\rangle 10,\ \bar{1}0\rangle 01\bigr\},$$
$$\overline{\mathcal{W}}_{\Theta_2}(\pi) = \bigl\{00\rangle 01,\ 1\bar{1}\rangle 10,\ 0\bar{1}\rangle 11,\ 1\bar{1}\rangle 01,\ 0\bar{1}\rangle 10\bigr\},$$

$$
\begin{aligned}
\overline{\mathcal{W}}_{\Theta_3}(\pi) &= \bigl\{00\rangle 01,\ 01\rangle \bar{1}0,\ 10\rangle \bar{1}1,\ 00\rangle \bar{1}2,\ 00\rangle 00,\ 01\rangle \bar{2}1,\ 10\rangle \bar{1}0,\ 00\rangle \bar{1}1\bigr\},\\
\overline{\mathcal{W}}_{\Theta_4}(\pi) &= \bigl\{00\rangle 0\bar{1},\ 00\rangle 1\bar{1},\ 10\rangle 0\bar{1},\ 00\rangle 00,\ 00\rangle 1\bar{2},\ 00\rangle 0\bar{1}\bigr\}
\end{aligned}
$$

and

$$
\begin{aligned}
q_{\pi,\Theta_1}(x;\lambda) &= \Bigl(x-\frac{1}{6}\lambda\Bigr)\Bigl(x-\frac{1}{2}\Bigr)\Bigl(x+\frac{1}{6}\lambda-\frac{4}{3}\Bigr),\\
q_{\pi,\Theta_2}(x;\lambda) &= \Bigl(x-\frac{1}{3}\lambda\Bigr)\Bigl(x-\frac{1}{6}\lambda-\frac{1}{3}\Bigr)\Bigl(x-\frac{3}{4}\Bigr)\Bigl(x+\frac{1}{6}\lambda-\frac{7}{6}\Bigr)\Bigl(x+\frac{1}{3}\lambda-\frac{5}{3}\Bigr),\\
q_{\pi,\Theta_3}(x;\lambda) &= \Bigl(x-\frac{1}{4}\lambda\Bigr)\Bigl(x-\frac{1}{6}\lambda-\frac{1}{6}\Bigr)\Bigl(x-\frac{1}{12}\lambda-\frac{5}{12}\Bigr)\Bigl(x-\frac{5}{6}\Bigr)\\
&\qquad\cdot\Bigl(x-1\Bigr)\Bigl(x+\frac{1}{12}\lambda-1\Bigr)\Bigl(x+\frac{1}{6}\lambda-\frac{4}{3}\Bigr)\Bigl(x+\frac{1}{4}\lambda-\frac{7}{4}\Bigr),\\
q_{\pi,\Theta_4}(x;\lambda) &= \Bigl(x-\frac{11}{6}\lambda\Bigr)\Bigl(x-\frac{1}{12}\lambda-\frac{1}{12}\Bigr)\Bigl(x-\frac{1}{2}\Bigr)\Bigl(x-1\Bigr)\\
&\qquad\cdot\Bigl(x+\frac{1}{12}\lambda-1\Bigr)\Bigl(x+\frac{1}{6}\lambda-\frac{11}{6}\Bigr).
\end{aligned}
$$

The extremal low weights of π with respect to Ψ' are as follows:

$$
\begin{aligned}
\varpi_{\alpha_1'} &= \varpi_4-\alpha_4-\alpha_3-\alpha_2 = \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\\
\varpi_{\alpha_2'} &= \varpi_4-\alpha_4-\alpha_3 = \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\\
\varpi_{\alpha_3'} &= \varpi_4-\alpha_4 = \alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\
\varpi_{\alpha_4'} &= \varpi_4 = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4.
\end{aligned}
$$

None of them is a member of $\Sigma(\mathfrak{g}_\Theta)\cup\{0\}$ for any $\Theta \subsetneq \Psi'$. Hence by Proposition 2.39 i) and Lemma 3.24, the functions $r_{\alpha_i'}(\lambda)$ $(i=1,2,3,4)$ are not identically zero.

Example 4.11 (G_2)**.**

$$
\begin{array}{cc}
\alpha_1 \quad \alpha_2 & \qquad\qquad 3 \quad 2 \\
\circ \Lleftarrow \circ & \qquad\qquad \circ \Lleftarrow \circ\!\!-\!\!\!-\!\!\!-\!\!\bullet
\end{array}
$$

$\Psi = \{\alpha_1 = \varepsilon_1-\varepsilon_2,\ \alpha_2 = -2\varepsilon_1+\varepsilon_2+\varepsilon_3\}$
$\rho = -\varepsilon_1-2\varepsilon_2+3\varepsilon_3 = 5\alpha_1+3\alpha_2$

i) $\pi = \varpi_1 := -\varepsilon_2+\varepsilon_3 = 2\alpha_1+\alpha_2$ (multiplicity free)
$\dim\varpi_1 = 7$
$C_{\varpi_1} = \frac{1}{2}\bigl(2\sum_{1\le i<j\le 3}(\varepsilon_i-\varepsilon_j,\varepsilon_1-\varepsilon_2)^2+(0,\varepsilon_1-\varepsilon_2)^2\bigr) = 6$
$(\varpi_1,\rho) = 5$
$q_{\varpi_1}(x) = \bigl(x-1\bigr)\prod_{1\le i<j\le 3}\bigl((x-\frac{5}{6})^2-(\varepsilon_i-\varepsilon_j)^2\bigr)$

ii) $\pi = \varpi_2 := -\varepsilon_1-\varepsilon_2+2\varepsilon_3 = 3\alpha_1+2\alpha_2$ (adjoint)
$\dim\varpi_2 = 14$
$C_{\varpi_2} = 24$
$(\varpi_2,\rho) = 9$
$q_{\varpi_2}(x) = (x-\frac{1}{2})\prod_{\substack{\alpha\in\Sigma(G_2)\\ |\alpha|=|\alpha_{\max}|}}\bigl(x-\alpha-\frac{3}{8}\bigr)\prod_{\substack{\beta\in\Sigma(G_2)\\ |\beta|<|\alpha_{\max}|}}\bigl(x-\beta-\frac{11}{24}\bigr)$

Consider the representation π with the highest weight ϖ_1. Then as is shown in [8], the weights of π are indicated by

$$
\varepsilon_2-\varepsilon_3 \xrightarrow{\alpha_1} \varepsilon_1-\varepsilon_3 \xrightarrow{\alpha_2} -\varepsilon_1+\varepsilon_2 \xrightarrow{\alpha_1} 0 \xrightarrow{\alpha_1} \varepsilon_1-\varepsilon_2 \xrightarrow{\alpha_2} -\varepsilon_1+\varepsilon_3 \xrightarrow{\alpha_1} -\varepsilon_2+\varepsilon_3
$$

and therefore

$$\begin{aligned}
\overline{\mathcal{W}}_{\{\alpha_1\}}(\pi) &= \{\varepsilon_2-\varepsilon_3,\ -\varepsilon_1+\varepsilon_2,\ -\varepsilon_1+\varepsilon_3\},\\
\overline{\mathcal{W}}_{\{\alpha_2\}}(\pi) &= \{\varepsilon_2-\varepsilon_3,\ \varepsilon_1-\varepsilon_3,\ 0,\ \varepsilon_1-\varepsilon_2,\ -\varepsilon_2+\varepsilon_3\}.
\end{aligned}$$

For $\lambda \in \mathfrak{a}_\Theta^*$ we put $\lambda_\Theta = \lambda_1\varpi_1 + \lambda_2\varpi_2$. Then $\lambda_1 = 0$ (resp. $\lambda_2 = 0$) if $\Theta = \{\alpha_1\}$ (resp. $\{\alpha_2\}$) and

$$\begin{aligned}
q_{\pi,\{\alpha_1\}}(x;\lambda) &= \Bigl(x+\frac{\lambda_2}{2}\Bigr)\Bigl(x-\frac{(\alpha_1+\alpha_2,\rho)}{6}\Bigr)\Bigl(x-\frac{\lambda_2}{2}-\frac{(3\alpha_1+2\alpha_2,\rho)}{6}\Bigr)\\
&= \Bigl(x+\frac{\lambda_2}{2}\Bigr)\Bigl(x-\frac{2}{3}\Bigr)\Bigl(x-\frac{\lambda_2}{2}-\frac{3}{2}\Bigr),\\
q_{\pi,\{\alpha_2\}}(x;\lambda) &= \Bigl(x+\frac{\lambda_1}{3}\Bigr)\Bigl(x+\frac{\lambda_1}{6}-\frac{(\alpha_1,\rho)}{6}\Bigr)\Bigl(x-1\Bigr)\\
&\qquad\cdot\Bigl(x-\frac{\lambda_1}{6}-\frac{(3\alpha_1+\alpha_2,\rho)}{6}\Bigr)\Bigl(x-\frac{\lambda_1}{3}-\frac{(4\alpha_1+2\alpha_2,\rho)}{6}\Bigr)\\
&= \Bigl(x+\frac{\lambda_1}{3}\Bigr)\Bigl(x+\frac{\lambda_1}{6}-\frac{1}{6}\Bigr)\Bigl(x-1\Bigr)\Bigl(x-\frac{\lambda_1}{6}-1\Bigr)\Bigl(x-\frac{\lambda_1}{3}-\frac{5}{3}\Bigr).
\end{aligned}$$

Moreover, from Remark 3.20, we get

$$\begin{aligned}
r_{\alpha_1}(\lambda) &= \langle\lambda_\Theta+\rho,(-\varpi_1+\alpha_1)-(-\varpi_1+\alpha_1+\alpha_2)\rangle\\
&\qquad\cdot\langle\lambda_\Theta+\rho,(-\varpi_1+\alpha_1)-(-\varpi_1+3\alpha_1+2\alpha_2)\rangle\\
&= 2\langle\lambda_\Theta+\rho,\alpha_2\rangle\langle\lambda_\Theta+\rho,\alpha_1+\alpha_2\rangle\\
&= \frac{1}{6}(\lambda_2+1)(3\lambda_2+4),\\
r_{\alpha_2}(\lambda) &= \Bigl(\langle\lambda_\Theta,(-\varpi_1+\alpha_1)-(-\varpi_1)\rangle-\langle\alpha_2,-\varpi_1+\alpha_1\rangle\Bigr)\\
&\qquad\cdot\Bigl(\langle\lambda_\Theta,(-\varpi_1+\alpha_1+\alpha_2)-0\rangle+\langle-\varpi_1+\alpha_1+\alpha_2,\alpha_1\rangle\Bigr)\\
&\qquad\cdot\langle\lambda_\Theta+\rho,(-\varpi_1+\alpha_1+\alpha_2)-(-\varpi_1+3\alpha_1+\alpha_2)\rangle\\
&\qquad\cdot\langle\lambda_\Theta+\rho,(-\varpi_1+\alpha_1+\alpha_2)-(-\varpi_1+4\alpha_1+2\alpha_2)\rangle\\
&= -\frac{2}{9}\Bigl(\langle\lambda_\Theta+\rho,\alpha_1\rangle\Bigr)\Bigl(\langle\lambda_\Theta+\rho,3\alpha_1+2\alpha_2\rangle\Bigr)\Bigl(\langle\lambda_\Theta+\rho,3\alpha_1+\alpha_2\rangle\Bigr)^2\\
&= -\frac{1}{216}(\lambda_1+1)(\lambda_1+2)^2(\lambda_1+3).
\end{aligned}$$

Here we have used the following relations:

$$\begin{cases}
-\langle\alpha_2,-\varpi_1+\alpha_1\rangle = -\langle\alpha_2,\alpha_1\rangle = \dfrac{\langle\rho,3\alpha_1+2\alpha_2\rangle}{3},\\[2ex]
\langle-\varpi_1+\alpha_1+\alpha_2,\alpha_1\rangle = -\langle\alpha_1,\alpha_1\rangle = -\dfrac{\langle\rho,3\alpha_1+\alpha_2\rangle}{3}.
\end{cases}$$

Note that $\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, 3\alpha_1+\alpha_2 \in \Sigma(\mathfrak{g})$ and $r_{\alpha_i}(\lambda)\neq 0$ if the condition ii) of Theorem 3.12 (we do not assume here that $\lambda_\Theta+\rho$ is dominant) is satisfied.

Let $S(\mathfrak{a})^{(m)}$ denote the space of the elements of the symmetric algebra over $\mathfrak{a}$ whose degree are at most m. Note that

$$(\operatorname{Trace} F_\pi^{2m})_{\mathfrak{a}} \equiv 2(\varepsilon_1-\varepsilon_2)^{2m}+2(\varepsilon_2-\varepsilon_3)^{2m}+2(\varepsilon_1-\varepsilon_3)^{2m} \mod S(\mathfrak{a})^{(2m-1)}$$

$$\equiv 2(\varepsilon_1-\varepsilon_2)^{2m}+2(\varepsilon_1+2\varepsilon_2)^{2m}+2(2\varepsilon_1+\varepsilon_2)^{2m} \quad \bmod S(\mathfrak{a})(\varepsilon_1+\varepsilon_2+\varepsilon_3),$$

$$(\operatorname{Trace} F_\pi^2)_{\mathfrak{a}} \equiv 12(\varepsilon_1^2+\varepsilon_1\varepsilon_2+\varepsilon_2^2) \quad \bmod S(\mathfrak{a})^{(1)}+S(\mathfrak{a})(\varepsilon_1+\varepsilon_2+\varepsilon_3),$$

$$(\operatorname{Trace} F_\pi^4)_{\mathfrak{a}} \equiv \frac{1}{4}((\operatorname{Trace} F_\pi^2)_{\mathfrak{a}})^2 \quad \bmod S(\mathfrak{a})^{(3)}+S(\mathfrak{a})(\varepsilon_1+\varepsilon_2+\varepsilon_3).$$

Moreover $(\operatorname{Trace} F_\pi^6)_{\mathfrak{a}}$ and $\big((\operatorname{Trace} F_\pi^2)_{\mathfrak{a}}\big)^3$ are linearly independent in

$$S(\mathfrak{a})/\Big(S(\mathfrak{a})^{(5)}+S(\mathfrak{a})(\varepsilon_1+\varepsilon_2+\varepsilon_3)\Big).$$

Thus we have

$$Z(\mathfrak{g})=\mathbb{C}[\operatorname{Trace} F_\pi^2, \operatorname{Trace} F_\pi^6]. \tag{91}$$

Proposition 4.12. *We denote by α_i the elements in $\Psi(\mathfrak{g})$ which are specified by the Dynkin diagrams in the examples in this section.*

For $\alpha\in\Psi(\mathfrak{g})$ define $\Lambda_\alpha\in\mathfrak{a}^*$ by

$$2\frac{\langle\Lambda_\alpha,\beta\rangle}{\langle\beta,\beta\rangle}=\begin{cases}1 & \text{if } \beta=\alpha,\\ 0 & \text{if } \beta\in\Psi(\mathfrak{g})\setminus\{\alpha\}.\end{cases} \tag{92}$$

Let π_α^* be the irreducible representation of $\mathfrak{g}$ with the lowest weight $-\Lambda_\alpha$ and let Λ_α^* be the highest weight of π_α^*.

i) *Suppose $\mathfrak{g}=\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{sp}_n$ or $\mathfrak{o}_{2n+1}$ and π is the natural representation of $\mathfrak{g}$. Then* (44) *holds for any Θ if the infinitesimal character of the Verma module $M(\lambda_\Theta)$ is regular, that is*

$$\langle\lambda_\Theta+\rho,\alpha\rangle\neq 0 \quad (\forall\alpha\in\Sigma(\mathfrak{g})). \tag{93}$$

If $\lambda_\Theta+\rho$ is dominant, then (44) *is equivalent to* (41). *Moreover in* Proposition 3.3 *we may put $A=\{i;\, d_i<\deg_x q_{\pi,\Theta}\}$.*

ii) *Suppose $\mathfrak{g}=G_2$ and π is the non-trivial minimal dimensional representation of $\mathfrak{g}$. Then the same statement as above holds.*

iii) *Suppose $\mathfrak{g}=\mathfrak{o}_{2n}$ with $n\geq 4$ and π is the natural representation of $\mathfrak{g}$.*

Suppose $\Theta\supset\{\alpha_{n-1},\alpha_n\}$. Then (44) *holds if $\lambda_\Theta+\rho$ is regular and* (44) *is equivalent to* (41) *if $\lambda_\Theta+\rho$ is dominant.*

Suppose $\Theta\cap\{\alpha_{n-1},\alpha_n\}=\emptyset$ and $\langle\lambda_\Theta,\alpha_n-\alpha_{n-1}\rangle=0$. In this case we may replace $q_{\pi,\Theta}(x;\lambda)$ in the definition of $I_{\pi,\Theta}$ by $q'_{\pi,\Theta}(x;\lambda)$ given in Example 4.6. *Then the same statement as the previous case holds. Note that $\deg_x q'_{\pi,\Theta}=\deg_x q_{\pi,\Theta}-1$.*

In other general cases, (44) *holds if the infinitesimal character of $M(\lambda_\Theta)$ is strongly regular, that is, $\lambda_\Theta+\rho$ is not fixed by any non-trivial element of the Weyl group of the non-connected Lie group $O(2n,\mathbb{C})$. In particular, if $\Theta\cap\{\alpha_{n-1},\alpha_n\}=\emptyset$, then* (44) *holds under the conditions* (93) *and*

$$\langle\lambda_\Theta+\rho, 2\alpha_i+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\rangle\neq 0 \quad \text{for } i=2,\ldots,n-1 \text{ satisfying } \alpha_{i-1}\in\Theta \text{ and } \alpha_i\notin\Theta. \tag{94}$$

Suppose $\Theta\cap\{\alpha_{n-1},\alpha_n\}=\{\alpha_{n-1}\}$. Then

$$J_\Theta(\lambda)=I_{\pi,\Theta}(\lambda)+I_{\pi^*_{\alpha_{n-1}},\Theta}(\lambda)+J(\lambda_\Theta) \tag{95}$$

if (93), (94) *and*

$$\langle \lambda_\Theta + \rho, \varpi + \Lambda_{\alpha_{n-1}} - \alpha_{n-1} \rangle \neq 0 \qquad \text{for any } \varpi \in \mathcal{W}_\Theta(\pi^*_{\alpha_{n-1}}) \text{ satisfying } \varpi > \alpha_{n-1} - \Lambda_{\alpha_{n-1}} \tag{96}$$

hold.

In Proposition 3.3 *we may put* $r = n$ *and* $\Delta_1, \ldots, \Delta_{n-1}$ *are invariant under the outer automorphism of* $\mathfrak{g}$ *corresponding to* $\varepsilon_n \mapsto -\varepsilon_n$ *and* $A = \{i; d_i < \deg_x q_{\pi,\Theta}\} \cup \{n\}$.

iv) *Suppose* $\mathfrak{g} = E_n$ *with* $n = 6$, 7 *or* 8 (*cf.* Example 4.7, 4.8, 4.9)*. For* $\alpha_i \in \Psi(\mathfrak{g})$ *put*

$$\iota(\alpha_i) = \begin{cases} \alpha_1 & \text{if } i = 1 \text{ or } 3, \\ \alpha_2 & \text{if } i = 2, \\ \alpha_n & \text{if } i \geq 4, \end{cases} \qquad \hat{\alpha}_i = \begin{cases} \alpha_i & \text{if } i = 1 \text{ or } 2, \\ \alpha_1 + \alpha_3 & \text{if } i = 3, \\ \alpha_i + \cdots + \alpha_n & \text{if } i \geq 4. \end{cases} \tag{97}$$

Here $\iota(\alpha_i)$ *satisfies* $\#\{\beta \in \Psi(\mathfrak{g}); \langle \iota(\alpha_i), \beta \rangle < 0\} \leq 1$ *and* $\hat{\alpha}$ *is the smallest root with* $\hat{\alpha} \geq \alpha$ *and* $\hat{\alpha} \geq \iota(\alpha)$*. Let* $\lambda \in \mathfrak{a}^*_\Theta$*. If* (93) *holds and moreover* λ *satisfies*

$$2\langle \lambda_\Theta + \rho, \varpi + \Lambda_{\iota(\alpha)} - \hat{\alpha} \rangle \neq \langle \varpi, \varpi \rangle - \langle \Lambda_{\iota(\alpha)}, \Lambda_{\iota(\alpha)} \rangle \qquad \text{for } \alpha \in \Theta \text{ and } \varpi \in \overline{\mathcal{W}}_\Theta(\pi^*_{\iota(\alpha)}) \text{ satisfying } \varpi > \hat{\alpha} - \Lambda_{\iota(\alpha)}, \tag{98}$$

then

$$J_\Theta(\lambda) = \sum_{\alpha \in \iota(\Theta)} I_{\pi^*_\alpha, \Theta}(\lambda) + J(\lambda_\Theta). \tag{99}$$

In particular, under the notation in Definition 2.20 *the condition*

$$2\frac{\langle \lambda_\Theta + \rho, \mu \rangle}{\langle \Lambda_{\iota(\alpha)}, \Lambda_{\iota(\alpha)} \rangle} \notin [-1, 0] \qquad \text{for } \alpha \in \Theta \text{ and } \mu \in R_+ \text{ with } 0 < \mu \leq \Lambda_{\iota(\alpha)} + \Lambda^*_{\iota(\alpha)} - \hat{\alpha} \tag{100}$$

assures (98)*. Moreover, if* $\pi = \pi^*_{\alpha_1}$ *or* $\pi^*_{\alpha_n}$*, we may put* $A = \{i;\, d_i < \deg_x q_{\pi,\Theta}\}$ *in* Proposition 3.3.

v) *Suppose* $\mathfrak{g} = F_4$*. For* $\alpha_i \in \Psi(\mathfrak{g})$ *put*

$$\iota(\alpha_i) = \begin{cases} \alpha_1 & \text{if } i \leq 2, \\ \alpha_4 & \text{if } i \geq 3, \end{cases} \qquad \hat{\alpha}_i = \begin{cases} \alpha_i & \text{if } i = 1 \text{ or } 4, \\ \alpha_1 + \alpha_2 & \text{if } i = 2, \\ \alpha_3 + \alpha_4 & \text{if } i = 3. \end{cases} \tag{101}$$

Then the same statement as iv) *holds for* $\pi = \pi^*_{\alpha_4}$ (*cf.* Example 4.10).

Proof. The statements i) and iii) are direct consequences of [22, Theorem 4.4] (or Theorem 3.21) and Theorem 3.12. The statement ii) is a consequence of Example 4.11.

Suppose $\mathfrak{g}$ is E_6, E_7, E_8, F_4 or G_2 and π is a minimal dimensional nontrivial irreducible representation of $\mathfrak{g}$. Then in Proposition 3.3 it follows from [17] that the elements $\sum_{\varpi \in \mathcal{W}(\pi)} m_\pi(\varpi)\varpi^{d_i}$ $(i = 1, \ldots, n)$ generate the algebra of the

W-invariants of $U(\mathfrak{a})$ (For G_2 we confirm it in Example 4.11) and hence we may put $A = \{i;\, d_i < \deg_x q_{\pi,\Theta}\}$.

Suppose $\mathfrak{g}$ is E_6, E_7, E_8 or F_4. Fix $\alpha \in \Theta$. Then Theorem 3.21 assures $X_{-\alpha} \in I_{\pi^*_{\iota(\alpha)},\Theta}(\lambda) + J(\lambda_\Theta)$ if $r_{\alpha,\varpi_\alpha}(\lambda) \neq 0$. Here $r_{\alpha,\varpi_\alpha}(\lambda)$ is defined by (80) with $\pi = \pi^*_{\iota(\alpha)}$ and $\varpi_\alpha = -\Lambda_{\iota(\alpha)} + (\hat{\alpha} - \alpha)$. Then the assumption of Remark 3.20 v) holds and therefore the second factor $\prod_{i=1}^{L}(\cdots)$ of $r_{\alpha,\varpi_\alpha}(\lambda)$ in (80) does not vanish under the condition (93). On the other hand, $\varpi \in \mathcal{W}(\pi^*_{\iota(\alpha)})$ which does not satisfy $\varpi \leq -\Lambda_{\iota(\alpha)} + \hat{\alpha}$ always satisfies $\varpi > -\Lambda_{\iota(\alpha)} + \hat{\alpha}$ because $\{\gamma_1, \ldots, \gamma_K\}$ in Remark 3.20 is of type A_K and $\langle \Lambda_{\iota(\alpha)}, \beta\rangle = \langle \gamma_i, \beta\rangle = 0$ for $i = 1, \ldots, K-1$ and $\beta \in \Psi(\mathfrak{g}) \setminus \{\gamma_1, \ldots, \gamma_K\}$. Hence (98) assures that the first factor of $r_{\alpha,\varpi_\alpha}(\lambda)$ does not vanish. Thus we have $X_{-\alpha} \in I_{\pi^*_{\iota(\alpha)},\Theta}(\lambda) + J(\lambda_\Theta)$. It implies (99). It is clear that (98) follows from (100) since $\langle \Lambda_{\iota(\alpha)}, \Lambda_{\iota(\alpha)}\rangle \geq \langle \varpi, \varpi\rangle$ for $\varpi \in \mathcal{W}(\pi^*_{\iota(\alpha)})$. ■

Remark 4.13. Suppose $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{g}$ is simple. In the preceding proposition we explicitly give a two sided ideal $I_\Theta(\lambda)$ of $U(\mathfrak{g})$ which satisfies $J_\Theta(\lambda) = I_\Theta(\lambda) + J(\lambda_\Theta)$ if at least

$$\operatorname{Re}\langle \lambda_\Theta + \rho, \alpha\rangle > 0 \quad \text{for } \alpha \in \Psi(\mathfrak{g}). \tag{102}$$

In particular, this condition is valid when $\lambda = 0$.

Remark 4.14. Suppose $\mathfrak{g} = \mathfrak{gl}_n$. Then in [20], the generator system of $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ is constructed for any Θ and λ through quantizations of elementary divisors. It shows that the zeros of the image of the Harish-Chandra homomorphism of $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ equals $\{w.\lambda_\Theta;\, w \in W(\Theta)\}$ and proves that (41) holds if and only if (103) is not valid for any positive numbers j and k which are smaller or equal to L. Here we note that this condition for (41) follows from this description of the zeros and Lemma 3.4 and the following Lemma with the notation in Example 4.2.

Lemma 4.15. *Let $n_0 = 0 < n_1 < n_2 < \cdots < n_L = n$ be a strictly increasing sequence of non-negative integers. Let $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathbb{C}^L$. Define $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{C}^n$ by*

$$\bar{\lambda}_\nu = \lambda_k + (\nu - 1) - \frac{n-1}{2} \text{ if } n_{k-1} < \nu \leq n_k$$

and put

$$\Lambda_k = \{\bar{\lambda}_{n_{k-1}+1}, \bar{\lambda}_{n_{k-1}+2}, \ldots, \bar{\lambda}_{n_k}\}.$$

Then there exists ν with $n_{j-1} < \nu < n_j$ satisfying $(\nu, \nu+1)\bar{\lambda} \in W(\Theta)\bar{\lambda}$ if and only if there exists $k \in \{1, \ldots, L\}$ such that

$$\Lambda_k \cap \Lambda_j \neq \emptyset,\ \Lambda_j \not\subset \Lambda_k \text{ and } \Bigl(\mu \in \Lambda_j \setminus \Lambda_k,\ \mu' \in \Lambda_k \Rightarrow (\mu' - \mu)(k - j) > 0\Bigr). \tag{103}$$

Here $(i, j) \in \mathfrak{S}_n$ is the transposition of i and j and

$$W(\Theta) = \{\sigma \in \mathfrak{S}_n;\, \sigma(i) < \sigma(j) \text{ if there exists } k \text{ with } n_{k-1} < i < j \leq n_k\},$$
$$\sigma\mu = (\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(n)}) \text{ for } \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n.$$

Proof. Suppose (103). Then there exists m such that

$$\begin{cases} j<k,\ 1\le m<n_j-n_{j-1} \text{ and } n_{k-1}+n_j-n_{j-1}-m\le n_k,\\ \bar\lambda_{n_{j-1}+\nu}=\bar\lambda_{n_{k-1}+\nu-m} \text{ for } m<\nu\le n_j-n_{j-1}\end{cases}$$

or

$$\begin{cases} j>k,\ 1\le m<n_j-n_{j-1} \text{ and } n_k-m+1>n_{k-1},\\ \bar\lambda_{n_{j-1}+\nu}=\bar\lambda_{n_k+\nu-m} \text{ for } 1\le\nu\le m.\end{cases}$$

Defining $\sigma\in W(\Theta)$ by

$$\sigma=(n_{j-1}+m,n_{j-1}+m+1)\prod_{m<\nu\le n_j-n_{j-1}}(n_{j-1}+\nu,n_{k-1}+\nu-m),$$

or

$$\sigma=(n_{j-1}+m,n_{j-1}+m+1)\prod_{1\le\nu\le m}(n_{j-1}+\nu,n_k+\nu-m),$$

respectively, we have $(\nu,\nu+1)\bar\lambda=\sigma\bar\lambda\in W(\Theta)\bar\lambda$ with $\nu=n_{j-1}+m$.

Conversely suppose $(\nu,\nu+1)\bar\lambda=\sigma\bar\lambda$ for suitable $\nu\in\{n_{j-1}+1,\dots,n_j-1\}$ and $\sigma\in W(\Theta)$. Put

$$\{\ell_1,\dots,\ell_m\}=\{\ell;\ \ell\le n_{j-1} \text{ and } \bar\lambda_\ell=\bar\lambda_{n_{j-1}+1}\},$$
$$\{\ell'_{m+2},\dots,\ell'_{m+m'+1}\}=\{\ell';\ \ell'>n_j \text{ and } \bar\lambda_{\ell'}=\bar\lambda_{n_j}\}$$

and define

$$\begin{cases}\ell'_i=\ell_i+(n_j-n_{j-1}-1) & \text{if } i\le m,\\ \ell_i=\ell'_i-(n_j-n_{j-1}-1) & \text{if } i\ge m+2,\\ \ell_{m+1}=n_{j-1}+1,\ \ell'_{m+1}=n_j.\end{cases}$$

Assume that (103) is not valid for any k. Then for $i\in I:=\{1,\dots,m+m'+1\}$, there exist integers N_i with $n_{N_{i-1}}<\ell_i<\ell'_i\le n_{N_i}$ and therefore $\bar\lambda_{\ell_i}=\bar\lambda_{n_{j-1}+1}$ and $\bar\lambda_{\ell'_i}=\bar\lambda_{n_j}$.

Note that $\#I_1\le m+1$ and $\#I_2\le m'$ by denoting

$$I_1=\{i\in I;\ \sigma(\ell_i)\le n_j\} \text{ and } I_2=\{i\in I;\ \sigma(\ell'_i)>n_j\}.$$

Since $\sigma(\ell_i)<\sigma(\ell'_i)$, we have $I_1\cup I_2=I$ and therefore $\#I_1=m+1$ and $\#I_2=m'$. Then there exists i_0 with $n_{j-1}<\sigma(\ell_{i_0})\le n_j$. Since $I_1\cap I_2=\emptyset$, we have $\sigma(\ell'_{i_0})\le n_j$, which implies $\sigma^{-1}(\nu')=\ell_{i_0}+\nu'-n_{j-1}-1$ for $n_{j-1}<\nu'\le n_j$. It contradicts to the assumption $(\nu,\nu+1)\bar\lambda=\sigma\bar\lambda$. ∎

Remark 4.16. Suppose $\mathfrak{g}=\mathfrak{gl}_n$ and π is its natural representation. Then the condition $r_\alpha(\lambda)\ne0$ for any $\alpha\in\Theta$ is necessary and sufficient for (44) (cf. [22, Remark 4.5]). Under the notation in the preceding lemma, it is easy to see that the condition is equivalent to the fact that

$$\Lambda_k\cap\Lambda_j\ne\emptyset,\ \Lambda_j\not\subset\Lambda_k \text{ and } \Big(\exists\mu\in\Lambda_j\setminus\Lambda_k,\ \exists\mu'\in\Lambda_k \text{ such that } (\mu'-\mu)(k-j)>0\Big)$$

does not hold for any positive numbers k and j smaller or equal to L.

A Infinitesimal Mackey's Tensor Product Theorem

In this appendix we explain *infinitesimal Mackey's tensor product theorem* following the method given in [16].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{p}$ a subalgebra of $\mathfrak{g}$. Let V and U be a $U(\mathfrak{g})$-module and a $U(\mathfrak{p})$-module, respectively. We denote by $V|_{\mathfrak{p}}$ and $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} U$ the restriction of the coefficient ring $U(\mathfrak{g})$ to $U(\mathfrak{p})$ and the induced representation $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U$ in the usual way.

Theorem A1 (infinitesimal Mackey's tensor product theorem)**.** *The map defined by*

$$\begin{aligned} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (U \otimes_{\mathbb{C}} V|_{\mathfrak{p}}) &\to (U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U) \otimes_{\mathbb{C}} V, \\ D \otimes_{U(\mathfrak{p})} (u \otimes_{\mathbb{C}} v) &\mapsto D \cdot [(1 \otimes_{U(\mathfrak{p})} u) \otimes_{\mathbb{C}} v] \end{aligned} \tag{104}$$

gives a canonical $U(\mathfrak{g})$-module isomorphism

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(U \otimes_{\mathbb{C}} V|_{\mathfrak{p}}) \simeq \left(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} U\right) \otimes_{\mathbb{C}} V. \tag{105}$$

To prove this we need two lemmas.

Lemma A2. *Let R be a ring and R-Mod the category of left R-modules. For $M, N \in R$-Mod consider $F_M : \cdot \mapsto \mathrm{Hom}_R(M, \cdot)$ and $F_N : \cdot \mapsto \mathrm{Hom}_R(N, \cdot)$, which are functors from R-Mod to the category of abelian groups. Suppose that F_M and F_N are naturally equivalent, namely, there exists an assignment $A \mapsto \tau_A$ for each object $A \in R$-Mod of an isomorphism $\tau_A : \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A)$ such that $F_N(f) \circ \tau_A = \tau_B \circ F_M(f)$ for each $f \in \mathrm{Hom}_R(A, B)$. Then $M \simeq N$ as R-modules.*

Proof. Put $\varphi = \tau_N^{-1}(\mathrm{id}_N) \in \mathrm{Hom}_R(M, N)$ and $\psi = \tau_M(\mathrm{id}_M) \in \mathrm{Hom}_R(N, M)$. Then $\varphi \circ \psi = F_N(\varphi)(\psi) = F_N(\varphi) \circ \tau_M(\mathrm{id}_M) = \tau_N \circ F_M(\varphi)(\mathrm{id}_M) = \tau_N(\varphi) = \mathrm{id}_N$. Similarly $\psi \circ \varphi = \mathrm{id}_M$. Hence $M \simeq N$. ■

Lemma A3. *Let (π_i, V_i) $(i = 1, 2, 3)$ be $U(\mathfrak{g})$-modules. Consider $\mathrm{Hom}_{\mathbb{C}}(V_2, V_3)$ as a $U(\mathfrak{g})$-module by $X\Phi = \pi_3(X) \circ \Phi - \Phi \circ \pi_2(X)$ for $\Phi \in \mathrm{Hom}_{\mathbb{C}}(V_2, V_3)$ and $X \in \mathfrak{g}$. Then naturally*

$$\mathrm{Hom}_{U(\mathfrak{g})}(V_1 \otimes_{\mathbb{C}} V_2, V_3) \simeq \mathrm{Hom}_{U(\mathfrak{g})}(V_1, \mathrm{Hom}_{\mathbb{C}}(V_2, V_3)).$$

Proof. We have only to define the mapping $\varphi \mapsto \Phi$ from the left-hand side to the right-hand side by $(\Phi(v_1))(v_2) = \varphi(v_1 \otimes v_2)$ for $v_1 \in V_1$ and $v_2 \in V_2$. ■

Proof of Theorem A1. Lemma A3 implies the following isomorphism for a given $U(\mathfrak{g})$-module A:

$$\begin{aligned} \mathrm{Hom}_{U(\mathfrak{g})}\left((U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U) \otimes_{\mathbb{C}} V, A\right) &\simeq \mathrm{Hom}_{U(\mathfrak{g})}\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U, \mathrm{Hom}_{\mathbb{C}}(V, A)\right) \\ &\simeq \mathrm{Hom}_{U(\mathfrak{p})}\left(U, \mathrm{Hom}_{\mathbb{C}}(V|_{\mathfrak{p}}, A|_{\mathfrak{p}})\right) \\ &\simeq \mathrm{Hom}_{U(\mathfrak{p})}\left(U \otimes_{\mathbb{C}} V|_{\mathfrak{p}}, A|_{\mathfrak{p}}\right) \\ &\simeq \mathrm{Hom}_{U(\mathfrak{g})}\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (U \otimes_{\mathbb{C}} V|_{\mathfrak{p}}), A\right). \end{aligned}$$

It gives a natural equivalence between $F_{(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}U)\otimes_{\mathbb{C}}V}$ and $F_{U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(U\otimes_{\mathbb{C}}V|_{\mathfrak{p}})}$ under the notation of Lemma A2 with $R = U(\mathfrak{g})$. Hence by Lemma A2, we have (105). It is easy to see the isomorphism is explicitly given by (104). ■

B Undesirable Cases

In this appendix we give counter examples stated in Remark 3.23. Let $\mathfrak{g} = \mathfrak{sl}_n$ and use the notation in §2. and §3.. Suppose the Dynkin diagram of the fundamental system $\Psi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is the same as in Example 4.2. Let $\{\Lambda_1, \ldots, \Lambda_{n-1}\}$ be the system of fundamental weights corresponding to Ψ. Let π be the irreducible representation of $\mathfrak{g}$ with lowest weight $\bar{\pi} = -m_1\Lambda_1 - m_2\Lambda_2$. Here m_1 and m_2 are positive integers. Then the multiplicity of the weight $\varpi' := \bar{\pi} + \alpha_1 + \alpha_2 \in \mathcal{W}(\pi)$ equals 2.

Now take $\Theta = \Psi \setminus \{\alpha_2\} = \{\alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_{n-1}\}$. Since the multiplicity of the weight $\bar{\pi} + \alpha_2$ is 1, both ϖ' and $\bar{\pi} + \alpha_2$ belong to $\overline{\mathcal{W}}_\Theta(\pi)$. On the other hand, by Remark 3.17, the weight $\varpi_{\alpha_{n-1}} := \bar{\pi} + \alpha_2 + \alpha_3 + \cdots + \alpha_{n-2}$ is a unique extremal low weight of π with respect to α_{n-1}. Note that $\{\varpi \in \overline{\mathcal{W}}_\Theta(\pi);\, \varpi \leq \varpi_{\alpha_{n-1}}\} = \{\bar{\pi}, \bar{\pi} + \alpha_2\}$ and the weight $\varpi'_{\alpha_{n-1}} := \bar{\pi} + \alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}$ satisfies $\varpi'_{\alpha_{n-1}}|_{\mathfrak{a}_\Theta} = \varpi'|_{\mathfrak{a}_\Theta} = (\bar{\pi} + \alpha_2)|_{\mathfrak{a}_\Theta} \neq \bar{\pi}|_{\mathfrak{a}_\Theta}$. Moreover, it follows from Lemma 2.22

$$D_\pi(\varpi') - D_\pi(\bar{\pi} + \alpha_2) = -\langle \bar{\pi} + \alpha_2, \alpha_1 \rangle = \frac{m_1 + 1}{2} \langle \alpha_1, \alpha_1 \rangle,$$

$$D_\pi(\varpi'_{\alpha_{n-1}}) - D_\pi(\bar{\pi} + \alpha_2) = -\langle \alpha_2, \alpha_3 \rangle - \cdots - \langle \alpha_{n-2}, \alpha_{n-1} \rangle = \frac{n-3}{2} \langle \alpha_1, \alpha_1 \rangle.$$

It shows the first factor of the function (80) with $(\alpha, \varpi_\alpha) = (\alpha_{n-1}, \varpi_{\alpha_{n-1}})$ is identically zero if $n = m_1 + 4$.

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