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Symplectic Submanifolds and Symplectic Ideals

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Abstract. Let N be a Poisson manifold with global coordinate functions. We define a symplectic ideal of the ring $C^\infty(N)$ of smooth functions and investigate a relationship between symplectic submanifolds of N and symplectic ideals of $C^\infty(N)$.
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Introduction

Poisson structures have recently played an important role in many branches of mathematics and have been studied by many mathematicians, for instance, in [1], [2], [5], [6], [7], [8] and [9]. It is well known that every Poisson manifold can be written as a disjoint union of maximal connected symplectic submanifolds, called symplectic leaves (see [9]), and Hodges-Levasseur-Toro and Vancliff proved in [5] and [8] that symplectic leaves of certain Poisson varieties correspond bijectively to primitive ideals of quantized algebras of the coordinate ring of a given Poisson variety if the Poisson structure is algebraic. But every symplectic leaf of a Poisson variety is not determined by a Poisson ideal (see Remark 2.3) and thus symplectic leaf is a concept of a Poisson manifold but not a Poisson variety. The purpose of this note is to give a definition for symplectic ideal in the ring of smooth functions on a Poisson manifold and see how closely the concept 'symplectic ideal' is related to the geometric concept 'symplectic submanifold'(see Definition 1.2 and Theorem 1.4), and find symplectic submanifolds for several Poisson manifolds using the main result Theorem 1.4.

Assume throughout that all spaces will be over the complex number field $\mathbb{C}$. Recall that a finite dimensional smooth manifold N is said to be a Poisson manifold if the ring $C^\infty(N)$ of its smooth functions is a Poisson algebra, that is, there exists a bilinear map $\{\cdot,\cdot\} : C^\infty(N) \times C^\infty(N) \longrightarrow C^\infty(N)$ such that, for any $f, g, h \in C^\infty(N)$,

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- $\{f,g\} = -\{g,f\}$ (Lie identity)
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)
- $\{fg,h\} = f\{g,h\} + \{f,h\}g$ (Leibniz rule)

Assume throughout the paper that N denotes a finite dimensional Poisson manifold with global coordinate functions $x_1, x_2, \cdots, x_n$.

1. Symplectic submanifolds

For any $f \in C^\infty(N)$, the linear map

$$h_f : C^\infty(N) \longrightarrow C^\infty(N), \quad h_f(g) = \{f,g\}$$

is a derivation which is called the Hamiltonian vector field induced by f. For each $p \in N$, there exists a natural bilinear form $\langle \cdot, \cdot \rangle_p$ on the cotangent space $T_p^*(N)$ defined by

$$\langle df, dg \rangle_p = \{f,g\}(p)$$

for all $f, g \in C^\infty(N)$. This is equivalent to the property that, for each $p \in N$, there exists a natural linear map

$$B_p : T_p^*(N) \longrightarrow T_p(N), \; B_p(df)(g) = \{f,g\}(p)$$

for all $f, g \in C^\infty(N)$. If the bilinear form $\langle \cdot, \cdot \rangle_p$ is non-degenerate for each $p \in N$ then N is said to be symplectic.

Define an equivalence relation $\sim$ on N as follows: For $p, q \in N$, $p \sim q$ if and only if there exists a piecewise smooth curve in N joining p to q, each smooth segment of which is part of an integral curve determined by Hamiltonian vector field. Then, by [9, Proposition 1.3], the equivalence classes are (immersed) symplectic submanifolds of N, called symplectic leaves. Moreover, if M is a symplectic leaf of N then

$$\operatorname{rank}(B_p) = \dim(M) \tag{1}$$

for each $p \in M$.

Note that $T_p^*(N)$ is canonically isomorphic to $\mathfrak{m}_p/\mathfrak{m}_p^2$ as vector spaces, where

$$\mathfrak{m}_p = \{f \in C^\infty(N) \mid f(p) = 0\}.$$

An ideal I of $C^\infty(N)$ is said to be a Poisson ideal if $\{I, C^\infty(N)\} \subseteq I$. For an ideal I of $C^\infty(N)$, denote by $(I : \mathcal{H}(C^\infty(N)))$ the largest Poisson ideal of $C^\infty(N)$ contained in I. That is, $(I : \mathcal{H}(C^\infty(N)))$ is the sum of all Poisson ideals contained in I.

Lemma 1.1. *For any $f \in (\mathfrak{m}_p : \mathcal{H}(C^\infty(N)))$, $df \in \ker B_p$.*

Proof. Since $(\mathfrak{m}_p : \mathcal{H}(C^\infty(N)))$ is a Poisson ideal contained in $\mathfrak{m}_p$, we have immediately that $\{f,g\} \in \mathfrak{m}_p$ for any $f \in (\mathfrak{m}_p : \mathcal{H}(C^\infty(N)))$ and $g \in C^\infty(N)$. It completes the proof. ■

The above lemma motivates the following definition.

Definition 1.2. A Poisson ideal P of $C^\infty(N)$ is said to be a symplectic ideal if there exists an element $p \in N$ such that $P = (\mathfrak{m}_p : \mathcal{H}(C^\infty(N)))$. That is, P is the largest Poisson ideal contained in a maximal ideal of the form $\mathfrak{m}_p$.

Setting $c_{ij} = \{x_i, x_j\}$, $i, j = 1, 2, \cdots, n$, the Lie identity and the Jacobi identity induce the following conditions:

$$c_{ij} = -c_{ji} \tag{2}$$

$$\sum_i c_{ir}\frac{\partial c_{st}}{\partial x_i} + c_{is}\frac{\partial c_{tr}}{\partial x_i} + c_{it}\frac{\partial c_{rs}}{\partial x_i} = 0 \tag{3}$$

since $\{f, x_r\} = \sum_i \frac{\partial f}{\partial x_i} c_{ir}$ for all $f \in C^\infty(N)$. Let I_d be the ideal of $C^\infty(N)$ generated by all $d \times d$-minors of the $n \times n$-matrix $[c_{ij}]$. Note that

$$C^\infty(N) \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

and that I_1 is a Poisson ideal. In fact, we have the following.

Lemma 1.3. *For each d, I_d is a Poisson ideal of $C^\infty(N)$.*

Proof. Let

$$C = \begin{bmatrix} c_{s_1t_1} & c_{s_1t_2} & \cdots & c_{s_1t_d} \\ c_{s_2t_1} & c_{s_2t_2} & \cdots & c_{s_2t_d} \\ \cdot & \cdot & \cdots & \cdot \\ c_{s_dt_1} & c_{s_dt_2} & \cdots & c_{s_dt_d} \end{bmatrix}$$

be a $d \times d$-submatrix of $[c_{ij}]$. Using (2) and (3), we have that, for each k,

$$\begin{aligned}
\{|C|, x_k\} &= \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)\{c_{s_1t_{\sigma(1)}}c_{s_2t_{\sigma(2)}}\cdots c_{s_dt_{\sigma(d)}}, x_k\} \\
&= \sum_{i=1}^n \sum_{\sigma\in S_d}\sum_{\ell=1}^d \operatorname{sgn}(\sigma)(c_{s_1t_{\sigma(1)}}\cdots \hat{c}_{s_\ell t_{\sigma(\ell)}}\cdots c_{s_dt_{\sigma(d)}})\frac{\partial c_{s_\ell t_{\sigma(\ell)}}}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{\sigma\in S_d}\sum_{\ell=1}^d \operatorname{sgn}(\sigma)(c_{s_1t_{\sigma(1)}}\cdots c_{s_\ell i}\cdots c_{s_dt_{\sigma(d)}})\frac{\partial c_{t_{\sigma(\ell)}k}}{\partial x_i} \\
&\qquad\qquad - \operatorname{sgn}(\sigma)(c_{s_1t_{\sigma(1)}}\cdots c_{it_{\sigma(\ell)}}\cdots c_{s_dt_{\sigma(d)}})\frac{\partial c_{ks_\ell}}{\partial x_i} \\
&= \sum_{i=1}^n\sum_{\ell=1}^d |C^{(\ell,i)}|\frac{\partial c_{t_\ell k}}{\partial x_i} - |C_{(\ell,i)}|\frac{\partial c_{ks_\ell}}{\partial x_i} \in I_d,
\end{aligned}$$

where $\hat{c}_{s_\ell t_{\sigma(\ell)}} = c_{ik}$ and $C_{(\ell,i)}$ (resp., $C^{(\ell,i)}$) is the $d\times d$-matrix obtained from C by replacing ℓ-row (resp., ℓ-column) by the row vector $(c_{it_1}, \cdots, c_{it_d})$ (resp., the column vector $(c_{s_1i}, \cdots, c_{s_di})^T$). It follows that I_d is a Poisson ideal of $C^\infty(N)$ since

$$\{|C|, f\} = \sum_{k=1}^n \{|C|, x_k\}\frac{\partial f}{\partial x_k}$$

for any $f \in C^\infty(N)$. ∎

For any symplectic ideal P of $C^\infty(N)$, set

$$\mathcal{S}(P) = \mathcal{V}(P) - \bigcup_{\substack{P \subset Q \\ \text{symplectic}}} \mathcal{V}(Q),$$

where

$$\mathcal{V}(I) = \{p \in N \mid f(p) = 0 \text{ for all } f \in I\}$$

for a subset $I \subseteq C^\infty(N)$.

Theorem 1.4. *For every symplectic ideal P of $C^\infty(N)$, $\mathcal{S}(P)$ is a symplectic submanifold of N and*

$$N = \bigsqcup_{P \text{ symplectic}} \mathcal{S}(P).$$

Proof. For any $p, q \in N$, define a relation

$$p \sim q \Leftrightarrow (\mathfrak{m}_p : \mathcal{H}(C^\infty(N))) = (\mathfrak{m}_q : \mathcal{H}(C^\infty(N))).$$

Then the relation $\sim$ is an equivalence relation and thus N is a disjoint union of the equivalence classes $\mathcal{S}(P)$, P symplectic ideals.

We begin with adapting the proof of [1, 3.5 Lemma]. For any element $p \in N$, the Poisson ideal $P = (\mathfrak{m}_p : \mathcal{H}(C^\infty(N))$ is symplectic, $p \in \mathcal{V}(P)$ and, for any element $f \in C^\infty(N)$, the canonical image of f in $C^\infty(N)/P$, denoted by $\overline{f}$, may be considered as a smooth function on $\mathcal{V}(P)$. Let $B(\epsilon)$ be the ϵ-ball and denote by

$$\phi : B(\epsilon) \longrightarrow N, \quad \overline{\phi} : B(\epsilon) \longrightarrow \mathcal{V}(P)$$

the integral curves to the respective Hamiltonian vector fields h_f and $h_{\overline{f}}$ such that $\phi(0) = p = \overline{\phi}(0)$. For any $g \in C^\infty(N)$, we have that

$$\frac{d}{dt}(g \circ \phi) = \{f, g\} \circ \phi \tag{4}$$

$$\frac{d}{dt}(\overline{g} \circ \overline{\phi}) = \{\overline{f}, \overline{g}\} \circ \overline{\phi} \tag{5}$$

by the definition of integral curve. The left hand side of (5) is $\frac{d}{dt}(g \circ \overline{\phi})$ and the right hand side of (5) is $\{f, g\} \circ \overline{\phi}$ since P is a Poisson ideal. Hence, comparing this with (4), we deduce from the uniqueness of integral curves in a neighborhood of 0 that $\phi = \overline{\phi}$. Since symplectic leaves are determined by integral curves to Hamiltonians, we conclude that the symplectic leaf containing p is contained in $\mathcal{V}(P)$.

Let M be a symplectic leaf of N with dimension m. Then the canonical map

$$j_M : C^\infty(N) \longrightarrow C^\infty(M), \quad j_M(f) = f|_M$$

is a Poisson homomorphism by [2, 1.1 B] and

$$\dim(\mathfrak{m}_q / \ker(j_M) + \mathfrak{m}_q^2) = \dim(M) = m \tag{6}$$

for every $q \in M$. Fix an element $p \in M$ and let

$$P = (\mathfrak{m}_p : \mathcal{H}(C^\infty(N)), \quad Q = (\mathfrak{m}_q : \mathcal{H}(C^\infty(N))),$$

where $q \in M$. Then $M \subseteq \mathcal{V}(P) \cap \mathcal{V}(Q)$ by the above paragraph, and thus $P = Q$ since $Q \subseteq \mathfrak{m}_p$ and $P \subseteq \mathfrak{m}_q$. It follows that $p \sim q$ and $M \subseteq \mathcal{S}(P)$. Moreover, we have that

$$\ker(j_M) \subseteq P = (\mathfrak{m}_p : \mathcal{H}(C^\infty(N)) \subseteq \mathfrak{m}_p$$

and thus

$$m = \mathrm{rank}(B_p) \leq \dim(\mathfrak{m}_p/P + \mathfrak{m}_p^2) \leq \dim(\mathfrak{m}_p/\ker(j_M) + \mathfrak{m}_p^2) = m$$

by (1), (6) and Lemma 1.1, where

$$B_p : T_p^*(N) = \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow T_p(N) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

is the canonical linear map. It follows that

$$\mathrm{rank}(B_p) = \dim(\mathfrak{m}_p/P + \mathfrak{m}_p^2) = \dim(\mathfrak{m}_p/\ker(j_M) + \mathfrak{m}_p^2). \tag{7}$$

Suppose that there exists $r \in \mathcal{S}(P)$ such that $\mathrm{rank}(B_r) \neq \mathrm{rank}(B_p)$, say $d-1 = \mathrm{rank}(B_r) < \mathrm{rank}(B_p)$. Let Z be the symplectic leaf containing r. Then $Z \subseteq \mathcal{S}(P)$ by the above paragraph and, by (7),

$$\dim(\mathfrak{m}_p/P + \mathfrak{m}_p^2) = \mathrm{rank}(B_p) \neq \mathrm{rank}(B_r) = \dim(\mathfrak{m}_r/P + \mathfrak{m}_r^2).$$

The ideal I_d of $C^\infty(N)$ generated by all $d \times d$-minors of the matrix $[c_{ij}]$ is a Poisson ideal contained in P by Lemma 1.3 since $r \in \mathcal{V}(I_d)$, and thus $p \in \mathcal{V}(I_d)$. It follows that $\mathrm{rank}(B_p) \leq d-1$, a contradiction. Hence, for each $q \in \mathcal{S}(P)$, $\mathrm{rank}(B_q)$ is constant m. For each point $q \in \mathcal{S}(P)$, we may find a local coordinate $y_1, \cdots, y_m, z_1, \cdots, z_m, u_1, \cdots, u_k$ of an open neighborhood U of q such that $\dim(N) = 2m + k$, $\{y_i, z_j\} = \delta_{ij} = -\{z_j, y_i\}$ and the brackets of the rest pairs are zero at the point q. Identify U and an open subset of $\mathbb{C}^n$ such that q is the origin in the identification. Note that

$$h_{y_i}(q) = \frac{\partial}{\partial z_i}|_q, \quad h_{z_i}(q) = -\frac{\partial}{\partial y_i}|_q, \quad h_{u_i}(q) = 0.$$

Denote by φ_i and φ_{m+i} the local flows of the respective Hamiltonian vector fields h_{y_i} and h_{z_i} around the origin $0 = p$. Then since

$$\frac{d}{dt}(\sum \varphi_i(a_i t) + \varphi_{m+i}(b_i t)) = \sum a_i h_{y_i} + \sum b_i h_{z_i} = h_v,$$

where $v = \sum a_i y_i + b_i z_i$, the map $\varphi_v(t) = \sum \varphi_i(a_i t) + \varphi_{m+i}(b_i t)$ is the local flow of the Hamiltonian vector field h_v. This shows that $\mathcal{S}(P)$ is an immersed submanifold with dimension $2m$ which is symplectic and the proof is complete. ∎

Remark 1.5. Let N be a Poisson variety. Then N is a smooth Poisson manifold and the coordinate ring $\mathbb{C}[N]$ of the Poisson variety N is contained in the ring $C^\infty(N)$ of all smooth functions. By Theorem 1.4 and Remark 2.3 below, we have that the symplectic submanifolds of N are determined by symplectic ideals of $C^\infty(N)$ but not of $\mathbb{C}[N]$. That is, the coordinate ring $\mathbb{C}[N]$ does not have sufficiently many elements enough to determine symplectic submanifolds.

2. Examples

Here we find symplectic submanifolds for several Poisson manifolds using Theorem 1.4.

Example 2.1. The 2-space $\mathbb{C}^2$ is a Poisson manifold such that $C^\infty(\mathbb{C}^2)$ has the Poisson bracket $\{x, y\} = xy$, where x, y are the coordinate functions. Since $yC^\infty(\mathbb{C}^2)$ is a Poisson ideal and $C^\infty(\mathbb{C}^2)/yC^\infty(\mathbb{C}^2)$ has the trivial Poisson bracket, $\mathfrak{m}_{(\alpha,0)}$ is a symplectic ideal for each $\alpha \in \mathbb{C}$. Similarly $\mathfrak{m}_{(0,\alpha)}$ is also a symplectic ideal for each $\alpha \in \mathbb{C}$.

Let $p = (\alpha, \beta) \in (\mathbb{C}^\times)^2$. Then $B_p : T_p^*(\mathbb{C}^2) \longrightarrow T_p(\mathbb{C}^2)$ is an isomorphism because the matrix

$$\left[\begin{array}{cc} \{x-\alpha, x-\alpha\} & \{x-\alpha, y-\beta\} \\ -\{x-\alpha, y-\beta\} & \{y-\beta, y-\beta\} \end{array}\right]_{|p} = \left[\begin{array}{cc} 0 & \alpha\beta \\ -\alpha\beta & 0 \end{array}\right]$$

is nonsingular. Hence $(\mathbb{C}^\times)^2$ is a symplectic submanifold. It follows that the maximal symplectic submanifolds are as follows:

0-dim. symp. : $\{(\gamma, 0)\}, \{(0, \gamma)\} \quad (\gamma \in \mathbb{C})$
2-dim. symp. : $\{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{C}^\times\}$

Example 2.2. The manifold $\mathbb{C}^3$ is a Poisson manifold such that $C^\infty(\mathbb{C}^3)$ has the Poisson bracket

$$\{x, y\} = 0,\ \{x, z\} = \alpha x,\ \{y, z\} = y \tag{8}$$

for some nonzero real number α, where x, y, z are the coordinate functions. Since the ideal $xC^\infty(\mathbb{C}^3) + yC^\infty(\mathbb{C}^3)$ is a Poisson ideal and the Poisson algebra $C^\infty(\mathbb{C}^3)/(xC^\infty(\mathbb{C}^3) + yC^\infty(\mathbb{C}^3))$ has the trivial Poisson bracket, we have immediately that $xC^\infty(\mathbb{C}^3) + yC^\infty(\mathbb{C}^3) + (z - c)C^\infty(\mathbb{C}^3)$ is a symplectic ideal for each $c \in \mathbb{C}$.

Since $xC^\infty(\mathbb{C}^3)$ and $yC^\infty(\mathbb{C}^3)$ are Poisson ideals, $X = 0 \times \mathbb{C} \times \mathbb{C}$ and $Y = \mathbb{C} \times 0 \times \mathbb{C}$ are Poisson submanifolds with the respective coordinate rings $C^\infty(\mathbb{C}^3)/xC^\infty(\mathbb{C}^3)$ and $C^\infty(\mathbb{C}^3)/yC^\infty(\mathbb{C}^3)$. For any $p = (0, \alpha, \beta) \in X$, $\alpha \neq 0$, the natural linear map $B_p : T_p^*(X) \longrightarrow T_p(X)$ is an isomorphism, and thus X is a symplectic submanifold. Similarly, Y is also a symplectic submanifold. Note that $(x - dy^\alpha)C^\infty(\mathbb{C}^3)$ is a Poisson ideal for each $d \in \mathbb{C}^\times$. Setting $Z_d = \{(dc^\alpha, c, e) \mid c \in \mathbb{C}^\times, e \in \mathbb{C}\}$ for each $d \in \mathbb{C}^\times$, Z_d is a symplectic submanifold since the natural linear map $B_p : T_p^*(Z_d) \longrightarrow T_p(Z_d)$ is an isomorphism. Therefore the maximal symplectic submanifolds are as follows:

$$\begin{array}{lll} \text{0-dim. symp.}: & \{(0,0,c)\} & (c \in \mathbb{C}) \\ \text{2-dim. symp.}: & \{(0,c,d) \mid c \in \mathbb{C}^\times, d \in \mathbb{C}\} & \\ & \{(c,0,d) \mid c \in \mathbb{C}^\times, d \in \mathbb{C}\} & \\ & \{(dc^\alpha, c, e) \mid c \in \mathbb{C}^\times, e \in \mathbb{C}\} & (d \in \mathbb{C}^\times) \end{array}$$

Remark 2.3. Suppose that α given in (8) is irrational. Then the smooth function $x - dy^\alpha \in C^\infty(\mathbb{C}^3)$ defining the symplectic submanifold Z_d is not an element of the polynomial ring $\mathbb{C}[x, y, z]$, hence $\mathbb{C}^3$ with Poisson bracket (8) is not algebraic. That is, all symplectic leaves of the Poisson variety $\mathbb{C}^3$ with Poisson bracket (8) are not induced by Poisson ideals of its coordinate ring $\mathbb{C}[x, y, z]$.

Example 2.4. Let A be the ring of smooth functions of the smooth manifold $M_2(\mathbb{C})$ of 2×2-matrices and let a, b, c, d be the coordinate functions. It is well-known that $M_2(\mathbb{C})$ is a Poisson manifold with Poisson bracket

$$\begin{array}{lll} \{a,b\} = 2ab & \{a,c\} = 2ac & \{b,c\} = 0 \\ \{b,d\} = 2bd & \{c,d\} = 2cd & \{a,d\} = 4bc \end{array}$$

Note that $(ad - bc - \mu)A$, $(b - \lambda c)A$, bA and cA are Poisson ideals of A for all $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}^\times$. Since $A/(bA + cA)$ is a Poisson algebra with trivial Poisson bracket, every symplectic ideal containing $bA + cA$ is of the form

$$(a - \alpha)A + bA + cA + (d - \beta)A \quad (\alpha, \beta \in \mathbb{C}).$$

Now A/cA is a Poisson algebra with Poisson bracket

$$\{\overline{a}, \overline{b}\} = 2\overline{a}\overline{b},\ \{\overline{a}, \overline{d}\} = 0,\ \{\overline{b}, \overline{d}\} = 2\overline{bd}$$

and its symplectic ideals contain $\overline{a}\overline{d} - \mu$, $\mu \in \mathbb{C}$. Similarly, A/bA is a Poisson algebra with Poisson bracket

$$\{\overline{a}, \overline{c}\} = 2\overline{ac},\ \{\overline{a}, \overline{d}\} = 0,\ \{\overline{c}, \overline{d}\} = 2\overline{c}\overline{d}$$

and its symplectic ideals contain $\overline{a}\overline{d} - \mu$, $\mu \in \mathbb{C}$.

Note that $(ad - bc - \mu)A + (b - \lambda c)A$, $\lambda \in \mathbb{C}^\times$, is a Poisson ideal of A and $X = \mathcal{V}((ad - bc - \mu)A + (b - \lambda c)A)$ is a Poisson submanifold. Let $p = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in X$ such that $\beta\gamma \neq 0$. Now the bilinear map B_p in X is an isomorphism since the induced matrix

$$\begin{bmatrix} \{\overline{a}, \overline{a}\} & \{\overline{a}, \overline{d}\} \\ \{\overline{d}, \overline{a}\} & \{\overline{d}, \overline{d}\} \end{bmatrix}_{|p} = \begin{bmatrix} 0 & 2\beta\gamma \\ -2\beta\gamma & 0 \end{bmatrix}$$

is nonsingular. Therefore the maximal symplectic submanifolds of $M_2(\mathbb{C})$ are as follows.

0-dim. symp. :

$$\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\},\ \{\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\},\ \{\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\} \qquad \alpha, \beta \in \mathbb{C}$$

2-dim. symp. :

$$\begin{array}{ll}
\{\begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\},\ \{\begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\} & \\
\{\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\},\ \{\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\} & \\
\{\begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1}\lambda \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\} & \lambda \in \mathbb{C}^\times \\
\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1}\lambda \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\} & \lambda \in \mathbb{C}^\times \\
\{\begin{pmatrix} \alpha & \lambda\gamma \\ \gamma & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{C}^\times, \alpha\beta - \lambda\gamma^2 = \mu\} & \mu \in \mathbb{C},\ \lambda \in \mathbb{C}^\times
\end{array}$$

Example 2.5. In the ring A of smooth functions on $M_2(\mathbb{C})$ given in Example 2.4, the ideal $(ad-bc-1)A$ is a Poisson ideal and thus the special linear group $SL_2(\mathbb{C})$ is a Poisson manifold with Poisson bracket induced from $M_2(\mathbb{C})$. By Example 2.4, the maximal symplectic submanifolds of $SL_2(\mathbb{C})$ are as follows (see [4, Theorem B.2.1]).

0-dim. symp. :

$$\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\} \quad \lambda \in \mathbb{C}^\times$$

2-dim. symp. :

$$\{\begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\}$$
$$\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times\}$$
$$\{\begin{pmatrix} \alpha & \lambda\gamma \\ \gamma & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{C}^\times, \alpha\beta - \lambda\gamma^2 = 1\} \quad \lambda \in \mathbb{C}^\times$$

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