On Centralizers of Elements in the Lie Algebra of the Special Cremona Group $SA_2(k)$

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Abstract. We give a description of maximal abelian subalgebras and centralizers of elements in the Lie algebra $sa_2(k) = \{D \in \text{Der} k[x, y] \mid \text{div} D = 0\}$ over an algebraically closed field $k$ of characteristic 0. This description is given in terms of closed polynomials.

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1. Introduction

The special affine Cremona group $SA_n(k)$ over a field $k$ consists of all automorphisms $F = (f_1, \ldots, f_n)$ of the polynomial algebra $k[x_1, \ldots, x_n]$ with $\text{det}(JF) = 1$, where $JF$ is the Jacobian matrix of $F$. From [4] it follows that the Lie algebra $sa_n(k)$ of the infinite dimensional algebraic group $SA_n(k)$ consists of all derivations $D = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}$, $a_i(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ of the algebra $k[x_1, \ldots, x_n]$ with $\text{div} D = \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0$.

The aim of this paper is to give a description of centralizers of elements in the Lie algebra $sa_2(k)$ and to describe all maximal abelian subalgebras of this algebra over an algebraically closed field $k$ of characteristic 0. The investigation of the structure of subalgebras in $sa_2(k)$ is of great interest, because many problems (in particular the Jacobian conjecture for $n = 2$) are closely connected with properties of subalgebras in $sa_2(k)$.

To describe centralizers of elements in $sa_2(k)$ we represent this Lie algebra as a quotient algebra of the Lie algebra $P_2(k)$ of all polynomials in two variables with multiplication rule $[f, g] = \text{det}(J(f, g))$, where $\text{det}(J(f, g))$ is the Jacobian of polynomials $f, g \in k[x, y]$. In fact, $P_2(k)$ is a Poisson algebra but we mainly consider it as a Lie algebra. Using results from [3], it is easy to obtain a description of centralizers of elements and of maximal abelian subalgebras in $P_2(k)$ (see also [5]). This description is given in terms of closed polynomials, i.e., polynomials...
Let \( f \in k[x, y] \) for which the subalgebra \( k[f] \) is integrally closed in \( k[x, y] \). Using some results from [1], one can replace here closed polynomials by irreducible ones.

Notations in the paper are standard. The ground field \( k \) is algebraically closed of characteristic 0. The center of a Lie algebra \( L \) is denoted by \( Z(L) \). It is easy to show that \( Z(P_2(k)) = k \), where \( k \) is considered as a subalgebra in \( P_2(k) \).

For a polynomial \( f \in k[x, y] \) we denote by \( k[f] \) the (associative) subalgebra in \( k[x, y] \) generated by \( f \). The one-dimensional vector subspace of \( k[x, y] \) spanned on \( f \) is denoted by \( kf \).

A polynomial \( f(x, y) \in k[x, y] \) is called a Jacobian polynomial if there exists a polynomial \( g \) such that \( [f, g] = \text{det}(J(f, g)) \in k^* \) (see, for example [2], p.245).

\[ 2. \text{ Closed polynomials} \]

**Lemma 2.1.** The Lie algebra \( \text{sa}_2(k) \) is isomorphic to the quotient algebra of \( P_2(k) \) by \( Z(P_2(k)) = k \), i.e.,

\[ \text{sa}_2(k) \cong P_2(k)/k. \]

**Proof.** Any element \( f(x, y) \) of the Lie algebra \( P_2(k) \) induces the inner derivation \( \text{ad} f : P_2(k) \to P_2(k) \), \( \text{ad} f(g) = [f, g] \) of the Lie algebra \( P_2(k) \). The linear mapping \( \text{ad} f \) is also a derivation of the associative algebra \( k[x, y] \). It is easy to see that the kernel of the homomorphism of Lie algebras \( \text{ad} : P_2(k) \to \text{Der}(k[x, y]) \) coincides with \( k \), where \( k \) is considered as a subalgebra in \( P_2(k) \). Since \( \text{ad} f = -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \), we get \( \text{div}(\text{ad} f) = -\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y} = 0 \). Therefore, \( \text{ad} f \in \text{sa}_2(k) \).

This proves \( \text{ad}(P_2(k)) \subseteq \text{sa}_2(k) \).

Let us show that \( \text{ad} \) is a surjective map. Let \( D = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \) be an element of \( \text{sa}_2(k) \). Then \( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \). This condition guarantees the existence of a polynomial \( \varphi(x, y) \) (a potential) such that \( \frac{\partial \varphi}{\partial x} = Q(x, y), \frac{\partial \varphi}{\partial y} = -P(x, y) \). For \( \varphi \) we obtain \( [\varphi, x] = -\frac{\partial \varphi}{\partial y} = P(x, y), [\varphi, y] = \frac{\partial \varphi}{\partial x} = Q(x, y) \), in other words \( \text{ad}(\varphi) = D \). This proves the surjectivity of the map \( \text{ad} \). Using that \( \ker \text{ad} = k \), we obtain \( P_2(k)/k \cong \text{sa}_2(k) \).

**Lemma 2.2.** 1) A polynomial \( f \in k[x_1, \ldots, x_n] \setminus k \) is closed if and only if \( k[f] \) is a maximal element in the partially ordered set (with respect to inclusion)

\[ \mathcal{M} = \{ k[h] \mid h \in k[x_1, \ldots, x_n] \setminus k \}. \]

2) Let \( D \) be a derivation of \( k[x, y] \), \( D \neq 0 \). Then \( \ker D = k[f] \) for some closed polynomial \( f \).

**Proof.** 1) See [3], Lemma 3.1.

2) From [3], Theorem 2.8, it follows that \( \ker D = k[f] \) for some polynomial \( f \). The subalgebra \( \ker D = k[f] \) is integrally closed in \( k[x, y] \) by the Lemma 2.1 from [3]. Therefore the polynomial \( f \) is closed.

Let \( f, h \in k[x_1, \ldots, x_n] \). We call a polynomial \( h \) a generative polynomial of \( f \) if \( h \) is closed and if \( f \in k[h] \), i.e., \( f = F(h) \) for some \( F(t) \in k[t] \).
Lemma 2.3. Let \( f \in k[x,y] \setminus k \). The polynomial \( f \) is closed in the following two cases:

1) when \( f \) is irreducible;
2) when \( f \) is a Jacobian polynomial.

Proof. 1) If \( f \) is not closed, then \( f = F(h) \) for some polynomials \( h \in k[x,y] \) and \( F(t) \in k[t], \deg F > 1 \). Since \( F \) is reducible (as a polynomial in one indeterminate) \( f \) is reducible as well.

2) Let \( f \) be Jacobian but not closed. Then there exists a polynomial \( F(t) \in k[t], \deg F \geq 2 \) such that \( f = F(h) \) for some polynomial \( h \in k[x,y] \). As \( f \) is Jacobian there exists a polynomial \( g \in k[x,y] \) with \( \det(J(f,g)) = c \in k^* \). Then

\[
\det(J(f,g)) = \det(J(F(h),g)) = F'(h) \det(J(h,g)) = c.
\]

This is impossible because \( \deg F'(h) \geq 1 \).

Lemma 2.4. 1) If polynomials \( f, g \in k[x,y] \setminus k \) are algebraically dependent, there exists a closed polynomial \( h \in k[x,y] \) such that \( f \in k[h] \) and \( g \in k[h] \);

2) For any polynomial \( f \in k[x,y] \setminus k \), there exists a generative polynomial. If \( h_1, h_2 \) are two generative polynomials of \( f \), there exist \( c_1 \in k^*, \ c_2 \in k \) such that \( h_2 = c_1h_1 + c_2 \);

3) In the set of all generative polynomials of a polynomial \( f \in k[x,y] \setminus k \) there exists at least one irreducible polynomial.

Proof. 1) If \( f \) and \( g \) are algebraically dependent, by Corollary 3 from [5] we obtain \([f, g] = 0\). By Lemma 2.2, we get \( \ker \text{ad} f = k[h] \) for some closed polynomial \( h(x,y) \). Since \( f \in \ker \text{ad} f \) and \( g \in \ker \text{ad} f \), one concludes \( f \in k[h] \) and \( g \in k[h] \).

2) Since from the inclusion \( k[f] \subsetneq k[g] \) it follows \( \deg g \leq \deg f \), \( f \) is contained in some maximal one-generated subalgebra \( k[h] \). By Lemma 2.2 \( h \) is a generative polynomial of \( f \). Suppose \( h_1 \) and \( h_2 \) are generative polynomials of \( f \). It means in particular that \( f \in k[h_1] \) and \( f \in k[h_2] \). Therefore, \( f = F_1(h_1) \) and \( f = F_2(h_2) \) for some polynomials \( F_1(t), F_2(t) \in k[t] \). Then \( F_1(h_1) - F_2(h_2) = 0 \) and this implies that \( h_1 \) and \( h_2 \) are algebraically dependent. By 1) we conclude \( h \in k[h], h_2 \in k[h] \) for some closed polynomial \( h \). Clearly \( k[h_1] = k[h] = k[h_2] \). Therefore \( h_2 = c_1h_1 + c_2 \) for some elements \( c_1 \in k^*, \ c_2 \in k \).

3) Let \( h \) be a generative polynomial of \( f \). Since \( h \) is closed it follows from [1] (see Théorème 8) that there exists \( c \in k \) such that \( h - c \) is an irreducible polynomial. Because \( k[h] = k[h - c] \), \( h - c \) is also a generative polynomial of \( f \). This proves the Lemma.

Corollary 2.5. If polynomials \( f(x,y) \) and \( g(x,y) \) are algebraically dependent then \( f = c_1g + c_2 \) for some \( c_1 \in k^*, \ c_2 \in k \).

Proof. Since \( f \) and \( g \) are algebraically dependent, by Lemma 2.4 there exists a closed polynomial \( h \) such that \( f \in k[h], \ g \in k[h] \). The irreducible polynomials \( f \) and \( g \) are closed by Lemma 2.3. Therefore \( k[f] = k[h] = k[g] \) and \( f = c_1g + c_2 \), for some \( c_1 \in k^*, \ c_2 \in k \).
Corollary 2.6. For any polynomial \( f \in k[x, y] \setminus k \) there exist an irreducible polynomial \( h(x, y) \) and a polynomial \( F(t) \in k[t] \) such that \( f = F(h) \).

Proof. By Lemma 2.4 there exists an irreducible polynomial \( h \) such that \( f \in k[h] \). This implies the required statement. \( \blacksquare \)

Lemma 2.7. 1) For any polynomial \( f \in P_2(k) \setminus k \) its centralizer \( C_{P_2(k)}(f) \) coincides with \( k[h] \) for any generative polynomial \( h \) of \( f \).

2) Let \( A \) be a maximal abelian subalgebra of the Lie algebra \( P_2(k) \). Then \( A = k[f] \) for some irreducible polynomial \( f \in P_2(k) \setminus k \). Conversely, for any irreducible polynomial \( f \in P_2(k) \setminus k \), the subalgebra \( k[f] \) is a maximal abelian subalgebra of \( P_2(k) \).

Proof. 1) Follows from Lemma 2.2, since \( C_{P_2(k)}(f) = \ker \text{ad } f \).

2) Let \( A \) be a maximal abelian subalgebra of the Lie algebra \( P_2(k) \) and let \( f \) be any non-constant polynomial from \( A \). Obviously, \( A \subseteq C_{P_2(k)}(f) = k[h] \) for some closed polynomial \( h \). Since \( k[h] \) is an abelian subalgebra, \( A = k[h] \). By Lemma 2.4, \( h \) can be chosen irreducible.

Now let \( f \) be an irreducible polynomial. The polynomial \( f \) is closed by Lemma 2.3. We shall show that \( k[f] \) is a maximal abelian subalgebra. Let \( g \) be a polynomial such that \( [f, g] = 0 \). Then as in the proof of Lemma 2.4 \( f \in k[h] \), \( g \in k[h] \) for some closed polynomial \( h \). Therefore, using that \( f \) is closed, we conclude \( g \in k[h] = k[f] \). We proved that all polynomials commuting with \( f \) belong to \( k[f] \). Therefore \( k[f] \) is a maximal abelian subalgebra in \( P_2(k) \). \( \blacksquare \)

3. Main results

Theorem 3.1. Let \( D = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \) be a non-zero element of the Lie algebra \( sa_2(k) \). Let \( f(x, y) \in k[x, y] \) be a polynomial such that \( \frac{\partial f}{\partial x} = Q(x, y) \), \( \frac{\partial f}{\partial y} = -P(x, y) \) and let \( f \) be a generative polynomial of \( f \). Then

1) if \( f(x, y) \) is not a Jacobian polynomial,

\[
C_{sa_2(k)}(D) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right);
\]

2) if \( f(x, y) \) is a Jacobian polynomial and \( g(x, y) \) is a polynomial such that \( \det(J(f, g)) \in k^* \),

\[
C_{sa_2(k)}(D) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right) + k \left( -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right).
\]

Proof. 1) By Lemma 2.7 \( C_{P_2(k)}(f) = k[f] \). The homomorphism \( \text{ad} : P_2(k) \to sa_2(k) \) takes the polynomial \( f \) to the derivation \( \text{ad } f = -\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \). Let \( D_1 = P_1(x, y) \frac{\partial}{\partial x} + Q_1(x, y) \frac{\partial}{\partial y} \) be an arbitrary non-zero element of \( C_{sa_2(k)}(D) \).

By Lemma 2.1 there exists a polynomial \( f_1(x, y) \) such that \( \text{ad } f_1 = D_1 \). Since \( \ker \text{ad} = k \), \( [f, f_1] \) lies in \( k \). But \( f \) is not a Jacobian polynomial, so we can
conclude $[f, f_1] = 0$. Therefore $f_1$ lies in $C_{P_2(k)}(f) = k[\bar{f}]$. This means that $\text{ad}^{-1}(C_{\text{sa}_2(k)}(D)) = k[\bar{f}]$. Using the surjectivity of the homomorphism $\text{ad}$ we obtain $C_{\text{sa}_2(k)}(D) = \text{ad}(k[\bar{f}]) = k[\bar{f}] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$.

2) Let $f$ be a Jacobian polynomial, i.e., there exists a polynomial $g$ such that $\det(J(f, g)) = c \in k^*$. By Lemma 2.3 the polynomial $f$ is closed, i.e., one can assume $\bar{f} = f$. Since $[\text{ad} f, \text{ad} g] = \text{ad} c = 0$, we have $\text{ad} g \in C_{\text{sa}_2(k)}(\text{ad} f) = C_{\text{sa}_2(k)}(D)$. It is easy to see that

$$\text{ad}^{-1}(C_{\text{sa}_2(k)}(D)) = \{ h \in P_2(k) | [f, h] \in k \} = k[\bar{f}] + kg = k[f] + kg.$$ Therefore,

$$C_{\text{sa}_2(k)}(D) = \text{ad}(k[f] + kg) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right) + k \left( -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right).$$

**Remark 3.2.** From Lemma 2.4 it follows that the polynomial $\bar{f}$ in Theorem 3.1 can be chosen irreducible.

**Remark 3.3.** From the description of centralizers of elements in Theorem 3.1 it follows that the centralizer of a derivation corresponding to a non-Jacobian polynomial is an abelian subalgebra, and the centralizer of a derivation corresponding to any Jacobian polynomial is solvable of derived length 2.

**Lemma 3.4.** Let $L = k[f] + kg$ be a subalgebra of the Lie algebra $P_2(k)$ with $\det(J(f, g)) = c \in k^*$. If $A$ is a nilpotent subalgebra of $L$ and the nilpotency class of $A$ is at most 2 then either $A \subseteq k[f]$ or $A$ is contained in the subalgebra $k + kf + k(g + p(f))$ for some $p(t) \in k[t]$.

**Proof.** Suppose that $A$ is not contained in $k[f]$. As $\dim L/k[f] = 1$ the $k-$subspace $A \cap k[f]$ is of codimension 1 in $A$. Therefore $A = (A \cap k[f]) + k(g + p(f))$ for some $p(t) \in k[t]$. Since $[q(f), g + p(f)] = q'(f) \cdot c$ for any polynomial $q(t) \in k[t]$ the subspace $A \cap k[f]$ may not contain polynomials of degree $> 1$. So the intersection $A \cap k[f]$ is contained in the subalgebra $k + kf$ and therefore $A \subseteq k + kf + k(g + p(f))$.

**Theorem 3.5.** Let $A$ be a maximal abelian subalgebra of the Lie algebra $\text{sa}_2(k)$. Then

1) if $\dim A = \infty$, then $A = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$, where $f(x, y)$ is an irreducible polynomial. Conversely, for any irreducible polynomial $f$, the algebra

$$k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$$

is a maximal abelian subalgebra in $\text{sa}_2(k)$;

2) if $\dim A < \infty$ then $A = kD_1 + kD_2$, where $D_1 = -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$, $D_2 = -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y}$ for some polynomials $f, g$ such that $\det(J(f, g)) \in k^*$. Conversely, for any two polynomials $f, g$ with condition $\det(J(f, g)) \in k^*$ the subalgebra $kD_1 + kD_2$, where $D_1$ and $D_2$ are defined as above, is a maximal abelian subalgebra of $\text{sa}_2(k)$. 
Proof. Let $D$ be an arbitrary non-zero element of $A$. Then $A \subseteq C_{sa_2(k)}(D)$ and clearly $A$ is a maximal abelian subalgebra of $C_{sa_2(k)}(D)$. By Theorem 3.1 either

$$C_{sa_2(k)}(D) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$$

or

$$C_{sa_2(k)}(D) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right) + k \left( -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right).$$

In the first case $f$ is a closed irreducible polynomial, in the second one the polynomials $f$ and $g$ satisfy the condition $\det(J(f,g)) \in k^*$. In the first case $C_{sa_2(k)}(D)$ is an abelian subalgebra. Thus $A = C_{sa_2(k)}(D) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$. Consider the second case. Denote $L = \text{ad}^{-1}(C_{sa_2(k)}(D))$ where $\text{ad} : P_2(k) \to sa_2(k)$ is the homomorphism from the Lemma 2.1. Then $\text{ad}^{-1}(A)$ is a subalgebra in $L$. It is easy to see that $L = k[f] + kg$. Since $\ker \text{ad} = Z(P_2(k)) = k$, we conclude that $\text{ad}^{-1}(A)$ is a nilpotent subalgebra of the nilpotency class $\leq 2$. By Lemma 3.4 it holds either $\text{ad}^{-1}(A) \subseteq k[f]$ or $\text{ad}^{-1}(A) \subseteq k + k[f] + k(g + p(f))$ for some $p(t) \in k[t]$. Since $A$ is a maximal abelian subalgebra of $sa_2(k)$ it follows from inclusion $\text{ad}^{-1}(A) \subseteq k[f]$ that $\text{ad}^{-1}(A) = k[f]$. Then we have $A = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right).

Let now $\text{ad}^{-1}(A) \subseteq k + k[f] + k(g + p(f))$. Applying the map $\text{ad}$ we get the inclusion $A \subseteq \text{ad}(k + k[f] + k(g + p(f))) = kD_1 + kD_2$, where $D_1 = \text{ad} f, D_2 = \text{ad}(g + p(f))$. The subalgebra $kD_1 + kD_2$ is abelian and therefore $A = kD_1 + kD_2$. Denoting $g + p(f)$ by $g$ we have $D_1 = \text{ad} f, D_2 = \text{ad} g$. So we have proved the necessary conditions for both statements of the Theorem.

Let $f$ be an irreducible polynomial. We will show that $k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$ is a maximal abelian subalgebra in $sa_2(k)$. Clearly, since $f$ is an irreducible polynomial, by Lemma 2.7 $k[f]$ is a maximal abelian subalgebra in $P_2(k)$. It is obvious that

$$\text{ad}(k[f]) = k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$$

is an abelian subalgebra in $sa_2(k)$. Suppose that $\text{ad}(k[f])$ is not maximal abelian. Then it is properly contained in some maximal abelian subalgebra $B$ of the algebra $sa_2(k)$. Since $\dim B = \infty$, as it was proved above there exists a closed polynomial $g$ such that $B = k[g] \left( -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right)$. From this one easily concludes that $k[f]$ is properly contained in $\text{ad}^{-1}(B) = k[g]$. This is impossible by Lemma 2.2, since $k[f]$ is a maximal in the set of subalgebras of the form $k[h]$ in $P_2(k)$. This proves that $k[f] \left( -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$ is a maximal abelian subalgebra in $sa_2(k)$.

Let now $f$ and $g$ be two polynomials from $k[x, y]$ such that $\det(J(f,g)) \in k^*$. Then the elements $D_1 = \text{ad} f$ and $D_2 = \text{ad} g$ commute. Therefore $A = kD_1 + kD_2$ is an abelian two-dimensional subalgebra in $sa_2(k)$. Suppose, $A$ is not a maximal abelian subalgebra of the algebra $sa_2(k)$. Then $A$ is contained in some maximal abelian subalgebra $B$ of $sa_2(k)$. If $\dim B = \infty$, by the above proved statement, $B = k[h] \left( -\frac{\partial h}{\partial y} \frac{\partial}{\partial x} + \frac{\partial h}{\partial x} \frac{\partial}{\partial y} \right)$ for some closed polynomial $h$. Then $\text{ad}^{-1}(B) = k[h]$ is an abelian subalgebra in $P_2(k)$ which contains the non-abelian subalgebra $k + k[f] + kg$. This is impossible and therefore $\dim B < \infty$. As above one
can obtain $\dim B = 2$. This implies $A = B$ which contradicts to our assumption. This contradiction proves that $A$ is a maximal abelian subalgebra in $so_2(k)$. The sufficient conditions for the both statements of the Theorem are proved. ■

References


