Journal of Lie Theory
Volume **16** (2006) 57–65

On the Irreducibility of the Commuting Variety of the Symmetric Pair $\mathbf{so}_{p+2}$, $\mathbf{so}_p\times\mathbf{so}_2$

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Communicated by J. Faraut

Abstract. In this paper, we prove that the commuting variety of the family of symmetric pairs $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$, $p \geq 2$, is irreducible.
Mathematics Subject Classification: 17B20, 14L30.
Key words and phrases: Symmetric pairs, Lie algebra, Commuting variety.

1. Introduction and notations

Let $\mathfrak{g}$ be a complex reductive Lie algebra and θ an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces with respect to θ, where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. In this case, we say that $(\mathfrak{g}, \mathfrak{k})$ is a *symmetric pair.*

Let G be the adjoint group of $\mathfrak{g}$ and K the connected algebraic subgroup of G whose Lie algebra is $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ consisting of semisimple elements. Any such subspace is called a *Cartan subspace* of $\mathfrak{p}$. All the Cartan subspaces are K-conjugate. Its dimension is called the *rank* of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

We define the commuting variety of $(\mathfrak{g}, \mathfrak{k})$ as the following set:

$$C(\mathfrak{p}) = \{(x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0\}.$$

We may also consider the commuting variety $C(\mathfrak{g})$ of $\mathfrak{g}$, defined in the same way. Richardson proved in [10] that, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $C(\mathfrak{g}) = \overline{G.(\mathfrak{h} \times \mathfrak{h})}$. In particular, the commuting variety $C(\mathfrak{g})$ is an irreducible algebraic variety.

On the other hand, the commuting variety of any semisimple symmetric pair is not irreducible in general. Panyushev showed in [7] that in the case of the symmetric pair $(\mathrm{sl}_n, \mathrm{gl}_{n-1})$, $n > 2$, associated to the involutive automorphism, defined via conjugation by the diagonal matrix $\mathrm{diag}(-1, \ldots, -1, 1)$, the corresponding commuting variety has three irreducible components of dimension, respectively, $2n-1$, $2n-2$, $2n-2$.

Nevertheless, in some cases, the irreducibility problem has been solved.

ISSN 0949–5932 / \$2.50

- As an obvious consequence of the classical case proved by Richardson, the symmetric pair $(\mathfrak{g} \times \mathfrak{g}, \Delta(\mathfrak{g}))$, associated to the automorphism $(X,Y) \mapsto (Y,X)$, has an irreducible commuting variety.

- If the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is equal to the semisimple rank of $\mathfrak{g}$ (called the maximal rank case), then Panyushev proved in [7] that the corresponding commuting variety is irreducible.

- The rank 1 case has been considered independently by the authors [11] and Panyushev [8]. In this case, it has been proved that $(\mathrm{so}_{m+1}, \mathrm{so}_m)$ is the only simple symmetric pair whose commuting variety is irreducible.

- In [8], Panyushev proves the irreducibility of the commuting variety for the symmetric pairs $(\mathrm{sl}_{2n}, \mathrm{sp}_{2n})$ and (E_6, F_4).

For a symmetric pair of rank strictly larger than one, we observe that due to the rank 1 case, the inductive arguments used by Richardson in the classical case [10] do not apply. However, if $\mathfrak{a}$ is a Cartan subspace, then it is well-known that $C_0 = \overline{K.(\mathfrak{a} \times \mathfrak{a})}$ is the unique irreducible component of $C(\mathfrak{p})$ of maximal dimension, which is equal to $\dim \mathfrak{p} + \dim \mathfrak{a}$. The main problem is therefore to determine if there exist components other than the maximal one.

In [8], it has been conjectured that $C(\mathfrak{p})$ is irreducible if the rank of the symmetric pair is greater than or equal to 2.

In this paper, by showing that an even nilpotent element in $\mathfrak{p}$ is contained in a K-sheet containing non-zero semisimple elements, we obtain that for the commuting variety of a symmetric pair to be irreducible, it suffices that $\mathfrak{p}$-distinguished elements in every symmetric subpair are even. We use this to prove that the commuting variety of the family $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$, $p \geq 2$, of rank 2 symmetric pairs is irreducible.

Let us point out that this family of symmetric pairs comes from a larger family of symmetric pairs associated to parabolic subalgebras with abelian nilpotent radical. For such a symmetric pair in this larger family, it is possible to obtain descriptions of symmetric subpairs associated to centralizers of semisimple elements of $\mathfrak{p}$ by considering a suitable Cartan subspace. Unfortunately, we are not able to apply the arguments used here.

We shall conserve the notations above in the sequel. The reader may refer to [12] for basic definitions and properties of symmetric pairs.

The authors would like to thank Patrice Tauvel and Abderrazak Bouaziz for many useful discussions, and the referee for his suggestions.

2. Sheets and commuting varieties

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair. Recall that the connected algebraic group K acts on $\mathfrak{p}$. For $n \in \mathbb{N}$, we set:

$$\mathfrak{p}^{(n)} = \{X \in \mathfrak{p} \ ; \ \dim K.X = n\}.$$

The set $\mathfrak{p}^{(n)}$ is locally closed, and an irreducible component of $\mathfrak{p}^{(n)}$ shall be called a K-*sheet* of $\mathfrak{p}$. Clearly, K-sheets are K-invariant, and by [12], each K-sheet contains a nilpotent element.

Let $\pi_1 : C(\mathfrak{p}) \to \mathfrak{p}$ be the projection $(X,Y) \mapsto X$. Recall the following result concerning the commuting variety of $\mathfrak{p}$.

Theorem 2.1. *There exist K-sheets $\mathcal{S}_1, \ldots, \mathcal{S}_r$ of $\mathfrak{p}$ such that $\overline{\pi_1^{-1}(\mathcal{S}_i)}$, $i = 1, \ldots, r$, are the irreducible components of $C(\mathfrak{p})$.*

The proof of Theorem 2.1 is a simple consequence of the following result. For the sake of completeness, we have included a proof.

Lemma 2.2. *Let V be a vector space, $E \subset V \times V$ a locally closed subvariety and for $i = 1, 2$, $\pi_i : E \to V$ be the projection $(x_1, x_2) \mapsto x_i$. Suppose that:*

1. *$\pi_1(E)$ is locally closed.*
2. *There exists $r \in \mathbb{N}$ such that for all $x \in \pi_1(E)$, $\pi_2(\pi_1^{-1}(x))$ is a vector subspace of dimension r.*

If $\pi_1(E)$ is irreducible, then so is E.

Proof. Let $\mathbf{G}$ be the Grassmann variety of r-dimensional subspaces of V, $x \in \pi_1(E)$ and $W = \pi_2(\pi_1^{-1}(x)) \in \mathbf{G}$. Fix a complementary subspace U of W in V and set:

$$\mathbf{F} = \{T \in \mathbf{G} \ ; \ T \cap U = \{0\}\} = \{T \in \mathbf{G} \ ; \ T + U = V\}.$$

Clearly, $\mathbf{F}$ is an open subset of $\mathbf{G}$ containing W. For $\tau \in \mathrm{Hom}(W, U)$ the set of linear maps from W to U, we define:

$$T(\tau) = \{w + \tau(w) \ ; \ w \in W\}.$$

Then we check easily that $T(\tau) \in \mathbf{F}$, and we have a map $\mathrm{Hom}(W, U) \to \mathbf{F}$, $\tau \mapsto T(\tau)$. We claim that this map is an isomorphism.

Since $w_1 + \tau_1(w_1) = w_2 + \tau_2(w_2)$ is equivalent to $w_1 - w_2 = \tau_2(w_2) - \tau_1(w_1)$, we deduce that the above map is injective.

Now if $T \in \mathbf{F}$, then for $w \in W$, we define $\tau(w)$ to be the unique element in U such that $w + \tau(w) \in T$. We then verify easily that $T(\tau) = T$. So we have proved our claim.

The map

$$\Phi : \pi_1(E) \to \mathbf{G} \ , \ y \mapsto \pi_2(\pi_1^{-1}(y))$$

is a morphism of algebraic varieties. So $F = \Phi^{-1}(\mathbf{F})$ is an open subset of $\pi_1(E)$ containing x. The above claim says that we have a well-defined map:

$$\Psi : F \times W \to E \ , \ (y, w) \mapsto (y, w + \tau(w))$$

where $T(\tau) = \Phi(y)$. It is then a straightforward verification that Ψ is an isomorphism of the algebraic varieties $F \times W$ and $\pi_1^{-1}(F)$.

It follows that the map $\pi_1 : E \to \pi_1(E)$ is an open map whose fibers are irreducible. Hence by a classical result on topology [3, T.5], if $\pi_1(E)$ is irreducible, then E is irreducible. ∎

Since the set of $\mathfrak{p}$-generic elements and the set $\mathfrak{p}_{\mathrm{reg}}$ of $\mathfrak{p}$-regular elements are open subsets of $\mathfrak{p}$, we have the following corollary:

Corollary 2.3. *Let $\mathfrak{a}$ be a Cartan subspace in $\mathfrak{p}$. The set*

$$C_0 = \overline{K.(\mathfrak{a} \times \mathfrak{a})} = \overline{\pi_1^{-1}(\mathfrak{p}_{\mathrm{reg}})} = \overline{\pi_2^{-1}(\mathfrak{p}_{\mathrm{reg}})}$$

is the unique irreducible component of $C(\mathfrak{p})$ of maximal dimension.

Let $X \in \mathfrak{p}$ be a nilpotent element, and $(H, Y) \in \mathfrak{k} \times \mathfrak{p}$ be such that (X, H, Y) is a normal sl_2-triple (called a normal S-triple in [12]). Recall that X is *even* if the eigenvalues of $\mathrm{ad}_{\mathfrak{g}} H$ are even. In fact, this is equivalent to the condition that the eigenvalues of $\mathrm{ad}_{\mathfrak{p}} H$ are even.

Proposition 2.4. *Let $X \in \mathfrak{p}$ be an even nilpotent element, then X belongs to a K-sheet containing semisimple elements.*

Proof. Let (X, H, Y) be a normal sl_2-triple and $\mathfrak{s} = \mathbb{C}X + \mathbb{C}H + \mathbb{C}Y$. Then $\mathfrak{g}$ decomposes into a direct sum of simple $\mathfrak{s}$-modules, say V_i, $i = 1, \dots, r$. Since X is even, $\dim V_i$ is odd for $i = 1, \dots, r$.

For $\lambda \in \mathbb{C}$, we set $X_\lambda = X + \lambda Y \in \mathfrak{p}$. If $\lambda \neq 0$, then X_λ is semisimple because X_λ is G-conjugate to a multiple of H. We claim that $\dim \mathfrak{p}^{X_\lambda} = \dim \mathfrak{p}^X$ for all $\lambda \in \mathbb{C}$.

First of all, observe that $\mathfrak{p}^{X_\lambda} = \bigoplus_{i=1}^r (V_i \cap \mathfrak{p})^{X_\lambda}$ because $V_i = (V_i \cap \mathfrak{k}) \oplus (V_i \cap \mathfrak{p})$. Moreover $\dim (V_i \cap \mathfrak{p})^{X_\lambda} \leq 1$.

Now if $(V_i \cap \mathfrak{p})^{X_\lambda} \neq \{0\}$, then a simple weight argument shows that $(V_i \cap \mathfrak{p})^X \neq \{0\}$.

Conversely, suppose that $(V_i \cap \mathfrak{p})^X \neq \{0\}$. Let $\dim V_i = 2n + 1$ and $v_{-n}, \dots, v_n$ be a basis of weight vectors of V_i such that $Hv_k = 2kv_k$, $k = -n, \dots, n$. Then $(V_i \cap \mathfrak{p})^X = \mathbb{C}v_n$.

So $v_k \in \mathfrak{k}$ (resp. $v_k \in \mathfrak{p}$) when $n - k$ is odd (resp. even). In particular, $v_{-n} \in \mathfrak{p}$. It follows that for k such that $n - k$ odd, $\lambda Y v_{k+1} = -a_k X v_{k-1}$ for some $a_k \in \mathbb{C}$. We may therefore renormalize the v_k's so that $v = v_{-n} + v_{-n+2} + \cdots + v_{n-2} + v_n$ verifies $X_\lambda v = 0$.

We have therefore proved that $\dim \mathfrak{p}^{X_\lambda} = \dim \mathfrak{p}^X$ for all λ.

Now, consider the morphism $\Phi : K \times \mathbb{C} \to \mathfrak{p}$, $(k, \lambda) \mapsto k.X_\lambda$. The image of Φ is irreducible and contains semisimple elements, so it contains strictly $K.X$. Consequently, $K.X$ is contained strictly in a K-sheet with semisimple elements. ∎

Recall that an element of $\mathfrak{p}$ is said to be $\mathfrak{p}$-distinguished if its centralizer in $\mathfrak{p}$ does not contain any non-zero semisimple element. In particular, a $\mathfrak{p}$-distinguished element is nilpotent. So the number of K-orbits of $\mathfrak{p}$-distinguished elements is finite.

Definition 2.5. We say that the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ satisfies *condition* $(\mathcal{E})$ if for every symmetric subpair $(\mathfrak{g}', \mathfrak{k}')$ of $(\mathfrak{g}, \mathfrak{k})$, all the $\mathfrak{p}'$-distinguished elements of $\mathfrak{p}'$ are even.

Theorem 2.6. *If $(\mathfrak{g}, \mathfrak{k})$ satisfies condition $(\mathcal{E})$, then the commuting variety $C(\mathfrak{p})$ is irreducible.*

Proof. We proceed as in the proof of Richardson in the case of semisimple Lie algebras (see [10]) by using inductive arguments. Let $(X,Y)\in C(\mathfrak{p})$.

1. If X is semisimple, then X commutes with a $\mathfrak{p}$-regular semisimple element Z. The line $\mathcal{L}_Z=\{(X,tY+(1-t)Z),t\in\mathbb{C}\}$ is contained in $C(\mathfrak{p})$. Since $\{tY+(1-t)Z,t\in\mathbb{C}\}$ meets the set of $\mathfrak{p}$-regular semisimple elements which is open in $\mathfrak{p}$, we conclude that $\mathcal{L}_Z$, and hence (X,Y), is contained in C_0 (Corollary 2.3).

2. We may assume that neither X nor Y is semisimple.

Suppose that X is not nilpotent. Let $X=X_s+X_n$ be the corresponding decomposition into semisimple and nilpotent components. Then $(X,Y)\in\mathfrak{g}^{X_s}$. Since $(\mathfrak{g}^{X_s},\mathfrak{k}^{X_s})$ is a symmetric subpair of $(\mathfrak{g},\mathfrak{k})$, we may apply induction to show that $(X,Y)\in C_0$.

3. So we may further assume that X and Y are both nilpotent. If X commutes with a non-zero semisimple element $Z\in\mathfrak{p}$, then the same argument as in 1) works because the set of non-nilpotent elements is open.

4. So we are reduced to the case where both X and Y are $\mathfrak{p}$-distinguished. Denote by $\pi_1 : C(\mathfrak{p})\to\mathfrak{p}$ the projection $(X_1,X_2)\mapsto X_1$, $\mathcal{O}$ the set of non $\mathfrak{p}$-distinguished elements in $\mathfrak{p}$, and $\Omega_1,\dots,\Omega_r$ the set of K-orbits of $\mathfrak{p}$-distinguished elements in $\mathfrak{p}$. Thus $\mathfrak{p}=\mathcal{O}\cup\Omega_1\cup\cdots\cup\Omega_r$, and $C(\mathfrak{p})=\pi_1^{-1}(\mathcal{O})\cup\pi_1^{-1}(\Omega_1)\cup\cdots\cup\pi_1^{-1}(\Omega_r)$.

¿From the previous paragraph, we obtain that $\overline{\pi_1^{-1}(\mathcal{O})}\subset C_0$. Consequently, $C(\mathfrak{p})$ is the union of C_0 with $\overline{\pi_1^{-1}(\Omega_{i_1})},\dots,\overline{\pi_1^{-1}(\Omega_{i_s})}$. Now we check easily that for $X\in\mathfrak{p}$, $\pi_1^{-1}(K.X)=K.(X,\mathfrak{p}^X)$ is an irreducible subset of $C(\mathfrak{p})$ of dimension $\dim\mathfrak{k}-\dim\mathfrak{k}^X+\dim\mathfrak{p}^X=\dim\mathfrak{p}$. It follows that all irreducible components of $C(\mathfrak{p})$ other than C_0, if they exist, are of dimension $\dim\mathfrak{p}$.

Suppose that $C(\mathfrak{p})$ is not irreducible. By the previous discussion, there exists a $\mathfrak{p}$-distinguished element X such that $\overline{\pi_1^{-1}(K.X)}$ is an irreducible component of dimension $\dim\mathfrak{p}$.

On the other hand, Condition $(\mathcal{E})$ and Proposition 2.4 say that X belongs to a K-sheet $\mathcal{S}$ containing non-zero semisimple elements. So $\dim\mathcal{S}>\dim K.X$. Now Lemma 2.2 says that $\pi_1^{-1}(\mathcal{S})$ is an irreducible subset of $C(\mathfrak{p})$ containing $\pi_1^{-1}(K.X)$ and $\dim\pi_1^{-1}(\mathcal{S})>\dim\mathfrak{p}$. We have therefore obtained a contradiction.

So the theorem follows. ■

3. The case of the symmetric pair $(\mathrm{so}_{p+2},\mathrm{so}_p\times\mathrm{so}_2)$

Let us fix an integer $p\geq 2$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}=\mathrm{so}_{p+2}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Denote by $R\supset R^+\supset\Pi$ the corresponding set of roots, positive roots and simple roots. Let us also fix root vectors X_α, $\alpha\in R$, and for $\alpha\in R$, we set $\mathfrak{g}_\alpha=\mathbb{C}X_\alpha$. The rank ℓ of $\mathfrak{g}$ is the integer part of $(p+2)/2$.

Let us first consider the case where $\mathfrak{g}$ is simple and not of type A_n, or equivalently, $p\neq 2,4$. We shall use the numbering of simple roots $\alpha_1,\dots,\alpha_\ell$ in [1]. Let $H\in\mathfrak{h}$ be such that

$$\alpha_i(H)=\begin{cases}1 & \text{if } i=1,\\ 0 & \text{if } i\neq 1.\end{cases}$$

Then it follows that $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ where $\mathfrak{g}_i=\{X\in\mathfrak{g};[H,X]=iX\}$.

Observe that $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is simply the maximal parabolic subalgebra associated to $\Pi \setminus \{\alpha_1\}$. Its nilpotent radical $\mathfrak{g}_1$ is abelian.

The above decomposition defines a symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ where $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. It is clear that this is precisely the rank 2 symmetric pair $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$.

Let $\mathfrak{a}$ be the vector space span of the elements $X_{\alpha_1} + X_{-\alpha_1}$ and $X_{\alpha_{\max}} + X_{-\alpha_{\max}}$ where $\alpha_{\max}$ denotes the largest root in R. Then $\mathfrak{a}$ is a 2-dimensional abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. So $\mathfrak{a}$ is a Cartan subspace in $\mathfrak{p}$.

Let $X \in \mathfrak{a}$. Then $\mathfrak{g}^X$ is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{l} = [\mathfrak{g}^X, \mathfrak{g}^X]$ the semisimple part of $\mathfrak{g}^X$, and set $\mathfrak{l}_+ = \mathfrak{l} \cap \mathfrak{k}^X$, $\mathfrak{l}_- = \mathfrak{l} \cap \mathfrak{p}^X$ and $\mathfrak{r}_+ = [\mathfrak{l}_-, \mathfrak{l}_-]$. Then the decompositions

$$\mathfrak{g}^X = \mathfrak{k}^X \oplus \mathfrak{p}^X \ , \ \mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_- \text{ and } \mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{l}_-$$

define symmetric subpairs of $(\mathfrak{g}, \mathfrak{k})$, and the ranks of the pairs $(\mathfrak{l}, \mathfrak{l}_+)$ and $(\mathfrak{r}, \mathfrak{r}_+)$ are strictly inferior to that of $(\mathfrak{g}, \mathfrak{k})$.

We shall determine the symmetric pair $(\mathfrak{r}, \mathfrak{r}_+)$ for any non-zero non $\mathfrak{p}$-regular element $X \in \mathfrak{a}$, *i.e.* $\mathfrak{p}^X$ contains a non-zero nilpotent element.

Let us recall the classification of simple symmetric pairs of rank 1.

$$\begin{array}{cc} (\mathrm{sl}_{n+1}, \mathrm{sl}_n \times \mathbb{C}), & (\mathrm{so}_{n+1}, \mathrm{so}_n), \\ (\mathrm{sp}_{2n}, \mathrm{sp}_{2n-1} \times \mathrm{sp}_2), & (F_4, B_4). \end{array}$$

Lemma 3.1. *Let $X \in \mathfrak{a}$ be a non-zero non $\mathfrak{p}$-regular element. There exists $m \in \mathbb{N}$ such that $(\mathfrak{r}, \mathfrak{r}_+) = (\mathrm{so}_{m+1}, \mathrm{so}_m)$.*

Proof. From the definition of $\mathfrak{a}$, we verify easily that $\mathfrak{a}$ commutes with the Lie subalgebra $\mathfrak{s}$ generated by the root vectors $X_{\pm\alpha}$, $\alpha \in \Pi \setminus \{\alpha_1, \alpha_2\}$. So $\mathfrak{k}^X$ contains $\mathfrak{s}$. Note that $\mathfrak{s} \simeq \mathrm{so}_{p-2}$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{s}$, then $\mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^X$ (and also of $\mathfrak{g}$). It follows that the root system of the semisimple part $\mathfrak{l}$ of $\mathfrak{g}^X$ contains as a subsystem the root system of $\mathfrak{s}$. In particular, the semisimple rank of $\mathfrak{l}$ is equal to $\ell - 2$ or $\ell - 1$ (where ℓ is the rank of $\mathfrak{g}$).

Moreover, since X is not $\mathfrak{p}$-regular, $\mathfrak{p}^X$ contains a non-zero nilpotent element, and so $\mathfrak{l}$ contains strictly $\mathfrak{s}$.

If $p+2 = 2\ell+1$, then the preceding discussion implies that the root system $R(\mathfrak{l})$ of $\mathfrak{l}$ is of one of the following types: $A_{\ell-1}$, $B_{\ell-1}$ or $A_1 \times B_{\ell-2}$.

Since $\mathfrak{l}_+$ contains a simple Lie subalgebra of type $B_{\ell-2}$, by considering the Satake diagram of the corresponding involution (see for example [4]), it follows easily that $R(\mathfrak{l})$ is of type $B_{\ell-1}$ or $A_1 \times B_{\ell-2}$. Consequently, we deduce from the classification of rank 1 symmetric pairs that $(\mathfrak{r}, \mathfrak{r}_+)$ has the required form.

If $p + 2 = 2\ell$, then the root system $R(\mathfrak{l})$ is of one of the following types: $A_{\ell-1}$, $D_{\ell-1}$ or $A_1 \times D_{\ell-2}$. The same argument as above applies, and again, we may conclude by using the classification of rank 1 symmetric pairs. ∎

Remark 3.2. Note that we may extend Lemma 3.1 to the symmetric pairs $(\mathrm{so}_4, \mathrm{so}_2 \times \mathrm{so}_2)$ and $(\mathrm{so}_6, \mathrm{so}_4 \times \mathrm{so}_2)$.

In the first case, we have $(\mathrm{so}_4, \mathrm{so}_2 \times \mathrm{so}_2) = (\mathrm{so}_3, \mathrm{so}_2) \times (\mathrm{so}_3, \mathrm{so}_2)$. Take $\mathfrak{a}$ to be the direct product of a rank 1 Cartan subspace $\mathfrak{a}_0$ of the symmetric pair $(\mathrm{so}_3, \mathrm{so}_2)$. Then $(X, Y) \in \mathfrak{a}$ is non $\mathfrak{p}$-regular if $X = 0$ or $Y = 0$. It follows that $\mathfrak{g}^X \simeq \mathrm{so}_3 \times \mathbb{C}$, and $(\mathfrak{r}, \mathfrak{r}_+) = (\mathrm{so}_3, \mathrm{so}_2)$.

In the second case, we have $(\mathrm{so}_6, \mathrm{so}_4 \times \mathrm{so}_2) = (\mathrm{sl}_4, \mathrm{sl}_2 \times \mathrm{sl}_2 \times \mathbb{C})$. Take $\mathfrak{a}$ to be the vector space span of the vectors $X_1 = X_{\alpha_1+\alpha_2+\alpha_3} + X_{-(\alpha_1+\alpha_2+\alpha_3)}$ and $X_2 = X_{\alpha_2} + X_{-\alpha_2}$. Then $\mathfrak{a}$ is a Cartan subspace, and $X = \lambda_1 X_1 + \lambda_2 X_2 \in \mathfrak{a}$ is non $\mathfrak{p}$-regular if and only if $\lambda_1\lambda_2 = 0$. A direct computation shows that $(\mathfrak{r}, \mathfrak{r}_+) = (\mathrm{so}_3, \mathrm{so}_2)$.

Summarizing, since Cartan subspaces are K-conjugate, we have therefore obtained the following result:

Proposition 3.3. *Let $(\mathfrak{g}, \mathfrak{k})$ be the symmetric pair $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$, $p \geq 2$. For any non-zero non $\mathfrak{p}$-regular semisimple element X in $\mathfrak{p}$, the subpair $(\mathfrak{r}, \mathfrak{r}_+)$ is of the form $(\mathrm{so}_{m+1}, \mathrm{so}_m)$ for some $m \in \mathbb{N}$.*

Corollary 3.4. *The commuting variety of $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$ is irreducible.*

Proof. By the previous proposition and theorem 2.6, it suffices to check that all $\mathfrak{p}$-distinguished elements are even for the symmetric pairs $(\mathrm{so}_{p+1}, \mathrm{so}_p)$ and $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$.

One may use the classification of Popov-Tevelev [9] of $\mathfrak{p}$-distinguished elements. However, it is not difficult to do the checking directly. The case $(\mathrm{so}_{p+1}, \mathrm{so}_p)$ is trivial because for any $X \in \mathfrak{p} \setminus \{0\}$, $\mathfrak{p}^X$ is $\Bbbk X$ (see for example [11, Proposition 3]). For $(\mathrm{so}_{p+2}, \mathrm{so}_p)$, via the Kostant-Sekiguchi correspondence as described in [2] using signed partitions, we observe that there are 6 (resp. 7) non-zero nilpotent K-orbits in $\mathfrak{p}$ for $p > 4$ (resp. $p = 4$). Then it is a simple verification that the $\mathfrak{p}$-distinguished nilpotent elements correspond to the orbits whose partition has parts all of the same parity. This in turn implies that $\mathfrak{p}$-distinguished elements are even. ∎

Remark 3.5. The above realization of the symmetric pair $(\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)$ and the construction of the specific Cartan subspace $\mathfrak{a}$ come from a more general construction.

Namely, let $\alpha \in \Pi$ be a simple root such that the corresponding standard maximal parabolic subalgebra has an abelian nilpotent radical $\mathfrak{n}$. Then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as above and we obtain a symmetric pair. Let $\mathcal{R}(\Pi)$ be the set of pairwise strongly orthogonal roots in R^+ constructed via the "cascade" construction of Kostant (see for example [5], [6] or [13]). Then the vector subspace spanned by the elements $X_\beta + X_{-\beta}$, $\beta \in \mathcal{R}(\Pi)$ and $X_\beta \in \mathfrak{n}$, is a Cartan subspace. The list of all such symmetric pairs is as follows:

$$(\mathrm{sl}_{p+1}, \mathrm{sl}_{p+1-i} \times \mathrm{sl}_i \times \mathbb{C})\ (i = 1, \dots, p)\ ,\ (\mathrm{so}_{p+2}, \mathrm{so}_p \times \mathrm{so}_2)\ (p \neq 2, 4)\ ,$$
$$(\mathrm{sp}_{2p}, \mathrm{gl}_p)\ ,\ (\mathrm{so}_{2n}, \mathrm{gl}_n)\ (n \geq 4)\ ,\ (E_6, D_5 \times \mathbb{C})\ ,\ (E_7, E_6 \times \mathbb{C}).$$

It is possible to describe in the same way symmetric subpairs associated to the centralizer of non $\mathfrak{p}$-regular semisimple elements for these symmetric pairs. For example, for $(E_7, E_6 \times \mathbb{C})$, the symmetric subpair $(\mathfrak{r}, \mathfrak{r}_+)$ is a product of symmetric pairs of the form $(\mathrm{so}_{m+1}, \mathrm{so}_m)$ or $(\mathrm{so}_{m+2}, \mathrm{so}_m \times \mathrm{so}_2)$. pairs of rank 2. However, this symmetric pair does not satisfy condition $(\mathcal{E})$ since there is a non-even $\mathfrak{p}$-distinguished element. Namely, the orbit corresponding to label 3 of Table 13 of [9].

Let us also point out that in all the other rank 2 cases listed above, there exists $X \in \mathfrak{a}$ non $\mathfrak{p}$-regular such that $\mathfrak{p}^X$ contains two non proportional commuting nilpotent elements. Hence by the result of [11, Proposition 3], the corresponding symmetric subpair cannot be of the form $(\mathrm{so}_{m+1}, \mathrm{so}_m)$.

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Received December 14, 2004
and in final form May 2, 2005