On the Irreducibility of the Commuting Variety of the Symmetric Pair $so_{p+2}, so_p \times so_2$

Hervé Sabourin and Rupert W.T. Yu

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Abstract. In this paper, we prove that the commuting variety of the family of symmetric pairs $(so_{p+2}, so_p \times so_2)$, $p \geq 2$, is irreducible.

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1. Introduction and notations

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta$ an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces with respect to $\theta$, where $\mathfrak{k} = \{ X \in \mathfrak{g} \mid \theta(X) = X \}$, $\mathfrak{p} = \{ X \in \mathfrak{g} \mid \theta(X) = -X \}$. In this case, we say that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair.

Let $G$ be the adjoint group of $\mathfrak{g}$ and $K$ the connected algebraic subgroup of $G$ whose Lie algebra is $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ consisting of semisimple elements. Any such subspace is called a Cartan subspace of $\mathfrak{p}$. All the Cartan subspaces are $K$-conjugate. Its dimension is called the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

We define the commuting variety of $(\mathfrak{g}, \mathfrak{k})$ as the following set:

$$C(\mathfrak{p}) = \{ (x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0 \}.$$ 

We may also consider the commuting variety $C(\mathfrak{g})$ of $\mathfrak{g}$, defined in the same way. Richardson proved in [10] that, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $C(\mathfrak{g}) = G(\mathfrak{h} \times \mathfrak{h})$. In particular, the commuting variety $C(\mathfrak{g})$ is an irreducible algebraic variety.

On the other hand, the commuting variety of any semisimple symmetric pair is not irreducible in general. Panyushev showed in [7] that in the case of the symmetric pair $(\mathfrak{sl}_n, \mathfrak{gl}_{n-1})$, $n > 2$, associated to the involutive automorphism, defined via conjugation by the diagonal matrix $\text{diag}(-1, \ldots, -1, 1)$, the corresponding commuting variety has three irreducible components of dimension, respectively, $2n - 1, 2n - 2, 2n - 2$. Nevertheless, in some cases, the irreducibility problem has been solved.
- As an obvious consequence of the classical case proved by Richardson, the symmetric pair \((\mathfrak{g} \times \mathfrak{g}, \Delta(\mathfrak{g}))\), associated to the automorphism \((X, Y) \mapsto (Y, X)\), has an irreducible commuting variety.

- If the rank of the symmetric pair \((\mathfrak{g}, \mathfrak{t})\) is equal to the semisimple rank of \(\mathfrak{g}\) (called the maximal rank case), then Panyushev proved in [7] that the corresponding commuting variety is irreducible.

- The rank 1 case has been considered independently by the authors [11] and Panyushev [8]. In this case, it has been proved that \((\mathfrak{so}_{m+1}, \mathfrak{so}_m)\) is the only simple symmetric pair whose commuting variety is irreducible.

- In [8], Panyushev proves the irreducibility of the commuting variety for the symmetric pairs \((\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})\) and \((\mathfrak{E}_6, \mathfrak{F}_4)\).

For a symmetric pair of rank strictly larger than one, we observe that due to the rank 1 case, the inductive arguments used by Richardson in the classical case [10] do not apply. However, if \(\mathfrak{a}\) is a Cartan subspace, then it is well-known that \(C_0 = K.(\mathfrak{a} \times \mathfrak{a})\) is the unique irreducible component of \(C(\mathfrak{p})\) of maximal dimension, which is equal to \(\dim \mathfrak{p} + \dim \mathfrak{a}\). The main problem is therefore to determine if there exist components other than the maximal one.

In [8], it has been conjectured that \(C(\mathfrak{p})\) is irreducible if the rank of the symmetric pair is greater than or equal to 2.

In this paper, by showing that an even nilpotent element in \(\mathfrak{p}\) is contained in a \(K\)-sheet containing non-zero semisimple elements, we obtain that for the commuting variety of a symmetric pair to be irreducible, it suffices that \(\mathfrak{p}\)-distinguished elements in every symmetric subpair are even. We use this to prove that the commuting variety of the family \((\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2), p \geq 2\), of rank 2 symmetric pairs is irreducible.

Let us point out that this family of symmetric pairs comes from a larger family of symmetric pairs associated to parabolic subalgebras with abelian nilpotent radical. For such a symmetric pair in this larger family, it is possible to obtain descriptions of symmetric subpairs associated to centralizers of semisimple elements of \(\mathfrak{p}\) by considering a suitable Cartan subspace. Unfortunately, we are not able to apply the arguments used here.

We shall conserve the notations above in the sequel. The reader may refer to [12] for basic definitions and properties of symmetric pairs.

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2. Sheets and commuting varieties

Let \((\mathfrak{g}, \mathfrak{t})\) be a symmetric pair. Recall that the connected algebraic group \(K\) acts on \(\mathfrak{p}\). For \(n \in \mathbb{N}\), we set:

\[
\mathfrak{p}^{(n)} = \{ X \in \mathfrak{p} ; \ \dim K.X = n \}.
\]

The set \(\mathfrak{p}^{(n)}\) is locally closed, and an irreducible component of \(\mathfrak{p}^{(n)}\) shall be called a \(K\)-sheet of \(\mathfrak{p}\). Clearly, \(K\)-sheets are \(K\)-invariant, and by [12], each \(K\)-sheet contains a nilpotent element.
Let \( \pi_1 : C(\mathfrak{p}) \to \mathfrak{p} \) be the projection \((X,Y) \mapsto X\). Recall the following result concerning the commuting variety of \( \mathfrak{p} \).

**Theorem 2.1.** There exist \( K \)-sheets \( \mathcal{S}_1, \ldots, \mathcal{S}_r \) of \( \mathfrak{p} \) such that \( \pi_1^{-1}(\mathcal{S}_i) \), \( i = 1, \ldots, r \), are the irreducible components of \( C(\mathfrak{p}) \).

The proof of Theorem 2.1 is a simple consequence of the following result. For the sake of completeness, we have included a proof.

**Lemma 2.2.** Let \( V \) be a vector space, \( E \subset V \times V \) a locally closed subvariety and for \( i = 1, 2 \), \( \pi_i : E \to V \) be the projection \((x_1, x_2) \mapsto x_i\). Suppose that:

1. \( \pi_1(E) \) is locally closed.
2. There exists \( r \in \mathbb{N} \) such that for all \( x \in \pi_1(E) \), \( \pi_2(\pi_1^{-1}(x)) \) is a vector subspace of dimension \( r \).

If \( \pi_1(E) \) is irreducible, then so is \( E \).

**Proof.** Let \( G \) be the Grassmann variety of \( r \)-dimensional subspaces of \( V \), \( x \in \pi_1(E) \) and \( W = \pi_2(\pi_1^{-1}(x)) \in G \). Fix a complementary subspace \( U \) of \( W \) in \( V \) and set:

\[
F = \{ T \in G \mid T \cap U = \{0\} \} = \{ T \in G \mid T + U = V \}.
\]

Clearly, \( F \) is an open subset of \( G \) containing \( W \). For \( \tau \in \text{Hom}(W, U) \) the set of linear maps from \( W \) to \( U \), we define:

\[
T(\tau) = \{ w + \tau(w) \mid w \in W \}.
\]

Then we check easily that \( T(\tau) \in F \), and we have a map \( \text{Hom}(W, U) \to F \), \( \tau \mapsto T(\tau) \). We claim that this map is an isomorphism.

Since \( w_1 + \pi_1(w_1) = w_2 + \pi_2(w_2) \) is equivalent to \( w_1 - w_2 = \pi_2(w_2) - \pi_1(w_1) \), we deduce that the above map is injective.

Now if \( T \in F \), then for \( w \in W \), we define \( \tau(w) \) to be the unique element in \( U \) such that \( w + \tau(w) \in T \). We then verify easily that \( T(\tau) = T \). So we have proved our claim.

The map

\[
\Phi : \pi_1(E) \to G, \; y \mapsto \pi_2(\pi_1^{-1}(y))
\]

is a morphism of algebraic varieties. So \( F = \Phi^{-1}(F) \) is an open subset of \( \pi_1(E) \) containing \( x \). The above claim says that we have a well-defined map:

\[
\Psi : F \times W \to E, \; (y, w) \mapsto (y, w + \tau(w))
\]

where \( T(\tau) = \Phi(y) \). It is then a straightforward verification that \( \Psi \) is an isomorphism of the algebraic varieties \( F \times W \) and \( \pi_1^{-1}(F) \).

It follows that the map \( \pi_1 : E \to \pi_1(E) \) is an open map whose fibers are irreducible. Hence by a classical result on topology [3, T.5], if \( \pi_1(E) \) is irreducible, then \( E \) is irreducible. \( \blacksquare \)

Since the set of \( \mathfrak{p} \)-generic elements and the set \( \mathfrak{p}_{\text{reg}} \) of \( \mathfrak{p} \)-regular elements are open subsets of \( \mathfrak{p} \), we have the following corollary:
Corollary 2.3. Let \( a \) be a Cartan subspace in \( p \). The set

\[
C_0 = K.(a \times a) = \pi_1^{-1}(p_{\text{reg}}) = \pi_2^{-1}(p_{\text{reg}})
\]

is the unique irreducible component of \( C(p) \) of maximal dimension.

Let \( X \in p \) be a nilpotent element, and \((H,Y) \in \mathfrak{k} \times p \) be such that \((X,H,Y)\) is a normal \( \mathfrak{sl}_2 \)-triple (called a normal \( S \)-triple in [12]). Recall that \( X \) is even if the eigenvalues of \( \text{ad}_p H \) are even. In fact, this is equivalent to the condition that the eigenvalues of \( \text{ad}_p H \) are even.

Proposition 2.4. Let \( X \in p \) be an even nilpotent element, then \( X \) belongs to a \( K \)-sheet containing semisimple elements.

Proof. Let \((X,H,Y)\) be a normal \( \mathfrak{sl}_2 \)-triple and \( \mathfrak{s} = CX + CH + CY \). Then \( \mathfrak{g} \) decomposes into a direct sum of simple \( \mathfrak{s} \)-modules, say \( V_i, i = 1, \ldots, r \). Since \( X \) is even, \( \dim V_i \) is odd for \( i = 1, \ldots, r \).

For \( \lambda \in \mathbb{C} \), we set \( X_\lambda = X + \lambda Y \in p \). If \( \lambda \neq 0 \), then \( X_\lambda \) is semisimple because \( X_\lambda \) is \( G \)-conjugate to a multiple of \( H \). We claim that \( \dim p^{X_\lambda} = \dim p^X \) for all \( \lambda \in \mathbb{C} \).

First of all, observe that \( p^{X_\lambda} = \bigoplus_{i=1}^{r} (V_i \cap p)^{X_\lambda} \) because \( V_i = (V_i \cap \mathfrak{k}) \oplus (V_i \cap p) \). Moreover \( \dim (V_i \cap p)^{X_\lambda} \leq 1 \).

Now if \( (V_i \cap p)^{X_\lambda} \neq \{0\} \), then a simple weight argument shows that \((V_i \cap p)^X \neq \{0\} \).

Conversely, suppose that \((V_i \cap p)^X \neq \{0\} \). Let \( \dim V_i = 2n+1 \) and \( v_{-n}, \ldots, v_n \) be a basis of weight vectors of \( V_i \) such that \( Hv_k = 2kv_k \), \( k = -n, \ldots, n \). Then \((V_i \cap p)^X = \mathbb{C}v_n \).

So \( v_k \in \mathfrak{k} \) (resp. \( v_k \in p \)) when \( n - k \) is odd (resp. even). In particular, \( v_{-n} \in p \). It follows that for \( k \) such that \( n - k \) odd, \( \lambda Y v_{k+1} = -a_k X v_{k-1} \) for some \( a_k \in \mathbb{C} \). We may therefore renormalize the \( v_k \)’s so that \( v = v_{-n} + v_{-n+2} + \cdots + v_{n-2} + v_n \) verifies \( X_\lambda v = 0 \).

We have therefore proved that \( \dim p^{X_\lambda} = \dim p^X \) for all \( \lambda \).

Now, consider the morphism \( \Phi : K \times \mathbb{C} \rightarrow p, (k,\lambda) \mapsto k.X_\lambda \). The image of \( \Phi \) is irreducible and contains semisimple elements, so it contains strictly \( K.X \). Consequently, \( K.X \) is contained strictly in a \( K \)-sheet with semisimple elements.

Recall that an element of \( p \) is said to be \( p \)-distinguished if its centralizer in \( p \) does not contain any non-zero semisimple element. In particular, a \( p \)-distinguished element is nilpotent. So the number of \( K \)-orbits of \( p \)-distinguished elements is finite.

Definition 2.5. We say that the symmetric pair \((\mathfrak{g}, \mathfrak{t})\) satisfies condition \((E)\) if for every symmetric subpair \((\mathfrak{g}', \mathfrak{t}')\) of \((\mathfrak{g}, \mathfrak{t})\), all the \( \mathfrak{p}' \)-distinguished elements of \( \mathfrak{p}' \) are even.

Theorem 2.6. If \((\mathfrak{g}, \mathfrak{t})\) satisfies condition \((E)\), then the commuting variety \( C(p) \) is irreducible.
Proof. We proceed as in the proof of Richardson in the case of semisimple Lie algebras (see [10]) by using inductive arguments. Let \((X,Y) \in \mathcal{C}(p)\).

1. If \(X\) is semisimple, then \(X\) commutes with a \(p\)-regular semisimple element \(Z\). The line \(L_Z = \{(X, tY + (1-t)Z), t \in \mathbb{C}\}\) is contained in \(\mathcal{C}(p)\). Since \(\{tY + (1-t)Z, t \in \mathbb{C}\}\) meets the set of \(p\)-regular semisimple elements which is open in \(p\), we conclude that \(L_Z\), and hence \((X,Y)\), is contained in \(C_0\) (Corollary 2.3).

2. We may assume that neither \(X\) nor \(Y\) is semisimple.

Suppose that \(X\) is not nilpotent. Let \(X = X_s + X_n\) be the corresponding decomposition into semisimple and nilpotent components. Then \((X,Y) \in g^{X_s}\). Since \((g^{X_s}, t^{X_s})\) is a symmetric subpair of \((g, t)\), we may apply induction to show that \((X,Y) \in C_0\).

3. So we may further assume that \(X\) and \(Y\) are both nilpotent. If \(X\) commutes with a non-zero semisimple element \(Z \in p\), then the same argument as in 1) works because the set of non-nilpotent elements is open.

4. So we are reduced to the case where both \(X\) and \(Y\) are \(p\)-distinguished. Denote by \(\pi_i: C(p) \rightarrow p\) the projection \((X_1, X_2) \mapsto X_1\), \(\mathcal{O}\) the set of non \(p\)-distinguished elements in \(p\), and \(\Omega_1, \ldots, \Omega_r\) the set of \(K\)-orbits of \(p\)-distinguished elements in \(p\). Thus \(p = \mathcal{O} \cup \Omega_1 \cup \cdots \cup \Omega_r\), and \(C(p) = \pi^{-1}_1(\mathcal{O}) \cup \pi^{-1}_1(\Omega_1) \cup \cdots \cup \pi^{-1}_1(\Omega_r)\).

From the previous paragraph, we obtain that \(\pi^{-1}_1(\mathcal{O}) \subset C_0\). Consequently, \(C(p)\) is the union of \(C_0\) with \(\pi^{-1}_1(\Omega_1), \ldots, \pi^{-1}_1(\Omega_r)\). Now we check easily that for \(X \in p\), \(\pi^{-1}_1(K.X) = K.(X, p^X)\) is an irreducible subset of \(C(p)\) of dimension \(\dim \mathfrak{t} - \dim \mathfrak{t}^X + \dim \mathfrak{p}^X = \dim \mathfrak{p}\). It follows that all irreducible components of \(C(p)\) other than \(C_0\), if they exist, are of dimension \(\dim \mathfrak{p}\).

Suppose that \(C(p)\) is not irreducible. By the previous discussion, there exists a \(p\)-distinguished element \(X\) such that \(\pi^{-1}_1(K.X)\) is an irreducible component of dimension \(\dim \mathfrak{p}\).

On the other hand, Condition (\(E\)) and Proposition 2.4 say that \(X\) belongs to a \(K\)-sheet \(\mathcal{S}\) containing non-zero semisimple elements. So \(\dim \mathcal{S} > \dim K.X\). Now Lemma 2.2 says that \(\pi^{-1}_1(\mathcal{S})\) is an irreducible subset of \(C(p)\) containing \(\pi^{-1}_1(K.X)\) and \(\dim \pi^{-1}_1(\mathcal{S}) > \dim \mathfrak{p}\). We have therefore obtained a contradiction.

So the theorem follows.

3. The case of the symmetric pair \((so_{p+2}, so_p \times so_2)\)

Let us fix an integer \(p \geq 2\), \(h\) a Cartan subalgebra of \(g = so_{p+2}\) and a Borel subalgebra \(b\) containing \(h\). Denote by \(R \supset R^+ \supset \Pi\) the corresponding set of roots, positive roots and simple roots. Let us also fix root vectors \(X_{\alpha}, \alpha \in R\), and for \(\alpha \in R\), we set \(g_{\alpha} = CX_{\alpha}\). The rank \(\ell\) of \(g\) is the integer part of \((p + 2)/2\).

Let us first consider the case where \(g\) is simple and not of type \(A_n\), or equivalently, \(p \neq 2, 4\). We shall use the numbering of simple roots \(\alpha_1, \ldots, \alpha_\ell\) in [1]. Let \(H \in h\) be such that

\[
\alpha_i(H) = \begin{cases} 
1 & \text{if } i = 1, \\
0 & \text{if } i \neq 1.
\end{cases}
\]

Then it follows that \(g = g_{-1} \oplus g_0 \oplus g_1\) where \(g_i = \{X \in g; [H, X] = iX\}\).
Observe that \( g_0 \oplus g_1 \) is simply the maximal parabolic subalgebra associated to \( \Pi \setminus \{ \alpha_1 \} \). Its nilpotent radical \( g_1 \) is abelian.

The above decomposition defines a symmetric pair \((g, g_0)\) where \( p = g_{-1} \oplus g_1 \). It is clear that this is precisely the rank 2 symmetric pair \((so_{p+2}, so_p \times so_2)\).

Let \( a \) be the vector space span of the elements \( X_{\alpha_1} + X_{-\alpha_1} \) and \( X_{\alpha_{\text{max}}} + X_{-\alpha_{\text{max}}} \) where \( \alpha_{\text{max}} \) denotes the largest root in \( R \). Then \( a \) is a 2-dimensional abelian subalgebra of \( g \) contained in \( p \). So \( a \) is a Cartan subspace in \( p \).

Let \( X \in a \). Then \( g^X \) is a Levi factor of a parabolic subalgebra of \( g \). Denote by \( l = [g^X, g^X] \) the semisimple part of \( g^X \), and set \( l_+ = l \cap t^X \), \( l_- = l \cap p^X \) and \( t_+ = [l_-, l_-] \). Then the decompositions

\[
g^X = t^X \oplus p^X, \quad l = l_+ \oplus l_- \quad \text{and} \quad t = t_+ \oplus l_-
\]

define symmetric subpairs of \((g, t)\), and the ranks of the pairs \((l, l_+)\) and \((t, t_+)\) are strictly inferior to that of \((g, t)\).

We shall determine the symmetric pair \((t, t_+)\) for any non-zero non \( p \)-regular element \( X \in a \), i.e. \( p^X \) contains a non-zero nilpotent element.

Let us recall the classification of simple symmetric pairs of rank 1.

\[
(sln_{n+1}, sl_n \times \mathbb{C}), \quad (so_{n+1}, so_n), \\
(sp_{2n}, sp_{2n-1} \times sp_2), \quad (F_4, B_4).
\]

**Lemma 3.1.** Let \( X \in a \) be a non-zero non \( p \)-regular element. There exists \( m \in \mathbb{N} \) such that \((t, t_+) = (so_{m+1}, so_m)\).

**Proof.** From the definition of \( a \), we verify easily that \( a \) commutes with the Lie subalgebra \( s \) generated by the root vectors \( X_{\pm \alpha}, \alpha \in \Pi \setminus \{ \alpha_1, \alpha_2 \} \). So \( t^X \) contains \( s \). Note that \( s \simeq so_{p-2} \).

Let \( t \) be a Cartan subalgebra of \( s \), then \( a \oplus t \) is a Cartan subalgebra of \( g^X \) (and also of \( g \) ). It follows that the root system of the semisimple part \( l \) of \( g^X \) contains as a subsystem the root system of \( s \). In particular, the semisimple rank of \( l \) is equal to \( \ell - 2 \) or \( \ell - 1 \) (where \( \ell \) is the rank of \( g \)).

Moreover, since \( X \) is not \( p \)-regular, \( p^X \) contains a non-zero nilpotent element, and so \( l \) contains strictly \( s \).

If \( p + 2 = 2\ell + 1 \), then the preceding discussion implies that the root system \( R(l) \) of \( l \) is of one of the following types: \( A_{\ell-1}, B_{\ell-1} \) or \( A_1 \times B_{\ell-2} \).

Since \( l_- \) contains a simple Lie subalgebra of type \( B_{\ell-2} \), by considering the Satake diagram of the corresponding involution (see for example [4]), it follows easily that \( R(l) \) is of type \( B_{\ell-1} \) or \( A_1 \times B_{\ell-2} \). Consequently, we deduce from the classification of rank 1 symmetric pairs that \((t, t_+)\) has the required form.

If \( p + 2 = 2\ell \), then the root system \( R(l) \) is of one of the following types: \( A_{\ell-1}, D_{\ell-1} \) or \( A_1 \times D_{\ell-2} \). The same argument as above applies, and again, we may conclude by using the classification of rank 1 symmetric pairs.

**Remark 3.2.** Note that we may extend Lemma 3.1 to the symmetric pairs \((so_4, so_2 \times so_2)\) and \((so_6, so_4 \times so_2)\).
In the first case, we have \((\mathfrak{so}_4, \mathfrak{so}_2 \times \mathfrak{so}_2) = (\mathfrak{so}_3, \mathfrak{so}_2) \times (\mathfrak{so}_3, \mathfrak{so}_2)\). Take \(\mathfrak{a}\) to be the direct product of a rank 1 Cartan subspace \(\mathfrak{a}_0\) of the symmetric pair \((\mathfrak{so}_3, \mathfrak{so}_2)\). Then \((X, Y) \in \mathfrak{a}\) is non \(\mathfrak{p}\)-regular if \(X = 0\) or \(Y = 0\). It follows that \(\mathfrak{g}^X \simeq \mathfrak{so}_3 \times \mathbb{C}\), and \((\mathfrak{r}, \mathfrak{r}_+) = (\mathfrak{so}_3, \mathfrak{so}_2)\).

In the second case, we have \((\mathfrak{so}_6, \mathfrak{so}_4 \times \mathfrak{so}_2) = (\mathfrak{sl}_4, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathbb{C})\). Take \(\mathfrak{a}\) to be the vector space span of the vectors \(X_1 = X_{\alpha_1+\alpha_2+\alpha_3} + X_{-(\alpha_1+\alpha_2+\alpha_3)}\) and \(X_2 = X_{\alpha_2} + X_{-\alpha_2}\). Then \(\mathfrak{a}\) is a Cartan subspace, and \(X = \lambda_1 X_1 + \lambda_2 X_2 \in \mathfrak{a}\) is non \(\mathfrak{p}\)-regular if and only if \(\lambda_1 \lambda_2 = 0\). A direct computation shows that \((\mathfrak{r}, \mathfrak{r}_+) = (\mathfrak{so}_3, \mathfrak{so}_2)\).

Summarizing, since Cartan subspaces are \(K\)-conjugate, we have therefore obtained the following result:

**Proposition 3.3.** Let \((\mathfrak{g}, \mathfrak{t})\) be the symmetric pair \((\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)\), \(p \geq 2\). For any non-zero non \(\mathfrak{p}\)-regular semisimple element \(X\) in \(\mathfrak{p}\), the subpair \((\mathfrak{r}, \mathfrak{r}_+)\) is of the form \((\mathfrak{so}_{m+1}, \mathfrak{so}_m)\) for some \(m \in \mathbb{N}\).

**Corollary 3.4.** The commuting variety of \((\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)\) is irreducible.

**Proof.** By the previous proposition and theorem 2.6, it suffices to check that all \(\mathfrak{p}\)-distinguished elements are even for the symmetric pairs \((\mathfrak{so}_{p+1}, \mathfrak{so}_p)\) and \((\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)\).

One may use the classification of Popov-Tevelev [9] of \(\mathfrak{p}\)-distinguished elements. However, it is not difficult to do the checking directly. The case \((\mathfrak{so}_{p+1}, \mathfrak{so}_p)\) is trivial because for any \(X \in \mathfrak{p} \setminus \{0\}\), \(\mathfrak{p}^X\) is \(\mathbb{k}X\) (see for example [11, Proposition 3]). For \((\mathfrak{so}_{p+2}, \mathfrak{so}_p)\), via the Kostant-Sekiguchi correspondence as described in [2] using signed partitions, we observe that there are 6 (resp. 7) non-zero nilpotent \(K\)-orbits in \(\mathfrak{p}\) for \(p > 4\) (resp. \(p = 4\)). Then it is a simple verification that the \(\mathfrak{p}\)-distinguished nilpotent elements correspond to the orbits whose partition has parts all of the same parity. This in turn implies that \(\mathfrak{p}\)-distinguished elements are even.

**Remark 3.5.** The above realization of the symmetric pair \((\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)\) and the construction of the specific Cartan subspace \(\mathfrak{a}\) come from a more general construction.

Namely, let \(\alpha \in \Pi\) be a simple root such that the corresponding standard maximal parabolic subalgebra has an abelian nilpotent radical \(\mathfrak{n}\). Then \(\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1\) as above and we obtain a symmetric pair. Let \(\mathcal{R}(\Pi)\) be the set of pairwise strongly orthogonal roots in \(R^+\) constructed via the “cascade” construction of Kostant (see for example [5], [6] or [13]). Then the vector subspace spanned by the elements \(X_{\beta} + X_{-\beta}\), \(\beta \in \mathcal{R}(\Pi)\) and \(X_{\beta} \in \mathfrak{n}\), is a Cartan subspace. The list of all such symmetric pairs is as follows:

\[
(\mathfrak{sl}_{p+1}, \mathfrak{sl}_{p+1-i} \times \mathfrak{sl}_i \times \mathbb{C}) \ (i = 1, \ldots, p) , \ (\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2) \ (p \neq 2, 4) , \\
(\mathfrak{sp}_{2p}, \mathfrak{gl}_p) , \ (\mathfrak{so}_{2n}, \mathfrak{gl}_n) \ (n \geq 4) , \ (\mathfrak{E}_6, \mathfrak{D}_5 \times \mathbb{C}) , \ (\mathfrak{E}_7, \mathfrak{E}_6 \times \mathbb{C}) .
\]

It is possible to describe in the same way symmetric subpairs associated to the centralizer of non \(\mathfrak{p}\)-regular semisimple elements for these symmetric pairs. For example, for \((\mathfrak{E}_7, \mathfrak{E}_6 \times \mathbb{C})\), the symmetric subpair \((\mathfrak{r}, \mathfrak{r}_+)\) is a product of symmetric
pairs of the form \((\text{so}_{m+1}, \text{so}_m)\) or \((\text{so}_{m+2}, \text{so}_m \times \text{so}_2)\). pairs of rank 2. However, this symmetric pair does not satisfy condition \((E)\) since there is a non-even \(p\)-distinguished element. Namely, the orbit corresponding to label 3 of Table 13 of [9].

Let us also point out that in all the other rank 2 cases listed above, there exists \(X \in \mathfrak{a}\) non \(p\)-regular such that \(p^X\) contains two non proportional commuting nilpotent elements. Hence by the result of [11, Proposition 3], the corresponding symmetric subpair cannot be of the form \((\text{so}_{m+1}, \text{so}_m)\).
References


Hervé Sabourin
UMR 6086 CNRS
Département de Mathématiques
Université de Poitiers
Boulevard Marie et Pierre Curie
86962 Futuroscope Chasseneuil cedex
France
sabourin@math.univ-poitiers.fr

Rupert W. T. Yu
UMR 6086 CNRS
Département de Mathématiques
Université de Poitiers
Boulevard Marie et Pierre Curie
86962 Futuroscope Chasseneuil cedex
France
yuyu@math.univ-poitiers.fr

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