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On the Poisson bracket on the free Lie algebra in two generators

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Abstract. We prove a combinatorial formula for the Poisson bracket of two elements of the free Lie algebra on two generators, which has a particularly nice cocycle form when the two elements are Lie monomials containing only one y. By relating this cocycle form with the period polynomials introduced by Eichler-Shimura and Zagier, we completely describe and classify a set of fundamental relations in Ihara's stable derivation algebra, generalizing the first few cases of these relations which he had observed and computed by hand.
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0. Introduction

The Poisson bracket on the elements of the free Lie algebra $\mathbb{L} = \mathbb{L}[x,y]$ on two generators is given by

$$\{f,g\} = [f,g] + D_f(g) - D_g(f),$$

where $[f,g] = fg - gf$, and to each element $f \in \mathbb{L}$ one associates a derivation D_f of $\mathbb{L}$ via $D_f(x) = 0$, $D_f(y) = [y,f]$. The main result of this article (theorem 3.1 of §3) is a formula expressing $D_f(g)$, for arbitrary elements $f, g \in \mathbb{L}$, in terms of a certain permutation action on monomials in x and y introduced in §2. This yields a formula for the Poisson bracket $\{f,g\}$, and as a consequence, we find a particularly simple "cocycle" form for $\{f,g\}$ when f and g are Lie monomials containing only one y (proposition 3.2).

The next part of the article, §4, contains an application of this cocycle form to answer a question which was originally raised by Y. Ihara (see [2]). Let $\mathbb{L}_n$ denote the graded part of $\mathbb{L}$ generated by Lie words of length n, so that $\mathbb{L} = \oplus_{n\geq 1}\mathbb{L}_n$, and set $\mathcal{F}^m\mathbb{L} = \oplus_{n\geq m}\mathbb{L}_n$. For each odd $n \geq 3$, choose an element $f_n \in \mathbb{L}_n$ satisfying the following condition: its expansion as a polynomial in the non-commutative variables x, y contains the monomial $x^{n-1}y$ with coefficient

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1. For even $n \geq 12$, let E_n denote the vector space of linear combinations

$$\{G = \sum_{\substack{i+j=n \\ i,j\geq 3 \text{ odd}}} a_{ij}\{f_i, f_j\} \mid G \equiv 0 \bmod \mathcal{F}^3\mathbb{L}.\}.$$

Ihara studied a certain very interesting graded Lie subalgebra of $\mathbb{L}$, called the stable derivation algebra $\mathcal{D}$, whose graded parts $\mathcal{D}_n = 0$ for $n = 1, 2, 4, 6$ and are 1-dimensional for $n = 3, 5, 7, 8, 9$. He chose one generator $\tilde{f}_n$ for each $\mathcal{D}_n$, $n = 3, 5, 7, 9$ (the graded part $\mathcal{D}_8$ being generated by $[\tilde{f}_3, \tilde{f}_5]$), however instead of normalizing them by taking the coefficient of the monomial $x^{n-1}y$ which appears in each one to be 1, he required $\tilde{f}_n$ to have integral coefficients. He then observed (cf. [2], I.5, (6.3)) the following surprising fact:

$$2\{\tilde{f}_3, \tilde{f}_9\} - 27\{\tilde{f}_5, \tilde{f}_7\} \equiv 0 \mod 691 = \text{numerator of the Bernoulli number } B_{12}.$$

Replacing $\tilde{f}_i$ by its scalar multiple f_i with the coefficient of $x^{n-1}y$ normalized to 1, this can be written as

$$\{f_3, f_9\} - 3\{f_5, f_7\} \equiv 0 \mod 691 \tag{0.1}$$

It is easy to see that the linear combination $\{f_3, f_9\} - 3\{f_5, f_7\}$ is the only linear combination of $\{f_3, f_9\}$ and $\{f_5, f_7\}$ whose polynomial expansion contains no monomials with less than three y's, i.e. which is $\equiv 0 \bmod \mathcal{F}^3\mathbb{L}$. It is therefore a natural question to determine, for each odd $n \geq 3$, all linear combinations of the brackets $\{f_i, f_j\}$ for $i+j=n$ which are $\equiv 0$ modulo $\mathcal{F}^3\mathbb{L}$, and then ask oneself whether they all satisfy similar surprising arithmetic properties with respect to numerators of Bernoulli numbers. Ihara and Takao found that there is no such linear combination when $n = 14$, i.e. $E_{14} = 0$, and that E_{16} is 1-dimensional, generated by

$$-2\{f_3, f_{13}\} + 7\{f_5, f_{11}\} - 11\{f_7, f_9\}. \tag{0.2}$$

The elements f_{11} and f_{13} are not uniquely determined, since the dimensions of $\mathcal{D}_{11}$ and $\mathcal{D}_{13}$ are not equal to 1, but in fact it is easy to compute that there are (non-unique) choices of these elements for which this linear combination is indeed congruent to 0 modulo 3617, the numerator of the Bernoulli number B_{16}.

This is where the theory of modular forms makes its surprising appearance. Ihara and Takao proved that the dimension of the space E_n is equal to the dimension of $S_n\big(\mathrm{SL}_2(\mathbb{Z})\big)$, the space of cusp forms of weight n on $\mathrm{SL}_2(\mathbb{Z})$, namely $[(n-4)/4] - [(n-2)/6]$ (cf. [2], II.4, Theorem 2). This result does not depend in any way on the actual elements $f_n \in \mathcal{D}_n$ they consider; it holds for any fixed choice of one $f_n \in \mathbb{L}_n$ for each odd $n \geq 3$, as long as f_n contains the monomial $x^{n-1}y$ with non-zero coefficient.

In §4 of this article, using their result on the dimension and the cocycle form mentioned above, we prove (theorem 4.1, corollary 4.2) that E_n is in fact canonically isomorphic to the space of the even period polynomials associated to holomorphic cusp forms by the Eichler-Shimura-Manin correspondence, which were also studied by Zagier. They are polynomials $P(X)$, of degree $n-4$ without constant terms, satisfying the period relations

$$P(X) + X^{n-2}P(\frac{-1}{X}) = 0$$

and

$$P(X) + X^{n-2}P(1-\frac{1}{X}) + (X-1)^{n-2}P(\frac{1}{1-X}) = 0.$$

The Eichler-Shimura-Manin correspondence shows that the vector space of these period polynomials is canonically isomorphic to $S_n\big(\mathrm{SL}_2(\mathbb{Z})\big)$. What we show is that a linear combination $G = \sum_{i+j=n} a_{ij}\{f_i, f_j\}$ lies in E_n if and only if the associated polynomial $P(X) = \sum_{i+j=n} a_{ij}(X^{j-1} - X^{i-1})$ is a period polynomial (see §4 and [3]). For instance, the first period polynomials are $(X^8 - X^2) - 3(X^6 - X^4)$ for $n = 12$, and $-2(X^{12} - X^2) + 7(X^{10} - X^4) - 11(X^8 - X^6)$ for $n = 16$; compare with (0.1) and (0.2) above. As the space of period polynomials is very easy to compute, by solving the small linear system given by the period relations above, the results of §4 mean that the spaces E_n can also easily be determined even for large n; the complete list up to $n = 22$ is given at the end of §4.

Hopefully, the complete knowledge of the elements of E_n, applied to the case where the f_n lie in $\mathcal{D}$, may provide some insight into the surprising appearance of the numerators of the Bernoulli numbers.

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1. The free Lie algebra $\mathbb{L}$ on two letters and the Poisson bracket

Let $\mathbb{L}$ denote the free Lie algebra over $\mathbb{Q}$ on two letters x and y; its elements lie in the ring $\mathbb{Q}\langle x, y\rangle$ of power series in the non-commutative variables x and y. There is a natural grading on this algebra obtained by letting $\deg(x) = \deg(y) = 1$; let $\mathbb{L}_n$ denote the graded part of weight n for $n \geq 1$.

A typical element of $\mathbb{L}$ is a linear combination of *Lie monomials*; these are the expressions formed from x and y by using only brackets, but not addition, such as for instance $[[x,y],[x,[[x,y],x]]]$. For $n \geq 2$, the dimension of the subspace $\mathbb{L}_n^d$ of $\mathbb{L}_n$ generated by Lie monomials containing d y's and $(n-d)$ x's is given by

$$\frac{1}{n}\Bigg(\sum_{a|d,n-d} \mu(a)\frac{(\frac{n}{a})!}{(\frac{n-d}{a})!(\frac{d}{a})!}\Bigg),$$

where μ denotes the Möbius function. In particular, the dimension of $\mathbb{L}_n^1$ is always equal to 1, and this space is generated by the Lie monomial $[x,[x,\cdots,[x,y]\cdots]]$, which we usually abbreviate as $[x^{n-1}y]$.

To each element f of the Lie algebra $\mathbb{L}$, one can associate a derivation D_f of $\mathbb{L}$ by setting $D_f(x) = 0$, $D_f(y) = [y,f]$. The set of derivations $\mathbb{L}_D$

thus obtained is naturally equipped with a law of addition, and defining the Lie bracket $[D_f, D_g] = D_f \circ D_g - D_g \circ D_f$ for composition of derivations makes $\mathbb{L}_D$ into a Lie algebra. Indeed, we have

$$[D_f, D_g](x) = 0$$

and

$$\begin{aligned}
[D_f, D_g](y) &= D_f([y,g]) - D_g([y,f]) \\
&= [D_f(y), g] + [y, D_f(g)] - [D_g(y), f] - [y, D_g(f)] \\
&= [[y,f],g] + [y, D_f(g)] - [[y,g],f] - [y, D_g(f)] \\
&= [[y,f],g] + [[g,y],f] + [y, D_f(g)] - [y, D_g(f)] \\
&= [y,[f,g]] + [y, D_f(g)] - [y, D_g(f)],
\end{aligned}$$

so that in fact

$$[D_f, D_g] = D_h$$

with $h = [f,g] + D_f(g) - D_g(f)$. Since $\mathbb{L}$ and $\mathbb{L}_D$ are bijective as sets, this leads us to introduce a new Lie bracket directly onto the vector space $\mathbb{L}$, namely the Poisson bracket defined by

$$\{f,g\} = [f,g] + D_f(g) - D_g(f).$$

Thus, the vector space $\mathbb{L}$ can be viewed as a Lie algebra in two different ways. Note that $\mathbb{L}$ cannot be viewed as being generated by x and y as a Lie algebra under the Poisson bracket; for instance $\{x,y\} = 0$.

2. Permutation action on elements of $\mathbb{L}$

Definition . Let $\mathbb{Q}^d\langle x,y\rangle$ denote the vector subspace of $\mathbb{Q}\langle x,y\rangle$ generated by all words (monomials) in the non-commutative variables x and y of depth d, i.e. containing d y's. Let $\mathbb{Q}_n\langle x,y\rangle$ denote the vector subspace generated by words of length n. Let $\mathbb{Q}_n^d\langle x,y\rangle$ be generated by words of length n and depth d.

There is a canonical S_{d+1} action on the set of monomials in $\mathbb{Q}^d\langle x,y\rangle$, which preserves $\mathbb{Q}_n^d\langle x,y\rangle$ for each n, given by letting $\sigma \in S_{d+1}$ map the word

$$x^{i_1}yx^{i_2}y\cdots x^{i_d}yx^{i_{d+1}} \quad \text{to} \quad x^{i_{\sigma^{-1}(1)}}yx^{i_{\sigma^{-1}(2)}}y\cdots x^{i_{\sigma^{-1}(d)}}yx^{i_{\sigma^{-1}(d+1)}}.$$

This action extends by linearity to an automorphism of the vector space $\mathbb{Q}^d\langle x,y\rangle$, i.e. it extends to polynomials by linearity.

Notation. Let us introduce some notation concerning this permutation action.

- For any $d \geq 0$, let $\omega_d \in S_{d+1}$ denote the cycle $(1,2,\ldots,d+1)$ (we index ω by d rather than $d+1$ because it will be acting on words of depth d).
- For $d \geq 0$ and $0 \leq i < j \leq d+1$, define the permutation $\theta_d^{i,j} \in S_{d+1}$ by

$$\theta_d^{i,j} = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & j & j+1 & \cdots & d & d+1 \\ 1 & \cdots & i & j & \cdots & i+1 & j+1 & \cdots & d & d+1 \end{pmatrix},$$

where the notation $a\cdots b$ denotes the increasing or decreasing consecutive sequence according to whether $a<b$ or $a>b$. In other words, the permutation $\theta_d^{i,j}$ lies in S_{d+1}, and breaks the sequence $1,\ldots,d+1$ into three subsequences $(1,\ldots,i)$, $(i+1,\ldots,j)$ and $(j+1,\ldots,d+1)$: it fixes the first subsequence, inverts the middle one and again fixes the last one. The case $i=0$ means that the first (fixed) subsequence is empty; the case $j=d+1$ means that the last (fixed) subsequence is empty. Thus, $\theta_d^{0,d+1}$ is simply the "backwards" permutation in S_{d+1}. We drop the superscript $0,d+1$ and simply write θ_d for the backwards permutation of length $d+1$.

• Let $f\in\mathbb{Q}_n^d\langle x,y\rangle$. Then for every word v of length n and depth d, we write (f,v) for the coefficient of the word v in f. Clearly we have

$$\big(\tau(f),\tau(v)\big)=\big(f,v\big),\quad \text{i.e.}\quad \big(f,\tau(v)\big)=\big(\tau^{-1}(f),v\big) \tag{2.1}$$

for every $\tau\in S_{d+1}$.

Lemma 2.1. *(i) For every element $f\in\mathbb{L}$ of homogeneous length ℓ_f and depth d_f, we have*

$$f=(-1)^{\ell_f-1}\,\theta_{d_f}(f). \tag{2.2}$$

(ii) For every pair of elements f and g as in (i), we have

$$fg=(-1)^{\ell_f+\ell_g}\,\theta_{d_f+d_g}(gf). \tag{2.3}$$

Proof. By linearity, it is enough to consider the case where f and g are both just Lie monomials. Let us prove (i) by induction on the length of f. If f is of length 1, i.e. equal to x or y, the statement $f=(-1)^{\ell_f-1}\theta_{d_f}(f)$ holds. Now let f be a Lie monomial of length ℓ_f; then f can be written as $AB-BA$ with A and B Lie words of length $<\ell_f$. Let ℓ_A and ℓ_B denote the lengths of A and B, so that $\ell_A+\ell_B=\ell_f$, and let d_A and d_B denote their depths, so that $d_f=d_A+d_B$; it is also the depth of AB. By induction, we assume that the first statement holds for A and B, i.e. $A=(-1)^{\ell_A-1}\theta_{d_A}(A)$ and $B=(-1)^{\ell_B-1}\theta_{d_B}(B)$. Consider the expansions of the Lie monomials A and B as polynomials in x and y. Then by (2.1) and the induction hypothesis, for every word v of length ℓ_A and depth d_A, we have

$$\big(\theta_{d_A}(A),v\big)=\big(A,\theta_{d_A}(v)\big)=(-1)^{\ell_A-1}\big(\theta_{d_A}(A),\theta_{d_A}(v)\big)=(-1)^{\ell_A-1}(A,v),$$

and similarly for B.

A given word v of length n will appear in the expansion of AB if and only if it is the concatenation $v=v_1v_2$ of two words v_1 and v_2, where v_1 appears in A, so is of length ℓ_A and depth d_A, and v_2 appears in B, so is of length ℓ_B and depth d_B. Then the coefficient (AB,v) of the word v in AB is equal to the product of the coefficients $(A,v_1)(B,v_2)$. By the induction hypothesis, this is equal to $(-1)^{\ell_A+\ell_B}(A,\theta_{d_A}(v_1))(B,\theta_{d_B}(v_2))$, and since $\theta_{d_f}(v)=\theta_{d_B}(v_2)\theta_{d_A}(v_1)$, we find that $(AB,v)=(-1)^{\ell_f}(BA,\theta_{d_f}(v))$. Exchanging the roles of A and

B, we also have $(BA,v) = (-1)^{\ell_f}(AB,\theta_{d_f}(v))$. Therefore, for every word v appearing in f, we have

$$\begin{aligned}(f,v) &= (AB-BA,v)\\ &= (AB,v)-(BA,v)\\ &= (-1)^{\ell_f}\big(BA,\theta_{d_f}(v)\big)-(-1)^{\ell_f}\big(AB,\theta_{d_f}(v)\big)\\ &= (-1)^{\ell_f-1}\big(AB-BA,\theta_{d_f}(v)\big)\\ &= (-1)^{\ell_f-1}\big(\theta_{d_f}(AB-BA),v\big)\\ &= \big((-1)^{\ell_f-1}\theta_{d_f}(f),v\big).\end{aligned}$$

Thus, for every word v appearing in f, the coefficients of f and $(-1)^{\ell_f-1}\theta_{d_f}(f)$ are equal, so these two polynomials are equal (given that they contain the same number of monomials), which proves (i).

The same argument also automatically proves the second statement, since now it is no longer an induction hypothesis, but a proven fact that $A = (-1)^{\ell_A-1}\theta_{d_A}(A)$ and $B=(-1)^{\ell_B-1}\theta_{d_B}(B)$ for every pair of Lie monomials A and B, and we saw that this led to

$$(BA,v)=(-1)^{\ell_A+\ell_B}\big(AB,\theta_{d_A+d_B}(v)\big)=\big((-1)^{\ell_A+\ell_B}\theta_{d_A+d_B}(AB),v\big)$$

for all v, i.e. $BA=(-1)^{\ell_A+\ell_B}\theta_{d_A+d_B}(AB)$. ∎

3. A formula for the Poisson bracket

We now come to the main theorem of this article; it gives an explicit expression for $D_f(g)$, which a fortiori gives rise to an explicit expression for $\{f,g\}$, in terms of permutation actions on the elements of $\mathbb{L}$.

Theorem 3.1. *Let f (resp. g) be an element of $\mathbb{L}$ of homogeneous length ℓ_f and depth d_f (resp. ℓ_g and d_g); assume that $d_g \geq 1$. Set $\ell=\ell_f+\ell_g$ and $d=d_f+d_g$. Then*

$$\begin{aligned}D_f(g) = {}&(-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(gf)-\theta_d^{0,d_f+1}(fg)\Big)\\ &+\sum_{i=1}^{d_g-1}\bigg((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^{i}(g)f)\Big)-\omega_d^{-i}(\omega_{d_g}^{i}(g)f)\bigg).\end{aligned}\tag{3.1}$$

In particular, when $d_g=1$, we have

$$D_f(g)=(-1)^{\ell_f-1}\Big(\theta_d^{1,d+1}(gf)-\theta_d^{0,d_f+1}(fg)\Big).\tag{3.2}$$

To prove this theorem, one proceeds by setting $E_f(x)=0$ and then, for all $g\in\mathbb{L}$ with $d_g\geq 1$, defining $E_f(g)$ to be the right-hand side of (3.1); one then shows that $E_f(y)=yf-fy$ (easy) and finally that E_f is a derivation, so that

$E_f(g) = D_f(g)$. The proof of this theorem is essentially a long computation using several properties of the permutation action; it has been relegated to the Appendix.

As we noted in §1, the space $\mathbb{L}_n^1$ of elements of $\mathbb{L}$ of length n and depth 1 is one-dimensional, generated by the Lie monomial $[x^{n-1}y]$. Let us now study the Poisson bracket of two such elements of odd length.

Proposition 3.2. *Let $f = [x^{n-1}y] \in \mathbb{L}_n^1$ and $g = [x^{m-1}y] \in \mathbb{L}_m^1$, with n, m odd ≥ 3. Let $\theta = (13)$ and $\omega = (123)$ in S_3. Then*

$$\{f, g\} = [f, g] + \omega([f, g]) + \omega^2([f, g]). \tag{3.3}$$

Proof. By (3.2) of theorem 3.1, we have

$$D_f(g) = \theta_2^{1,3}(gf) - \theta_2^{0,2}(fg), \qquad D_g(f) = \theta_2^{1,3}(fg) - \theta_2^{0,2}(gf),$$

so

$$\{f, g\} = [f, g] + \theta_2^{1,3}(gf) - \theta_2^{0,2}(fg) - \theta_2^{1,3}(fg) + \theta_2^{0,2}(gf).$$

Writing out the permutations, we have $\theta_2^{1,3} = (23)$ and $\theta_2^{0,2} = (12)$, so

$$\{f, g\} = [f, g] + (23)(gf) - (12)(fg) - (23)(fg) + (12)(gf).$$

By lemma 2.1 (ii), we have $fg = \theta_2(gf)$, but $\theta_2 = (13)$, so

$$\begin{aligned}
\{f, g\} &= [f, g] + (23)(13)(fg) - (12)(13)(gf) - (23)(13)(gf) + (12)(13)(fg) \\
&= [f, g] + (123)(fg) - (132)(gf) - (123)(gf) + (132)(fg) \\
&= [f, g] + (123)(fg - gf) + (132)(fg - gf) \\
&= [f, g] + \omega([f, g]) + \omega^2([f, g]).
\end{aligned}$$

which proves the result. ■

4. Relation with period polynomials

Definition . Let n be a positive even number. An even polynomial $P(X)$ of degree $n-4$ with no constant term is said to be a *period polynomial* if it satisfies the period relations

$$P(X) + X^{n-2}P(\frac{-1}{X}) = 0$$

and

$$P(X) + X^{n-2}P(1 - \frac{1}{X}) + (X - 1)^{n-2}P(\frac{1}{1 - X}) = 0.$$

It is known [3] that these polynomials are in bijection with cusp forms of weight n on $\mathrm{SL}_2(\mathbb{Z})$, and therefore the vector space of such polynomials has dimension equal to $\dim S_n\big(\mathrm{SL}_2(\mathbb{Z})\big)$.

Theorem 4.1. *Fix an even number $n \geq 12$ and set*

$$F = \sum_{i=1}^{(n-4)/2} a_i [x^{2i} y][x^{n-2-2i} y]; \tag{4.1}$$

consider the S_3 action on the polynomial expansion of F in $\mathbb{Q}_n^2\langle x, y\rangle$. Let $\theta = (13)$ and $\omega = (123) \in S_3$. Then F satisfies the cocycle relations

$$F + \theta(F) = 0 \quad \textit{and} \quad F + \omega(F) + \omega^2(F) = 0$$

if and only if the associated one-variable polynomial

$$P(X) = \sum_{i=1}^{(n-4)/2} a_i X^{n-2-2i},$$

which is even of degree $n-4$ with no constant term, is a period polynomial.

Proof. By lemma 2.1 (ii), the condition $F + \theta(F) = 0$ is satisfied if and only if

$$a_{n/2-1-i} = -a_i$$

for $1 \leq i \leq [(n-4)/4]$, i.e. if F can be written

$$F = \sum_{i=1}^{[(n-4)/4]} a_i \big[[x^{2i} y], [x^{n-2-2i} y] \big].$$

This is exactly equivalent to the first cocycle relation $P(X) + X^{n-2}P(-1/X) = 0$ on $P(X)$. The proof is more complicated for the second cocycle condition. Assume that F and $P(X)$ satisfy the first cocycle condition.

Set $Q(X) = P(X) + X^{n-2}P(1-1/X) + (X-1)^{n-2}P(1/(1-X))$ and expand $Q(X)$ in powers of X. We easily see that the degree of $Q(X)$ is equal to $n-3$ and that it has no constant term, so if $Q(X) = 0$, we obtain $n-3$ linear equations in the a_i. By symmetry, the coefficient of X^i is equal to the opposite of that of X^{n-2-i}, so that it is enough to consider the coefficients of $X, X^2, \ldots, X^{(n-4)/2}$ (even this is redundant). For $j = 1, \ldots, (n-4)/2$, let b_j be the coefficient of X^j in $Q(X)$, so that b_j is a linear combination of $a_1, \ldots, a_{[(n-4)/4]}$.

Let us compute the b_j explicitly. Each b_j is a linear combination of $a_1, \ldots,$ $a_{[(n-4)/4]}$ which is the coefficient of X^j in $Q(X)$. There are three separate contributions to this coefficient coming from the X^j terms of the three polynomials

$$P(X), \quad X^{n-2}P(1-1/X), \quad (X-1)^{n-2}P(1/(1-X)).$$

So we compute the coefficient of a_i in the X^j term of each of these three polynomials. We find:

- The contribution from $P(X)$ is equal to

$$\begin{cases} -1 & \text{for } j = 2i \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

- The contribution from $X^{n-2}P(1-1/X)$ is equal to

$$(-1)^j \binom{n-2-2i}{j-2i}. \tag{4.3}$$

- The contribution from $(X-1)^{n-2}P(1/(1-X))$ is equal to

$$(-1)^j \binom{2i}{j} - (-1)^j \binom{n-2-2i}{j}. \tag{4.4}$$

(To check this last equality, it is easier to write $P(X) = a_1X^{n-4}+\cdots+a_{(n-4)/2}X^2$ and separately consider the contribution from a_i, which is equal to $(-1)^j\binom{2i}{j}$ and from $a_{(n-2-2i)/2} = -a_i$, which is equal to $(-1)^j\binom{n-2-2i}{j}$.)

As a total, we find that the coefficient of a_i in b_j for $1 \leq j \leq (n-4)/2$ is equal to the sum of (4.2), (4.3), (4.4), i.e. to

$$-\delta_{j,2i} + (-1)^j\binom{n-2-2i}{j-2i} + (-1)^j\binom{2i}{j} - (-1)^j\binom{n-2-2i}{j}. \tag{4.5}$$

Now consider F satisfying $F + \theta(F) = 0$, so that F is of the form

$$\sum_{i=1}^{[(n-4)/4]} a_i\big[\,[x^{2i}y], [x^{n-2-2i}y]\,\big],$$

and set $G = F + \omega(F) + \omega^2(F)$. If we expand G as a homogeneous polynomial of length n and depth 2, then the coefficient of each word occurring in G is a linear combination in $a_1, \ldots, a_{[(n-4)/4]}$, so that if $G = 0$ we have a very large and redundant linear system in the a_i.

We will now show that the solutions of this linear system are the same as those of the linear system $b_1 = 0, b_2 = 0, \ldots, b_{(n-4)/2} = 0$ coming from the second cocycle relation $Q(X) = 0$. We proceed in two steps; first we show that the linear combinations b_j coming from $Q(X) = 0$ occur explicitly in the linear system coming from $G = 0$, as the (negatives of the) coefficients of the words $x^jy^2x^{n-2-j}$ for $1 \leq j \leq (n-4)/2$ in the expansion of G. Then we conclude that the two systems coming from $Q(X) = 0$ and from $G = 0$ are equivalent, by using known results on the dimensions of the space of solutions.

For the first step, we need to compute the coefficient of the word $x^jy^2x^{n-2-j}$ in G for $1 \leq j \leq (n-4)/2$. Like b_j, this coefficient has three contributions, which are the coefficients of the word $x^jy^2x^{n-2-j}$ coming from F, $\omega(F)$ and $\omega^2(F)$ respectively. But by the definition of ω, these are equal to the coefficients of the three words $x^jy^2x^{n-2-j}$, $x^{n-2-j}yx^jy$ and $yx^{n-2-j}yx^j$ in F. For all three coefficients, we directly use the expansions

$$[x^{2i}y]=\sum_{u=0}^{2i}(-1)^u\binom{2i}{u}x^uyx^{2i-u} \tag{4.6}$$

and

$$[x^{n-2-2i}y]=\sum_{v=0}^{n-2-2i}(-1)^v\binom{n-2-2i}{v}x^{n-2-2i-v}yx^v. \tag{4.7}$$

Let us first compute the coefficient in F of $x^jy^2x^{n-2-j}$ for $1\le j\le (n-4)/2$. We first compute the coefficient of $x^jy^2x^{n-2-j}$ in the product $[x^{2i}y]\cdot[x^{n-2-2i}y]$; it comes from the term $u=j$, $v=n-2-j$ and thus has coefficient

$$(-1)^j\binom{2i}{j}\cdot(-1)^{n-2-j}\binom{n-2-2i}{n-2-j}=\binom{2i}{j}\binom{n-2-2i}{n-2-j}. \tag{4.8}$$

Clearly this quantity is thus equal to

$$\begin{cases}0 & j>2i\\ 1 & j=2i\\ 0 & j<2i.\end{cases} \tag{4.9}$$

Similarly, we compute the coefficient of $x^jy^2x^{n-2-j}$ in the product $[x^{n-2-2i}y]\cdot[x^{2i}y]$ and obtain

$$(-1)^{n-2-j}\binom{2i}{n-2-j}\cdot(-1)^j\binom{n-2-2i}{j}=\binom{2i}{n-2-j}\binom{n-2-2i}{j}.$$

Now, we have $1\le i\le[(n-4)/4]$ and $1\le j\le(n-2)/4$ so $2i+j\le(n-4)/2+(n-2)/4=(3n-10)/4<n-2$. Thus $2i<n-2-j$ and the first binomial coefficient is zero, so we have shown that the coefficient of $x^jy^2x^{n-2-j}$ in $[x^{n-2-2i}y]\cdot[x^{2i}y]$, is always zero, and therefore the coefficient of $x^jy^2x^{n-2-j}$ in $\big[[x^{2i}y],[x^{n-2-2i}y]\big]$ is given by (4.9), i.e.

• coefficient of $x^jy^2x^{n-2-j}$ in F:

$$\begin{cases}0 & j>2i\\ 1 & j=2i\\ 0 & j<2i.\end{cases} \tag{4.10}$$

We are sorry this is taking so long. Fortunately, it is easier to compute the coefficient of $x^jy^2x^{n-2-j}$ in $\omega(F)$. By computing the coefficient of $yx^{n-2-j}yx^j$ in F, again using (4.6) and (4.7), we directly obtain

• coefficient of $x^jy^2x^{n-2-j}$ in $\omega(F)$:

$$(-1)^j\binom{n-2-2i}{j}-(-1)^j\binom{2i}{j}. \tag{4.11}$$

Finally, we compute the coefficient of $x^jy^2x^{n-2-j}$ in $\omega^2(F)$ by computing the coefficient of $x^{n-2-j}yx^jy$ in F. The same direct computation yields

$$(-1)^j\binom{2i}{n-2-j} - (-1)^j\binom{n-2-2i}{n-2-j}.$$

But we saw above that $2i+j<n-2$ so $2i<n-2-j$ so the first term is zero, and using $\binom{a}{b}=\binom{a}{a-b}$, we finally obtain

• coefficient of $x^jy^2x^{n-2-j}$ in $\omega^2(F)$:

$$-(-1)^j\binom{n-2-2i}{j-2i}. \tag{4.12}$$

The sum of (4.10), (4.11) and (4.12) give the coefficient of a_i in the linear combination of $a_1,\ldots,a_{[(n-4)/4]}$ which is the coefficient of $x^jy^2x^{n-2-j}$ in G, and it is equal to the negative of the quantity in (4.5). Therefore, we have shown that the coefficient of the word $x^jy^2x^{n-2-j}$ in $G=F+\omega(F)+\omega^2(F)$ is equal to $-b_j$, where we recall that b_j is the coefficient of X^j in

$$Q(X)=P(X)+X^{n-2}P(1-1/X)+(X-1)^{n-2}P(1/(1-X)).$$

We thus see that the linear system given by the coefficients of G strictly contains the much smaller linear system given by the coefficients of Q. Our second step is to show that the sets of solutions coincide by a dimension count. Indeed, Zagier [3] has shown that the dimension of the space of solutions of $Q(X)=0$ for $P(X)$ of even degree $n\geq 12$ is equal to $\dim S_n(\mathrm{SL}_2(\mathbb{Z}))$. But by proposition 3.2, the space of solutions to $G=0$ is equal to the space of solutions of linear combinations of the form

$$\sum_{i=0}^{[(n-4)/4]} a_i\big\{[x^{2i}y],[x^{n-2-2i}y]\big\}=0$$

and Ihara and Takao (see [2], Theorem 2 of II.4) have shown that the dimension of this space of solutions is also equal to $\dim S_n(\mathrm{SL}_2(\mathbb{Z}))$. This concludes the proof of theorem 4.1. ■

For each odd $n\geq 3$, let $\mathcal{F}^n\mathbb{L}$ denote the subspace of $\mathbb{L}$ generated by all Lie monomials containing n or more y's.

Corollary 4.2. *For each odd $n\geq 3$, choose an element f_n of $\mathbb{L}_n$ whose polynomial expansion contains the monomial $x^{n-1}y$ with coefficient 1. Let*

$$G=\sum_{i=1}^{[(n-4)/4]} a_i\big\{f_{2i+1},f_{n-1-2i}\big\}\in\mathbb{L}_n.$$

Then

$$G\equiv 0 \mod \mathcal{F}^3\mathbb{L}$$

if and only if the polynomial

$$P(X) = \sum_{i=1}^{[(n-4)/4]} a_i(X^{n-2-2i} - X^{2i})$$

is a period polynomial.

Proof. Clearly $G \equiv 0 \mod \mathcal{F}^3\mathbb{L}$ if and only if $G' = 0$, where

$$G' = \sum_{i=1}^{[(n-4)/4]} a_i \big\{ [x^{2i}y], [x^{n-2-2i}y] \big\} \in \mathbb{L}_n.$$

By proposition 3.2, G' is equal to

$$\sum_{i=1}^{[(n-4)/4]} a_i \Bigg(\big[[x^{2i}y], [x^{n-2-2i}y] \big] + \omega\big(\big[[x^{2i}y], [x^{n-2-2i}y] \big] \big) + \omega^2\big(\big[[x^{2i}y], [x^{n-2-2i}y] \big] \big) \Bigg)$$

which is equal to $F + \omega(F) + \omega^2(F)$ for $F = \sum_{i=1}^{[(n-4)/4]} a_i \big[[x^{2i}y], [x^{n-2-2i}y] \big]$. But we saw that for such an F, we have $F + \theta(F) = 0$. Therefore $G' = 0$ if and only if F satisfies the two cocycle relations, which by theorem 4.2 can happen if and only if the associated polynomial $P(X)$ is a period polynomial. ∎

Examples . For $n = 12, 16, 18, 20, 22$ we have $\dim S_n(\mathrm{SL}_2(\mathbb{Z})) = 1$, so up to scalar multiple, there is exactly one relation for each of these values of n, and exactly one even period polynomial of degree $n-4$. They are given by

1) $n = 12$: $\{f_3, f_9\} - 3\{f_5, f_7\} \equiv 0 \mod \mathcal{F}^3\mathbb{L}$
$P(X) = X^8 - 3X^6 + 3X^4 - X^2$.

2) $n = 16$: $-2\{f_3, f_{13}\} + 7\{f_5, f_{11}\} - 11\{f_7, f_9\} \equiv 0 \mod \mathcal{F}^3\mathbb{L}$
$P(X) = -2X^{12} + 7X^{10} - 11X^8 + 11X^6 - 7X^4 + 2X^2$.

3) $n = 18$: $8\{f_3, f_{15}\} - 25\{f_5, f_{13}\} + 26\{f_7, f_{11}\} \equiv 0 \mod \mathcal{F}^3\mathbb{L}$
$P(X) = 8X^{14} - 25X^{12} + 26X^{10} - 26X^6 + 25X^4 - 8X^2$.

4) $n = 20$: $3\{f_3, f_{17}\} - 10\{f_5, f_{15}\} + 14\{f_7, f_{13}\} - 13\{f_9, f_{11}\} \equiv 0 \mod \mathcal{F}^3\mathbb{L}$
$P(X) = 3X^{16} - 10X^{14} + 14X^{12} - 13X^{10} + 13X^8 - 14X^6 + 10X^4 - 3X^2$.

5) $n = 22$: $32\{f_3, f_{19}\} - 105\{f_5, f_{17}\} + 136\{f_7, f_{15}\} - 85\{f_9, f_{13}\} \equiv 0 \mod \mathcal{F}^3\mathbb{L}$
$P(X) = 32X^{18} - 105X^{16} + 136X^{14} - 85X^{12} + 85X^8 - 136X^6 + 105X^4 - 32X^2$.

Appendix: Proof of Theorem 3.1

Let us begin by assembling some properties of the action of the permutations $\theta_d^{i,j}$ on products of words; they are all immediate, so we do not include the proofs.

Lemma A.1. *Let f be a homogeneous polynomial of length ℓ_f and depth d_f; let $0 \le i < j \le d+1$. Then*

$$x\,\theta^{i,j}_{d_f}(f) = \theta^{i,j}_{d_f}(xf) \quad \text{if} \quad i > 0, \tag{A.1}$$

$$\theta^{i,j}_{d_f}(f)x = \theta^{i,j}_{d_f}(fx) \quad \text{if} \quad j < d_f + 1, \tag{A.2}$$

$$x\,\theta_{d_f}(f) = \theta_{d_f}(fx) \quad \text{and} \quad \theta_{d_f}(f)x = \theta_{d_f}(xf) \tag{A.3}$$

$$y\,\theta^{i,j}_{d_f}(f) = \theta^{i+1,j+1}_{d_f+1}(yf) \tag{A.4}$$

$$\theta^{i,j}_{d_f}(f)y = \theta^{i,j}_{d_f+1}(fy) \tag{A.5}$$

If g is homogeneous of length ℓ_g and depth d_g, then

$$x\,\theta^{0,d_f+1}_{d_f+d_g}(fg) = \theta^{0,d_f+1}_{d_f+d_g}(fxg) \tag{A.6}$$

$$\theta^{d_f,d_f+d_g+1}_{d_f+d_g}(fg)\,x = \theta^{d_f,d_f+d_g+1}_{d_f+d_g}(fxg) \tag{A.7}$$

Some of these equalities can be generalized to entire words instead of just x and y. Let h be a word of length ℓ_h and depth d_h. Then

$$h\,\theta^{i,j}_{d_f}(f) = \theta^{i+d_h,j+d_h}_{d_f+d_h}(hf) \quad \text{if } i > 0 \tag{A.8}$$

$$\theta^{i,j}_{d_f}(f)\,h = \theta^{i,j}_{d_f+d_h}(fh) \quad \text{if } j < d_f + 1 \tag{A.9}$$

Lemma A.2. *For $1 \le i \le d_g$, we have*

$$\omega_d^{-i}\big(\omega^i_{d_g}(g)f\big)x = \omega_d^{-i}\big(\omega^i_{d_g}(gx)f\big) \quad \text{and} \quad x\omega_d^{-i}\big(\omega^i_{d_g}(g)f\big) = \omega_d^{-i}\big(\omega^i_{d_g}(xg)f\big), \tag{A.10}$$

and more generally,

$$\begin{cases} \omega^{-i}_{d_g+d_f}\big(\omega^i_{d_g}(g)f\big)h = \omega^{-i-d_h}_{d_g+d_f+d_h}\big(\omega^{i+d_h}_{d_g+d_h}(gh)f\big) \\ h\omega^{-i}_{d_f+d_g}\big(\omega^i_{d_g}(g)f\big) = \omega^{-i}_{d_f+d_g+d_h}\big(\omega^i_{d_g+d_h}(hg)f\big). \end{cases} \tag{A.11}$$

Lemma A.3. *We have the following relations between θ's and ω's:*

$$g\,\theta^{0,d_f+1}_{d_f+d_h}(fh) = (-1)^{\ell_f-1}\omega_d^{-d_h}\big(\omega^{d_h}_{d_g+d_h}(gh)f\big) \tag{A.12}$$

and

$$\theta^{d_h,d_f+d_h+1}_{d_f+d_h}(hf)\,g = \theta^{d_h,d_f+d_h+1}_{d}\Big(\omega_d^{-d_g}\big(\omega^{d_g}_{d_g+d_h}(hg)f\big)\Big) \tag{A.13}$$

Theorem 3.1. *Let f be an element of $\mathbb{L}$ of homogeneous length ℓ_f and depth d_f, and similarly, let g be an element of $\mathbb{L}$ of homogeneous length ℓ_g and depth d_g; assume that $d_g \geq 1$. Set $\ell = \ell_f + \ell_g$ and $d = d_f + d_g$. Then*

$$\begin{aligned}D_f(g) &= (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(gf) - \theta_d^{0,d_f+1}(fg)\Big) \\ &\quad + \sum_{i=1}^{d_g-1}\bigg((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big) - \omega_d^{-i}(\omega_{d_g}^i(g)f)\bigg).\end{aligned} \tag{$*$}$$

Proof. For all g of depth ≥ 1, let $E_f(g)$ be the right-hand side of (*), and set $E_f(x) = 0$. In order to prove that $E_f(g) = D_f(g)$, it suffices to show first that $E_f(y) = D_f(y)$ and then that E_f is a derivation.

For $g = y$, we have

$$\begin{aligned}E_f(y) &= (-1)^{\ell_f-1}\Big(\theta_{d_f+1}^{1,d_f+2}(yf) - \theta_{d_f+1}^{0,d_f+1}(fy)\Big) \\ &= (-1)^{\ell_f-1}\Big(y\,\theta_{d_f}^{0,d_f+1}(f) - \theta_{d_f}^{0,d_f+1}(f)\,y\Big) \qquad \text{(by (A.4) and (A.5))} \\ &= yf - fy \qquad \text{(by lemma 2.1 (i))} \\ &= D_f(y).\end{aligned}$$

Let us now show that E_f is a derivation, i.e. that

$$E_f([g,h]) = [E_f(g),h] + [g,E_f(h)]$$

for all elements g, h; by linearity, we may assume that g and h are Lie monomials. We first expedite the case where one of g,h is equal to x (these two cases are equivalent since $[g,h] = -[h,g]$. Assume thus that $h = x$; then $E_f(h) = 0$, so we need to show that

$$[E_f(g),x] = E_f([g,x]).$$

We begin by computing $[E_f(g),x] = E_f(g)x - xE_f(g)$. Using (*) and identities (A.1), (A.2), (A.3), (A.6) and (A.7) from lemma A.1, we compute

$$\begin{aligned}E_f(g)x = &\ (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(gf)x - \theta_d^{0,d_f+1}(fg)x\Big) \\ &+ \sum_{i=1}^{d_g-1}\bigg((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big)x - \omega_d^{-i}(\omega_{d_g}^i(g)f)x\bigg) \\ = &\ (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(gxf) - \theta_d^{0,d_f+1}(fgx)\Big) \\ &+ \sum_{i=1}^{d_g-1}\bigg((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(g)f)x\Big) - \omega_d^{-i}(\omega_{d_g}^i(g)f)x\bigg),\end{aligned}$$

and

$$\begin{aligned} xE_f(g) = & \ (-1)^{\ell_f-1}\Big(x\theta_d^{d_g,d+1}(gf) - x\theta_d^{0,d_f+1}(fg)\Big) \\ & + \sum_{i=1}^{d_g-1}\Big((-1)^{\ell_f-1}x\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big) - x\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big) \\ = & \ (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(xgf) - \theta_d^{0,d_f+1}(fxg)\Big) \\ & + \sum_{i=1}^{d_g-1}\Big((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(x\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big) - x\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big), \end{aligned}$$

Thus the difference $[E_f(g), x] = E_f(g)x - xE_f(g)$ is given by

$$\begin{aligned} & (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(gxf) - \theta_d^{0,d_f+1}(fgx)\Big) \\ & - (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}(xgf) + \theta_d^{0,d_f+1}(fxg)\Big) \\ & + \sum_{i=1}^{d_g-1}\Big((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(g)f)x\Big) - \omega_d^{-i}(\omega_{d_g}^i(g)f)x \\ & \qquad - (-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(x\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big) + x\omega_d^{-i}(\omega_{d_g}^i(g)f)\Big). \end{aligned}$$

Using (A.10) from lemma A.2 and combining terms, this reduces to

$$\begin{aligned} & [E_f(g), x] \\ & = E_f(g)x - xE_f(g) = (-1)^{\ell_f-1}\Big(\theta_d^{d_g,d+1}((gx-xg)f) - \theta_d^{0,d_f+1}(f(gx-xg))\Big) \\ & + \sum_{i=1}^{d_g-1}\Big((-1)^{\ell_f-1}\theta_d^{d_g-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g}^i(gx-xg)f)\Big) - \omega_d^{-i}(\omega_{d_g}^i(gx-xg)f)\Big) \\ & = E_f(gx-xg) \end{aligned}$$

as desired.

Let us now pass to the case where neither g nor h is equal to x, i.e. $d_g, d_h \geq 1$. We will compute

$$[E_f(g), h] + [g, E_f(h)]$$

and show that it is equal to $E_f([g,h])$. In fact, we begin by computing

$$\begin{aligned} & (-1)^{\ell_f-1}\Big([E_f(g), h] + [g, E_f(h)]\Big) \\ & = \Big[\theta_{d_f+d_g}^{d_g,d_f+d_g+1}(gf) - \theta_{d_f+d_g}^{0,d_f+1}(fg) \\ & \ + \sum_{i=1}^{d_g-1}\Big(\theta_{d_f+d_g}^{d_g-i,d_g+d_f-i+1}\Big(\omega_{d_g+d_f}^{-i}(\omega_{d_g}^i(g)f)\Big) - (-1)^{\ell_f-1}\omega_{d_g+d_f}^{-i}(\omega_{d_g}^i(g)f)\Big)\, ,\, h\Big] \\ & + \Big[\, g\, ,\, \theta_{d_f+d_h}^{d_h,d_f+d_h+1}(hf) - \theta_{d_f+d_h}^{0,d_f+1}(fh) \\ & \ + \sum_{i=1}^{d_h-1}\Big(\theta_{d_h+d_f}^{d_h-i,d_h+d_f-i+1}\Big(\omega_{d_h+d_f}^{-i}(\omega_{d_g}^i(g)f)\Big) - (-1)^{\ell_f-1}\omega_{d_h+d_f}^{-i}(\omega_{d_g}^i(g)f)\Big)\Big] \end{aligned}$$

$$= \Bigg(\theta^{d_g,d_f+d_g+1}_{d_f+d_g}(gf)\,h - \theta^{0,d_f+1}_{d_f+d_g}(fg)\,h - h\,\theta^{d_g,d_f+d_g+1}_{d_f+d_g}(gf) + h\,\theta^{0,d_f+1}_{d_f+d_g}(fg)$$

$$+g\,\theta^{d_h,d_f+d_h+1}_{d_f+d_h}(hf) - g\,\theta^{0,d_f+1}_{d_f+d_h}(fh) - \theta^{d_h,d_f+d_h+1}_{d_f+d_h}(hf)\,g + \theta^{0,d_f+1}_{d_f+d_h}(fh)\,g\Bigg)$$

$$+\sum_{i=1}^{d_g-1}\Bigg(\theta^{d_g-i,d_g+d_f-i+1}_{d_f+d_g}\Big(\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)\Big)h - (-1)^{\ell_f-1}\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)h$$

$$-h\theta^{d_g-i,d_g+d_f-i+1}_{d_f+d_g}\Big(\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)\Big) + (-1)^{\ell_f-1}h\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)\Bigg)$$

$$+\sum_{i=1}^{d_h-1}\Bigg(g\theta^{d_h-i,d_h+d_f-i+1}_{d_h+d_f}\Big(\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)\Big) - (-1)^{\ell_f-1}g\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)$$

$$-\theta^{d_h-i,d_h+d_f-i+1}_{d_h+d_f}\Big(\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)\Big)g + (-1)^{\ell_f-1}\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)g\Bigg),$$

which by (A.8)-(A.9) gives

$$\begin{aligned} = \Bigg(&\theta^{d_g,d_f+d_g+1}_{d_f+d_g}(gf)\,h - \theta^{0,d_f+1}_{d_f+d_g+d_h}(fgh)\\ &- \theta^{d_g+d_h,d_f+d_g+d_h+1}_{d_f+d_g+d_h}(hgf) + h\,\theta^{0,d_f+1}_{d_f+d_g}(fg)\\ &+ \theta^{d_h+d_g,d_f+d_g+d_h+1}_{d_f+d_h+d_g}(ghf) - g\,\theta^{0,d_f+1}_{d_f+d_g}(fh)\\ &- \theta^{d_h,d_f+d_h+1}_{d_f+d_h}(hf)\,g + \theta^{0,d_f+1}_{d_f+d_g+d_h}(fhg)\Bigg)\end{aligned}$$

$$+\sum_{i=1}^{d_g-1}\Bigg(\theta^{d_g-i,d_g+d_f-i+1}_{d_f+d_g+d_h}\Big(\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)h\Big) - (-1)^{\ell_f-1}\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)h$$

$$-\theta^{d_g+d_h-i,d_g+d_f+d_h-i+1}_{d_f+d_g+d_h}\Big(h\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)\Big) + (-1)^{\ell_f-1}h\omega^{-i}_{d_g+d_f}(\omega^i_{d_g}(g)f)\Bigg)$$

$$+\sum_{i=1}^{d_h-1}\Bigg(\theta^{d_h+d_g-i,d_h+d_f+d_g-i+1}_{d_h+d_f+d_g}\Big(g\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)\Big) - (-1)^{\ell_f-1}g\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)$$

$$-\theta^{d_h-i,d_h+d_f-i+1}_{d_h+d_f+d_g}\Big(\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)g\Big) + (-1)^{\ell_f-1}\omega^{-i}_{d_h+d_f}(\omega^i_{d_h}(h)f)g\Bigg).$$

Applying (A.11) to the terms in the sum part yields

$$= \theta^{d_g,d_f+d_g+1}_{d_f+d_g}(gf)\,h - \theta^{0,d_f+1}_{d_f+d_g+d_h}(fgh) - \theta^{d_g+d_h,d_f+d_g+d_h+1}_{d_f+d_g+d_h}(hgf) + h\,\theta^{0,d_f+1}_{d_f+d_g}(fg)$$

$$+\theta^{d_h+d_g,d_f+d_g+d_h+1}_{d_f+d_h+d_g}(ghf) - g\,\theta^{0,d_f+1}_{d_f+d_g}(fh) - \theta^{d_h,d_f+d_h+1}_{d_f+d_h}(hf)\,g + \theta^{0,d_f+1}_{d_f+d_g+d_h}(fhg)$$

$$+ \sum_{i=1}^{d_g-1} \Big(\theta_{d_f+d_g+d_h}^{d_g-i,d_g+d_f-i+1} \Big(\omega_{d_g+d_f+d_h}^{-i-d_h} (\omega_{d_g+d_h}^{i+d_h}(gh)f) \Big)$$

$$- (-1)^{\ell_f-1} \omega_{d_g+d_f+d_h}^{-i-d_h} (\omega_{d_g+d_h}^{i+d_h}(gh)f)$$

$$- \theta_{d_f+d_g+d_h}^{d_g+d_h-i,d_g+d_f+d_h-i+1} \Big(\omega_{d_g+d_f+d_h}^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big)$$

$$+ (-1)^{\ell_f-1} \omega_{d_g+d_f+d_h}^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big)$$

$$+ \sum_{i=1}^{d_h-1} \Big(\theta_{d_h+d_f+d_g}^{d_h+d_g-i,d_h+d_f+d_g-i+1} \Big(\omega_{d_h+d_f+d_g}^{-i} (\omega_{d_h+d_g}^{i}(gh)f) \Big)$$

$$- (-1)^{\ell_f-1} \omega_{d_h+d_f+d_g}^{-i} (\omega_{d_h+d_g}^{i}(gh)f)$$

$$- \theta_{d_h+d_f+d_g}^{d_h-i,d_h+d_f-i+1} \Big(\omega_{d_h+d_f+d_g}^{-i-d_g} (\omega_{d_h+d_g}^{i+d_g}(hg)f) \Big)$$

$$+ (-1)^{\ell_f-1} \omega_{d_h+d_f+d_g}^{-i} (\omega_{d_h+d_g}^{i}(hg)f) \Big).$$

Pairing similar terms, setting $d = d_f + d_g + d_h$, and reindexing part of the sum yields

$$= \theta_d^{d_h+d_g,d+1}(ghf) - \theta_d^{d_g+d_h,d+1}(hgf) - \theta_d^{0,d_f+1}(fgh) + \theta_d^{0,d_f+1}(fhg)$$

$$+\theta_{d_f+d_g}^{d_g,d_f+d_g+1}(gf)\,h + h\,\theta_{d_f+d_g}^{0,d_f+1}(fg) - g\,\theta_{d_f+d_g}^{0,d_f+1}(fh) - \theta_{d_f+d_h}^{d_h,d_f+d_h+1}(hf)\,g$$

$$+ \sum_{i=d_h+1}^{d_g+d_h-1} \Big(\theta_d^{d_g+d_h-i,d-i+1} \Big(\omega_d^{-i} (\omega_{d_g+d_h}^{i}(gh)f) \Big) - (-1)^{\ell_f-1} \omega_d^{-i} (\omega_{d_g+d_h}^{i}(gh)f) \Big)$$

$$- \sum_{i=1}^{d_g-1} \Big(\theta_d^{d_g+d_h-i,d-i+1} \Big(\omega_d^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big) - (-1)^{\ell_f-1} \omega_d^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big)$$

$$+ \sum_{i=1}^{d_h-1} \Big(\theta_d^{d_g+d_h-i,d-i+1} \Big(\omega_d^{-i} (\omega_{d_g+d_h}^{i}(gh)f) \Big) - (-1)^{\ell_f-1} \omega_d^{-i} (\omega_{d_g+d_h}^{i}(gh)f) \Big)$$

$$- \sum_{i=d_g+1}^{d_g+d_h-1} \Big(\theta_d^{d_g+d_h-i,d_f+d_h-i+1} \Big(\omega_d^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big)$$

$$- (-1)^{\ell_f-1} \omega_d^{-i} (\omega_{d_g+d_h}^{i}(hg)f) \Big). \qquad (A.14)$$

Recall that this quantity is equal to $(-1)^{\ell_f-1}([E_f(g),h]+[g,E_f(h)])$. In order for E_f to be a derivation, it should be equal to $(-1)^{\ell_f-1}E_f([g,h])$, which according to (*) is given by

$$(-1)^{\ell_f-1}E_f([g,h]) = \theta_d^{d_g+d_h,d+1}([g,h]f) - \theta_d^{0,d_f+1}(f[g,h])$$

$$+ \sum_{i=1}^{d_g+d_h-1} \Big(\theta_d^{d_g+d_h-i,d-i+1} \Big(\omega_d^{-i} (\omega_{d_g+d_h}^{i}([g,h])f) \Big) - \omega_d^{-i} (\omega_{d_g+d_h}^{i}([g,h])f) \Big)$$

$$= \theta_d^{d_g+d_h,d+1}([g,h]f) - \theta_d^{0,d_f+1}(f[g,h]) \qquad (A.15a)$$

$$+ \sum_{i=1}^{d_g+d_h-1} \theta_d^{d_g+d_h-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g+d_h}^i(gh)f)\Big) \qquad (A.15b)$$

$$-(-1)^{\ell_f-1} \sum_{i=1}^{d_g+d_h-1} \omega_d^{-i}(\omega_{d_g+d_h}^i(gh)f) \qquad (A.15c)$$

$$- \sum_{i=1}^{d_g+d_h-1} \theta_d^{d_g+d_h-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g+d_h}^i(hg)f)\Big) \qquad (A.15d)$$

$$+(-1)^{\ell_f-1} \sum_{i=1}^{d_g+d_h-1} \omega_d^{-i}(\omega_{d_g+d_h}^i(hg)f). \qquad (A.15e)$$

Let us successively subtract off the terms of (A.14) from this and show that we obtain zero. We first subtract off the first four terms (first line) of (A.14), noting that they are equal to

$$\theta_d^{d_g+d_h,d+1}\big((gh-hg)f\big) - \theta_d^{0,d_f+1}\big(f(gh-hg)\big)$$
$$= \theta_d^{d_g+d_h,d+1}\big([g,h]f\big) - \theta_d^{0,d_f+1}\big(f[g,h]\big),$$

which is exactly (A.15a). We now subtract the third sum (fifth line) of (A.14) from (A.15b,c) and the second sum (fourth line) from (A.15d,e), yielding

$$+ \sum_{i=d_h}^{d_g+d_h-1} \theta_d^{d_g+d_h-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g+d_h}^i(gh)f)\Big) \qquad (A.15b')$$

$$-(-1)^{\ell_f-1} \sum_{i=d_h}^{d_g+d_h-1} \omega_d^{-i}(\omega_{d_g+d_h}^i(gh)f) \qquad (A.15c')$$

$$- \sum_{i=d_g}^{d_g+d_h-1} \theta_d^{d_g+d_h-i,d-i+1}\Big(\omega_d^{-i}(\omega_{d_g+d_h}^i(hg)f)\Big) \qquad (A.15d')$$

$$+(-1)^{\ell_f-1} \sum_{i=d_g}^{d_g+d_h-1} \omega_d^{-i}(\omega_{d_g+d_h}^i(hg)f). \qquad (A.15e')$$

Now subtract the first sum (third line) of (A.14) from (A.15b',c') and the fourth sum (sixth line) from (A.15d',e'); we obtain

$$\theta_d^{d_g,d_f+d_g+1}\Big(\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big)\Big) - (-1)^{\ell_f-1}\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big)$$

$$-\theta_d^{d_h,d_f+d_h+1}\Big(\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big)\Big) + (-1)^{\ell_f-1}\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big).$$

To finish, we need to subtract off the remaining (second) line of (A.14) from this and show that the result is zero, i.e. that

$$\theta_d^{d_g,d_f+d_g+1}\Big(\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big)\Big) - (-1)^{\ell_f-1}\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big)$$
$$-\theta_d^{d_h,d_f+d_h+1}\Big(\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big)\Big) + (-1)^{\ell_f-1}\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big)$$
$$-\theta_{d_f+d_g}^{d_g,d_f+d_g+1}(gf)\,h - h\,\theta_{d_f+d_g}^{0,d_f+1}(fg) + g\,\theta_{d_f+d_g}^{0,d_f+1}(fh) + \theta_{d_f+d_h}^{d_h,d_f+d_h+1}(hf)\,g = 0.$$

In fact, the eight terms of this sum cancel out in pairs as follows.

The second and seventh terms

$$-(-1)^{\ell_f-1}\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big) + g\,\theta_{d_f+d_h}^{0,d_f+1}(fh)$$

sum to zero, using (A.12) from lemma A.3. The fourth and sixth terms

$$(-1)^{\ell_f-1}\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big) - h\,\theta_{d_f+d_g}^{0,d_f+1}(fg)$$

sum to zero, again by (A.12) but exchanging the roles of g and h.

The first and fifth terms give:

$$\theta_d^{d_g,d_f+d_g+1}\Big(\omega_d^{-d_h}\big(\omega_{d_g+d_h}^{d_h}(gh)f\big)\Big) - \theta_{d_f+d_g}^{d_g,d_f+d_g+1}(gf)\,h$$

sum to zero by (A.13), and the third and eighth terms

$$-\theta_d^{d_h,d_f+d_h+1}\Big(\omega_d^{-d_g}\big(\omega_{d_g+d_h}^{d_g}(hg)f\big)\Big) + \theta_{d_f+d_h}^{d_h,d_f+d_h+1}(hf)\,g$$

sum to zero by (A.13) with g and h exchanged. This concludes the proof. ■

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