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Parabolic Induction of Projective Orbits and Subalgebras

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Abstract. The concepts of parabolic induction of subalgebras and varieties appeared independently in works of different authors. Here we introduce and study the parabolic induction of projective orbits. This concept turns out to be closely related to the above ones, and can be used as a method to study them. As an application, we describe all subalgebras of a semisimple Lie algebra that contain a given subalgebra obtained by parabolic induction.
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1. Introduction

Let G be a reductive algebraic group over the field of characteristic zero and P be a proper parabolic subgroup of G. One says that a G-variety X is obtained *by parabolic induction* from a P-variety Y if the radical of P acts trivially on Y, and there exists a P-equivariant injective morphism $\psi : Y \to X$ giving rise to a birational surjective morphism $G \times_P Y \to X$. The concept of the parabolic induction of varieties has been used by Akhiezer ([1], Th. 4) and Cupit-Foutou ([2], Th. 1.3) in the study of two-orbit varieties, in the work of Wasserman ([8], pp. 378–379) and in the work of Luna ([3], pr. 3.4) where he classified the spherical varieties of type A. A similar concept has been studied in the work of Kempf ([4]).

Let L be a Levi subgroup of P. The action $G : X$ shares many properties with the action $L : Y$. For example, the complexities and the ranks of these actions coincide. In particular, the property of being spherical is preserved under parabolic induction. This is the reason why in many problems it is natural to study *irreducible* actions, those that cannot be obtained by a non-trivial parabolic induction.

Let $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{p}$ be the tangent algebras of G, L and P respectively and $\phi : \mathfrak{p} \to \mathfrak{l}$ be the normal projection. Then we have $\mathfrak{g}_{\psi(y)} = \mathfrak{p}_y = \phi^{-1}(\mathfrak{l}_y)$ for a generic point $y \in Y$. This is an example where the parabolic induction of subalgebras appears. Let us recall this notion.

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Let $\mathfrak{g}$ be an algebraic reductive Lie algebra, $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, and ϕ be a homomorphism from $\mathfrak{p}$ onto some algebraic reductive Lie algebra $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{h}}$ be a subalgebra of $\tilde{\mathfrak{g}}$. Set $\mathfrak{h} = \phi^{-1}(\tilde{\mathfrak{h}})$. The pair $(\mathfrak{g}, \mathfrak{h})$ is said to be obtained from the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ *by parabolic induction* via the parabolic subalgebra $\mathfrak{p}$. We will denote this by $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \overrightarrow{_{\mathfrak{p}}} (\mathfrak{g}, \mathfrak{h})$. In the case of parabolic induction of varieties we have $(\mathfrak{l}, \mathfrak{l}_y) \overrightarrow{_{\mathfrak{p}}} (\mathfrak{g}, \mathfrak{g}_{\psi(y)})$.

The parabolic induction of subalgebras is an interesting concept in itself. For example, it was used in the work of Wasserman ([8], see Definition 2.3, the notion of a prime subgroup) devoted to classification of wonderful varieties. The parabolic induction of subalgebras preserves some properties of pairs (for example, the complexity). Therefore, being interested in subalgebras of reductive algebras having certain properties, one can reduce the study to subalgebras $\mathfrak{h} \subset \mathfrak{g}$ such that the pair $(\mathfrak{g}, \mathfrak{h})$ is *irreducible*, i.e. cannot be obtained by a non-trivial parabolic induction.

Below we introduce the parabolic induction of projective orbits. This concept appeared in a special case in the work of the author ([5], prop. 2.15, 2.16) where it was applied to the study of two-orbit varieties with their embeddings in projective spaces. It serves as a good model for both the parabolic induction of varieties and subalgebras and can be used to prove the properties of theirs.

Let G be the connected algebraic group whose tangent algebra is $\mathfrak{g}$, T be a maximal torus in G, $\mathfrak{t}$ be its tangent algebra (a Cartan subalgebra of $\mathfrak{g}$), $\mathfrak{t}(\mathbb{R})$ be the real form of $\mathfrak{t}$ consisting of elements with real eigenvalues and $\mathbb{E} = \mathfrak{t}(\mathbb{R})^*$. We will fix a G-invariant inner product on $\mathfrak{g}$ and identify $\mathfrak{t}$ with $\mathfrak{t}^*$ (so that $\mathbb{E} \subset \mathfrak{t}$). For any representation space V of $\mathfrak{g}$ let $\Phi = \Phi(V) \subset \mathbb{E}$ be the system of weights of T in V and $V = \oplus_{\lambda \in \Phi} V_\lambda$ be the weight decomposition. For any subset $R \subset \mathbb{E}$ set $V_R := \oplus_{\lambda \in \Phi \cap R} V_\lambda$ and call V_R the *restriction* of V to R. (In particular, $V_{\{\psi\}} = V_\psi$ for any weight ψ). The convex hull of Φ in $\mathbb{E}$ will be called *the weight polytope* of V and denoted by $M(V)$. Let $V = \oplus_\phi V(\phi)$ be the decomposition into isotypic components (relative to the G-action) and $v = \sum_\phi v(\phi)$, where $v(\phi) \in V(\phi)$.

Let Γ be a face of $M(V)$. Denote the affine hull of Γ with $\operatorname{Aff} \Gamma$ and the center of mass of Γ with z_Γ. For any root α parallel to Γ, we have $r_\alpha \Gamma = \Gamma$, therefore $r_\alpha z_\Gamma = z_\Gamma$ (where r_α is the reflection relative to α). Thus z_Γ is orthogonal to all roots parallel to Γ. We will call the subalgebra of $\mathfrak{g}$ generated by the linear span of Γ and the root vectors corresponding to the roots parallel to Γ the *reductive subalgebra associated with* Γ and denote it with $\mathfrak{g}_\Gamma$. We will denote the subalgebra of $\mathfrak{g}$ generated by the orthogonal complement to the linear span of Γ and the root vectors corresponding to the roots orthogonal to the linear span of Γ with $\mathfrak{g}_\Gamma^\perp$. We will call the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{t}$ and the root vectors corresponding to the roots having non-negative inner products with z_Γ the *parabolic subalgebra associated with* Γ and denote it with $\mathfrak{p}_\Gamma$. We will denote the corresponding subgroups of G with G_Γ and P_Γ. The following statement holds:

$$\mathfrak{p}_\Gamma = (\mathfrak{g}_\Gamma \oplus \mathfrak{g}_\Gamma^\perp) \oplus \mathfrak{p}_\Gamma^{\mathrm{u}},$$

where $\mathfrak{p}_\Gamma^{\mathrm{u}}$ is the unipotent radical of $\mathfrak{p}_\Gamma$. Also set $\mathfrak{l}_\Gamma = \mathfrak{g}_\Gamma \oplus \mathfrak{g}_\Gamma^\perp$. It is a Levi subalgebra of $\mathfrak{p}_\Gamma$. The space V_Γ can be considered as a representation space of $\mathfrak{g}_\Gamma$. If V is $\mathfrak{g}$-irreducible, then V_Γ is $\mathfrak{g}_\Gamma$-irreducible (e.g. [6], Prop. 8).

For any vector $v = \sum_{\lambda \in \Phi} v_\lambda$, define the set of its weights by $\Phi_v := \{\lambda \in \Phi | v_\lambda \neq 0\}$. The convex hull of this set will be called the *support* of v and denoted by $\operatorname{supp} v$. For any point $\langle v\rangle \in \mathbb{P}(V)$ set $\operatorname{supp}\langle v\rangle = \operatorname{supp} v$. We will call a face Γ of $M(V)$ *minimal* for the orbit Gv if there is a point in this orbit whose support lies in Γ, but there is no point whose support lies in any proper face of Γ. If $\operatorname{supp} v \subset \Gamma$ and Γ is minimal for the orbit Gv, we will say that the orbit $G\langle v\rangle$ is obtained *by parabolic induction* from the orbit $G_\Gamma\langle v\rangle$.

The following theorem summarizes the properties of the parabolic induction of projective orbits.

Theorem 1.1. *a) Any two minimal faces for the orbit Gv are equivalent under the Weyl group.*
b) The intersection of any G-orbit in V with V_Γ is a single G_Γ-orbit or empty.
c) If $\operatorname{supp} v \subset \Gamma$, $\operatorname{supp} gv \subset \Gamma$ and Γ is minimal for the orbit Gv, then $g \in P_\Gamma$ (in particular, the stabilizer of $\langle v\rangle$ lies in P_Γ).
d) If $\operatorname{supp} v \in \Gamma$ and $\Gamma_1 \subset \Gamma$ is minimal for the orbit $G_\Gamma v$ then it is minimal for the orbit Gv as well.

Theorem 1.2. *Let $\operatorname{supp} v \subset \Gamma$ and Γ be minimal for the orbit Gv. Then*
1) The G-variety $\overline{G\langle v\rangle}$ is obtained by parabolic induction from the P_Γ-variety $\overline{G_\Gamma\langle v\rangle}$,
2) Let T' be a torus acting on V permutably with G, $\mathfrak{t}'$ be its tangent algebra and let $\mathfrak{h}$ be the projection of $(\mathfrak{g}\oplus\mathfrak{t}')_v$ onto $\mathfrak{g}$. Then $(\mathfrak{g}_\Gamma, \mathfrak{g}_\Gamma\cap\mathfrak{h}) \xrightarrow[\mathfrak{p}_\Gamma]{} (\mathfrak{g},\mathfrak{h})$ In particular, $(\mathfrak{g}_\Gamma, (\mathfrak{g}_\Gamma)_{\langle v\rangle}) \xrightarrow[\mathfrak{p}_\Gamma]{} (\mathfrak{g}, \mathfrak{g}_{\langle v\rangle})$ and $(\mathfrak{g}_\Gamma, (\mathfrak{g}_\Gamma)_v) \xrightarrow[\mathfrak{p}_\Gamma]{} (\mathfrak{g}, \mathfrak{g}_v)$.

It follows from these theorems that the map $G\times_P V_\Gamma \to GV_\Gamma$ is birational, therefore, according to the theorem of Kempf ([4]), the variety GV_Γ always has rational singularities.

According to Theorem 1.2, the parabolic induction of orbits turns out to be a convenient model for studying both the parabolic induction of varieties and subalgebras.

Moreover, this construction can be applied to the study of projective orbits satisfying certain conditions for the following reason: the orbit obtained by parabolic induction shares many properties with the orbit it has been obtained from. As it was said earlier, this approach was used by the author ([5], prop. 2.15, 2.16) in the study of two-orbit varieties.

As one more application, we are going to prove a theorem on the parabolic induction of subalgebras.

Let us introduce some notation. We will call a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ *admissible* for a reductive algebra $\tilde{\mathfrak{g}}$ if there exists a homomorphism ϕ from $\mathfrak{p}$ onto $\tilde{\mathfrak{g}}$; a parabolic subalgebra $\mathfrak{p}_1 \supset \mathfrak{p}$ of $\mathfrak{g}$, an *admissible enlargement* of $\mathfrak{p}$ for ϕ if ϕ extends to a homomorphism ϕ_1 from $\mathfrak{p}_1$ onto $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{h}}$ be a subalgebra of $\tilde{\mathfrak{g}}$ and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \xrightarrow[\mathfrak{p}]{} (\mathfrak{g},\mathfrak{h})$.

Let us study the subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$. There are two obvious ways to construct such subalgebras:
(1) *enlarging the subalgebra of* $\tilde{\mathfrak{g}}$: take any subalgebra $\tilde{\mathfrak{h}}_1 \supset \tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$ and let $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}_1) \xrightarrow[\mathfrak{p}]{} (\mathfrak{g},\mathfrak{h}_1)$; then $\mathfrak{h}_1 \supset \mathfrak{h}$.
(2) *enlarging the parabolic subalgebra:* take any admissible enlargement $\mathfrak{p}_1 \supset \mathfrak{p}$ for ϕ and let $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \xrightarrow[\mathfrak{p}_1]{} (\mathfrak{g},\mathfrak{h}_1)$; then $\mathfrak{h}_1 \supset \mathfrak{h}$.

These two operations are nearly enough to construct any subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We shall need only one more operation not changing $\mathfrak{h}$ but $\tilde{\mathfrak{g}}$:

(3) *passing to a deeper parabolic induction:* suppose that the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is reducible, so it can be obtained from a pair $(\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{h}}_1)$ by parabolic induction via a parabolic subalgebra $\tilde{\mathfrak{p}}_1$ of $\tilde{\mathfrak{g}}$ (with a homomorphism $\tilde{\phi}_1$). Then we can replace $\tilde{\mathfrak{g}}$ by $\tilde{\mathfrak{g}}_1$ so that $(\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{h}}_1) \xrightarrow[\mathfrak{p}_1]{} (\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{p}_1 = \phi^{-1}(\tilde{\mathfrak{p}}_1)$ and $\phi_1 = \tilde{\phi}_1 \circ \phi$.

Obviously, any composition of these operations also leads to a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The theorem we are going to prove claims that these three operations are enough to obtain all subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$.

Theorem 1.3. *Let $\tilde{\mathfrak{g}}$ be an reductive algebraic Lie algebra, $\mathfrak{p}$ be an admissible for $\tilde{\mathfrak{g}}$ parabolic subalgebra of $\mathfrak{g}$, $\tilde{\mathfrak{h}}$ be a subalgebra of $\tilde{\mathfrak{g}}$ and the pair $(\mathfrak{g}, \mathfrak{h})$ be obtained from $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ by parabolic induction via $\mathfrak{p}$. Then the subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are exactly the subalgebras obtained by enlarging the subalgebra of $\tilde{\mathfrak{g}}$, then passing to a deeper parabolic induction and finally enlarging the parabolic subalgebra.*

Theorem 1.3 can be used to classify the subalgebras of semisimple algebras satisfying certain conditions. For example, it was used by the author to classify the maximal non-horospherical subalgebras of a semisimple Lie algebra (unpublished).

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Notation:

G:	a connected reductive algebraic group;
$\mathfrak{g}$:	the tangent algebra of G;
T:	a maximal torus of G;
$\mathfrak{t}$:	the tangent algebra of T;
$\mathfrak{b}$:	a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$;
Δ:	the system of roots of $\mathfrak{g}$;
$\Delta^+ \subset \Delta$:	the system of positive roots of $\mathfrak{g}$;
$\mathfrak{p}^{\mathrm{u}}$:	the unipotent radical of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$;
$\mathfrak{u} = \mathfrak{b}^{\mathrm{u}}$:	the maximal unipotent subalgebra of $\mathfrak{b}$ and $\mathfrak{g}$;
W:	the Weyl group of the pair (G, T);
r_α:	the reflection relative to a root α;
$\overline{n}$:	the element of W corresponding to an element $n \in N(T)$;
$\langle M \rangle$:	the linear span of a set M in a vector space;
$\mathfrak{g}_x$:	the stabilizing subalgebra of a point x.

The vector space $\mathbb{E} = \mathfrak{t}(\mathbb{R})^*$ will be in certain cases considered as an affine space and its elements as points.

If $\alpha \in \mathbb{E}$, then h_α is an element of $\mathfrak{t}$ such that h_α is orthogonal to $\operatorname{Ker} \alpha$ and $\alpha(h_\alpha) = 2$.

If $\alpha \in \Delta$, then e_α will denote the root vector, corresponding to α (defined up to a scalar factor). We will claim that $[e_\alpha, e_{-\alpha}] = h_\alpha$ (thus, $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ will be an $\mathfrak{sl}_2$-triple).

The notation $x \sim y$ means that both vectors x and y are non-zero and are proportional to each other.

We shall call a face of the weight polytope dominant, if its center of mass lies in the dominant (closed) Weyl chamber.

For a hyperplane H in an affine space A we will denote the two closed half-spaces, into which H divides A, with $H^{\pm}$. In case when $0 \notin H$ we will assume that $0 \in H^-$.

A hyperplane H in an affine space A is called *a hyperplane of support of* a set $M \subset A$, if the intersection $H \cap M$ is nonempty, and M is contained in H^+ or H^-.

2. Parabolic induction of projective orbits

In this section we will prove Theorems 1.1 and 1.2.

First we will study some properties of convex polytopes invariant under the Weyl group.

Let $M \subset \mathbb{E}$ be such a polytope and Γ be a face of it. Let $\Delta_{\|}(\Gamma)$ be the set of roots parallel to Γ. As we explained above, $\Delta_{\|}(\Gamma) \perp z_\Gamma$. Let $\Delta_{\perp}(\Gamma)$ be the set of roots orthogonal to z_Γ and not parallel to Γ, $\Delta_+(\Gamma)$ be the set of roots having positive inner product with z_Γ.

The polytope M is an intersection of some (closed) half-spaces H_i^-. Let

$$S_{\pm}(\Gamma) = ((\cap_{\Gamma \subset H_i} H_i^{\pm}) \setminus \operatorname{Aff} \Gamma) - p$$

for any point $p \in \Gamma$. Clearly $S^+(\Gamma)$ and $S^-(\Gamma)$ do not depend on p. They are opposite convex cones. Let $S_0(\Gamma)$ be the complementary set to $S_-(\Gamma) \cup S_+(\Gamma)$.

The following three lemmas generalize some results of Vinberg (see [7], p. 9-12).

Lemma 2.1. *1) If $\alpha \in \Delta_{\perp}(\Gamma)$ then α is orthogonal to the linear span $\langle \Gamma \rangle$ of Γ in $\mathbb{E}$;*

2) $\Delta_{\|}(\Gamma) \perp \Delta_{\perp}(\Gamma)$;

3) $\Delta_{\|}(\Gamma) \cup \Delta_{\perp}(\Gamma) = \Delta \cap S_0(\Gamma)$;

4) $\Delta_+(\Gamma) = \Delta \cap S_+(\Gamma)$.

Proof. Let α be a root. If α is orthogonal to z_Γ then $r_\alpha \Gamma = \Gamma$, therefore either $\alpha \parallel \Gamma$, or $\alpha \perp \langle \Gamma \rangle$. This proves 1) and 2).

In both cases $r_\alpha S_{\pm}(\Gamma) = S_{\pm}(\Gamma)$, but $r_\alpha \alpha = -\alpha$, therefore $\alpha \notin S_{\pm}(\Gamma)$. Thus $\Delta_{\|}(\Gamma) \cup \Delta_{\perp}(\Gamma) \subset S_0(\Gamma)$.

On the other hand, if $\alpha \in S_0(\Gamma)$, then $r_\alpha \Gamma = \Gamma$, therefore $\alpha \perp z_\Gamma$. This proves 3).

Now let α be a root having a positive inner product with z_Γ. We have $r_\alpha z_\Gamma \in M$, therefore $z_\Gamma - r_\alpha z_\Gamma \in S_+(\Gamma)$ (this vector does not lie in $S_0(\Gamma)$ because it is proportional to α, so has a non-zero inner product with z_Γ). Since $(\alpha, z_\Gamma) > 0$, $z_\Gamma - r_\alpha z_\Gamma = c\alpha$ for some positive c. Consequently, $\alpha \in S_+(\Gamma)$.

Therefore $\alpha \in \Delta_+(\Gamma)$ if and only if $\alpha \in S_+(\Gamma)$ (if and only if $-\alpha \in S_-(\Gamma)$). This proves 4). ■

Choose a system of positive roots $\Delta_+ \supset \Delta_+(\Gamma)$. Let $\mathcal{C}$ be the corresponding dominant Weyl chamber. Denote the group generated by r_α for $\alpha \in \Delta_{\|}(\Gamma)$ with W_Γ. Let

$$\mathcal{C}_\Gamma = \{\lambda \in \operatorname{Aff} \Gamma : (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta_+ \cap \Delta_{\|}(\Gamma)\}.$$

Obviously, $z_\Gamma \in \mathcal{C}_\Gamma$, and $\mathcal{C}_\Gamma$ is a fundamental region for the action $W_\Gamma : \operatorname{Aff} \Gamma$.

Lemma 2.2. $\mathcal{C} \cap \Gamma = \mathcal{C}_\Gamma \cap \Gamma$.

Proof. Obviously, $\mathcal{C} \cap \Gamma \subset \mathcal{C} \cap \operatorname{Aff} \Gamma \subset \mathcal{C}_\Gamma$. On the other hand, for any $\alpha \in \Delta_+(\Gamma)$ and $p \in \Gamma$ we have $(\alpha, p) \geq (\alpha, z_\Gamma) \geq 0$. Therefore

$$\mathcal{C}_\Gamma \cap \Gamma \subset \{\lambda \in \operatorname{Aff} \Gamma : (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta_+ \cap \Delta_\|(\Gamma)\} \cap \Gamma \subset$$
$$\subset \{\lambda \in E : (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta_+ \cap \Delta_\|(\Gamma)\} \cap \{\lambda \in E : (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta_+(\Gamma)\} =$$
$$= \{\lambda \in E : (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta_+\} = \mathcal{C}. \ \blacksquare$$

Lemma 2.3. *For any face $\Gamma_1 \subset \Gamma$ there is an element $w \in W_\Gamma$ such that $w\Gamma_1$ is dominant.*

Proof. According to the choice of Δ_+, $z_\Gamma \in \mathcal{C}_\Gamma \cap \Gamma \subset \mathcal{C}$. Choose an element $w \in W_\Gamma$ such that $wz_{\Gamma_1} \in \mathcal{C}_\Gamma$. Since $z_{\Gamma_1} \in \Gamma$, $wz_{\Gamma_1} \in \Gamma$, thus according to Lemma 2.2, $wz_{\Gamma_1} \in \mathcal{C}_\Gamma \cap \Gamma = \mathcal{C} \cap \Gamma \subset \mathcal{C}$. ■

Now let $M = M(V)$ be a weight polytope. Then W_Γ is the Weyl group of G_Γ and $\langle \Gamma \rangle = \mathfrak{g}_\Gamma \cap \mathbb{E}$.

Lemma 2.4. *Let $v \in V$ be a vector such that* $\operatorname{supp} v \subset \Gamma$. *Then*
1) If $\alpha \in \Delta_+(\Gamma)$ then $e_\alpha \in \mathfrak{g}_v$;
2) If $\alpha \in \Delta_\perp(\Gamma)$ then $e_\alpha \in \mathfrak{g}_v$ and $h_\alpha \in \mathfrak{g}_v$.

Proof. Due to Lemma 2.1, for all $p \in \Gamma$ and $\alpha \in \Delta_+(\Gamma) \cup \Delta_\perp(\Gamma)$ the vector $p + \alpha$ is not a weight. Therefore $e_\alpha \in \mathfrak{g}_v$. If $\alpha \in \Delta_\perp(\Gamma)$ then the same is true for $-\alpha$, therefore the whole $\mathfrak{sl}_2$-triple $\{e_{-\alpha}, h_\alpha, e_\alpha\}$ stabilizes v. ■

According to Lemma 2.4, both the unipotent radical $\mathfrak{p}_\Gamma^{\mathfrak{u}}$ of $\mathfrak{p}$ and $\mathfrak{g}_\Gamma^\perp$ act trivially on V_Γ. This proves the following:

Lemma 2.5. *If* $\operatorname{supp} v \subset \Gamma$ *and $b \in P_\Gamma$ then there is $g \in G_\Gamma$ such that $bv = gv$.*

We will need one more lemma:

Lemma 2.6. *Let $n \in N(T)$ be an element such that $w = \overline{n}$ stabilizes the face Γ. Then $n = n_1 n_2$, where $n_1 \in N(T) \cap G_\Gamma$ and $n_2 \in N(T) \cap G_\Gamma^\perp$. In particular, $n \in P_\Gamma$.*

Proof. The element w can be decomposed as a product of two commuting elements: $w = w_1 w_2$, where w_1 (resp. w_2) is a product of reflections relative to roots lying in $\Delta_\|(\Gamma)$ (resp. $\Delta_\perp(\Gamma)$).

Let n_1 and n_2 be some representatives of w_1 and w_2 in $N(T) \cap G_\Gamma$ and $N(T) \cap G_\Gamma^\perp$, respectively. Then n differs from $n_1 n_2$ by an element of T, and one can correct n_1 and n_2 so that $n = n_1 n_2$. ■

Proof. (of Theorem 1.1) For convenience suppose that Γ is dominant.

a) Let $v_1, v_2 \in Gv$ be two vectors whose supports lie in Γ_1 and Γ_2, respectively, and $v_2 = gv_1$. Apply the Bruhat decomposition and decompose g as $g = b_2 n b_1$, where $b_1 \in P_{\Gamma_1}$, $b_2 \in P_{\Gamma_2}$ and $n \in N(T)$. Then the face Γ_1 (resp. Γ_2) is the minimal face containing the support of $b_1 v_1$ (resp. of $b_2^{-1} v_2$), consequently $\overline{n}\Gamma_1 = \Gamma_2$.

b) Let $\operatorname{supp} v_1 \subset \Gamma$, $\operatorname{supp} v_2 \subset \Gamma$, and $v_2 \sim gv_1$. Apply the Bruhat decomposition and decompose g as $g = b_2 n b_1$, where $b_1, b_2 \in P_\Gamma$ and $n \in N(T)$. Then $b_2^{-1} v_2 \sim n b_1 v_1$. The supports of $b_2^{-1} v_2$ and $b_1 v_1$ lie in Γ. According to Lemma 2.5 we can assume that $b_1, b_2 \in G_\Gamma$ and prove the theorem for the case when $b_1 = b_2 = e$, so $v_2 \sim nv_1$.

Let $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma$ be the minimal faces containing the supports of v_1 and v_2 respectively. Then $\Gamma_2 = \overline{n}\Gamma_1$. According to Lemma 2.3, there are elements $w_1, w_2 \in W_\Gamma$ such that the faces $w_1\Gamma_1$ and $w_2\Gamma_2$ are dominant, therefore coincide. Let $w = w_2^{-1} w_1$. Then $\Gamma_2 = w\Gamma_1$ and $w \in W_\Gamma$. The element w has a representative if G_Γ, therefore we can reduce the theorem to the case when $\Gamma_1 = \Gamma_2$.

Now we have an element $n \in N(T)$ so that $\overline{n}\Gamma_1 = \Gamma_1$. According to Lemma 2.6, one can choose an element n' of $N(T) \cap G_{\Gamma_1}$ such that $n'v_1 = nv_1$. Now it is enough to note that $G_{\Gamma_1} \subset G_\Gamma$.

c) Decompose g as $g = b_2 n b_1$, where $b_1, b_2 \in P_\Gamma$ and $n \in N(T)$. Then $b_2^{-1} gv = n b_1 v$.

Note that the supports of $b_2^{-1} gv$ and $b_1 v$ lie in Γ, and, moreover, the support of $b_1 v$ is not contained in any proper face of Γ. Hence, $\overline{n}$ stabilizes the face Γ, therefore, according to Lemma 2.6, n is contained in P_Γ; consequently, $g \in P_\Gamma$.

d) Suppose that Γ_1 is not minimal for Gv. Then there is an element $g \in G$ such that the support of gv is a proper face of Γ_1. But according to point b) there is an element $g' \in G_\Gamma$ such that $g'v = gv$, and we come to a contradiction. ■

Proof. (of Theorem 1.2) To prove the first part of the theorem it is enough to consider the map $\psi : G \times_P \overline{G_\Gamma \langle v \rangle} \to \overline{G_\Gamma \langle v \rangle}$, $\psi(g \times x) = gx$ and notice that it is proper.

Now let H be the connected subgroup of G whose tangent algebra is $\mathfrak{h}$ and $g \in H$. Then $\operatorname{supp} gv = \operatorname{supp} v$, therefore $g \in P_\Gamma$ according to the point c) of Theorem 1.1. On the other hand, $\mathfrak{h}$ contains $\mathfrak{p}_\Gamma^{\mathrm{u}}$ and $\mathfrak{g}_\Gamma^\perp$ by Lemma 2.4. ■

3. An application to the parabolic induction of subalgebras

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}^{\mathrm{u}}$ be its Levi decomposition such that $\mathfrak{l} \supset \mathfrak{t}$, and $h_\mathfrak{p} \in \mathfrak{t}$ be an element such that $\alpha(h_\mathfrak{p}) > 0$ and $\alpha(h_\mathfrak{p}) \in \mathbb{Z}$ for all roots of $\mathfrak{p}^{\mathrm{u}}$ and $\alpha(h_\mathfrak{p}) = 0$ for all roots of $\mathfrak{l}$. For a weight ϕ let $H(h_\mathfrak{p}, \phi) = \{\lambda \in \mathbb{E} : \lambda(h_\mathfrak{p}) = \phi(h_\mathfrak{p})\}$.

The following lemma is known (see, e.g. [6]), but for convenience of the reader we give the proof here.

Lemma 3.1. *Let $G : V$ be an irreducible representation and ϕ be its highest weight, Then $H(h_{\mathfrak{p}}, \phi)$ is a hyperplane of support of $M(V)$ and*

$$\{v \in V : \mathfrak{p}^{\mathrm{u}} v \ = \ 0\} = V_{H(h_{\mathfrak{p}}, \phi)}.$$

Proof. Let L be the connected subgroup of G whose tangent algebra is $\mathfrak{l}$. Take a vector $W \in V$ such that $\mathfrak{p}^{\mathrm{u}} w = 0$. Consider the subspace $\langle Lw \rangle \subset V$. Obviously, $\mathfrak{p}^{\mathrm{u}} v = 0$ for any $v \in \langle Lw \rangle$. It follows that if v is a highest weight vector with respect to L, then it is a highest vector with respect to G. Therefore, $v \in \langle LV_{\phi} \rangle$, whence $\operatorname{supp} v \subset H(h_{\mathfrak{p}}, \phi)$.

On the other hand, $V = \langle GV_{\phi} \rangle$ and $\mathfrak{p}^{\mathrm{u}} V_{\phi} = 0$ implies that $M(V) \subset H(h_{\mathfrak{p}}, \phi)^{-}$. ■

Lemma 3.2. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_1 \supset \mathfrak{p}^{\mathrm{u}}$. Then there is a vector space V, a vector $v \subset V$, a hyperplane of support H of $M(V)$ orthogonal to $h_{\mathfrak{p}}$ and a one-dimensional torus T' acting on V permutably with G, such that*
1) $\operatorname{supp} v \subset H$*;*
2) $\mathfrak{h}_1$ is the projection of $(\mathfrak{g} \oplus \mathfrak{t}')_v$ onto $\mathfrak{g}$, where $\mathfrak{t}'$ is the tangent algebra of T'.

Proof. According to the Chevalley theorem, there exists a representation space U of G and a vector $u \in U$ such that $\mathfrak{g}_{\langle u \rangle} = \mathfrak{h}_1$. We may also assume that there is no proper G-invariant subspace of U containing u.

Since $\mathfrak{h}_1 \supset \mathfrak{p}^u$, then by Lemma 3.1, all the numbers $\phi(h_{\mathfrak{p}})$ for all ϕ such that $u(\phi) \neq 0$ are non-negative. Suppose that for all ϕ such that $u(\phi) \neq 0$ we have $\phi(h_{\mathfrak{p}}) = 0$. Then the statement of the lemma holds for $V = U$, $v = u$ and $H = H(h_{\mathfrak{p}}, 0)$.

Otherwise choose some ψ such that $u(\psi) \neq 0$ and $\psi(h_{\mathfrak{p}}) > 0$. Set $U' = U \otimes U(\psi)$ and $u' = u \otimes u(\psi)$. Obviously for all ϕ such that $u'(\phi) \neq 0$ we have $\phi(h_{\mathfrak{p}}) > 0$. Let us prove that $G_{\langle u' \rangle} = G_{\langle u \rangle}$.

If $g \in G_{\langle u \rangle}$ then $gu \sim u$, so $gu(\psi) \sim u(\psi)$ for all ψ. Therefore $G_{\langle u \rangle} \subset G_{\langle u' \rangle}$. Let us prove the inverse inclusion. If $g \in G_{\langle u' \rangle}$ then $g(u \otimes u(\psi)) \sim u \otimes u(\psi)$, hence $gu \sim u$, so $g \in G_{\langle u \rangle}$.

Choose some natural numbers $n(\phi)$ so that all the numbers $n(\phi)\phi(h_{\mathfrak{p}})$ coincide for all ϕ such that $u'(\phi) \neq 0$. Define $V = \oplus (U'(\phi)^{\otimes n(\phi)})$ and $v = \sum (u'(\phi)^{\otimes n(\phi)}) \in V$. Let T' be a one-dimensional torus acting on V via

$$t \cdot (\sum (u'(\phi)^{\otimes n(\phi)}) = \sum (t^n u'(\phi)^{\otimes n(\phi)})$$

Set $\Gamma = H \cap M(V)$; then $\operatorname{supp} v \subset \Gamma$. and we are done. ■

Lemma 3.3. *Let $G : V$ be a representation, H be a hyperplane of support of $M(V)$ orthogonal to $h_{\mathfrak{p}}$, and $\Gamma = M(V) \cap H$. Then there is an ideal $\mathfrak{n} \subset \mathfrak{g}_{\Gamma}^{\perp}$ such that $\mathfrak{l} = \mathfrak{g}_{\Gamma} \oplus \mathfrak{n}$.*

Proof. Let us first prove that $\mathfrak{g}_\Gamma \subset \mathfrak{l} \subset \mathfrak{l}_\Gamma$ The subalgebras $\mathfrak{g}_\Gamma$, $\mathfrak{l}$ and $\mathfrak{l}_\Gamma$ are $\mathfrak{l}_\Gamma$ are regular, the second and the third contain $\mathfrak{t}$, thus it is enough to prove the inclusion for the root vectors.

If α is a root of $\mathfrak{g}_\Gamma$ then $\alpha \parallel \Gamma$, therefore $\alpha \perp h_\mathfrak{p}$, proving the first inclusion. Now if α is a root of $\mathfrak{l}$ then $\alpha \perp h_\mathfrak{p}$. Notice that $\Gamma = M(V) \cap H$ and both $M(V)$ and H are r_α-invariant, therefore $r_\alpha \Gamma = \Gamma$ hence $\alpha \perp z_\Gamma$.

This proves the inclusion of subalgebras. Now it is enough to notice that the set of roots of $\mathfrak{l}$ can be represented as an intersection of a vector space with the set Δ. ∎

Now we are going to prove Theorem 1.3. Let $\mathfrak{h}_1$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The idea of the proof is to construct a certain representation space V of G and a vector $v \in V$ such that
1) The stabilizer $\mathfrak{g}_{\langle v \rangle}$ is "almost" equal to $\mathfrak{h}_1$ (they may differ by a one-dimensional subalgebra of $\mathfrak{t}$);
2) The support of v is contained in a face Γ of $M(V)$ such that $\mathfrak{g}_\Gamma \subset \mathfrak{p}$.

Together these two statements will allow us to use Theorem 1.1 and describe $\mathfrak{h}_1$ as stated in Theorem 1.3.

Proof. (of Theorem 1.3) Let V, v, H and Γ be the same with Lemma 3.2.

According to Lemma 3.3, $\mathfrak{l} = \mathfrak{g}_\Gamma \oplus \mathfrak{n}$, where $\mathfrak{n}$ is an ideal in $\mathfrak{g}_\Gamma^\perp$, hence the space V_Γ is a representation space of $\mathfrak{l}$.

We have $\mathfrak{l} \simeq \tilde{\mathfrak{g}} \oplus (\mathrm{Ker}\phi \cap \mathfrak{l})$. Since $V = \langle Gv \rangle$, $\langle P_\Gamma^- v \rangle$ is open in $\langle Gv \rangle$ (P_Γ^- is the opposite parabolic subgroup to P_Γ) and $(P_\Gamma^- v)_\Gamma = G_\Gamma v$, then $V_\Gamma = \langle G_\Gamma v \rangle$.

Consequently $\mathrm{Ker}\phi \cap \mathfrak{l}$ acts trivially on V_Γ. Therefore after identifying $\tilde{\mathfrak{g}}$ with a complement to $(\mathrm{Ker}\phi \cap \mathfrak{l})$ in $\mathfrak{l}$ we can consider V_Γ to be a representation space of $\tilde{\mathfrak{g}}$. Moreover there is an ideal $\mathfrak{n}' \in \mathfrak{n}$ such that $\tilde{\mathfrak{g}} = \mathfrak{g}_\Gamma \oplus \mathfrak{n}'$.

Let $\Gamma_1 \subset \Gamma$ be a minimal face of $M(V_\Gamma)$ for the orbit $G_\Gamma v$. According to point d) of Theorem 1.1, it is a minimal face of $M(V)$ for the orbit Gv as well. Note that if the statement of the theorem is true for $\mathfrak{h} \subset \mathfrak{h}_1$, it is true for $g \cdot \mathfrak{h} \subset g \cdot \mathfrak{h}_1$ for any $g \in \tilde{G}$, where $\tilde{G}$ is the connected subgroup of G whose tangent algebra is $\tilde{\mathfrak{g}}$. We have just shown that $G_\Gamma \subset \tilde{G}$, therefore it suffices to prove the theorem for the case when $\mathrm{supp}\,(v) \subset \Gamma_1$.

Due to theorem 1.2, The pair $(\mathfrak{g}, \mathfrak{h}_1)$ is obtained from $(\mathfrak{g}_{\Gamma_1}, \mathfrak{h}_1 \cap \mathfrak{g}_{\Gamma_1})$ by parabolic induction via $\mathfrak{p}_{\Gamma_1}$. We have a map ϕ from $\mathfrak{p}_{\Gamma_1}$ on $\mathfrak{g}_{\Gamma_1}$ such that $\phi^{-1}(\mathfrak{g}_{\Gamma_1}, \mathfrak{h}_1 \cap \mathfrak{g}_{\Gamma_1}) = \mathfrak{h}_1$. Restricting this map to $\tilde{\mathfrak{g}}$ we deduce that the pair $(\tilde{\mathfrak{g}}, \mathfrak{h}_1 \cap \tilde{\mathfrak{g}})$ is obtained from $(\mathfrak{g}_{\Gamma_1}, \mathfrak{h}_1 \cap \mathfrak{g}_{\Gamma_1})$ by parabolic induction via $\mathfrak{p}_{\Gamma_1} \cap \tilde{\mathfrak{g}}$.

Thus the following three operations lead from $(\mathfrak{g}, \mathfrak{h})$ to $(\mathfrak{g}, \mathfrak{h}_1)$:

1) replace $\tilde{\mathfrak{h}}$ with $\mathfrak{h}_1 \cap \tilde{\mathfrak{g}}$;

2) replace $\tilde{\mathfrak{g}}$ with $\mathfrak{g}_{\Gamma_1}$, $\mathfrak{h}_1 \cap \tilde{\mathfrak{g}}$ with $\mathfrak{h}_1 \cap \mathfrak{g}_{\Gamma_1}$ and $\mathfrak{p}$ with $(\mathfrak{p}_{\Gamma_1} \cap \tilde{\mathfrak{g}}) \oplus \mathfrak{p}^{\mathrm{u}}$;

3) enlarge the parabolic subalgebra to $\mathfrak{p}_{\Gamma_1}$. ∎

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