Parabolic Induction of Projective Orbits and Subalgebras

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Abstract. The concepts of parabolic induction of subalgebras and varieties appeared independently in works of different authors. Here we introduce and study the parabolic induction of projective orbits. This concept turns out to be closely related to the above ones, and can be used as a method to study them. As an application, we describe all subalgebras of a semisimple Lie algebra that contain a given subalgebra obtained by parabolic induction.

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1. Introduction

Let $G$ be a reductive algebraic group over the field of characteristic zero and $P$ be a proper parabolic subgroup of $G$. One says that a $G$-variety $X$ is obtained by parabolic induction from a $P$-variety $Y$ if the radical of $P$ acts trivially on $Y$, and there exists a $P$-equivariant injective morphism $\psi : Y \to X$ giving rise to a birational surjective morphism $G \times_P Y \to X$. The concept of the parabolic induction of varieties has been used by Akhiezer ([1], Th. 4) and Cupit-Foutou ([2], Th. 1.3) in the study of two-orbit varieties, in the work of Wasserman ([8], pp. 378–379) and in the work of Luna ([3], pr. 3.4) where he classified the spherical varieties of type A. A similar concept has been studied in the work of Kempf ([4]).

Let $L$ be a Levi subgroup of $P$. The action $G : X$ shares many properties with the action $L : Y$. For example, the complexities and the ranks of these actions coincide. In particular, the property of being spherical is preserved under parabolic induction. This is the reason why in many problems it is natural to study irreducible actions, those that cannot be obtained by a non-trivial parabolic induction.

Let $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{p}$ be the tangent algebras of $G$, $L$ and $P$ respectively and $\phi : \mathfrak{p} \to \mathfrak{l}$ be the normal projection. Then we have $\mathfrak{g}_{\psi(y)} = \mathfrak{p}_y = \phi^{-1}(\mathfrak{l}_y)$ for a generic point $y \in Y$. This is an example where the parabolic induction of subalgebras appears. Let us recall this notion.

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Let \( \mathfrak{g} \) be an algebraic reductive Lie algebra, \( \mathfrak{p} \) be a parabolic subalgebra of \( \mathfrak{g} \), and \( \phi \) be a homomorphism from \( \mathfrak{p} \) onto some algebraic reductive Lie algebra \( \tilde{\mathfrak{g}} \). Let \( \tilde{\mathfrak{h}} \) be a subalgebra of \( \tilde{\mathfrak{g}} \). Set \( \mathfrak{h} = \phi^{-1}(\tilde{\mathfrak{h}}) \). The pair \((\mathfrak{g}, \mathfrak{h})\) is said to be obtained from the pair \((\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})\) by parabolic induction via the parabolic subalgebra \( \mathfrak{p} \). We will denote this by \((\mathfrak{g}, \mathfrak{h}) = \phi^{-1}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})\). In the case of parabolic induction of varieties we have \((\mathfrak{g}, \mathfrak{h}) = \phi^{-1}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})\).

The parabolic induction of subalgebras is an interesting concept in itself. For example, it was used in the work of Wasserman ([8], see Definition 2.3, the notion of a prime subgroup) devoted to classification of wonderful varieties. The parabolic induction of subalgebras preserves some properties of pairs (for example, the complexity). Therefore, being interested in subalgebras of reductive algebras having certain properties, one can reduce the study to subalgebras \( \mathfrak{h} \subset \mathfrak{g} \) such that the pair \((\mathfrak{g}, \mathfrak{h})\) is irreducible, i.e. cannot be obtained by a non-trivial parabolic induction.

Below we introduce the parabolic induction of projective orbits. This concept appeared in a special case in the work of the author ([5], prop. 2.15, 2.16) where it was applied to the study of two-orbit varieties with their embeddings in projective spaces. It serves as a good model for both the parabolic induction of varieties and subalgebras and can be used to prove the properties of theirs.

Let \( G \) be the connected algebraic group whose tangent algebra is \( \mathfrak{g} \), \( T \) be a maximal torus in \( G \), \( t \) be its tangent algebra (a Cartan subalgebra of \( \mathfrak{g} \)), \( t(\mathbb{R}) \) be the real form of \( t \) consisting of elements with real eigenvalues and \( E = t(\mathbb{R})^* \). We will fix a \( G \)-invariant inner product on \( \mathfrak{g} \) and identify \( t \) with \( t^* \) (so that \( \mathbb{C} \subset t \)). For any representation space \( V \) of \( \mathfrak{g} \) let \( \Phi = \Phi(V) \subset E \) be the system of weights of \( T \) in \( V \) and \( V = \bigoplus_{\lambda \in \Phi} V_\lambda \) be the weight decomposition. For any subset \( R \subset \Phi \) set \( V_R := \bigoplus_{\lambda \in \Phi \cap R} V_\lambda \) and call \( V_R \) the restriction of \( V \) to \( R \). (In particular, \( V_{\{\psi\}} = V_\psi \) for any weight \( \psi \)). The convex hull of \( \Phi \) in \( E \) will be called the weight polytope of \( V \) and denoted by \( M(V) \). Let \( V = \bigoplus_{\lambda \in \Phi} V(\phi(\lambda)) \) be the decomposition into isotypic components (relative to the \( G \)-action) and \( v = \sum_{\phi(\lambda) \in \Phi} v(\phi(\lambda)) \), where \( v(\phi(\lambda)) \in V(\lambda) \).

Let \( \Gamma \) be a face of \( M(V) \). Denote the affine hull of \( \Gamma \) with \( \text{Aff} \Gamma \) and the center of mass of \( \Gamma \) with \( z_\Gamma \). For any root \( \alpha \) parallel to \( \Gamma \), we have \( r_\alpha \Gamma = \Gamma \), therefore \( r_\alpha z_\Gamma = z_\Gamma \) (where \( r_\alpha \) is the reflection relative to \( \alpha \)). Thus \( z_\Gamma \) is orthogonal to all roots parallel to \( \Gamma \). We will call the subalgebra of \( \mathfrak{g} \) generated by the linear span of \( \Gamma \) and the root vectors corresponding to the roots parallel to \( \Gamma \) the reductive subalgebra associated with \( \Gamma \) and denote it with \( \mathfrak{g}_\Gamma \). We will denote the subalgebra of \( \mathfrak{g} \) generated by the orthogonal complement to the linear span of \( \Gamma \) and the root vectors corresponding to the roots orthogonal to the linear span of \( \Gamma \) with \( \mathfrak{g}^\perp_\Gamma \). We will call the subalgebra of \( \mathfrak{g} \) generated by \( t \) and the root vectors corresponding to the roots having non-negative inner products with \( z_\Gamma \) the parabolic subalgebra associated with \( \Gamma \) and denote it with \( \mathfrak{p}_\Gamma \). We will denote the corresponding subgroups of \( G \) with \( G_\Gamma \) and \( P_\Gamma \). The following statement holds:

\[
\mathfrak{p}_\Gamma = (\mathfrak{g}_\Gamma \oplus \mathfrak{g}^\perp_\Gamma) \oplus \mathfrak{p}^u_\Gamma,
\]

where \( \mathfrak{p}^u_\Gamma \) is the unipotent radical of \( \mathfrak{p}_\Gamma \). Also set \( \ell_\Gamma = \mathfrak{g}_\Gamma \oplus \mathfrak{g}^\perp_\Gamma \). It is a Levi subalgebra of \( \mathfrak{p}_\Gamma \). The space \( V_\Gamma \) can be considered as a representation space of \( \mathfrak{g}_\Gamma \). If \( V \) is \( \mathfrak{g} \)-irreducible, then \( V_\Gamma \) is \( \mathfrak{g}_\Gamma \)-irreducible (e.g. [6], Prop. 8).

For any vector \( v = \sum_{\lambda \in \Phi} v_\lambda \), define the set of its weights by \( \Phi_v := \{ \lambda \in \Phi | v_\lambda \neq 0 \} \). The convex hull of this set will be called the support of \( v \) and denoted
by supp v. For any point ⟨v⟩ ∈ ℙ(V) set supp ⟨v⟩ = supp v. We will call a face Γ of M(V) minimal for the orbit Gv if there is a point in this orbit whose support lies in Γ, but there is no point whose support lies in any proper face of Γ. If supp v ⊂ Γ and Γ is minimal for the orbit Gv, we will say that the orbit G⟨v⟩ is obtained by parabolic induction from the orbit GΓ⟨v⟩.

The following theorem summarizes the properties of the parabolic induction of projective orbits.

**Theorem 1.1.**  

a) Any two minimal faces for the orbit Gv are equivalent under the Weyl group.  
b) The intersection of any G-orbit in V with VΓ is a single GΓ-orbit or empty.  
c) If supp v ⊂ Γ, supp gv ⊂ Γ and Γ is minimal for the orbit Gv, then g ∈ PΓ (in particular, the stabilizer of ⟨v⟩ lies in PΓ).  
d) If supp v ∈ Γ and Γ1 ⊂ Γ is minimal for the orbit GΓv then it is minimal for the orbit Gv as well.

**Theorem 1.2.** Let supp v ⊂ Γ and Γ be minimal for the orbit Gv. Then  
1) The G-variety G⟨v⟩ is obtained by parabolic induction from the PΓ-variety GΓ⟨v⟩,  
2) Let T′ be a torus acting on V permutably with G, T′ be its tangent algebra and let h be the projection of (g ⊕ T′)v onto g. Then (gΓ, gΓ ∩ h) → (g, h). In particular, (gΓ, (gΓ)v) → (g, gΓv) and (gΓ, (gΓ)v) → (g, gΓv).

It follows from these theorems that the map G × P VΓ → GVΓ is birational, therefore, according to the theorem of Kempf ([4]), the variety GVΓ always has rational singularities.

According to Theorem 1.2, the parabolic induction of orbits turns out to be a convenient model for studying both the parabolic induction of varieties and subalgebras.

Moreover, this construction can be applied to the study of projective orbits satisfying certain conditions for the following reason: the orbit obtained by parabolic induction shares many properties with the orbit it has been obtained from. As it was said earlier, this approach was used by the author ([5], prop. 2.15, 2.16) in the study of two-orbit varieties.

As one more application, we are going to prove a theorem on the parabolic induction of subalgebras.

Let us introduce some notation. We will call a parabolic subalgebra p of g admissible for a reductive algebra ˜g if there exists a homomorphism φ from p onto ˜g; a parabolic subalgebra p1 ⊃ p of g, an admissible enlargement of p for φ if φ extends to a homomorphism φ1 from p1 onto ˜g. Let h be a subalgebra of ˜g and (˜g, h) → (g, h).

Let us study the subalgebras of g containing h. There are two obvious ways to construct such subalgebras:  

1) enlarging the subalgebra of ˜g: take any subalgebra ˜h1 ⊃ h of ˜g and let (˜g, ˜h1) → (g, h); then h1 ⊃ h.  
2) enlarging the parabolic subalgebra: take any admissible enlargement p1 ⊃ p for φ and let (g, h) → (g, h); then h1 ⊃ h.

These two operations are nearly enough to construct any subalgebra of g containing h. We shall need only one more operation not changing h but ˜g:
passing to a deeper parabolic induction: suppose that the pair \((\tilde{g}, \tilde{h})\) is reducible, so it can be obtained from a pair \((\tilde{g}_1, \tilde{h}_1)\) by parabolic induction via a parabolic subalgebra \(\hat{p}_1\) of \(\tilde{g}\) (with a homomorphism \(\tilde{\phi}_1\)). Then we can replace \(\tilde{g}\) by \(\tilde{g}_1\) so that \((\tilde{g}_1, \tilde{h}_1)\rightarrow p_1^*(g, h)\), where \(p_1 = \phi^{-1}(\hat{p}_1)\) and \(\phi_1 = \tilde{\phi}_1 \circ \phi\).

Obviously, any composition of these operations also leads to a subalgebra of \(g\) containing \(h\). The theorem we are going to prove claims that these three operations are enough to obtain all subalgebras of \(g\) containing \(h\).

**Theorem 1.3.** Let \(\tilde{g}\) be a reductive algebraic Lie algebra, \(p\) be an admissible for \(\tilde{g}\) parabolic subalgebra of \(g\), \(h\) be a subalgebra of \(\tilde{g}\) and the pair \((g, h)\) be obtained from \((\tilde{g}, \tilde{h})\) by parabolic induction via \(p\). Then the subalgebras of \(g\) containing \(h\) are exactly the subalgebras obtained by enlarging the subalgebra of \(\tilde{g}\), then passing to a deeper parabolic induction and finally enlarging the parabolic subalgebra.

Theorem 1.3 can be used to classify the subalgebras of semisimple algebras satisfying certain conditions. For example, it was used by the author to classify the maximal non-horospherical subalgebras of a semisimple Lie algebra (unpublished).

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**Notation:**
- \(G\): a connected reductive algebraic group;
- \(g\): the tangent algebra of \(G\);
- \(T\): a maximal torus of \(G\);
- \(t\): the tangent algebra of \(T\);
- \(b\): a Borel subalgebra of \(g\) containing \(t\);
- \(\Delta\): the system of roots of \(g\);
- \(\Delta^+ \subset \Delta\): the system of positive roots of \(g\);
- \(p^u\): the unipotent radical of a parabolic subalgebra \(p\) of \(g\);
- \(u = b^u\): the maximal unipotent subalgebra of \(b\) and \(g\);
- \(W\): the Weyl group of the pair \((G, T)\);
- \(r_\alpha\): the reflection relative to a root \(\alpha\);
- \(\pi\): the element of \(W\) corresponding to an element \(n \in N(T)\);
- \(\langle M \rangle\): the linear span of a set \(M\) in a vector space;
- \(g_x\): the stabilizing subalgebra of a point \(x\).

The vector space \(E = t(\mathbb{R})^*\) will be in certain cases considered as an affine space and its elements as points.

If \(\alpha \in E\), then \(h_\alpha\) is an element of \(t\) such that \(h_\alpha\) is orthogonal to \(\text{Ker} \alpha\) and \(\alpha(h_\alpha) = 2\).

If \(\alpha \in \Delta\), then \(e_\alpha\) will denote the root vector, corresponding to \(\alpha\) (defined up to a scalar factor). We will claim that \([e_\alpha, e_{-\alpha}] = h_\alpha\) (thus, \(\{h_\alpha, e_\alpha, e_{-\alpha}\}\) will be an \(\mathfrak{sl}_2\)-triple).

The notation \(x \sim y\) means that both vectors \(x\) and \(y\) are non-zero and are proportional to each other.

We shall call a face of the weight polytope dominant, if its center of mass lies in the dominant (closed) Weyl chamber.
For a hyperplane $H$ in an affine space $A$ we will denote the two closed half-spaces, into which $H$ divides $A$, with $H^\pm$. In case when $0 \notin H$ we will assume that $0 \in H^-$.

A hyperplane $H$ in an affine space $A$ is called a hyperplane of support of a set $M \subset A$, if the intersection $H \cap M$ is nonempty, and $M$ is contained in $H^+$ or $H^-$. 

2. Parabolic induction of projective orbits

In this section we will prove Theorems 1.1 and 1.2.

First we will study some properties of convex polytopes invariant under the Weyl group.

Let $M \subset E$ be such a polytope and $\Gamma$ be a face of it. Let $\Delta || (\Gamma)$ be the set of roots parallel to $\Gamma$. As we explained above, $\Delta || (\Gamma) \perp z_{\Gamma}$. Let $\Delta \perp (\Gamma)$ be the set of roots orthogonal to $z_{\Gamma}$ and not parallel to $\Gamma$, $\Delta_+ (\Gamma)$ be the set of roots having positive inner product with $z_{\Gamma}$.

The polytope $M$ is an intersection of some (closed) half-spaces $H_i^-$. Let $S_{\pm} (\Gamma) = (\cap_{\Gamma \subset H_i} H_i^\pm) \setminus \text{Aff } \Gamma$ for any point $p \in \Gamma$. Clearly $S^+ (\Gamma)$ and $S^- (\Gamma)$ do not depend on $p$. They are opposite convex cones. Let $S_0 (\Gamma)$ be the complementary set to $S_0 (\Gamma)$. The following three lemmas generalize some results of Vinberg (see [7], p. 9-12).

**Lemma 2.1.** 1) If $\alpha \in \Delta \perp (\Gamma)$ then $\alpha$ is orthogonal to the linear span $\langle \Gamma \rangle$ of $\Gamma$ in $E$; 
2) $\Delta || (\Gamma) \cap \Delta \perp (\Gamma)$; 
3) $\Delta || (\Gamma) \cup \Delta \perp (\Gamma) = \Delta \cap S_0 (\Gamma)$; 
4) $\Delta_+ (\Gamma) = \Delta \cap S_0 (\Gamma)$.

**Proof.** Let $\alpha$ be a root. If $\alpha$ is orthogonal to $z_{\Gamma}$ then $r_\alpha \Gamma = \Gamma$, therefore either $\alpha \parallel \Gamma$, or $\alpha \perp (\Gamma)$. This proves 1) and 2).

In both cases $r_\alpha S_\pm (\Gamma) = S_\pm (\Gamma)$, but $r_\alpha \alpha = -\alpha$, therefore $\alpha \notin S_\pm (\Gamma)$.

Thus $\Delta || (\Gamma) \cup \Delta \perp (\Gamma) \subset S_0 (\Gamma)$.

On the other hand, if $\alpha \in S_0 (\Gamma)$, then $r_\alpha \Gamma = \Gamma$, therefore $\alpha \perp z_{\Gamma}$. This proves 3).

Now let $\alpha$ be a root having a positive inner product with $z_{\Gamma}$. We have $r_\alpha z_{\Gamma} \in M$, therefore $z_{\Gamma} - r_\alpha z_{\Gamma} \in S_+ (\Gamma)$ (this vector does not lie in $S_0 (\Gamma)$ because it is proportional to $\alpha$, so has a non-zero inner product with $z_{\Gamma}$). Since $(\alpha, z_{\Gamma}) > 0$, $z_{\Gamma} - r_\alpha z_{\Gamma} = c \alpha$ for some positive $c$. Consequently, $\alpha \in S_+ (\Gamma)$.

Therefore $\alpha \in \Delta_+ (\Gamma)$ if and only if $\alpha \in S_+ (\Gamma)$ (if and only if $-\alpha \in S_- (\Gamma)$). This proves 4).

Choose a system of positive roots $\Delta_+ \supset \Delta_+ (\Gamma)$. Let $C$ be the corresponding dominant Weyl chamber. Denote the group generated by $r_\alpha$ for $\alpha \in \Delta || (\Gamma)$ with $W_{\Gamma}$. Let $C_{\Gamma} = \{ \lambda \in \text{Aff } \Gamma : (\lambda, \alpha) \geq 0 \forall \alpha \in \Delta_+ \cap \Delta || (\Gamma) \}$.

Obviously, $z_{\Gamma} \in C_{\Gamma}$, and $C_{\Gamma}$ is a fundamental region for the action $W_{\Gamma} : \text{Aff } \Gamma$. 

Lemma 2.2. \( C \cap \Gamma = C_{\Gamma} \cap \Gamma \).

Proof. Obviously, \( C \cap \Gamma \subset C \cap \text{Aff} \Gamma \subset C_{\Gamma}. \) On the other hand, for any \( \alpha \in \Delta_+ (\Gamma) \) and \( p \in \Gamma \) we have \((\alpha, p) \geq (\alpha, z_{\Gamma}) \geq 0. \) Therefore

\[
C_{\Gamma} \cap \Gamma \subset \{ \lambda \in \text{Aff} \Gamma : (\lambda, \alpha) \geq 0 \forall \alpha \in \Delta_+ \cap \Delta_\parallel (\Gamma) \} \cap \Gamma \subset \{ \lambda \in E : (\lambda, \alpha) \geq 0 \forall \alpha \in \Delta_+ (\Gamma) \} = \{ \lambda \in E : (\lambda, \alpha) \geq 0 \forall \alpha \in \Delta_+ \} = C. \]

Lemma 2.3. For any face \( \Gamma_1 \subset \Gamma \) there is an element \( w \in W_{\Gamma} \) such that \( w\Gamma_1 \) is dominant.

Proof. According to the choice of \( \Delta_+ \), \( z_{\Gamma_1} \in C_{\Gamma} \cap \Gamma \subset C. \) Choose an element \( w \in W_{\Gamma} \) such that \( wz_{\Gamma_1} \in C_{\Gamma} \). Since \( z_{\Gamma_1} \in \Gamma, wz_{\Gamma_1} \in \Gamma, \) thus according to Lemma 2.2, \( wz_{\Gamma_1} \in C_{\Gamma} \cap \Gamma = C \cap \Gamma \subset C. \]

Now let \( M = M(V) \) be a weight polytope. Then \( W_{\Gamma} \) is the Weyl group of \( G_{\Gamma} \) and \( \langle \Gamma \rangle = g_{\Gamma} \cap E. \)

Lemma 2.4. Let \( v \in V \) be a vector such that \( \text{supp} \, v \subset \Gamma. \) Then

1) If \( \alpha \in \Delta_+ (\Gamma) \) then \( e_\alpha \in g_v; \)
2) If \( \alpha \in \Delta_\perp (\Gamma) \) then \( e_\alpha \in g_v \) and \( h_\alpha \in g_v. \)

Proof. Due to Lemma 2.1, for all \( p \in \Gamma \) and \( \alpha \in \Delta_+ (\Gamma) \cup \Delta_\perp (\Gamma) \) the vector \( p + \alpha \) is not a weight. Therefore \( e_\alpha \in g_v. \) If \( \alpha \in \Delta_\perp (\Gamma) \) then the same is true for \(-\alpha, \) therefore the whole \( \mathfrak{sl}_2\)-triple \( \{ e_{-\alpha}, h_\alpha, e_\alpha \} \) stabilizes \( v. \) 

According to Lemma 2.4, both the unipotent radical \( p_{\Gamma}^u \) of \( p \) and \( g_{\Gamma}^\perp \) act trivially on \( V_{\Gamma}. \) This proves the following:

Lemma 2.5. If \( \text{supp} \, v \subset \Gamma \) and \( b \in P_{\Gamma} \) then there is \( g \in G_{\Gamma} \) such that \( bv = gv. \)

We will need one more lemma:

Lemma 2.6. Let \( n \in N(T) \) be an element such that \( w = \overline{n} \) stabilizes the face \( \Gamma. \) Then \( n = n_1n_2, \) where \( n_1 \in N(T) \cap G_{\Gamma} \) and \( n_2 \in N(T) \cap G_{\Gamma}^\perp. \) In particular, \( n \in P_{\Gamma}. \)

Proof. The element \( w \) can be decomposed as a product of two commuting elements: \( w = w_1w_2, \) where \( w_1 \) (resp. \( w_2 \)) is a product of reflections relative to roots lying in \( \Delta_\parallel (\Gamma) \) (resp. \( \Delta_\perp (\Gamma) \)).

Let \( n_1 \) and \( n_2 \) be some representatives of \( w_1 \) and \( w_2 \) in \( N(T) \cap G_{\Gamma} \) and \( N(T) \cap G_{\Gamma}^\perp, \) respectively. Then \( n \) differs from \( n_1n_2 \) by an element of \( T, \) and one can correct \( n_1 \) and \( n_2 \) so that \( n = n_1n_2. \)
Proof. (of Theorem 1.1) For convenience suppose that $\Gamma$ is dominant.

a) Let $v_1, v_2 \in Gv$ be two vectors whose supports lie in $\Gamma_1$ and $\Gamma_2$, respectively, and $v_2 = gv_1$. Apply the Bruhat decomposition and decompose $g$ as $g = b_2nb_1$, where $b_1 \in P_{\Gamma_1}$, $b_2 \in P_{\Gamma_2}$ and $n \in N(T)$. Then the face $\Gamma_1$ (resp. $\Gamma_2$) is the minimal face containing the support of $b_1v_1$ (resp. of $b_2^{-1}v_2$), consequently $\pi\Gamma_1 = \Gamma_2$.

b) Let $\text{supp} v_1 \subset \Gamma$, $\text{supp} v_2 \subset \Gamma$, and $v_2 \sim gv_1$. Apply the Bruhat decomposition and decompose $g$ as $g = b_2nb_1$, where $b_1, b_2 \in P_{\Gamma}$ and $n \in N(T)$. Then $b_2^{-1}v_2 \sim nb_1v_1$. The supports of $b_2^{-1}v_2$ and $b_1v_1$ lie in $\Gamma$. According to Lemma 2.5 we can assume that $b_1, b_2 \in G_{\Gamma}$ and prove the theorem for the case when $b_1 = b_2 = e$, so $v_2 \sim nv_1$.

Let $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma$ be the minimal faces containing the supports of $v_1$ and $v_2$ respectively. Then $\Gamma_2 = \pi\Gamma_1$. According to Lemma 2.3, there are elements $w_1, w_2 \in W_\Gamma$ such that the faces $w_1\Gamma_1$ and $w_2\Gamma_2$ are dominant, therefore coincide. Let $w = w_2^{-1}w_1$. Then $\Gamma_2 = w\Gamma_1$ and $w \in W_\Gamma$. The element $w$ has a representative if $G_{\Gamma}$, therefore we can reduce the theorem to the case when $\Gamma_1 = \Gamma_2$.

Now we have an element $n \in N(T)$ so that $\pi\Gamma_1 = \Gamma_1$. According to Lemma 2.6, one can choose an element $n'$ of $N(T) \cap G_{\Gamma_1}$ such that $n'v_1 = nv_1$. Now it is enough to note that $G_{\Gamma_1} \subset G_{\Gamma}$.

c) Decompose $g$ as $g = b_2nb_1$, where $b_1, b_2 \in P_{\Gamma}$ and $n \in N(T)$. Then $b_2^{-1}gv = nb_1v$.

Note that the supports of $b_2^{-1}gv$ and $b_1v$ lie in $\Gamma$, and, moreover, the support of $b_1v$ is not contained in any proper face of $\Gamma$. Hence, $\pi$ stabilizes the face $\Gamma$, therefore, according to Lemma 2.6, $n$ is contained in $P_{\Gamma}$; consequently, $g \in P_{\Gamma}$.

d) Suppose that $\Gamma_1$ is not minimal for $Gv$. Then there is an element $g \in G$ such that the support of $gv$ is a proper face of $\Gamma_1$. But according to point b) there is an element $g' \in G_{\Gamma}$ such that $g'v = gv$, and we come to a contradiction. ■

Proof. (of Theorem 1.2) To prove the first part of the theorem it is enough to consider the map $\psi : G \times P G_{\Gamma}(v) \to G_{\Gamma}(v), \psi(g \times x) = gx$ and notice that it is proper.

Now let $H$ be the connected subgroup of $G$ whose tangent algebra is $\mathfrak{h}$ and $g \in H$. Then $\text{supp} gv = \text{supp} v$, therefore $g \in P_{\Gamma}$ according to the point c) of Theorem 1.1. On the other hand, $\mathfrak{h}$ contains $\mathfrak{p}_T^u$ and $\mathfrak{g}_T^\perp$ by Lemma 2.4. ■

3. An application to the parabolic induction of subalgebras

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, $\mathfrak{p} = \mathfrak{t} \oplus \mathfrak{p}^u$ be its Levi decomposition such that $\mathfrak{t} \supset \mathfrak{t}$, and $h_p \in \mathfrak{t}$ be an element such that $\alpha(h_p) > 0$ and $\alpha(h_p) \in \mathbb{Z}$ for all roots of $\mathfrak{p}^u$ and $\alpha(h_p) = 0$ for all roots of $\mathfrak{t}$. For a weight $\phi$ let $H(h_p, \phi) = \{ \lambda \in \mathbb{E} : \lambda(h_p) = \phi(h_p) \}$.

The following lemma is known (see, e.g. [6]), but for convenience of the reader we give the proof here.
Lemma 3.1. Let $G : V$ be an irreducible representation and $\phi$ be its highest weight, Then $H(h_p, \phi)$ is a hyperplane of support of $M(V)$ and

$$\{ v \in V : p^n v = 0 \} = V_{H(h_p, \phi)}.$$

Proof. Let $L$ be the connected subgroup of $G$ whose tangent algebra is $l$. Take a vector $W \in V$ such that $p^n w = 0$. Consider the subspace $\langle Lw \rangle \subset V$. Obviously, $p^n v = 0$ for any $v \in \langle Lw \rangle$. It follows that if $v$ is a highest weight vector with respect to $L$, then it is a highest vector with respect to $G$. Therefore, $v \in \langle LV_\phi \rangle$, whence $\supp v \subset H(h_p, \phi)$.

On the other hand, $V = \langle GV_\phi \rangle$ and $p^n V_\phi = 0$ implies that $M(V) \subset H(h_p, \phi)$. \hfill \blacksquare

Lemma 3.2. Let $p$ be a parabolic subalgebra of $g$ and $h_1 \supset p^n$. Then there is a vector space $V$, a vector $v \subset V$, a hyperplane of support $H$ of $M(V)$ orthogonal to $h_p$ and a one-dimensional torus $T'$ acting on $V$ permutably with $G$, such that

1) $\langle v \rangle \subset H$;
2) $h_1$ is the projection of $(g \oplus t')_v$ onto $g$, where $t'$ is the tangent algebra of $T'$.

Proof. According to the Chevalley theorem, there exists a representation space $U$ of $G$ and a vector $u \in U$ such that $g_{(u)} = h_1$. We may also assume that there is no proper $G$-invariant subspace of $U$ containing $u$.

Since $h_1 \supset p^n$, then by Lemma 3.1, all the numbers $\phi(h_p)$ for all $\phi$ such that $u(\phi) \neq 0$ are non-negative. Suppose that for all $\phi$ such that $u(\phi) \neq 0$ we have $\phi(h_p) = 0$. Then the statement of the lemma holds for $V = U$, $v = u$ and $H = H(h_p, 0)$.

Otherwise choose some $\psi$ such that $u(\psi) \neq 0$ and $\psi(h_p) > 0$. Set $U' = U \otimes U (\psi)$ and $u' = u \otimes u (\psi)$. Obviously for all $\phi$ such that $u'(\phi) \neq 0$ we have $\phi(h_p) > 0$. Let us prove that $G_{(u')} = G_{(u)}$.

If $g \in G_{(u)}$ then $gu \sim u$, so $gu(\psi) \sim u(\psi)$ for all $\psi$. Therefore $G_{(u')} \subset G_{(u')}$. Let us prove the inverse inclusion. If $g \in G_{(u')}$ then $g(u \otimes u (\psi)) \sim u \otimes u (\psi)$, hence $gu \sim u$, so $g \in G_{(u)}$.

Choose some natural numbers $n(\phi)$ so that all the numbers $n(\phi) \phi(h_p)$ coincide for all $\phi$ such that $u'(\phi) \neq 0$. Define $V = \oplus (U'(\phi)^{\otimes n(\phi)})$ and $v = \sum (u'(\phi)^{\otimes n(\phi)}) \in V$. Let $T'$ be a one-dimensional torus acting on $V$ via

$$t \cdot \sum (u'(\phi)^{\otimes n(\phi)}) = \sum (t^n u'(\phi)^{\otimes n(\phi)})$$

Set $\Gamma = H \cap M(V)$; then $\supp v \subset \Gamma$. and we are done. \hfill \blacksquare

Lemma 3.3. Let $G : V$ be a representation, $H$ be a hyperplane of support of $M(V)$ orthogonal to $h_p$, and $\Gamma = M(V) \cap H$. Then there is an ideal $n \subset g_{\Gamma}$ such that $l = g_{\Gamma} \oplus n$. 

Proof. Let us first prove that \( g_\Gamma \subset I \subset I_\Gamma \). The subalgebras \( g_\Gamma \), \( I \) and \( I_\Gamma \) are \( I_\Gamma \) are regular, the second and the third contain \( t \), thus it is enough to prove the inclusion for the root vectors.

If \( \alpha \) is a root of \( g_\Gamma \) then \( \alpha \parallel \Gamma \), therefore \( \alpha \perp h_p \), proving the first inclusion. Now if \( \alpha \) is a root of \( I \) then \( \alpha \perp h_p \). Notice that \( \Gamma = M(V) \cap H \) and both \( M(V) \) and \( H \) are \( r_n \)-invariant, therefore \( r_n\Gamma = \Gamma \) hence \( \alpha \perp z_\Gamma \).

This proves the inclusion of subalgebras. Now it is enough to notice that the set of roots of \( I \) can be represented as an intersection of a vector space with the set \( \Delta \).

Now we are going to prove Theorem 1.3. Let \( h_1 \) be a subalgebra of \( g \) containing \( h \). The idea of the proof is to construct a certain representation space \( V \) of \( G \) and a vector \( v \in V \) such that
1) The stabilizer \( g_\Gamma \) is “almost” equal to \( h_1 \) (they may differ by a one-dimensional subalgebra of \( t \));
2) The support of \( v \) is contained in a face \( \Gamma \) of \( M(V) \) such that \( g_\Gamma \subset p \).

Together these two statements will allow us to use Theorem 1.1 and describe \( h_1 \) as stated in Theorem 1.3.

Proof. (of Theorem 1.3) Let \( V, v, H \) and \( \Gamma \) be the same with Lemma 3.2.

According to Lemma 3.3, \( I = g_\Gamma \oplus n \), where \( n \) is an ideal in \( g_\Gamma^+ \), hence the space \( V_\Gamma \) is a representation space of \( I \).

We have \( I \simeq \tilde{g} \oplus (\ker \rho \cap I) \). Since \( V = \langle Gv \rangle \), \( \langle P^-_\Gamma v \rangle \) is open in \( \langle Gv \rangle \) \( (P^-_\Gamma \) is the opposite parabolic subgroup to \( P_\Gamma \)) and \( \langle P^-_\Gamma v \rangle_\Gamma = G_{\Gamma\Gamma}v \), then \( V_\Gamma = \langle G_{\Gamma\Gamma}v \rangle \).

Consequently \( \ker \rho \cap I \) acts trivially on \( V_\Gamma \). Therefore after identifying \( \tilde{g} \) with a complement to \( \ker \rho \cap I \) in \( I \) we can consider \( V_\Gamma \) to be a representation space of \( \tilde{g} \). Moreover there is an ideal \( n' \in n \) such that \( \tilde{g} = g_\Gamma \oplus n' \).

Let \( \Gamma_1 \subset \Gamma \) be a minimal face of \( M(V_\Gamma) \) for the orbit \( G_{\Gamma\Gamma}v \). According to point d) of Theorem 1.1, it is a minimal face of \( M(V) \) for the orbit \( Gv \) as well. Note that if the statement of the theorem is true for \( h \subset h_1 \), it is true for \( g \cdot h \subset g \cdot h_1 \) for any \( g \in G \), where \( G \) is the connected subgroup of \( G \) whose tangent algebra is \( \tilde{g} \). We have just shown that \( G_{\Gamma\Gamma} \subset G \), therefore it suffices to prove the theorem for the case when \( \operatorname{supp}(v) \subset \Gamma_1 \).

Due to theorem 1.2, The pair \( (g_\Gamma, h_1) \) is obtained from \( (g_{\Gamma_1}, h_1 \cap g_{\Gamma_1}) \) by parabolic induction via \( p_{\Gamma_1} \). We have a map \( \phi \) from \( p_{\Gamma_1} \) on \( g_{\Gamma_1} \) such that \( \phi^{-1}(g_{\Gamma_1}, h_1 \cap g_{\Gamma_1}) = h_1 \). Restricting this map to \( \tilde{g} \) we deduce that the pair \( (\tilde{g}, h_1 \cap \tilde{g}) \) is obtained from \( (g_{\Gamma_1}, h_1 \cap g_{\Gamma_1}) \) by parabolic induction via \( p_{\Gamma_1} \cap \tilde{g} \).

Thus the following three operations lead from \( (g, h_1) \) to \( (\tilde{g}, h_1) \):
1) replace \( h \) with \( h_1 \cap \tilde{g} \);
2) replace \( \tilde{g} \) with \( g_{\Gamma_1} \), \( h_1 \cap \tilde{g} \) with \( h_1 \cap g_{\Gamma_1} \) and \( p \) with \( (p_{\Gamma_1} \cap \tilde{g}) \oplus p^n \);
3) enlarge the parabolic subalgebra to \( p_{\Gamma_1} \).

References


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