Central Extension, Derivations and Automorphism Group for Lie Algebras Arising from the 2-Dimensional Torus

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Abstract. Let $A = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ be the ring of Laurent polynomials and $B$ the set of skew derivations of $A$. Set $\tilde{L} = A \oplus B$. In this paper, we study the automorphism group, derivations and universal central extension of the derived Lie subalgebra of $\tilde{L}$.

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1. Introduction

The rank one Heisenberg-Virasoro Lie algebra is a central extension of the Lie algebra $\{ f(t) \frac{d}{dt} + g(t) [ f, g \in \mathbb{C}[t, t^{-1}] \}$ of differential operators on a circle of order at most one. Arbarello et al. studied the rank one Heisenberg-Virasoro algebra and established a connection between the second cohomology of certain moduli spaces of curves and that of the Lie algebra of differential operators of order at most one in [1]. Billig constructed a class of irreducible representations for the Heisenberg-Virasoro Lie algebra of level zero in [2].

In this paper, we generalize the rank one Heisenberg-Virasoro Lie algebra to the rank two case. Let $A = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$, $B$ be the set of skew derivations of $A$ spanned by the elements of the form $x^{\vec{m}}(m_2d_1 - m_1d_2)$, where $\vec{m} = (m_1, m_2) \in \mathbb{Z}^2 \setminus \{0\}$ and $d_1, d_2$ are the degree derivations of $A$. We set $\tilde{L} = A \oplus B$ and write $L$ for the derived Lie subalgebra of $\tilde{L}$. One can easily check that $L$ is a perfect Lie algebra. Our purpose in this paper is to study the automorphism group (Theorem 2.10 in section two), derivations (Theorem 3.3 in section three) and the universal central extension (Theorem 4.10 in section four) of the Lie algebra $L$.

Let $\Gamma = \mathbb{Z}^2 \setminus \{0\} \subset \mathbb{Z}e_1 + \mathbb{Z}e_2$. For $\vec{m} = m_1e_1 + m_2e_2$, $\vec{n} = n_1e_1 + n_2e_2 \in \Gamma$, we set $x^{\vec{m}} = x^{m_1}x^{m_2} = x_1^{m_1}x_2^{m_2} \in A$, and denote the skew derivation $x^{\vec{m}}(m_2d_1 - m_1d_2)$ of $A$ by $E(\vec{m})$, then the Lie algebra $L$ is an infinite dimensional vector space spanned by elements of the form $x^{\vec{m}}$, $E(\vec{m})$, for $\vec{m} \in \Gamma$, together with the following Lie bracket relations:

$$[x^{\vec{m}}, E(\vec{n})] = -[E(\vec{n}), x^{\vec{m}}] = g(\vec{m}, \vec{n})x^{\vec{m} + \vec{n}}$$

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\[ [E(\bar{m}), E(\bar{n})] = g(\bar{m}, \bar{n})E(\bar{m} + \bar{n}), \quad [x^\bar{m}, x^\bar{n}] = 0 \]
where \( g(\bar{m}, \bar{n}) = (m_2n_1 - m_1n_2) \).

2. Automorphism Group

Denoted by \( AutL \) the automorphism group of the Lie algebra \( L \). In this section, we first construct two classes of special automorphisms which form subgroups of the automorphism group \( AutL \), then we give the structure of the \( AutL \).

It is clear that the Lie algebra \( L \) is \( \mathbb{Z}^2 \)-graded. Indeed \( L = \bigoplus_{\bar{m} \in \Gamma} L_{\bar{m}} = \bigoplus_{\bar{m} \in \Gamma} (x^\bar{m} \oplus E(\bar{m})) \). Let \( \pi_A : L \to A, \pi_B : L \to B \) be the natural projective maps, one can easily show that the map \( \pi_B \) is a Lie algebra homomorphism.

Lemma 2.1. If \( \sigma \in AutL \), then \( \sigma A \subset A \).

Proof. We prove the lemma by contradiction. Assume \( \sigma(x) \notin A \) for some \( x \in A \). First we show there exists \( y \in L \), such that \( [[\sigma(x), \sigma(y)], \sigma(x)] \neq 0 \). We define an order for \( \Gamma \) as follows:

\[ \bar{m} < \bar{n} \iff m_1 < n_1 \text{ or } m_1 = n_1 \text{ and } m_2 < n_2. \]

Note that \( \sigma(x) \notin A \), we can set \( \pi_B \sigma(x) = \sum_{i=1}^{k} \lambda_i E(\bar{m}_i) \), with the lowest term \( \lambda_1 E(\bar{m}) \neq 0 \). With loss of generality, we can assume the first component \( m_{11} \) of \( \bar{m}_1 \) is nonzero. Since \( \sigma \) is surjective, we can choose \( y \in L \), so that \( \sigma(y) = E(e_2) \).

By an easy calculation we have
\[ [[\pi_B \sigma(x), \pi_B \sigma(y)], \pi_B \sigma(x)] = -\lambda_1^2 m_{11}^2 E(2\bar{m} + e_2) + \text{higher order terms} \neq 0. \]

Hence \( [[\sigma(x), \sigma(y)], \sigma(x)] \neq 0 \). But \( \sigma[[x, y], x] \in [[A, L], A] \subset [A, A] = 0 \), which is a contradiction. Therefore we have \( \sigma A \subset A \).

Lemma 2.2. Let \( \sigma \in AutL \), then \( \pi_B \sigma|_B \in AutB \), and for any \( \bar{m} \in \Gamma \), there exists \( \bar{n} \in \Gamma \), so that \( \pi_B \sigma(E(\bar{m})) \in \mathbb{C} E(\bar{n}) \).

Proof. By Lemma 2.1, we have \( \pi_B \sigma|_B \) is surjective. Now we show that \( \pi_B \sigma|_B \) is injective. If \( y \in B \) and \( \pi_B \sigma(y) = 0 \), then \( \sigma(y) \in A \). This implies \( y \in \sigma^{-1} A \subset A \), and \( y \in A \cap B = \{0\} \), hence \( \pi_B \sigma|_B \) is injective. This shows that \( \pi_B \sigma|_B \in AutB \).

[4] shows that any automorphism of the Lie algebra \( B \) is a graded automorphism. Hence there exists \( \bar{n} \in \Gamma \), such that \( \pi_B \sigma(E(\bar{m})) \in \mathbb{C} E(\bar{n}) \).

Lemma 2.3. Let \( \pi_B \sigma(E(\bar{m})) \in \mathbb{C} E(\bar{n}), \) then \( \sigma(x^\bar{m}) \in \mathbb{C} x^{\bar{n}} \).

Proof. Since \( [E(\bar{m}), x^\bar{m}] = 0 \), we have \( \sigma(x^\bar{m}) \in \bigoplus_{g(\bar{r}, \bar{n}) = 0} \mathbb{C} x^{\bar{n}} \). Choose any \( \bar{p} \in \Gamma \) satisfying \( g(\bar{m}, \bar{p}) \neq 0 \). Let \( \pi_B \sigma(E(\bar{p})) \in \mathbb{C} E(\bar{q}) \), then
\[ \sigma(x^{\bar{m}+\bar{p}}) = \frac{1}{g(\bar{m}, \bar{p})} [\sigma(x^{\bar{m}}), \sigma(E(\bar{p}))] \in \bigoplus_{g(\bar{r}, \bar{n}) = 0} \mathbb{C} x^{\bar{r}+\bar{q}}. \]
On the other hand, since
\[ \pi_B \sigma(E(\vec{m} + \vec{p})) = \frac{1}{g(\vec{m}, \vec{p})} \pi_B [\sigma(E(\vec{m})), \sigma(E(\vec{p}))] \in C E(\vec{m} + \vec{q}), \]
we get \( \sigma(x^{\vec{m} + \vec{p}}) \in \bigoplus_{g(\vec{m} + \vec{p}) = 0} \mathbb{C} x^x, \) hence
\[
\sigma(x^{\vec{m} + \vec{p}}) \in \left( \bigoplus_{g(\vec{n}', \vec{q}) = 0} \mathbb{C} x^{\vec{n} + \vec{q}} \right) \cap \left( \bigoplus_{g(\vec{n}, \vec{q}) = 0} \mathbb{C} x^{\vec{n} + \vec{q}} \right) \\
= \left( \bigoplus_{g(\vec{n}', \vec{q}) = 0} \mathbb{C} x^{\vec{n} + \vec{q}} \right) \cap \left( \bigoplus_{g(\vec{n}, \vec{q}) = 0} \mathbb{C} x^{\vec{n} + \vec{q}} \right)
\]
Note that \( g(\vec{n}', \vec{q}) E(\vec{n} + \vec{q}) = g(\vec{n}, \vec{q}) E(\vec{n} + \vec{q}) \neq 0, \) hence \( g(\vec{n}, \vec{q}) \neq 0. \) Therefore \( g(\vec{n}', \vec{n}) = g(\vec{q}, \vec{n} - \vec{n}') \neq 0 \) and which also finishes the most of the lemma.

Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \) \( \Omega = \{(A, a, b, c) | A \in GL_2(\mathbb{Z}), a, b, c \in \mathbb{C}^*\}. \) We define a multiplication on \( \Omega \) as follows:
\[
(B, x_2, y_2, z_2) \cdot (A, x_1, y_1, z_1) = (BA, x_1 a_1 + y_1 a_2, y_1 x_2 a_1 + y_2 a_2, z_2 z_1)
\] (2.2)
where \( A = (a_{ij})_{2 \times 2}. \) It is easy to check that the multiplication satisfies the associativity law, the element \( (I, 1, 1, 1) \) is the identity and
\[
(A, a, b, c)^{-1} = (A^{-1}, a^{-1} b^{-1} a^{-1}, a^{-1} b^{-1} a^{-1}, c^{-1})
\]
is the inverse of \( (A, a, b, c) \in \Omega. \) Hence \( \Omega \) forms a group.

**Lemma 2.4.** \( \Omega \cong GL_2(\mathbb{Z}) \times (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*) \)

**Proof.** The two sets
\[
\Omega_1 = \{(A, 1, 1, 1) | A \in G(2, \mathbb{Z})\} \text{ and } \Omega_2 = \{(I, a, b, c) | A \in \mathbb{C}^*\}
\]
are subgroups of \( \Omega. \) It is easy to see that \( \Omega_1 \cap \Omega_2 = (I, 1, 1, 1), \) \( \Omega_2 \triangleleft \Omega \) and \( \Omega_1 \cdot \Omega_2 = \Omega, \) hence \( \Omega = \Omega_1 \times \Omega_2 \cong GL_2(\mathbb{Z}) \times (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*) \)

Now we study the structure of \( AutL. \) First we construct a subgroup of \( AutL. \)

**Lemma 2.5.** Let \( \Theta_1 = \{\overline{\sigma} | \overline{\sigma} \in AutL, \pi_A \overline{\sigma} |_B = 0\}. \) Then \( \Theta_1 \) is a subgroup of \( AutL, \) and \( \Theta_1 \cong GL_2(\mathbb{Z}) \times (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*) \)

**Proof.** For any \( \overline{\sigma} \in \Theta_1, \) by Lemma 2.2 we see \( \overline{\sigma} |_B = \pi_B \overline{\sigma} |_B \in AutB. \) Moreover, from [4], there exists \( A = (a_{ij}) \in GL_2(\mathbb{Z}), a, b \in \mathbb{C}^*, \) so that
\[
\overline{\sigma}(E(\vec{m})) = |A|a^{m_1}b^{m_2}E((m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2).
\]
Now we give the action of \( \overline{\sigma} \) on \( A. \) By Lemma 2.3, we can set
\[
\overline{\sigma}(x^{\vec{m}}) = k(\vec{m}) x^{(m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2},
\]
where \( k : \Gamma \to \mathbb{C}^* \) is a mapping. Since \( \sigma[E(\vec{m}), \vec{x}] = [\sigma(E(\vec{m})), \sigma(\vec{x})] \), we have

\[
g(\vec{m}, \vec{n})(k(\vec{m} + \vec{n}) - a^{m_1}b^{m_2}k(\vec{n})) = 0.
\]

For any fixed \( \vec{m}, \vec{n} \) satisfying \( g(\vec{m}, \vec{n}) \neq 0 \), the above identity implies

\[
k(\vec{m} + \vec{n}) = a^{m_1}b^{m_2}k(\vec{n})) = a^{n_1}b^{n_2}k(\vec{m}).
\]

(2.3)

Take \( \vec{m} = e_1, \vec{n} = e_2 \), the above identity gives \( \frac{k(e_1)}{a} = \frac{k(e_2)}{b} \). For any \( \vec{m} \in \Gamma \), we have \( g(\vec{m}, e_1) \neq 0 \) or \( g(\vec{m}, e_2) \neq 0 \). This gives \( \frac{k(\vec{m})}{a^{m_1}b^{m_2}} = \frac{k(e_1)}{a} \) or \( \frac{k(\vec{m})}{a^{m_1}b^{m_2}} = \frac{k(e_2)}{b} \) from (2.3). Hence \( k(\vec{m}) = c \cdot a^{m_1}b^{m_2} \), where \( c = \frac{k(e_1)}{a} = \frac{k(e_2)}{b} \) is a nonzero constant which is independent of \( \vec{m} \).

Now we prove the first part of the lemma. For any \( \sigma \in \Theta_1 \), then

\[
\sigma(E(\vec{m})) = |A|a^{m_1}b^{m_2}E((m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2)
\]

\[
\sigma(x^m) = c \cdot a^{m_1}b^{m_2}x^{(m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2}.
\]

We can easily check the following identities:

\[
\sigma^{-1}(E(\vec{m})) = |A|a^{m_1}b^{m_2}E((m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2)
\]

\[
\sigma^{-1}(x^m) = c^{-1} \cdot a^{m_1}b^{m_2}x^{(m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2},
\]

and also \( \sigma \sigma^{-1} \in \Theta_1 \) for any \( \sigma' \sigma \in \Theta_1 \). Hence \( \Theta_1 \) is a subgroup of \( AutL \).

We define a map \( \tau : \Omega \to \Theta_1 \) as follows, for \( (A, a, b, c) \in \Omega \),

\[
\tau(A, a, b, c)(E(\vec{m})) = |A|a^{m_1}b^{m_2}E((m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2)
\]

\[
\tau(A, a, b, c)(x^m) = c \cdot a^{m_1}b^{m_2}x^{(m_1a_{11} + m_2a_{12})e_1 + (m_1a_{21} + m_2a_{22})e_2}.
\]

Then it is easy to check that \( \tau \) is a bijective map. Moreover by (2.2), we have

\[
\tau((B, a_1, b_1, c_1)(A, a_2, b_2, c_2)) = \tau(B, a_1, b_1, c_1)\tau(A, a_2, b_2, c_2)
\]

Hence \( \tau \) is a group isomorphism. 

\[\blacksquare\]

**Remark 2.6.** Define a map \( \langle \cdot | \cdot \rangle : \mathbb{Z}^2 \times \mathbb{C}^2 \to \mathbb{C} \), by \( \langle \vec{m} | \vec{c} \rangle = m_1c_1 + m_2c_2 \), where \( \vec{m} = m_1e_1 + m_2e_2, \vec{c} = (c_1, c_2) \). Denote

\[\mathbb{C}^2_{\infty} = \{(\vec{c}_i)_{i,j} \in \mathbb{C}^2 | \langle \vec{c}_i \rangle_{i,j} \notin \mathbb{C}^2, \langle ie_1 + je_2 | \vec{c}_i \rangle = 0, \text{ and only finitely many } \vec{c}_i \neq 0\}.\]

It is clear \( \mathbb{C}^2_{\infty} \) is an abelian group under the usual addition. The following two lemmas give another class of the automorphisms of \( AutL \).

**Lemma 2.7.** For \( \hat{\sigma} \in AutL \), if \( \pi_B \hat{\sigma} \vert_B = id \vert_B \), then there exists finite nonzero elements \( \vec{c}_f \in \mathbb{C}^2 \), such that

\[
\hat{\sigma}(E(\vec{m})) = \sum_{\vec{r} \in \mathbb{Z}^2} \langle \vec{m} | \vec{c}_f \rangle x^{\vec{m}+\vec{r}} + E(\vec{m})
\]

where \( \langle \vec{r} | \vec{c}_f \rangle = 0 \) for all \( \vec{r} \in \mathbb{Z}^2 \).
Proof. Set \( \hat{\sigma}(E(\vec{m})) = \sum_{\vec{r} \in \mathbb{Z}^2} f_\vec{r}(\vec{m}) x^{\vec{m}+\vec{r}} + E(\vec{m}) \). Take \( \vec{c}_\vec{r} := (c^1_\vec{r}, c^2_\vec{r}) = (f_\vec{r}(e_1), f_\vec{r}(e_2)) \). From \( \hat{\sigma}[E(\vec{m}'), E(\vec{m}'')] = [\hat{\sigma}(E(\vec{m}')), \hat{\sigma}(E(\vec{m}''))] \) we have
\[
g(\vec{m}', \vec{m}'') f_\vec{r}(\vec{m}' + \vec{r} + \vec{m}'') = f_\vec{r}(\vec{m}') g(\vec{m}' + \vec{r}, \vec{m}'') + f_\vec{r}(\vec{m}'') g(\vec{m}', \vec{m}' + \vec{r}) \quad (2.4)
\]

Now we prove \( f_\vec{r}(\vec{m}) = \langle \vec{m}|\vec{c}_\vec{r} \rangle \) with \( \langle \vec{r}|\vec{c}_\vec{r} \rangle = 0 \) for all \( \vec{r} \in \mathbb{Z}^2 \). The argument is divided in three cases.

(A). For \( \vec{r} = r_1 e_1 + r_2 e_2 \) and \( r_1 r_2 \neq 0 \). Let \( \vec{m}' = \vec{m}, \vec{m}'' = l\vec{m} \), then (2.4) gives
\[
g(\vec{m}, \vec{r})(f_\vec{r}(l\vec{m}) - lf_\vec{r}(\vec{m})) = 0 \quad (2.5)
\]
Let \( m = e_1, e_2 \) respectively, (2.5) gives
\[
f_\vec{r}(le_1) = lc^1_\vec{r}, \quad f_\vec{r}(le_2) = lc^2_\vec{r} \quad (2.6)
\]
Let \( \vec{m}' = m_1 e_1, \vec{m}'' = m_2 e_2, m_1 m_2 \neq 0 \), (2.4) gives
\[
f_\vec{r}(\vec{m}) = \langle \vec{r} + \vec{m}|\vec{c}_\vec{r} \rangle \quad (2.7)
\]
In particular we choose \( \vec{m} \) satisfying \( g(\vec{m}, \vec{r}) \neq 0 \) in (2.7), then by (2.6) we have \( \langle \vec{r}|\vec{c}_\vec{r} \rangle = 0 \). Hence (2.7) induces \( f_\vec{r}(\vec{m}) = \langle \vec{m}|\vec{c}_\vec{r} \rangle \) for \( m_1 m_2 \neq 0 \). Thus the result follows from this and (2.6).

(B). For \( \vec{r} = r_1 e_1 + r_2 e_2 \) and \( r_1 = 0, r_2 \neq 0 \) or \( r_2 = 0, r_1 \neq 0 \). Without loss of generality, we assume \( r_1 = 0, r_2 \neq 0 \). By a similar argument as (2.6), we have \( f_\vec{r}(m_1 e_1) = m_1 c^1_\vec{r} \). Let \( \vec{m}' = e_1 + ke_2, \vec{m}'' = e_1, (2.4) \) gives
\[
k f_\vec{r}(e_1 + ke_2) = f_\vec{r}(e_1 + ke_2)(k + r_2) + c^1_\vec{r}(k - r_2) \quad (2.8)
\]
Let \( \vec{m}' = 2e_1, \vec{m}'' = ke_2, (2.4) \) gives
\[
2k f_\vec{r}(e_1 + ke_2) = 4c^1_\vec{r}k + 2f_\vec{r}(ke_2)(k + r_2) \quad (2.9)
\]
Let \( \vec{m}' = e_1, \vec{m}'' = ke_2, (2.4) \) gives
\[
k f_\vec{r}(e_1 + ke_2) = c^1_\vec{r}k + f_\vec{r}(ke_2)(k + r_2) \quad (2.10)
\]
Solving the equations (2.8)(2.9)(2.10) we have
\[
f_\vec{r}(ke_2)(k + r_2) = 0 \quad (2.11)
\]
Note that \( f_\vec{r}(\vec{m}) \) is the coefficient of \( x^{\vec{m}+\vec{r}} \) in \( \hat{\sigma}(E(\vec{m})) \), we have \( f_\vec{r}(-r_2 e_2) = f_\vec{r}(-\vec{r}) = 0 \). Hence for any \( k \in \mathbb{Z}\{0\}, f_\vec{r}(ke_2) = 0 \) from this and (2.11). In particular take \( k = 1 \), then \( c^2_\vec{r} = 0 \).

Taking \( \vec{m}' = m_1 e_1, \vec{m}'' = m_2 e_2, m_1 m_2 \neq 0 \), (2.4) gives \( f_\vec{r}(\vec{m}) = c^1_\vec{r}m_1 \).
Hence, for any \( \vec{m} \in \Gamma, f_\vec{r}(\vec{m}) = c^1_\vec{r}m_1 \). Immediately we have \( f_\vec{r}(\vec{m}) = \langle \vec{m}|\vec{c}_\vec{r} \rangle \) and \( \langle \vec{r}|\vec{c}_\vec{r} \rangle = 0 \).

(C). For \( \vec{r} = 0, (2.4) \) induces
\[
g(\vec{m}, \vec{r})(f_0(\vec{m}) + f_0(\vec{m} + \vec{r})) = 0 \quad (2.12)
\]

Similarly as the proof of case (A) and (B), we can get \( f_0(\vec{m}) = \langle \vec{m}|\vec{c}_0 \rangle \).
Theorem 3.3. \( \text{Der} L = \{ \hat{\sigma} | \sigma \in \text{Aut} L, \pi_B \hat{\sigma} | B = 0 \} \) is an abelian subgroup of \( \text{Aut} L \), and \( \Theta_2 \cong \mathbb{C}_2^\infty \).

Proof. By the Lemma 2.7, one can easily check that \( \Theta_2 \) is an abelian subgroup under composition of automorphisms in \( \Theta_2 \). Define \( \rho : \mathbb{C}_2^\infty \rightarrow \Theta_2 \) by \( (\check{c}_{ij})_{ij} \rightarrow \rho((\check{c}_{ij})_{ij}) \), where \( \rho((\check{c}_{ij})_{ij}) \in \Theta_2 \), such that
\[
\rho((\check{c}_{ij})_{ij})E(m) = \sum_{ij \in \mathbb{Z}} \langle m|\check{c}_{ij}|x^{\check{m}}+ix_1+jx_2 \rangle + E(m)
\]
\[
\rho((\check{c}_{ij})_{ij})x^{\check{m}} = x^{m}.
\]
One can check \( \rho((\check{c}_{ij})_{ij}) \) is bijective. Moreover, for any \((\check{b}_{ij})_{ij}, (\check{c}_{ij})_{ij} \in \mathbb{C}_2^\infty \), we have
\[
\rho((\check{b}_{ij})_{ij} + (\check{c}_{ij})_{ij}) = \rho((\check{b}_{ij})_{ij})\rho((\check{c}_{ij})_{ij})
\]
which implies \( \rho \) is a group isomorphism. 

Remark 2.9. For \( x^r \in A \), since \( (ad(x^r))^2 = 0 \) for \( r \in \Gamma \), we know that \( \exp ad(x^r) = 1 + ad(x^r) \) is an automorphism of \( L \). Indeed, we have
\[
\exp(ad(x^r))E(m) = \langle m|\check{c}_{r'}|x^{\check{m}}+x^{\check{r}} \rangle + E(m)
\]
where \( \check{c}_{r'} = (r_2,-r_1) \) for \( \check{r} = (r_1,r_2) \in \Gamma \). One may compare this with the automorphism \( \hat{\sigma} \) defined in Lemma 2.7 to see that the abelian subgroup \( \Theta_2 \) can also be constructed in this way.

Theorem 2.10. \( \text{Aut} L \cong (GL_2\mathbb{Z}) \ltimes ((\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)) \ltimes \mathbb{C}_2^\infty \)

Proof. First we show that \( \text{Aut} L = \Theta_2 \Theta_1 \). For any \( \sigma_0 \in \text{Aut} L \), define a map \( \sigma_1 \) by \( \sigma_1|_B = \pi_B \sigma_0|_B, \sigma_1|_A = 0 \), we have \( \sigma_1 \in \Theta_1, \sigma_0 \sigma_1^{-1} \in \Theta_2 \), so \( \text{Aut} L = \Theta_2 \Theta_1 \). Next we prove that the decomposition is unique. In fact, if \( \sigma_1 \in \Theta_1, \sigma'_1 \in \Theta_2 \) is another such a decomposition, then we have \( \sigma_2 \sigma_1 = \sigma'_2 \sigma'_1 \), this induces \( \sigma_1^{-1} \sigma_1' = \sigma_2^{-1} \sigma_2 \). Since \( \Theta_1 \cap \Theta_2 = id, \) we set \( \sigma_1 = \sigma'_1, \sigma_2 = \sigma'_2 \). Moreover, one can check \( \Theta_2 \) is a normal subgroup of \( \text{Aut} L \), thus \( \text{Aut} L = \Theta_1 \ltimes \Theta_2 \). By using Lemma 2.5 and Lemma 2.8, we get \( \text{Aut} L \cong (GL_2\mathbb{Z}) \ltimes ((\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)) \ltimes \mathbb{C}_2^\infty \).

3. Derivation Algebra

In this section, we study the derivation Lie algebra \( \text{Der}(L) \) of the Lie algebra \( L = [L,L] \) with \( L = A \oplus B \).

Lemma 3.1. \( L \) is a finitely generated \( \mathbb{Z}^2 \)-graded Lie algebra.

Theorem 3.2[5]. Let \( G \) be an abelian group. If \( A = \bigoplus_{\alpha \in G} A_\alpha \) is a finitely generated \( G \)-graded Lie algebra, then \( \text{Der}(A) = \bigoplus_{\alpha \in G} \text{Der}(A)_\alpha \) is also a \( G \)-graded Lie algebra. Moreover, if \( d \in \text{Der}(A)_\alpha, \) \( d(A_\beta) \subset A_{\alpha+\beta} \).

Theorem 3.3. \( \text{Der}(L) = \text{Inder}(L) \oplus \text{Outder}(L), \) where
\[ \text{Inder}(L) = \bigoplus_{\check{r} \in \Gamma} (\text{Cad}E(\check{r}) \oplus \text{Cad}E(\check{r})) \]
\[ \text{Outder}(L) = C_1 \oplus C_2 \oplus C_1 \oplus C_2 \oplus C_1 \]
and \( d_1, d_2 \) are degree derivations of \( L \), \( p_1, p_2 \), are determined by \( p_1 E(m) = m_1 x^{m}, \)
\( p_2 E(m) = m_2 x^{m}, \) \( p_1 x^m = p_2 x^m = 0, \) and \( \delta \) is determined by \( \delta E(m) = 0, \)
\( \delta x^m = x^m. \)
Proof. By Lemma 3.1 and Theorem 3.2, we set \( \text{Der}(L) = \bigoplus_{\mathbf{r} \in \mathbb{Z}^2} \text{Der}(L)_\mathbf{r} \). For \( \mathbf{r} \in \text{Der}(L)_\mathbf{r} \), let
\[
\begin{align*}
\gamma_{\mathbf{r}}(x^\mathbf{m}) &= \varphi_1(\mathbf{r}, \mathbf{m})x^{\mathbf{m}+\mathbf{r}} + \varphi_2(\mathbf{r}, \mathbf{m})E(\mathbf{m} + \mathbf{r}) \\
\gamma_{\mathbf{r}}(E(\mathbf{m})) &= \psi_1(\mathbf{r}, \mathbf{m})x^{\mathbf{m}+\mathbf{r}} + \psi_2(\mathbf{r}, \mathbf{m})E(\mathbf{m} + \mathbf{r}),
\end{align*}
\]
where \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) are maps from \( \mathbb{Z}^2 \) to \( \mathbb{C} \). For convenience of notations, we set
\[
\varphi_1(\mathbf{r}, -\mathbf{r}) = 0, \quad \varphi_2(\mathbf{r}, -\mathbf{r}) = 0, \quad \psi_1(\mathbf{r}, -\mathbf{r}) = 0, \quad \psi_2(\mathbf{r}, -\mathbf{r}) = 0
\]
Using the property of derivation, \( \gamma_{\mathbf{r}}[a, b] = [\gamma_{\mathbf{r}}(a), b] + [a, \gamma_{\mathbf{r}}(b)] \), we have
\[
\begin{align*}
\psi_1(\mathbf{r}, \mathbf{m} + \mathbf{n})g(\mathbf{m}, \mathbf{n}) &= \psi_1(\mathbf{r}, \mathbf{m})g(\mathbf{m} + \mathbf{r}, \mathbf{n}) + \psi_1(\mathbf{r}, \mathbf{n})g(\mathbf{m}, \mathbf{n} + \mathbf{r}), \quad (3.1) \\
\psi_2(\mathbf{r}, \mathbf{m} + \mathbf{n})g(\mathbf{m}, \mathbf{n}) &= \psi_2(\mathbf{r}, \mathbf{m})g(\mathbf{m} + \mathbf{r}, \mathbf{n}) + \psi_2(\mathbf{r}, \mathbf{n})g(\mathbf{m}, \mathbf{n} + \mathbf{r}), \quad (3.2) \\
\varphi_1(\mathbf{r}, \mathbf{m})g(\mathbf{m} + \mathbf{r}, \mathbf{n}) + \psi_2(\mathbf{r}, \mathbf{n})g(\mathbf{m}, \mathbf{n} + \mathbf{r}) &= \varphi_1(\mathbf{r}, \mathbf{m} + \mathbf{n})g(\mathbf{m}, \mathbf{n}), \quad (3.3) \\
\varphi_2(\mathbf{r}, \mathbf{m})g(\mathbf{m} + \mathbf{r}, \mathbf{n}) + \varphi_2(\mathbf{r}, \mathbf{n})g(\mathbf{m}, \mathbf{n} + \mathbf{r}) &= 0. \quad (3.4)
\end{align*}
\]
Now we prove that \( \varphi_2(\mathbf{r}, \mathbf{m}) \equiv 0 \). We may assume \( \mathbf{m} \neq -\mathbf{r} \). Interchange \( \mathbf{m}, \mathbf{n} \) in (3.4), then
\[
\varphi_2(\mathbf{r}, \mathbf{n})g(\mathbf{n} + \mathbf{r}, \mathbf{m}) = \varphi_2(\mathbf{r}, \mathbf{m} + \mathbf{n})g(\mathbf{n}, \mathbf{m})
\]
(3.6)\]
Solving the equations (3.4) through (3.6), we have \( \varphi_2(\mathbf{r}, \mathbf{m})g(\mathbf{m} + \mathbf{r}, \mathbf{n}) = 0 \). Choose \( \mathbf{n} \in \Gamma \) satisfying \( g(\mathbf{m} + \mathbf{r}, \mathbf{n}) \neq 0 \), then \( \varphi_2(\mathbf{r}, \mathbf{m}) = 0 \) as required.

Now we divide the proof into two cases. First for \( \mathbf{r} = 0 \), we solve the equations (3.1) through (3.3) to get
\[
\begin{align*}
g(\mathbf{m}, \mathbf{n})(\psi_1(0, \mathbf{m} + \mathbf{n}) - \psi_1(0, \mathbf{n})) &= 0 \quad (3.7) \\
g(\mathbf{m}, \mathbf{n})(\psi_2(0, \mathbf{m} + \mathbf{n}) - \psi_2(0, \mathbf{n})) &= 0 \quad (3.8) \\
g(\mathbf{m}, \mathbf{n})(\varphi_1(0, \mathbf{m}) + \varphi_2(0, \mathbf{m} + \mathbf{n}) - \varphi_1(0, \mathbf{m} + \mathbf{n})) &= 0. \quad (3.9)
\end{align*}
\]
\]From (3.7),(3.8) and (3.9) one can easily deduce the following identities:
\[
\begin{align*}
\psi_2(0, \mathbf{m}) &= m_1\psi_2(0, e_1) + m_2\psi_2(0, e_2) \\
\psi_1(0, \mathbf{m}) &= m_1\psi_1(0, e_1) + m_2\psi_1(0, e_2) \\
\varphi_1(0, \mathbf{m}) &= \psi_2(0, \mathbf{m}) + c, \quad c \in \mathbb{C},
\end{align*}
\]
where the constant \( c = \psi_1(0, e_2) - \varphi_1(0, e_2) \). Hence \( \text{dim Der}(L)_{\mathbf{0}} \leq 5 \).

Let \( d_1, d_2 \) be degree derivations of \( A \), \( p_1, p_2 \) and \( \delta \) are linear maps on \( L \), satisfy \( p_1E(\mathbf{m}) = m_1x^\mathbf{m}, \ p_2E(\mathbf{m}) = m_2x^\mathbf{m}, \ p_1x^\mathbf{m} = p_2x^\mathbf{m} = 0, \ \delta E(\mathbf{m}) = 0, \ \delta x^\mathbf{m} = x^\mathbf{m} \). It is easy to check \( d_1, d_2, p_1, p_2, \delta \) are linearly independent and belong to \( \text{Der}(L)_{\mathbf{0}} \), then we get \( \text{dim Der}(L)_{\mathbf{0}} = 5 \) and \( \text{Der}(L)_{\mathbf{0}} = \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \mathbb{C}\delta \).

Next we consider the case for \( \mathbf{r} \neq 0 \). Without loss of generality, we assume \( r_1 \neq 0 \). For \( \mathbf{m} = e_2, \ \mathbf{n} = m_2e_2 \), (3.1) gives
\[
\varphi_1(\mathbf{r}, m_2e_2) = m_2\varphi_1(\mathbf{r}, e_2)
\]
(3.10)\]
For \( \vec{m} = m_1 e_1, \vec{n} = e_2 \), (3.1) gives
\[
\varphi_1(\vec{r}, m_1 e_1 + e_2) m_1 = (m_1 + r_1) \varphi_1(\vec{r}, m_1 e_1) + m_1 (r_2 + 1) \varphi_1(\vec{r}, e_2) \tag{3.11}
\]
For \( \vec{m} = m_1 e_1 + e_2, \vec{n} = -e_2 \), (3.1) gives
\[
\varphi_1(\vec{r}, m_1 e_1) m_1 = (m_1 + r_1) \varphi_1(\vec{r}, m_1 e_1 + e_2) + (r_1 + m_1 - m_1 r_2) \varphi_1(\vec{r}, -e_2) \tag{3.12}
\]
By (3.10), we have \( \varphi_1(\vec{r}, -e_2) = -\varphi_1(\vec{r}, e_2) \), thus the above identity deduces
\[
\varphi_1(\vec{r}, m_1 e_1) m_1 = (m_1 + r_1) \varphi_1(\vec{r}, m_1 e_1 + e_2) - (r_1 + m_1 - m_1 r_2) \varphi_1(\vec{r}, e_2) \tag{3.12}
\]
Solving (3.11)(3.12) we get
\[
\psi_1(\vec{r}, m_1 e_1) = -\frac{r_2}{r_1} m_1 \psi_1(\vec{r}, e_2) \tag{3.13}
\]
For \( \vec{m} = m_1 e_1, \vec{n} = m_2 e_2 \), (3.1) gives
\[
\varphi_1(\vec{r}, \vec{m}) m_1 m_2 = m_2 (m_1 + r_1) \varphi_1(\vec{r}, m_1 e_1) + (r_2 + m_2) m_1 \varphi_1(\vec{r}, m_2 e_2) \tag{3.14}
\]
Substitute (3.10)(3.13) into (3.14) we get
\[
\psi_1(\vec{r}, \vec{m}) = (m_2 - \frac{r_2}{r_1} m_1) \psi_1(\vec{r}, e_2) \tag{3.15}
\]
By a similar argument, we have
\[
\psi_2(\vec{r}, \vec{m}) = (m_2 - \frac{r_2}{r_1} m_1) \psi_2(\vec{r}, e_2) \tag{3.16}
\]
Now we prove \( \varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m}) \). Let \( \vec{n} = \vec{m} \), (2.3) gives
\[
g(\vec{m}, \vec{r})(\varphi_1(\vec{r}, \vec{m}) - \psi_2(\vec{r}, \vec{m})) = 0
\]
This implies that if \( g(\vec{m}, \vec{r}) \neq 0 \), \( \varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m}) \). For the case of \( g(\vec{m}, \vec{r}) = 0 \) and \( \vec{m} + \vec{r} \neq 0 \), we choose \( \vec{n} \in \Gamma \) satisfying \( g(\vec{m} + \vec{r}, \vec{n}) \neq 0 \). \( g(\vec{m}, \vec{r} + \vec{n}) \neq 0 \), (3.2),(3.3) give us \( \varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m}) \), where we have used the fact that \( \varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m}) \) whenever \( g(\vec{m}, \vec{r}) \neq 0 \). This implies \( d_x \) is determined by \( \psi_1(\vec{r}, e_2) \) and \( \psi_2(\vec{r}, e_2) \), so \( \dim \text{Der}(L)_r \leq 2 \). Note that \( \text{ad}(x^\vec{r}), \text{ad}(E(\vec{r})) \) are linearly independent derivations in \( \text{Der}(L)_r \), hence \( \dim \text{Der}(L)_r = 2 \) and \( \text{Der}(L)_r = \mathbb{C} \text{ad}x^\vec{r} \oplus \mathbb{C} \text{ad}E(\vec{r}) \). Then we get the result.

4. Universal Central Extension

Let \( 0 \to K \to \hat{L} \xrightarrow{\lambda \tau} L \to 0 \) be a central extension of the Lie algebra \( L \), and \( \text{id}_L \) be the identity map on the Lie algebra \( L \). We can define a linear map \( \tau : L \to \hat{L} \), such that \( \lambda \tau = \text{id}_L \). Set \( \tau^* : L \times L \to K, \tau^*(a, b) = [\tau(a), \tau(b)] - \tau([a, b]) \) for \( a, b \in L \). Obviously we have
\[
\tau^*(a, [b, c]) + \tau^*(b, [c, a]) + \tau^*(c, [a, b]) = 0 \tag{4.1}
\]
and \( \tau^*(a, b) = -\tau^*(b, a) \). In this section, we study the universal central extension of the Lie algebra \( L \). For this purpose, we need the following three propositions.
Proposition 4.1. \( \tau^*(E(\vec{m}), E(\vec{n})) = \)
\[
\begin{align*}
\left\{
\begin{array}{l}
- \frac{m_2 n_1 - m_1 n_2}{(m_1 + n_1)(m_2 + n_2)} \tau^*(E((m_1 + n_1)e_1), E((m_2 + n_2)e_2)), \quad m_1 + n_1 \neq 0, m_2 + n_2 \neq 0 \\
(m_2 n_1 - m_1 n_2) \tau^*(E(e_1 + (m_2 + n_2)e_2), E(-e_1)), \quad m_1 + n_1 = 0, m_2 + n_2 \neq 0 \\
- (m_2 n_1 - m_1 n_2) \tau^*(E((m_1 + n_1)e_1 + e_2), E(-e_2)), \quad m_1 + n_1 \neq 0, m_2 + n_2 = 0 \\
m_1 \tau^*(E(e_1), E(-e_1)) + m_2 \tau^*(E(e_2), E(-e_2)), \quad m_1 + n_1 = 0, m_2 + n_2 = 0
\end{array}
\right.
\end{align*}
\]

Proposition 4.2. \( \tau^*(x^\vec{m}, E(\vec{n})) = \)
\[
\begin{align*}
\left\{
\begin{array}{l}
- \frac{m_2 n_1 - m_1 n_2}{(m_1 + n_1)(m_2 + n_2)} \tau^*(x^{m_1+n_1,0}, E((m_2 + n_2)e_2)), \quad m_1 + n_1 \neq 0, m_2 + n_2 \neq 0 \\
m_1 \tau^*(x^{1, m_2+n_2}, E(-e_1)), \quad m_1 + n_1 = 0, m_2 + n_2 \neq 0 \\
m_2 \tau^*(x^{m_1+n_1,1}, E(-e_2)), \quad m_1 + n_1 \neq 0, m_2 + n_2 = 0 \\
m_1 \tau^*(x^{1,0}, E(-e_1)) + m_2 \tau^*(x^{0,1}, E(-e_2)), \quad m_1 + n_1 = 0, m_2 + n_2 = 0
\end{array}
\right.
\end{align*}
\]

Proposition 4.3. \( \tau^*(x^\vec{m}, x^\vec{n}) = 0. \) Proposition 4.1 is given in [12]. Now we prove Proposition 4.2 by a series of lemmas. Proposition 4.3 will be proved in the last part of the section. Set \( a = x^\vec{m}, b = E(\vec{n}), c = E(\vec{r}) \), (4.1) gives
\[
g(\vec{n}', \vec{r}') \tau^*(x^{\vec{m}', E(\vec{n}'+\vec{r}')}) + g(\vec{r}', \vec{m}') \tau^*(E(\vec{n}'), x^{\vec{m}'+\vec{n}'}) + g(\vec{n}', \vec{n}) \tau^*(x^{\vec{m}-\vec{n}}, E(\vec{r}')) = 0 \quad (4.2)
\]

Lemma 4.4. \( \tau^*(x^\vec{m}, E(\vec{m})) = 0 \) for \( \vec{m} \in \Gamma. \)

Proof. Let \( \vec{m}' = \vec{r}' = \vec{m}, \vec{n}' = \vec{n} - \vec{m} \) satisfying \( g(\vec{m}, \vec{n}) \neq 0, (4.2) \) gives
\[
g(\vec{m}, \vec{n}) (\tau^*(x^\vec{m}, E(\vec{n})) + \tau^*(x^\vec{n}, E(\vec{m}))) = 0.
\]
Since \( g(\vec{m}, \vec{n}) \neq 0, \) we have
\[
\tau^*(x^\vec{m}, E(\vec{n})) = - \tau^*(x^\vec{n}, E(\vec{m})) = \tau^*(E(\vec{m}), x^\vec{n})
\]
Immediately we get
\[
\tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = \tau^*(E(2\vec{m} - \vec{n}), x^\vec{n}) \quad (4.3)
\]
For \( \vec{m}' = \vec{m}, \vec{n}' = \vec{n}, \vec{r}' = -\vec{n} + \vec{m}, (4.2) \) gives
\[
\tau^*(x^\vec{m}, E(\vec{m})) + \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) + \tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = 0 \quad (4.4)
\]
For \( \vec{r}' = \vec{m}, \vec{m}' = \vec{n}, \vec{n}' = -\vec{n} + \vec{m}, (4.2) \) gives
\[
\tau^*(x^\vec{m}, E(\vec{m})) - \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) + \tau^*(E(2\vec{m} - \vec{n}), x^\vec{n}) = 0 \quad (4.5)
\]
Solving the equations (4.2) through (4.5), we obtain
\[
\tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) = 0 \quad (4.6)
\]
Taking \( \vec{m} = \vec{m}', \vec{n} = \vec{n}' - \vec{n}' \) in (4.6) we have
\[
\tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = 0 \quad (4.7)
\]
Substitute (4.6)(4.7) to (4.4) we get \( \tau^*(x^\vec{m}, E(\vec{m})) = 0. \)
Lemma 4.5.  
(1) If $m_1 + n_1 \neq 0$, then $\tau^*(x^{m_1,0}, E(n_1 e_1)) = 0$.

(2) If $m_2 + n_2 \neq 0$, then $\tau^*(x^{0,m_2}, E(n_2 e_2)) = 0$, for nonzero integers $m_i, n_i, i = 1, 2$.

Proof. As a direct consequence of Lemma 4.4, we have

$$\tau^*(x^{m_1,0}, E(m_1,0)) = 0$$

We prove (1) for $m_1 \neq n_1$. For $\vec{m}' = n_1 e_1 + e_2, \vec{n}' = -e_2, \vec{r}' = m_1 e_1, (4.2)$ gives

$$m_1 \tau^*(x^{m_1,1}, E(m_1 e_1 - e_2)) + m_1 \tau^*(E(-e_2), x^{m_1+1,1}) + n_1 \tau^*(x^{n_1,0}, E(m_1 e_1)) = 0$$

(4.8)

For $\vec{m}' = n_1 e_1, \vec{n}' = m_1 e_1 - e_2, \vec{r}' = e_2, (4.2)$ gives

$$-m_1 \tau^*(x^{m_1,0}, E(m_1 e_1)) + n_1 \tau^*(E(m_1 e_1 - e_2), x^{n_1,1}) - n_1 \tau^*(x^{m_1+n_1,-1}, E(e_2)) = 0$$

(4.9)

For $\vec{m}' = -e_2, \vec{n}' = e_2, \vec{r}' = (m_1 + n_1) e_1, (4.2)$ gives

$$(m_1 + n_1) \tau^*(x^{0,-1}, E((m_1 + n_1) e_1 + e_2)) + (m_1 + n_1) \tau^*(E(e_2), x^{m_1+n_1,-1}) = 0$$

(4.10)

For $\vec{m}' = \vec{n}' = -e_2, \vec{r}' = (m_1 + n_1) e_1 + 2e_2, (4.2)$ gives

$$\tau^*(x^{0,-1}, E((m_1 + n_1) e_1 + e_2)) = \tau^*(E(-e_2), x^{m_1+n_1,1})$$

(4.11)

Solving (4.8) through (4.11), we get that, if $m_1 \neq n_1$, $\tau^*(x^{m_1,0}, E(n_1 e_1) = 0$. This finishes the proof of (1). (2) follows from (1) by symmetry.

Lemma 4.6. For $m_1 + n_1 \neq 0, m_2 + n_2 \neq 0$, we have

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = -\frac{m_2 n_1 - m_1 n_2}{(m_1 + n_1)(m_2 + n_2)} \tau^*(x^{m_1+n_1,0}, E((m_2 + n_2)e_2))$$

Proof. We divide the argument into three cases.

Case 1. For $n_2 = 0$, let $\vec{m}' = (m_1 + n_1) e_1, \vec{n}' = n_1 e_1, \vec{r}' = -n_1 e_1 + m_2 e_2, (3.2)$ gives

$$\tau^*(x^{\vec{m}}, E(n_1 e_1)) = -\frac{n_1}{m_1 + n_1} \tau^*(x^{m_1+n_1,0}, E(m_2 e_2))$$

as required. In particular, this gives

$$\tau^*(x^{m_1,m_2+n_2}, E(n_1 e_1)) = -\frac{n_1}{m_1 + n_1} \tau^*(x^{m_1+n_1,0}, E((m_2 + n_2)e_2))$$

(4.12)

Case 2. For $n_1 = 0$, let $\vec{m}' = (m_2 + n_2) e_2, \vec{n}' = n_2 e_2, \vec{r}' = m_1 e_1 - n_2 e_2, (4.2)$ gives

$$\tau^*(x^{\vec{m}}, E(n_2 e_2)) = -\frac{n_2}{m_2 + n_2} \tau^*(x^{0,m_2+n_2}, E(m_1 e_1))$$

By (4.12) we have

$$\tau^*(x^{\vec{m}}, E(n_2 e_2)) = \frac{n_2}{m_2 + n_2} \tau^*(x^{m_1,0}, E((m_2 + n_2)e_2))$$
as required. Also this implies
\[ \tau^*(x^{m_1+n_1,m_2}, E(n_2 e_2)) = \frac{n_2}{m_2 + n_2} \tau^*(x^{m_1+n_1,0}, E((m_2 + n_2)e_2)) \] (4.13)

Case 3. For \( n_1 \neq 0, n_2 \neq 0 \), let \( \vec{m}' = m_1 e_1 + m_2 e_2, \vec{n}' = n_1 e_1, r = n_2 e_2 \), (4.2) gives
\[ -n_1 n_2 \tau^*(x^{\vec{m}'}, E(\vec{n}')) + n_2 m_1 \tau^*(E(n_1 e_1), x^{m_1,m_2+n_2}) - n_1 m_2 \tau^*(x^{m_1+n_1,m_2}, E(n_2 e_2)) = 0. \]
Solving this equation together with (4.12)(4.13) we have
\[ \tau^*(x^{\vec{m}'}, E(\vec{n}')) = -\frac{m_2 n_1 - n_1 m_2}{(m_1 + n_1)(m_2 + n_2)} \tau^*(x^{m_1+n_1,0}, E((m_2 + n_2)e_2)). \]
This finishes the proof of the lemma.

**Lemma 4.7.** For \( m_1 + n_1 = 0, m_2 + n_2 \neq 0 \), we have
\[ \tau^*(x^{\vec{m}'}, E(\vec{n}')) = m_1 \tau^*(x^{1,m_2+n_2}, E(-e_1)) \]

**Proof.** We divide the proof into two cases.

Case 1. For \( m_2 = 0 \), let \( \vec{m}' = m_1 e_1, \vec{n}' = -e_1, \vec{r}' = (1 - m_1)e_1 + n_2 e_2 \), (4.2) gives
\[ \tau^*(x^{m_1,0}, E(-m_1 e_1 + n_2 e_2)) = m_1 \tau^*(x^{1,n_2}, E(-e_1)) \] (4.14)

Case 2. For \( m_2 \neq 0 \), let \( \vec{m}' = m_1 e_1, \vec{n}' = \vec{n}, \vec{r}' = m_2 e_2 \), (4.2) gives
\[ -n_1 m_2 \tau^*(x^{m_1,0}, E(n_1 e_1 + (m_2 + n_2)e_2)) + m_1 m_2 \tau^*(E(\vec{n}), x^{\vec{m}'}) + n_2 m_1 \tau^*(x^{0,n_2}, E(m_2 e_2)) = 0 \]
applying lemma 4.5 and (4.14) we have \( \tau^*(x^{\vec{m}'}, E(\vec{n}')) = m_1 \tau^*(x^{1,m_2+n_2}, E(-e_1)) \).

Similarly as Lemma 4.7 we get

**Lemma 4.8.** For \( m_1 + n_1 \neq 0, m_2 + n_2 = 0 \), we have
\[ \tau^*(x^{\vec{m}'}, E(\vec{n}')) = m_2 \tau^*(x^{m_1+n_1,1}, E(-e_2)) \]

**Lemma 4.9.** For \( m_1 + n_1 = 0, m_2 + n_2 = 0 \), we have
\[ \tau^*(x^{\vec{m}'}, E(\vec{n}')) = m_1 \tau^*(x^{1,0}, E(-e_1)) + m_2 \tau^*(x^{0,1}, E(-e_2)) \]
Proof. Let \( \vec{m}' = \vec{m}, \vec{n}' = \vec{n}, \vec{r}' = -\vec{m} - \vec{n} \), (4.2) gives

\[
(m_2 n_1 - m_1 n_2) (\tau^*(x^{\vec{m}}, E(-\vec{m})) + \tau^*(E(\vec{n}), x^{-\vec{n}}) - \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{m} - \vec{n}))) = 0
\]  
(4.15)

Let \( \vec{n} = -m_1 e_1 \neq 0 \), (4.15) gives

\[
\tau^*(x^{\vec{m}}, E(-\vec{m})) = \tau^*(x^{0,m_2}, E(-m_2 e_2)) + \tau^*(x^{m_1:0}, E(-m_1 e_1))
\]  
(4.16)

Let \( \vec{m} = m_1 e_1, \vec{n} = m_2 e_2 \neq 0 \), (4.15) gives

\[
\tau^*(x^{\vec{m}}, E(-\vec{m})) = \tau^*(x^{m_1:0}, E(-m_1 e_1)) + \tau^*(E(m_2 e_2), x^{0,-m_2})
\]  
(4.17)

Now we have to determine \( \tau^*(x^{m_1:0}, E(-m_1 e_1)) \) and \( \tau^*(E(m_2 e_2), x^{0,-m_2}) \). Solve (4.16)(4.17) and take \( m_2 = 1 \), we have

\[
\tau^*(x^{0,1}, E(-e_2)) = -\tau^*(x^{0,-1}, E(e_2))
\]

Similarly

\[
\tau^*(x^{1,0}, E(-e_1)) = -\tau^*(x^{-1,0}, E(e_1))
\]  
(4.18)

Let \( m_1 = -1, m_2 = 1 \), (4.17) gives

\[
\tau^*(x^{-1,1}, E(e_1 - e_2)) = \tau^*(x^{-1,0}, E(e_1)) + \tau^*(E(e_2), x^{0,-1})
\]  
(4.19)

Let \( \vec{m} = m_1 e_1, \vec{n} = -e_2 \), (4.15) gives

\[
\tau^*(x^{m_1:0}, E(-m_1 e_1)) - \tau^*(x^{0,1}, E(-e_2)) = \tau^*(x^{m_1:-1}, E(-m_1 e_1 + e_2))
\]  
(4.20)

Let \( \vec{m} = (m_1 - 1)e_1, \vec{n} = e_1 - e_2 \), then

\[
\tau^*(x^{m_1-1,0}, E(-m_1 + 1)e_1)) - \tau^*(x^{-1,1}, E(e_1 - e_2)) = \tau^*(x^{m_1:-1}, E(-m_1 e_1 + e_2))
\]  
(4.21)

equations (4.18)-(4.21) give the following formula

\[
\tau^*(x^{m_1:0}, E(-m_1 e_1)) - \tau^*(x^{m_1-1,0}, E(-m_1 + 1)e_1)) = \tau^*(x^{1,0}, E(-e_1))
\]

By introduction on \( m_1 \), we have

\[
\tau^*(x^{m_1:0}, E(-m_1 e_1)) = m_1 \tau^*(x^{1,0}, E(-e_1))
\]  
(4.22)

By symmetry,

\[
\tau^*(x^{0,m_2}, E(-m_2 e_2)) = m_2 \tau^*(x^{0,1}, E(-e_2))
\]  
(4.23)

Finally, substitute (4.22)(4.23) into (4.16), we obtain the result of this lemma. ■

Proof of Proposition 4.2: The result of Proposition 4.2 follows from Lemma 4.6 through 4.9. ■

Proof of Proposition 4.3: Take \( a = x^{\vec{m}'}, b = x^{\vec{n}'}, c = E(\vec{r}') \) in (4.1), then

\[
g(\vec{m}', \vec{r}') \tau^*(x^{\vec{m}'}, x^{\vec{r}'+\vec{r}'}) + g(\vec{r}', \vec{m}') \tau^*(x^{\vec{r}'}, x^{\vec{m}'+\vec{r}'}) = 0
\]  
(4.24)

Now we divide the proof into three cases.

Case 1. For \( g(\vec{m}, \vec{n}) \neq 0 \), take \( \vec{m}' = \vec{r}' = \vec{m}, \vec{n}' = \vec{n} - \vec{m} \) in (4.24). By a simple calculation, we have \( \tau^*(x^{\vec{m}}, x^{\vec{n}}) = 0 \).
Case 2. For \( g(\vec{m}, \vec{n}) = 0 \) and \( \vec{m} = -\vec{n} \), let \( \vec{r'} = -\vec{n'} + \vec{m}, \vec{n'} = \vec{n} \). \( \vec{m}' = \vec{m} \), \( \tau'^{x} = 0 \) gives \( g(\vec{m}, \vec{n}') = 0 \) \( (\ref{eq:4.24}) \) gives

\[
g(\vec{m}, \vec{n}') (\tau^* (x^m, x^{-\vec{m}}) + \tau^* (x^{\vec{n}}, x^{-\vec{n}})) = 0 \tag{4.25}
\]

For \( \vec{m} = e_1 + e_2 \) and \( \vec{n}' = e_1, e_2 \) respectively, \( \tau'^{x} (x^{1,0}, x^{-1,0}) = \tau^* (x^{1,1}, x^{-1,1}) = \tau^* (x^{0,1}, x^{0,-1}) \) \( \tag{4.26} \)

Moreover for \( \vec{m} = e_1, \vec{n}' = e_2 \), \( (\ref{eq:4.25}) \) gives

\[
\tau^* (x^{1,0}, x^{-1,0}) = \tau^* (x^{0,1}, x^{0,-1}) \tag{4.27}
\]

Hence \( (\ref{eq:4.26}) (\ref{eq:4.27}) \) imply \( \tau^* (x^{1,0}, x^{-1,0}) = \tau^* (x^{0,1}, x^{0,-1}) = 0 \).

Therefore for \( \vec{m} \in \Gamma \), from \( (\ref{eq:4.25}) \) we have \( \tau^* (x^m, x^{-\vec{m}}) = -\tau^* (x^{1,0}, x^{-1,0}) \) or \( \tau^* (x^m, x^{-\vec{m}}) = -\tau^* (x^{0,1}, x^{0,-1}) \). This implies \( \tau^* (x^m, x^{-\vec{m}}) = 0 \) for any \( \vec{m} \in \Gamma \).

Case 3. For \( g(\vec{m}, \vec{n}) = 0 \) and \( \vec{m} \neq -\vec{n} \), we set \( \vec{m}' = \vec{m}, \vec{r}' = \vec{r}, \vec{n}' = \vec{n} - \vec{r} \) so that \( g(\vec{m}, \vec{r}) \neq 0, g(\vec{n}, \vec{r}) \neq 0 \), then \( (\ref{eq:4.24}) \) gives

\[
g(\vec{n}, \vec{r}) \tau^* (x^\vec{m}, x^\vec{n}) + g(\vec{r}, \vec{m}) \tau^* (x^{\vec{n}-\vec{r}}, x^{\vec{m}+\vec{r}}) = 0
\]

Since \( g(\vec{n} - \vec{r}, \vec{m} + \vec{r}) = g(\vec{n} + \vec{m}, \vec{r}) \neq 0 \) from the assumption on \( \vec{m} \) and \( \vec{n} \), case 1 shows that \( \tau^* (x^{\vec{n}-\vec{r}}, x^{\vec{m}+\vec{r}}) = 0 \). Hence \( \tau^* (x^\vec{m}, x^\vec{n}) = 0 \). This finishes the proof of Proposition 4.3. \hfill \blacksquare

Denote the mapping \( \mu : L \rightarrow K \) as follows:

\[
\mu(x^\vec{m}) = \left\{ \begin{array}{ll}
-\frac{1}{m_1m_2} \tau^* (x^{m_1,0}, E(m_2e_2)), & m_1 \neq 0, m_2 \neq 0 \\
-\frac{1}{m_2} \tau^* (x^{1,m_2}, E(-e_1)), & m_1 = 0, m_2 \neq 0 \\
\frac{1}{m_1} \tau^* (x^{m_1,1}, E(-e_2)), & m_1 \neq 0, m_2 = 0
\end{array} \right.
\]

\[
\mu(E(\vec{m})) = \left\{ \begin{array}{ll}
-\frac{1}{m_1m_2} \tau^* (E(m_1e_2), E(m_2e_2)), & m_1 \neq 0, m_2 \neq 0 \\
\frac{1}{m_1} \tau^* (E(m_1e_1 + e_2), E(-e_2)), & m_1 \neq 0, m_2 = 0
\end{array} \right.
\]

Define Lie bracket [a, b] on the vector space \( L = L \oplus (K_1, K_2, K_3, K_4) \):

\[
[K, L] = 0, [x^\vec{m}, x^\vec{n}] = 0
\]

\[
[E(\vec{m})], (E(\vec{n})) = (m_2n_1 - m_1n_2)x^{\vec{m}+\vec{n}} + \delta_{\vec{m}+\vec{n},0} h(\vec{m})
\]

\[
[E(\vec{m})], E(\vec{n}) = (m_2n_1 - m_1n_2)E(\vec{m} + \vec{n}) + \delta_{\vec{m}+\vec{n},0} f(\vec{m})
\]

where \( i = 1, 2, 3, 4 \). \( h(\vec{m}) = m_1K_1 + m_2K_2, f(\vec{m}) = m_1K_3 + m_2K_4 \). We can easily check that \( L \) is a perfect lie algebra. Let \( \pi : L \rightarrow L \) be the projective homomorphism, then \( (L, \pi) \) is a central extension of \( L \), and \( ker\pi = (K_1, K_2, K_3, K_4) \). Indeed we have

**Theorem 4.10.** \( (L, \pi) \) is the universal central extension of the Lie algebra \( L \)

**Proof.** Let \( 0 \rightarrow K \rightarrow \hat{L} \overset{\lambda}{\rightarrow} L \rightarrow 0 \) be any central extension of \( L \), \( \tau, \tau^* \) are defined as before. We set \( \bar{\pi} : L \rightarrow \hat{L} \), such that

\[
\begin{align*}
\bar{\pi}(K_1) &= \tau^* (x^{1,0}, E(-e_1)) ; \\
\bar{\pi}(K_2) &= \tau^* (x^{0,1}, E(e_2)) ; \\
\bar{\pi}(K_3) &= \tau^* (E(e_1), E(-e_1)) ; \\
\bar{\pi}(K_4) &= \tau^* (E(e_2), E(-e_2)) ; \\
\bar{\pi}(a) &= \tau(a) + \mu(a), \forall a \in L
\end{align*}
\]
It’s easy to check that \( \overline{\mu} \) is a Lie homomorphism, and \( \lambda \overline{\mu} = \pi \). Since \( \overline{L} \) is perfect, for any \( a \in \overline{L} \), there exist \( b, c \in \overline{L} \), such that \( [b, c'] = a \). If \( \lambda \mu' = \pi = \lambda \overline{\mu} \), where \( \mu' \) is another homomorphism from \( \overline{L} \) to \( \hat{L} \), we have \( \overline{\mu}(b) - \mu'(b) \in K \) and \( \overline{\mu}(c) - \mu'(c) \in K \). So \( [\overline{\mu}(b), \overline{\mu}(c)] = [\mu'(b), \mu'(c)] \), \( \overline{\mu}(a) = \mu'(a) \). This proves that \( \overline{\mu} \) is unique, and also completes the proof the theorem.

\[ \text{Corollary 4.11.} \quad \text{The second cohomology } H^2(L, \mathbb{C}) \text{ of the Lie algebra } L \text{ is 4-dimensional.} \]

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References


