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Central Extension, Derivations and Automorphism Group for Lie Algebras Arising from the 2-Dimensional Torus

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Abstract. Let $A = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ be the ring of Laurant polynomials and B the set of skew derivations of A. Set $\tilde{L} = A \oplus B$. In this paper, we study the automorphism group, derivations and universal central extension of the derived Lie subalgebra of $\tilde{L}$.
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1. Introduction

The rank one Heisenberg-Virasoro Lie algebra is a central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) | f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators on a circle of order at most one. Arbarello et al. studied the rank one Heisenberg-Virasoro algebra and established a connection between the second cohomology of certain moduli spaces of curves and that of the Lie algebra of differential operators of order at most one in [1]. Billig constructed a class of irreducible representations for the Heisenberg-Virasoro Lie algebra of level zero in [2].

In this paper, we generalize the rank one Heisenberg-Virasoro Lie algebra to the rank two case. Let $A = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$, B be the set of skew derivations of A spanned by the elements of the form $x^{\vec{m}}(m_2 d_1 - m_1 d_2)$, where $\vec{m} = (m_1, m_2) \in \mathbb{Z}^2 \backslash \{\vec{0}\}$ and d_1, d_2 are the degree derivations of A. We set $\tilde{L} = A \oplus B$ and write L for the derived Lie subalgebra of $\tilde{L}$. One can easily check that L is a perfect Lie algebra. Our purpose in this paper is to study the automorphism group (Theorem 2.10 in section two), derivations (Theorem 3.3 in section three) and the universal central extension (Theorem 4.10 in section four) of the Lie algebra L.

Let $\Gamma = \mathbb{Z}^2 \backslash \{0\} \subset \mathbb{Z}e_1 + \mathbb{Z}e_2$. For $\vec{m} = m_1 e_1 + m_2 e_2$, $\vec{n} = n_1 e_1 + n_2 e_2 \in \Gamma$, we set $x^{\vec{m}} = x^{m_1, m_2} = x_1^{m_1} x_2^{m_2} \in A$, and denote the skew derivation $x^{\vec{m}}(m_2 d_1 - m_1 d_2)$ of A by $E(\vec{m})$, then the Lie algebra L is an infinite dimensional vector space spanned by elements of the form $x^{\vec{m}}$, $E(\vec{m})$, for $\vec{m} \in \Gamma$, together with the following Lie bracket relations:

$$[x^{\vec{m}}, E(\vec{n})] = -[E(\vec{n}), x^{\vec{m}}] = g(\vec{m}, \vec{n}) x^{\vec{m}+\vec{n}}$$

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$$[E(\vec{m}), E(\vec{n})] = g(\vec{m}, \vec{n})E(\vec{m} + \vec{n}), \qquad [x^{\vec{m}}, x^{\vec{n}}] = 0$$

where $g(\vec{m}, \vec{n}) = (m_2 n_1 - m_1 n_2)$.

2. Automorphism Group

Denoted by $AutL$ the automorphism group of the Lie algebra L. In this section, we first construct two classes of special automorphisms which form subgroups of the automorphism group $AutL$, then we give the structure of the $AutL$.

It is clear that the Lie algebra L is $\mathbb{Z}^2$-graded. Indeed $L = \bigoplus_{\vec{m} \in \Gamma} L_{\vec{m}} = \bigoplus_{\vec{m} \in \Gamma} \langle x^{\vec{m}} \oplus E(\vec{m}) \rangle$. Let $\pi_A : L \to A, \pi_B : L \to B$ be the natural projective maps, one can easily show that the map π_B is a Lie algebra homomorphism.

Lemma 2.1. *If $\sigma \in AutL$, then $\sigma A \subset A$.*

Proof. We prove the lemma by contradiction. Assume $\sigma(x) \notin A$ for some $x \in A$. First we show there exists $y \in L$, such that $[[\sigma(x), \sigma(y)], \sigma(x)] \neq 0$. We define an order for Γ as follows:

$$\vec{m} < \vec{n} \Longleftrightarrow m_1 < n_1 \;\; or \;\; m_1 = n_1 \; and \; m_2 < n_2.$$

Note that $\sigma(x) \notin A$, we can set $\pi_B \sigma(x) = \sum_{i=1}^{k} \lambda_i E(\vec{m}_i)$, with the lowest term $\lambda_1 E(\vec{m}) \neq 0$. With loss of generality, we can assume the first component m_{11} of $\vec{m_1}$ is nonzero. Since σ is surjective, we can choose $y \in L$, so that $\sigma(y) = E(e_2)$. By an easy calculation we have

$$[[\pi_B \sigma(x), \pi_B \sigma(y)], \pi_B \sigma(x)] = -\lambda_1^2 m_{11}^2 E(2\vec{m} + e_2) + higher\ order\ terms \neq 0.$$

Hence $[[\sigma(x), \sigma(y)], \sigma(x)] \neq 0$. But $\sigma[[x, y], x] \in [[A, L], A] \subset [A, A] = 0$, which is a contradiction. Therefore we have $\sigma A \subset A$. ■

Lemma 2.2. *Let $\sigma \in AutL$, then $\pi_B \sigma|_B \in AutB$, and for any $\vec{m} \in \Gamma$, there exists $\vec{n} \in \Gamma$, so that $\pi_B \sigma(E(\vec{m})) \in \mathbb{C}E(\vec{n})$.*

Proof. By Lemma 2.1, we have $\pi_B \sigma|_B$ is surjective. Now we show that $\pi_B \sigma|_B$ is injective. If $y \in B$ and $\pi_B \sigma(y) = 0$, then $\sigma(y) \in A$. This implies $y \in \sigma^{-1} A \subset A$, and $y \in A \cap B = \{0\}$, hence $\pi_B \sigma|_B$ is injective. This shows that $\pi_B \sigma|_B \in AutB$. [4] shows that any automorphism of the Lie algebra B is a graded automorphism. Hence there exists $\vec{n} \in \Gamma$, such that $\pi_B \sigma(E(\vec{m})) \in \mathbb{C}E(\vec{n})$. ■

Lemma 2.3. *Let $\pi_B \sigma(E(\vec{m})) \in \mathbb{C}E(\vec{n})$, then $\sigma(x^m) \in \mathbb{C}x^{\vec{n}}$.*

Proof. Since $[E(\vec{m}), x^{\vec{m}}] = 0$, we have $\sigma(x^{\vec{m}}) \in \bigoplus_{g(\vec{n}', \vec{n})=0} \mathbb{C}x^{\vec{n}'}$. Choose any $\vec{p} \in \Gamma$ satisfying $g(\vec{m}, \vec{p}) \neq 0$. Let $\pi_B \sigma(E(\vec{p})) \in \mathbb{C}E(\vec{q})$, then

$$\sigma(x^{\vec{m}+\vec{p}}) = \frac{1}{g(\vec{m}, \vec{p})} [\sigma(x^{\vec{m}}), \sigma(E(\vec{p}))] \in \bigoplus_{g(\vec{n}', \vec{n})=0} \mathbb{C}x^{\vec{n}'+\vec{q}}.$$

On the other hand, since

$$\pi_B\sigma(E(\vec{m}+\vec{p})) = \frac{1}{g(\vec{m},\vec{p})}\pi_B[\sigma(E(\vec{m})),\sigma(E(\vec{p}))] \in \mathbb{C}E(\vec{n}+\vec{q}),$$

we get $\sigma(x^{\vec{m}+\vec{p}}) \in \bigoplus_{g(\vec{s},\vec{n}+\vec{q})=0} \mathbb{C}x^{\vec{s}}$, hence

$$\begin{aligned}\sigma(x^{\vec{m}+\vec{p}}) &\in (\bigoplus_{g(\vec{n}',\vec{n})=0} \mathbb{C}x^{\vec{n}'+\vec{q}}) \cap (\bigoplus_{g(\vec{s},\vec{n}+\vec{q})=0} \mathbb{C}x^{\vec{s}}) \\ &= (\bigoplus_{g(\vec{n}',\vec{n})=0} \mathbb{C}x^{\vec{n}'+\vec{q}}) \cap (\bigoplus_{g(\vec{n}'+\vec{q},\vec{n}+\vec{q})=0} \mathbb{C}x^{\vec{n}'+\vec{q}}) \\ &= \bigoplus_{g(\vec{n}',\vec{n})=g(\vec{q},\vec{n}-\vec{n}')=0} \mathbb{C}x^{\vec{n}'+\vec{q}}\end{aligned} \tag{2.1}$$

Note that $\frac{g(\vec{n},\vec{q})E(\vec{n}+\vec{q})}{g(\vec{m},\vec{p})} = \frac{[E(\vec{n}),E(\vec{q})]}{g(\vec{m},\vec{p})} \neq 0$, hence $g(\vec{n},\vec{q}) \neq 0$. Therefore $g(\vec{n}',\vec{n}) = g(\vec{q},\vec{n}-\vec{n}') = 0$ force $\vec{n}' = \vec{n}$, and which also finishes the most of the lemma. ■

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\Omega = \{(A,a,b,c) | A \in GL_2(\mathbb{Z}), a,b,c \in \mathbb{C}^*\}$. We define a multiplication on Ω as follows:

$$(B,x_2,y_2,z_2)\cdot(A,x_1,y_1,z_1) = (BA, x_1x_2^{a_{11}}y_2^{a_{21}}, y_1x_2^{a_{12}}y_2^{a_{22}}, z_2z_1) \tag{2.2}$$

where $A = (a_{ij})_{2\times 2}$. It is easy to check that the multiplication satisfies the associativity law, the element $(I,1,1,1)$ is the identity and

$$(A,a,b,c)^{-1} = (A^{-1}, a^{-\frac{a_{22}}{|A|}}b^{\frac{a_{21}}{|A|}}, a^{\frac{a_{12}}{|A|}}b^{-\frac{a_{11}}{|A|}}, c^{-1})$$

is the inverse of $(A,a,b,c) \in \Omega$. Hence Ω forms a group.

Lemma 2.4. $\Omega \cong GL_2(\mathbb{Z}) \propto (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)$

Proof. The two sets

$$\Omega_1 = \{(A,1,1,1) | A \in G(2,\mathbb{Z})\} \text{ and } \quad \Omega_2 = \{(I,a,b,c) | A \in \mathbb{C}^*\}$$

are subgroups of Ω. It is easy to see that $\Omega_1 \cap \Omega_2 = (I,1,1,1)$, $\Omega_2 \lhd \Omega$ and $\Omega_1 \cdot \Omega_2 = \Omega$, hence $\Omega = \Omega_1 \propto \Omega_2 \cong GL_2(\mathbb{Z}) \propto (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)$ ■

Now we study the structure of $AutL$. First we construct a subgroup of $AutL$.

Lemma 2.5. *Let $\Theta_1 = \{\overline{\sigma} | \overline{\sigma} \in AutL, \pi_A\overline{\sigma}|_B = 0\}$. Then Θ_1 is a subgroup of $AutL$, and $\Theta_1 \cong GL_2(\mathbb{Z}) \propto (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)$*

Proof. For any $\overline{\sigma} \in \Theta_1$, by Lemma 2.2 we see $\overline{\sigma}|_B = \pi_B\overline{\sigma}|_B \in AutB$. Moreover, from [4], there exists $A = (a_{ij}) \in GL_2(\mathbb{Z}), a,b \in \mathbb{C}^*$, so that

$$\overline{\sigma}(E(\vec{m})) = |A|a^{m_1}b^{m_2}E((m_1a_{11}+m_2a_{12})e_1 + (m_1a_{21}+m_2a_{22})e_2).$$

Now we give the action of $\overline{\sigma}$ on A. By Lemma 2.3, we can set

$$\overline{\sigma}(x^{\vec{m}}) = k(\vec{m})x^{(m_1a_{11}+m_2a_{12})e_1+(m_1a_{21}+m_2a_{22})e_2},$$

where $k:\Gamma\to\mathbb{C}^*$ is a mapping. Since $\overline{\sigma}[E(\vec{m}),x^{\vec{n}}]=[\overline{\sigma}(E(\vec{m})),\overline{\sigma}(x^{\vec{n}})]$, we have

$$g(\vec{m},\vec{n})(k(\vec{m}+\vec{n})-a^{m_1}b^{m_2}k(\vec{n}))=0.$$

For any fixed $\vec{m},\vec{n}$ satisfying $g(\vec{m},\vec{n})\neq 0$, the above identity implies

$$k(\vec{m}+\vec{n})=a^{m_1}b^{m_2}k(\vec{n}))=a^{n_1}b^{n_2}k(\vec{m})) \tag{2.3}$$

Take $\vec{m}=e_1,\vec{n}=e_2$, the above identity gives $\frac{k(e_1)}{a}=\frac{k(e_2)}{b}$. For any $\vec{m}\in\Gamma$, we have $g(\vec{m},e_1)\neq 0$ or $g(\vec{m},e_2)\neq 0$. This gives $\frac{k(\vec{m})}{a^{m_1}b^{m_2}}=\frac{k(e_1)}{a}$ or $\frac{k(\vec{m})}{a^{m_1}b^{m_2}}=\frac{k(e_2)}{b}$ from (2.3). Hence $k(\vec{m})=c\cdot a^{m_1}b^{m_2}$, where $c=\frac{k(e_1)}{a}=\frac{k(e_2)}{b}$ is a nonzero constant which is independent of $\vec{m}$.

Now we prove the first part of the lemma. For any $\overline{\sigma}\in\Theta_1$, then

$$\overline{\sigma}(E(\vec{m}))=|A|a^{m_1}b^{m_2}E((m_1a_{11}+m_2a_{12})e_1+(m_1a_{21}+m_2a_{22})e_2)$$
$$\overline{\sigma}(x^{\vec{m}})=c\cdot a^{m_1}b^{m_2}x^{(m_1a_{11}+m_2a_{12})e_1+(m_1a_{21}+m_2a_{22})e_2}.$$

We can easily check the following identities:

$$\overline{\sigma}^{-1}(E(\vec{m}))=|A|a^{\frac{a_{12}m_2-a_{22}m_1}{|A|}}b^{\frac{a_{21}m_1-a_{11}m_2}{|A|}}E(\tfrac{m_1a_{22}-m_2a_{12}}{|A|}e_1+\tfrac{m_2a_{11}-m_1a_{21}}{|A|}e_2)$$
$$\overline{\sigma}^{-1}(x^{\vec{m}})=c^{-1}a^{\frac{a_{12}m_2-a_{22}m_1}{|A|}}b^{\frac{a_{21}m_1-a_{11}m_2}{|A|}}x^{\frac{m_1a_{22}-m_2a_{12}}{|A|}e_1+\frac{m_2a_{11}-m_1a_{21}}{|A|}e_2},$$

and also $\overline{\sigma}'\overline{\sigma}^{-1}\in\Theta_1$ for any $\overline{\sigma}',\overline{\sigma}\in\Theta_1$. Hence Θ_1 is a subgroup of $AutL$.

We define a map $\tau:\Omega\to\Theta_1$ as follows, for $(A,a,b,c)\in\Omega$,

$$\tau(A,a,b,c)(E(\vec{m}))=|A|a^{m_1}b^{m_2}E((m_1a_{11}+m_2a_{12})e_1+(m_1a_{21}+m_2a_{22})e_2)$$
$$\tau(A,a,b,c)(x^{\vec{m}})=c\cdot a^{m_1}b^{m_2}x^{(m_1a_{11}+m_2a_{12})e_1+(m_1a_{21}+m_2a_{22})e_2}.$$

Then it is easy to check that τ is a bijective map. Moreover by (2.2), we have

$$\tau((B,a_1,b_1,c_1)(A,a_2,b_2,c_2))=\tau(B,a_1,b_1,c_1)\tau(A,a_2,b_2,c_2)$$

Hence τ is a group isomorphism. ■

Remark 2.6. Define a map $\langle\cdot|\cdot\rangle:\mathbb{Z}^2\times\mathbb{C}^2\to\mathbb{C}$, by $\langle\vec{m}|\vec{c}\rangle=m_1c^1+m_2c^2$, where $\vec{m}=m_1e_1+m_2e_2,\vec{c}=(c^1,c^2)$. Denote

$$\mathbb{C}^2_\infty=\{(\vec{c}_{ij})_{i,j\in\mathbb{Z}}|\vec{c}_{ij}\in\mathbb{C}^2,\langle ie_1+je_2|\vec{c}_{ij}\rangle=0,\text{ and only finitely many }\vec{c}_{ij}\neq 0\}.$$

It is clear $\mathbb{C}^2_\infty$ is an abelian group under the usual addition. The following two lemmas give another class of the automorphisms of $AutL$.

Lemma 2.7. *For $\hat{\sigma}\in AutL$, if $\pi_B\hat{\sigma}|_B=id|_B$, then there exists finite nonzero elements $\vec{c}_{\vec{r}}\in\mathbb{C}^2$, such that*

$$\hat{\sigma}(E(\vec{m}))=\sum_{\vec{r}\in Z^2}\langle\vec{m}|\vec{c}_{\vec{r}}\rangle x^{\vec{m}+\vec{r}}+E(\vec{m})$$

where $\langle\vec{r}|\vec{c}_{\vec{r}}\rangle=0$ for all $\vec{r}\in\mathbb{Z}^2$.

Proof. Set $\hat{\sigma}(E(\vec{m})) = \sum_{\vec{r}\in Z^2} f_{\vec{r}}(\vec{m})x^{\vec{m}+\vec{r}} + E(\vec{m})$. Take $\vec{c}_{\vec{r}} := (c^1_{\vec{r}}, c^2_{\vec{r}}) = (f_{\vec{r}}(e_1), f_{\vec{r}}(e_2))$. From $\hat{\sigma}[E(\vec{m}'), E(\vec{n}')] = [\hat{\sigma}(E(\vec{m}')), \hat{\sigma}(E(\vec{m}'))]$ we have

$$g(\vec{m}', \vec{n}')f_{\vec{r}}(\vec{m}' + \vec{n}') = f_{\vec{r}}(\vec{m}')g(\vec{m}' + \vec{r}, \vec{n}') + f_{\vec{r}}(\vec{n}')g(\vec{m}', \vec{n}' + \vec{r}) \tag{2.4}$$

Now we prove $f_{\vec{r}}(\vec{m}) = \langle \vec{m}|\vec{c}_{\vec{r}}\rangle$ with $\langle \vec{r}|\vec{c}_{\vec{r}}\rangle = 0$ for all $\vec{r} \in \mathbb{Z}^2$. The argument is divided in three cases.

(A). For $\vec{r} = r_1e_1 + r_2e_2$ and $r_1r_2 \neq 0$. Let $\vec{m}' = \vec{m}, \vec{n}' = l\vec{m}$, then (2.4) gives

$$g(\vec{m}, \vec{r})(f_{\vec{r}}(l\vec{m}) - lf_{\vec{r}}(\vec{m})) = 0 \tag{2.5}$$

Let $m = e_1,\ e_2$ respectively, (2.5) gives

$$f_{\vec{r}}(le_1) = lc^1_{\vec{r}},\ f_{\vec{r}}(le_2) = lc^2_{\vec{r}} \tag{2.6}$$

Let $\vec{m}' = m_1e_1, \vec{n}' = m_2e_2, m_1m_2 \neq 0$, (2.4) gives

$$f_{\vec{r}}(\vec{m}) = \langle \vec{r} + \vec{m}|\vec{c}_{\vec{r}}\rangle \tag{2.7}$$

In particular we choose $\vec{m}$ satisfying $g(\vec{m}, \vec{r}) \neq 0$ in (2.7), then by (2.6) we have $\langle \vec{r}|\vec{c}_{\vec{r}}\rangle = 0$. Hence (2.7) induces $f_{\vec{r}}(\vec{m}) = \langle \vec{m}|\vec{c}_{\vec{r}}\rangle$ for $m_1m_2 \neq 0$. Thus the result follows from this and (2.6).

(B). For $\vec{r} = r_1e_1 + r_2e_2$ and $r_1 = 0, r_2 \neq 0$ or $r_2 = 0, r_1 \neq 0$. Without loss of generality, we assume $r_1 = 0$ and $r_2 \neq 0$. By a similar argument as (2.6), we have $f_{\vec{r}}(m_1e_1) = m_1c^1_{\vec{r}}$. Let $\vec{m}' = e_1 + ke_2, \vec{n}' = e_1$, (2.4) gives

$$kf_{\vec{r}}(2e_1 + ke_2) = f_{\vec{r}}(e_1 + ke_2)(k + r_2) + c^1_{\vec{r}}(k - r_2) \tag{2.8}$$

Let $\vec{m}' = 2e_1, \vec{n}' = ke_2$, (2.4) gives

$$2kf_{\vec{r}}(2e_1 + ke_2) = 4c^1_{\vec{r}}k + 2f_{\vec{r}}(ke_2)(k + r_2) \tag{2.9}$$

Let $\vec{m}' = e_1, \vec{n}' = ke_2$, (2.4) gives

$$kf_{\vec{r}}(e_1 + ke_2) = c^1_{\vec{r}}k + f_{\vec{r}}(ke_2)(k + r_2) \tag{2.10}$$

Solving the equations (2.8)(2.9)(2.10) we have

$$f_{\vec{r}}(ke_2)(k + r_2) = 0 \tag{2.11}$$

Note that $f_{\vec{r}}(\vec{m})$ is the coefficient of $x^{\vec{m}+\vec{r}}$ in $\hat{\sigma}(E(\vec{m}))$, we have $f_{\vec{r}}(-r_2e_2) = f_{\vec{r}}(-\vec{r}) = 0$. Hence for any $k \in \mathbb{Z}\backslash\{0\}$, $f_{\vec{r}}(ke_2) = 0$ from this and (2.11). In particular take $k = 1$, then $c^2_{\vec{r}} = 0$.

Taking $\vec{m}' = m_1e_1, \vec{n}' = m_2e_2, m_1m_2 \neq 0$, (2.4) gives $f_{\vec{r}}(\vec{m}) = c^1_{\vec{r}}m_1$. Hence, for any $\vec{m} \in \Gamma, f_{\vec{r}}(\vec{m}) = c^1_{\vec{r}}m_1$. Immediately we have $f_{\vec{r}}(\vec{m}) = \langle \vec{m}|\vec{c}_{\vec{r}}\rangle$ and $\langle \vec{r}|\vec{c}_{\vec{r}}\rangle = 0$.

(C). For $\vec{r} = 0$, (2.4) induces

$$g(\vec{m}, \vec{n})(f_{\vec{0}}(\vec{m}) + f_{\vec{0}}(\vec{n}) - f_{\vec{0}}(\vec{m} + \vec{n})) = 0 \tag{2.12}$$

Similarly as the proof of case (A) and (B), we can get $f_{\vec{0}}(\vec{m}) = \langle \vec{m}|\vec{c}_{\vec{0}}\rangle$. ■

Lemma 2.8. *$\Theta_2 = \{\hat{\sigma}|\hat{\sigma} \in AutL, \pi_B\hat{\sigma}|_B = id|_B, \hat{\sigma}|_A = id|_A\}$ is an abelian subgroup of $AutL$, and $\Theta_2 \cong \mathbb{C}^2_\infty$.*

Proof. By the Lemma 2.7, one can easily check that Θ_2 is an abelian subgroup under composition of automorphisms in Θ_2. Define $\rho : \mathbb{C}^2_\infty \to \Theta_2$ by $(\vec{c}_{ij})_{ij} \to \rho((\vec{c}_{ij})_{ij})$, where $\rho((\vec{c}_{ij})_{ij}) \in \Theta_2$, such that

$$\rho((\vec{c}_{ij})_{ij})E(\vec{m}) = \sum_{i,j\in Z} \langle m|\vec{c}_{ij}\rangle x^{\vec{m}+ie_1+je_2} + E(\vec{m})$$
$$\rho((\vec{c}_{ij})_{ij})x^{\vec{m}} = x^{\vec{m}}$$

One can check $\rho((\vec{c}_{ij})_{ij})$ is bijective. Moreover, for any $(\vec{b}_{ij})_{ij}, (\vec{c}_{ij})_{ij} \in \mathbb{C}^2_\infty$, we have $\rho((\vec{b}_{ij})_{ij} + (\vec{c}_{ij})_{ij}) = \rho((\vec{b}_{ij})_{ij})\rho((\vec{c}_{ij})_{ij})$, which implies ρ is a group isomorphism. ■

Remark 2.9. For $x^{\vec{r}} \in A$, since $(ad(x^{\vec{r}}))^2 = 0$ for $\vec{r} \in \Gamma$, we know that $\exp ad(x^{\vec{r}}) = 1 + ad(x^{\vec{r}})$ is an automorphism of L. Indeed, we have

$$\exp(ad(x^{\vec{r}}))E(\vec{m}) = \langle \vec{m}|\vec{c}_{\vec{r}}\rangle x^{\vec{m}+\vec{r}} + E(\vec{m})$$

where $\vec{c}_{\vec{r}} = (r_2, -r_1)$ for $\vec{r} = (r_1, r_2) \in \Gamma$. One may compare this with the automorphism $\hat{\sigma}$ defined in Lemma 2.7 to see that the abelian subgroup Θ_2 can also be constructed in this way.

Theorem 2.10. *$AutL \cong (GL_2\mathbb{Z}) \propto ((\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)) \propto \mathbb{C}^2_\infty$*

Proof. First we show that $AutL = \Theta_2\Theta_1$. For any $\sigma_0 \in AutL$, define a map σ_1 by $\sigma_1|_B = \pi_B\sigma_0|_B, \sigma_1|_A = \sigma_0|_A$, we have $\sigma_1 \in \Theta_1, \sigma_0\sigma_1^{-1} \in \Theta_2$, so $AutL = \Theta_2\Theta_1$. Next we prove that the decomposition is unique. In fact, if $\sigma_1' \in \Theta_1, \sigma_2' \in \Theta_2$ is another such a decomposition, then we have $\sigma_2\sigma_1 = \sigma_2'\sigma_1'$, this induces $\sigma_1'\sigma_1^{-1} = {\sigma_2'}^{-1}\sigma_2$. Since $\Theta_1 \cap \Theta_2 = id$, we set $\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$. Moreover, one can check Θ_2 is a normal subgroup of $AutL$, thus $AutL = \Theta_1 \propto \Theta_2$. By using Lemma 2.5 and Lemma 2.8, we get $AutL \cong (GL_2\mathbb{Z}) \propto (\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)) \propto \mathbb{C}^2_\infty$. ■

3. Derivation Algebra

In this section, we study the derivation Lie algebra $Der(L)$ of the Lie algebra $L = [\tilde{L}, \tilde{L}]$ with $\tilde{L} = A \oplus B$.

Lemma 3.1. *L is a finitely generated $\mathbb{Z}^2$-graded Lie algebra.*

Theorem 3.2[5]. *Let G be an abelian group. If $A = \bigoplus_{\alpha\in G} A_\alpha$ is a finitely generated G-graded Lie algebra, then $Der(A) = \bigoplus_{\alpha\in G} Der(A)_\alpha$ is also a G-graded Lie algebre. Moreover, if $d \in Der(A)_\alpha$, $d(A_\beta) \subset A_{\alpha+\beta}$.*

Theorem 3.3. *$Der(L) = Inder(L) \oplus Outder(L)$, where*

$$Inder(L) = \bigoplus_{\vec{r}\in\Gamma}(\mathbb{C}adx^{\vec{r}} \oplus \mathbb{C}adE(\vec{r}))$$
$$Outder(L) = \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \mathbb{C}\delta$$

and d_1, d_2 are degree derivations of L, p_1, p_2 are determined by $p_1E(\vec{m}) = m_1x^{\vec{m}}$, $p_2E(\vec{m}) = m_2x^{\vec{m}}$, $p_1x^{\vec{m}} = p_2x^{\vec{m}} = 0$, and δ is determined by $\delta E(\vec{m}) = 0$, $\delta x^{\vec{m}} = x^{\vec{m}}$.

Proof. By Lemma 3.1 and Theorem 3.2, we set $Der(L) = \bigoplus_{\vec{r}\in\mathbb{Z}^2} Der(L)_{\vec{r}}$. For $d_{\vec{r}} \in Der(L)_{\vec{r}}$, let

$$d_{\vec{r}}(x^{\vec{m}}) = \varphi_1(\vec{r},\vec{m})x^{\vec{m}+\vec{r}} + \varphi_2(\vec{r},\vec{m})E(\vec{m}+\vec{r})$$
$$d_{\vec{r}}(E(\vec{m})) = \psi_1(\vec{r},\vec{m})x^{\vec{m}+\vec{r}} + \psi_2(\vec{r},\vec{m})E(\vec{m}+\vec{r}),$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are maps from $\mathbb{Z}^2$ to $\mathbb{C}$. For convenience of notations, we set

$$\varphi_1(\vec{r},-\vec{r}) = 0,\ \varphi_2(\vec{r},-\vec{r}) = 0,\ \psi_1(\vec{r},-\vec{r}) = 0,\ \psi_2(\vec{r},-\vec{r}) = 0$$

Using the property of derivation, $d_{\vec{r}}[a,b] = [d_{\vec{r}}(a),b] + [a,d_{\vec{r}}(b)]$, we have

$$\psi_1(\vec{r},\vec{m}+\vec{n})g(\vec{m},\vec{n}) = \psi_1(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) + \psi_1(\vec{r},\vec{n})g(\vec{m},\vec{n}+\vec{r}), \tag{3.1}$$

$$\psi_2(\vec{r},\vec{m}+\vec{n})g(\vec{m},\vec{n}) = \psi_2(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) + \psi_2(\vec{r},\vec{n})g(\vec{m},\vec{n}+\vec{r}), \tag{3.2}$$

$$\varphi_1(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) + \psi_2(\vec{r},\vec{n})g(\vec{m},\vec{n}+\vec{r}) = \varphi_1(\vec{r},\vec{m}+\vec{n})g(\vec{m},\vec{n}), \tag{3.3}$$

$$\varphi_2(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) = \varphi_2(\vec{r},\vec{m}+\vec{n})g(\vec{m},\vec{n}), \tag{3.4}$$

$$\varphi_2(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) + \varphi_2(\vec{r},\vec{n})g(\vec{m},\vec{n}+\vec{r}) = 0. \tag{3.5}$$

Now we prove that $\varphi_2(\vec{r},\vec{m}) \equiv 0$. We may assume $\vec{m} \neq -\vec{r}$. Interchange $\vec{m},\vec{n}$ in (3.4), then

$$\varphi_2(\vec{r},\vec{n})g(\vec{n}+\vec{r},\vec{m}) = \varphi_2(\vec{r},\vec{m}+\vec{n})g(\vec{n},\vec{m}) \tag{3.6}$$

Solving the equations (3.4) through (3.6), we have $\varphi_2(\vec{r},\vec{m})g(\vec{m}+\vec{r},\vec{n}) = 0$. Choose $\vec{n} \in \Gamma$ satisfying $g(\vec{m}+\vec{r},\vec{n}) \neq 0$, then $\varphi_2(\vec{r},\vec{m}) = 0$ as required.

Now we divide the proof into two cases. First for $\vec{r} = 0$, we solve the equations (3.1) through (3.3) to get

$$g(\vec{m},\vec{n})(\psi_1(0,\vec{m}+\vec{n}) - \psi_1(0,\vec{m}) - \psi_1(0,\vec{n})) = 0 \tag{3.7}$$

$$g(\vec{m},\vec{n})(\psi_2(0,\vec{m}+\vec{n}) - \psi_2(0,\vec{m}) - \psi_2(0,\vec{n})) = 0 \tag{3.8}$$

$$g(\vec{m},\vec{n})(\varphi_1(0,\vec{m}) + \psi_2(0,\vec{n}) - \varphi_1(0,\vec{m}+\vec{n})) = 0. \tag{3.9}$$

¿From (3.7),(3.8) and (3.9) one can easily deduce the following identities:

$$\psi_2(0,\vec{m}) = m_1\psi_2(0,e_1) + m_2\psi_2(0,e_2)$$

$$\psi_1(0,\vec{m}) = m_1\psi_1(0,e_1) + m_2\psi_1(0,e_2)$$

$$\varphi_1(0,\vec{m}) = \psi_2(0,\vec{m}) + c, \quad c \in \mathbb{C},$$

where the constant $c = \psi_1(0,e_2) - \varphi_1(0,e_2)$. Hence $dimDer(L)_{\vec{0}} \leq 5$.

Let d_1, d_2 be degree derivations of A, p_1, p_2 and δ are linear maps on L, satisfy $p_1E(\vec{m}) = m_1x^{\vec{m}}$, $p_2E(\vec{m}) = m_2x^{\vec{m}}$, $p_1x^{\vec{m}} = p_2x^{\vec{m}} = 0$, $\delta E(\vec{m}) = 0, \delta x^{\vec{m}} = x^{\vec{m}}$. It is easy to check $d_1, d_2, p_1, p_2, \delta$ are linearly independent and belong to $Der(L)_{\vec{0}}$, then we get $dimDer(L)_{\vec{0}} = 5$ and $Der(L)_{\vec{0}} = \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \mathbb{C}\delta$.

Next we consider the case for $\vec{r} \neq 0$. Without loss of generality, we assume $r_1 \neq 0$. For $\vec{m} = e_2$, $\vec{n} = m_2e_2$, (3.1) gives

$$\varphi_1(\vec{r},m_2e_2) = m_2\varphi_1(\vec{r},e_2) \tag{3.10}$$

For $\vec{m} = m_1e_1,\ \vec{n} = e_2$, (3.1) gives

$$\varphi_1(\vec{r}, m_1e_1 + e_2)m_1 = (m_1 + r_1)\varphi_1(\vec{r}, m_1e_1) + m_1(r_2 + 1)\varphi_1(\vec{r}, e_2) \tag{3.11}$$

For $\vec{m} = m_1e_1 + e_2,\ \vec{n} = -e_2$, (3.1) gives

$$\varphi_1(\vec{r}, m_1e_1)m_1 = (m_1 + r_1)\varphi_1(\vec{r}, m_1e_1 + e_2) + (r_1 + m_1 - m_1r_2)\varphi_1(\vec{r}, -e_2)$$

By (3.10), we have $\varphi_1(\vec{r}, -e_2) = -\varphi_1(\vec{r}, e_2)$, thus the above identity deduces

$$\varphi_1(\vec{r}, m_1e_1)m_1 = (m_1 + r_1)\varphi_1(\vec{r}, m_1e_1 + e_2) - (r_1 + m_1 - m_1r_2)\varphi_1(\vec{r}, e_2) \tag{3.12}$$

Solving (3.11)(3.12) we get

$$\psi_1(\vec{r}, m_1e_1) = -\frac{r_2}{r_1}m_1\psi_1(\vec{r}, e_2) \tag{3.13}$$

For $\vec{m} = m_1e_1,\ \vec{n} = m_2e_2$, (3.1) gives

$$\varphi_1(\vec{r}, \vec{m})m_1m_2 = m_2(m_1 + r_1)\varphi_1(\vec{r}, m_1e_1) + (r_2 + m_2)m_1\varphi_1(\vec{r}, m_2e_2) \tag{3.14}$$

Substitute (3.10)(3.13) into (3.14) we get

$$\psi_1(\vec{r}, \vec{m}) = (m_2 - \frac{r_2}{r_1}m_1)\psi_1(\vec{r}, e_2) \tag{3.15}$$

By a similar argument, we have

$$\psi_2(\vec{r}, \vec{m}) = (m_2 - \frac{r_2}{r_1}m_1)\psi_2(\vec{r}, e_2) \tag{3.16}$$

Now we prove $\varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m})$. Let $\vec{n} = -\vec{m}$, (2.3) gives

$$g(\vec{m}, \vec{r})(\varphi_1(\vec{r}, \vec{m}) - \psi_2(\vec{r}, \vec{m})) = 0$$

This implies that if $g(\vec{m}, \vec{r}) \neq 0$, $\varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m})$. For the case of $g(\vec{m}, \vec{r}) = 0$ and $\vec{m} + \vec{r} \neq 0$, we choose $\vec{n} \in \Gamma$ satisfying $g(\vec{m} + \vec{r}, \vec{n}) \neq 0$, $g(\vec{r}, \vec{m} + \vec{n}) \neq 0$, (3.2),(3.3) give us $\varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m})$, where we have used the fact that $\varphi_1(\vec{r}, \vec{m}) = \psi_2(\vec{r}, \vec{m})$ whenever $g(\vec{m}, \vec{r}) \neq 0$. This implies $d_{\vec{r}}$ is determined by $\psi_1(\vec{r}, e_2)$ and $\psi_2(\vec{r}, e_2)$, so $dimDer(L)_{\vec{r}} \leq 2$. Note that $\mathrm{ad}(x^{\vec{r}})$, $\mathrm{ad}(E(\vec{r}))$ are linearly independent derivations in $Der(L)_{\vec{r}}$, hence $dimDer(L)_{\vec{r}} = 2$ and $Der(L)_{\vec{r}} = \mathbb{C}\mathrm{ad}x^{\vec{r}} \oplus \mathbb{C}\mathrm{ad}E(\vec{r})$. Then we get the result. ∎

4. Universal Central Extension

Let $0 \to K \to \hat{L} \xrightarrow{\lambda} L \to 0$ be a central extension of the Lie algebra L, and id_L be the identity map on the Lie algebra L. We can define a linear map $\tau : L \to \hat{L}$, such that $\lambda\tau = \mathrm{id}_L$. Set $\tau^* : L \times L \to K, \tau^*(a, b) = [\tau(a), \tau(b)] - \tau([a, b])$ for $a,\ b \in L$. Obviously we have

$$\tau^*(a, [b, c]) + \tau^*(b, [c, a]) + \tau^*(c, [a, b]) = 0 \tag{4.1}$$

and $\tau^*(a, b) = -\tau^*(b, a)$. In this section, we study the universal central extension of the Lie algebra L. For this purpose, we need the following three propositions.

Proposition 4.1. $\tau^*(E(\vec{m}), E(\vec{n})) =$

$$\begin{cases} -\frac{m_2n_1-m_1n_2}{(m_1+n_1)(m_2+n_2)}\tau^*(E((m_1+n_1)e_1), E((m_2+n_2)e_2), & m_1+n_1 \neq 0, m_2+n_2 \neq 0 \\ (m_2n_1-m_1n_2)\tau^*(E(e_1+(m_2+n_2)e_2, E(-e_1)), & m_1+n_1 = 0, m_2+n_2 \neq 0 \\ -(m_2n_1-m_1n_2)\tau^*(E((m_1+n_1)e_1+e_2), E(-e_2)), & m_1+n_1 \neq 0, m_2+n_2 = 0 \\ m_1\tau^*(E(e_1), E(-e_1)) + m_2\tau^*(E(e_2), E(-e_2)), & m_1+n_1 = 0, m_2+n_2 = 0 \end{cases}$$

Proposition 4.2. $\tau^*(x^{\vec{m}}, E(\vec{n})) =$

$$\begin{cases} -\frac{m_2n_1-m_1n_2}{(m_1+n_1)(m_2+n_2)}\tau^*(x^{m_1+n_1,0}, E((m_2+n_2)e_2), & m_1+n_1 \neq 0, m_2+n_2 \neq 0 \\ m_1\tau^*(x^{1,m_2+n_2}, E(-e_1)), & m_1+n_1 = 0, m_2+n_2 \neq 0 \\ m_2\tau^*(x^{m_1+n_1,1}, E(-e_2)), & m_1+n_1 \neq 0, m_2+n_2 = 0 \\ m_1\tau^*(x^{1,0}, E(-e_1)) + m_2\tau^*(x^{0,1}, E(-e_2)), & m_1+n_1 = 0, m_2+n_2 = 0 \end{cases}$$

Proposition 4.3. $\tau^*(x^{\vec{m}}, x^{\vec{n}}) = 0$. Proposition 4.1 is given in [12]. Now we prove Proposition 4.2 by a series of lemmas. Proposition 4.3 will be proved in the last part of the section. Set $a = x^{\vec{m}'}, b = E(\vec{n}'), c = E(\vec{r}')$, (4.1) gives

$$g(\vec{n}', \vec{r}')\tau^*(x^{\vec{m}'}, E(\vec{n}' + \vec{r}')) + g(\vec{r}', \vec{m}')\tau^*(E(\vec{n}'), x^{\vec{m}'+\vec{r}'}) \\ +g(\vec{n}', \vec{m}')\tau^*(x^{\vec{m}'+\vec{n}'}, E(\vec{r}')) = 0 \tag{4.2}$$

Lemma 4.4. $\tau^*(x^{\vec{m}}, E(\vec{m})) = 0$ *for* $\vec{m} \in \Gamma$.

Proof. Let $\vec{m}' = \vec{r}' = \vec{m}$, $\vec{n}' = \vec{n} - \vec{m}$ satisfying $g(\vec{m}, \vec{n}) \neq 0$, (4.2) gives

$$g(\vec{m}, \vec{n})(\tau^*(x^{\vec{m}}, E(\vec{n})) + \tau^*(x^{\vec{n}}, E(\vec{m}))) = 0.$$

Since $g(\vec{m}, \vec{n}) \neq 0$, we have

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = -\tau^*(x^{\vec{n}}, E(\vec{m})) = \tau^*(E(\vec{m}), x^{\vec{n}})$$

Immediately we get

$$\tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = \tau^*(E(2\vec{m} - \vec{n}), x^{\vec{n}}) \tag{4.3}$$

For $\vec{m}' = \vec{m}, \vec{n}' = \vec{n}, r' = -\vec{n} + \vec{m}$, (4.2) gives

$$\tau^*(x^{\vec{m}}, E(\vec{m})) + \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) + \tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = 0 \tag{4.4}$$

For $r' = \vec{m}, \vec{m}' = \vec{n}, \vec{n}' = -\vec{n} + \vec{m}$, (4.2) gives

$$\tau^*(x^{\vec{m}}, E(\vec{m})) - \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) + \tau^*(E(2\vec{m} - \vec{n}), x^{\vec{n}}) = 0 \tag{4.5}$$

Solving the equations (4.2) through (4.5), we obtain

$$\tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{n} + \vec{m})) = 0 \tag{4.6}$$

Taking $\vec{m} = \vec{m}'$, $\vec{n} = \vec{m}' - \vec{n}'$ in (4.6) we have

$$\tau^*(x^{2\vec{m}-\vec{n}}, E(\vec{n})) = 0 \tag{4.7}$$

Substitute (4.6)(4.7) to (4.4) we get $\tau^*(x^{\vec{m}}, E(\vec{m})) = 0$. ∎

Lemma 4.5. *(1) If $m_1+n_1\neq 0$, then $\tau^*(x^{m_1,0},E(n_1e_1))=0$.*

(2) If $m_2+n_2\neq 0$, then $\tau^*(x^{0,m_2},E(n_2e_2))=0$, for nonzero integers $m_i,n_i,i=1,2$.

Proof. As a direct consequence of Lemma 4.4, we have

$$\tau^*(x^{m_1,0},E(m_1,0))=0$$

We prove (1) for $m_1\neq n_1$. For $\vec{m}'=n_1e_1+e_2,\vec{n}'=-e_2,\vec{r}'=m_1e_1$, (4.2) gives

$$m_1\tau^*(x^{n_1,1},E(m_1e_1-e_2))+m_1\tau^*(E(-e_2),x^{m_1+n_1,1})+n_1\tau^*(x^{n_1,0},E(m_1e_1))=0 \tag{4.8}$$

For $\vec{m}'=n_1e_1,\vec{n}'=m_1e_1-e_2,\vec{r}'=e_2$, (4.2) gives

$$-m_1\tau^*(x^{n_1,0},E(m_1e_1))+n_1\tau^*(E(m_1e_1-e_2),x^{n_1,1})-n_1\tau^*(x^{m_1+n_1,-1},E(e_2))=0 \tag{4.9}$$

For $\vec{m}'=-e_2,\vec{n}'=e_2,\vec{r}'=(m_1+n_1)e_1$, (4.2) gives

$$(m_1+n_1)\tau^*(x^{0,-1},E((m_1+n_1)e_1+e_2))+(m_1+n_1)\tau^*(E(e_2),x^{m_1+n_1,-1})=0 \tag{4.10}$$

For $\vec{m}'=\vec{n}'=-e_2,\vec{r}=(m_1+n_1)e_1+2e_2$, (4.2) gives

$$\tau^*(x^{0,-1},E((m_1+n_1)e_1+e_2)=\tau^*(E(-e_2),x^{m_1+n_1,1}) \tag{4.11}$$

Solving (4.8) through (4.11), we get that, if $m_1\neq n_1$, $\tau^*(x^{m_1,0},E(n_1e_1)=0$. This finishes the proof of (1). (2) follows from (1) by symmetry. ∎

Lemma 4.6. *For $m_1+n_1\neq 0,m_2+n_2\neq 0$, we have*

$$\tau^*(x^{\vec{m}},E(\vec{n}))=-\frac{m_2n_1-m_1n_2}{(m_1+n_1)(m_2+n_2)}\tau^*(x^{m_1+n_1,0},E((m_2+n_2)e_2))$$

Proof. We divide the argument into three cases.
Case 1. For $n_2=0$, let $\vec{m}'=(m_1+n_1)e_1,\vec{n}'=n_1e_1,r'=-n_1e_1+m_2e_2$, (3.2) gives

$$\tau^*(x^{\vec{m}},E(n_1e_1))=-\frac{n_1}{m_1+n_1}\tau^*(x^{m_1+n_1,0},E(m_2e_2))$$

as required. In particular, this gives

$$\tau^*(x^{m_1,m_2+n_2},E(n_1e_1))=-\frac{n_1}{m_1+n_1}\tau^*(x^{m_1+n_1,0},E((m_2+n_2)e_2)) \tag{4.12}$$

Case 2. For $n_1=0$, let $\vec{m}'=(m_2+n_2)e_2,\vec{n}'=n_2e_2,r'=m_1e_1-n_2e_2$, (4.2) gives

$$\tau^*(x^{\vec{m}},E(n_2e_2))=-\frac{n_2}{m_2+n_2}\tau^*(x^{0,m_2+n_2},E(m_1e_1))$$

By (4.12) we have

$$\tau^*(x^{\vec{m}},E(n_2e_2))=\frac{n_2}{m_2+n_2}\tau^*(x^{m_1,0},E((m_2+n_2)e_2))$$

as required. Also this implies

$$\tau^*(x^{m_1+n_1,m_2}, E(n_2e_2)) = \frac{n_2}{m_2+n_2}\tau^*(x^{m_1+n_1,0}, E((m_2+n_2)e_2)) \tag{4.13}$$

Case 3. For $n_1 \neq 0, n_2 \neq 0$, let $\vec{m}' = m_1e_1 + m_2e_2, \vec{n}' = n_1e_1, r = n_2e_2$, (4.2) gives

$$-n_1n_2\tau^*(x^{\vec{m}}, E(\vec{n})) + n_2m_1\tau^*(E(n_1e_1), x^{m_1,m_2+n_2}) - n_1m_2\tau^*(x^{m_1+n_1,m_2}, E(n_2e_2))$$

$= 0$. Solving this equation together with (4.12)(4.13) we have

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = -\frac{m_2n_1 - m_1n_2}{(m_1+n_1)(m_2+n_2)}\tau^*(x^{m_1+n_1,0}, E((m_2+n_2)e_2).$$

This finishes the proof of the lemma. ∎

Lemma 4.7. *For $m_1 + n_1 = 0, m_2 + n_2 \neq 0$, we have*

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = m_1\tau^*(x^{1,m_2+n_2}, E(-e_1))$$

Proof. We divide the proof into two cases.
Case 1. For $m_2 = 0$, let $\vec{m}' = m_1e_1, \vec{n}' = -e_1, \vec{r}' = (1-m_1)e_1 + n_2e_2$, (4.2) gives

$$\tau^*(x^{m_1,0}, E(-m_1e_1 + n_2e_2)) = m_1\tau^*(x^{1,n_2}, E(-e_1)) \tag{4.14}$$

Case 2. For $m_2 \neq 0$, let $\vec{m}' = m_1e_1, \vec{n}' = \vec{n}, \vec{r}' = m_2e_2$, (4.2) gives

$$-n_1m_2\tau^*(x^{m_1,0}, E(n_1e_1 + (m_2+n_2)e_2)) + m_1m_2\tau^*(E(\vec{n}), x^{\vec{m}})$$

$$+n_2m_1\tau^*(x^{0,n_2}, E(m_2e_2)) = 0$$

applying lemma 4.5 and (4.14) we have $\tau^*(x^{\vec{m}}, E(\vec{n})) = m_1\tau^*(x^{1,m_2+n_2}, E(-e_1))$. ∎

Similarly as Lemma 4.7 we get

Lemma 4.8. *For $m_1 + n_1 \neq 0, m_2 + n_2 = 0$, we have*

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = m_2\tau^*(x^{m_1+n_1,1}, E(-e_2))$$

Lemma 4.9. *For $m_1 + n_1 = 0, m_2 + n_2 = 0$, we have*

$$\tau^*(x^{\vec{m}}, E(\vec{n})) = m_1\tau^*(x^{1,0}, E(-e_1)) + m_2\tau^*(x^{0,1}, E(-e_2))$$

Proof. Let $\vec{m}' = \vec{m}, \vec{n}' = \vec{n}, \vec{r}' = -\vec{m} - \vec{n}$, (4.2) gives

$$(m_2n_1 - m_1n_2)(\tau^*(x^{\vec{m}}, E(-\vec{m})) + \tau^*(E(\vec{n}), x^{-\vec{n}}) - \tau^*(x^{\vec{m}+\vec{n}}, E(-\vec{m} - \vec{n}))) = 0 \tag{4.15}$$

Let $\vec{n} = -m_1e_1 \neq 0$, (4.15) gives

$$\tau^*(x^{\vec{m}}, E(-\vec{m})) = \tau^*(x^{0,m_2}, E(-m_2e_2)) + \tau^*(x^{m_1,0}, E(-m_1e_1)) \tag{4.16}$$

Let $\vec{m} = m_1e_1, \vec{n} = m_2e_2 \neq 0$, (4.15) gives

$$\tau^*(x^{\vec{m}}, E(-\vec{m})) = \tau^*(x^{m_1,0}, E(-m_1e_1)) + \tau^*(E(m_2e_2), x^{0,-m_2}) \tag{4.17}$$

Now we have to determine $\tau^*(x^{m_1,0}, E(-m_1e_1))$ and $\tau^*(E(m_2e_2), x^{0,-m_2})$. Solve (4.16)(4.17) and take $m_2 = 1$, we have

$$\tau^*(x^{0,1}, E(-e_2)) = -\tau^*(x^{0,-1}, E(e_2))$$

Similarly

$$\tau^*(x^{1,0}, E(-e_1)) = -\tau^*(x^{-1,0}, E(e_1)) \tag{4.18}$$

Let $m_1 = -1, m_2 = 1$, (4.17) gives

$$\tau^*(x^{-1,1}, E(e_1 - e_2)) = \tau^*(x^{-1,0}, E(e_1)) + \tau^*(E(e_2), x^{0,-1}) \tag{4.19}$$

Let $\vec{m} = m_1e_1, \vec{n} = -e_2$, (4.15) gives

$$\tau^*(x^{m_1,0}, E(-m_1e_1)) - \tau^*(x^{0,1}, E(-e_2)) = \tau^*(x^{m_1,-1}, E(-m_1e_1 + e_2)) \tag{4.20}$$

Let $\vec{m} = (m_1 - 1)e_1, \vec{n} = e_1 - e_2$, then

$$\tau^*(x^{m_1-1,0}, E(-m_1 + 1)e_1)) - \tau^*(x^{-1,1}, E(e_1 - e_2)) = \tau^*(x^{m_1,-1}, E(-m_1e_1 + e_2)) \tag{4.21}$$

equations (4.18)-(4.21) give the following formula

$$\tau^*(x^{m_1,0}, E(-m_1e_1)) - \tau^*(x^{m_1-1,0}, E(-m_1 + 1)e_1)) = \tau^*(x^{1,0}, E(-e_1))$$

By introduction on m_1, we have

$$\tau^*(x^{m_1,0}, E(-m_1e_1)) = m_1\tau^*(x^{1,0}, E(-e_1)) \tag{4.22}$$

By symmetry,

$$\tau^*(x^{0,m_2}, E(-m_2e_2)) = m_2\tau^*(x^{0,1}, E(-e_2)) \tag{4.23}$$

Finally, substitute (4.22)(4.23) into (4.16), we obtain the result of this lemma. ∎

Proof of Proposition 4.2: The result of Proposition 4.2 follows from Lemma 4.6 through 4.9. ∎

Proof of Proposition 4.3: Take $a = x^{\vec{m}'}, b = x^{\vec{n}'}, c = E(\vec{r}')$ in (4.1), then

$$g(\vec{n}', \vec{r}')\tau^*(x^{\vec{m}'}, x^{\vec{n}'+\vec{r}'}) + g(\vec{r}', \vec{m}')\tau^*(x^{\vec{n}'}, x^{\vec{m}'+\vec{r}'}) = 0 \tag{4.24}$$

Now we divide the proof into three cases.

Case 1. For $g(\vec{m}, \vec{n}) \neq 0$, take $\vec{m}' = \vec{r}' = \vec{m}, \vec{n}' = \vec{n} - \vec{m}$ in (4.24). By a simple calculation, we have $\tau^*(x^{\vec{m}}, x^{\vec{n}}) = 0$.

Case 2. For $g(\vec{m},\vec{n})=0$ and $\vec{m}=-\vec{n}$, let $\vec{r}'=-\vec{n}'-\vec{m},\vec{n}'=\vec{n}',\vec{m}'=\vec{m}$, (4.24) gives

$$g(\vec{m},\vec{n}')(\tau^*(x^{\vec{m}},x^{-\vec{m}})+\tau^*(x^{\vec{n}},x^{-\vec{n}}))=0 \tag{4.25}$$

For $\vec{m}=e_1+e_2$ and $\vec{n}'=e_1,e_2$ respectively, (4.25) gives

$$\tau^*(x^{1,0},x^{-1,0})=-\tau^*(x^{1,1},x^{-1,-1})=\tau^*(x^{0,1},x^{0,-1}) \tag{4.26}$$

Moreover for $\vec{m}=e_1,\ \vec{n}'=e_2$, (4.25) gives

$$\tau^*(x^{1,0},x^{-1,0})=-\tau^*(x^{0,1},x^{0,-1}) \tag{4.27}$$

Hence (4.26)(4.27) imply $\tau^*(x^{1,0},x^{-1,0})=\tau^*(x^{0,1},x^{0,-1})=0$.

Therefore for $\vec{m}\in\Gamma$, from (4.25) we have $\tau^*(x^{\vec{m}},x^{-\vec{m}})=-\tau^*(x^{1,0},x^{-1,0})$ or $\tau^*(x^{\vec{m}},x^{-\vec{m}})=-\tau^*(x^{0,1},x^{0,-1})$. This implies $\tau^*(x^{\vec{m}},x^{-\vec{m}})=0$ for any $\vec{m}\in\Gamma$.
Case 3. For $g(\vec{m},\vec{n})=0$ and $\vec{m}\neq-\vec{n}$, we set $\vec{m}'=\vec{m},\vec{r}'=\vec{r},\vec{n}'=\vec{n}-\vec{r}$ so that $g(\vec{m},\vec{r})\neq 0, g(\vec{n},\vec{r})\neq 0$, then (4.24) gives

$$g(\vec{n},\vec{r})\tau^*(x^{\vec{m}},x^{\vec{n}})+g(\vec{r},\vec{m})\tau^*(x^{\vec{n}-\vec{r}},x^{\vec{m}+\vec{r}})=0$$

Since $g(\vec{n}-\vec{r},\vec{m}+\vec{r})=g(\vec{n}+\vec{m},\vec{r})\neq 0$ from the assumption on $\vec{m}$ and $\vec{n}$, case 1 shows that $\tau^*(x^{\vec{n}-\vec{r}},x^{\vec{m}+\vec{r}})=0$. Hence $\tau^*(x^{\vec{m}},x^{\vec{n}})=0$. This finishes the proof of Proposition 4.3. ■

Denote the mapping $\mu: L\to K$ as follows:

$$\mu(x^{\vec{m}})=\begin{cases} -\frac{1}{m_1m_2}\tau^*(x^{m_1,0},E(m_2e_2)), & m_1\neq 0, m_2\neq 0\\ -\frac{1}{m_2}\tau^*(x^{1,m_2},E(-e_1)), & m_1=0, m_2\neq 0\\ \frac{1}{m_1}\tau^*(x^{m_1,1},E(-e_2)), & m_1\neq 0, m_2=0\end{cases}$$

$$\mu(E(\vec{m}))=\begin{cases} -\frac{1}{m_1m_2}\tau^*(E(m_1e_2),E(m_2e_2)), & m_1\neq 0, m_2\neq 0\\ -\frac{1}{m_2}\tau^*(E(e_1+m_2e_2),E(-e_1)), & m_1=0, m_2\neq 0\\ \frac{1}{m_1}\tau^*(E(m_1e_1+e_2),E(-e_2)), & m_1\neq 0, m_2=0\end{cases}$$

Define Lie bracket $[a,b]'$ on the vector space $\overline{L}=L\oplus\langle K_1,K_2,K_3,K_4\rangle$:

$$\begin{aligned}&[K_i,\overline{L}]'=0, [x^{\vec{m}},x^{\vec{n}}]'=0\\ &[x^{\vec{m}},E(\vec{n})]'=(m_2n_1-m_1n_2)x^{\vec{m}+\vec{n}}+\delta_{\vec{m}+\vec{n},0}h(\vec{m})\\ &[E(\vec{m}),E(\vec{n})]'=(m_2n_1-m_1n_2)E(\vec{m}+\vec{n})+\delta_{\vec{m}+\vec{n},0}f(\vec{m})\end{aligned}$$

where i=1,2,3,4. $h(\vec{m})=m_1K_1+m_2K_2, f(\vec{m})=m_1K_3+m_2K_4$. We can easily check that $\overline{L}$ is a perfect lie algebra. Let $\pi:\overline{L}\to L$ be the projective homomorphism, then $(\overline{L},\pi)$ is a central extension of L, and $ker\pi=\langle K_1,K_2,K_3,K_4\rangle$. Indeed we have

Theorem 4.10. $(\overline{L},\pi)$ *is the universal central extension of the Lie algebra* L

Proof. Let $0\to K\to\hat{L}\xrightarrow{\lambda}L\to 0$ be any central extension of L, τ,τ^* are defined as before. We set $\overline{\mu}:\overline{L}\to\hat{L}$, such that

$$\begin{aligned}&\overline{\mu}(K_1)=\tau^*(x^{1,0},E(-e_1));\overline{\mu}(K_2)=\tau^*(x^{0,1},E(e_2));\\ &\overline{\mu}(K_3)=\tau^*(E(e_1),E(-e_1));\overline{\mu}(K_4)=\tau^*(E(e_2),E(-e_2));\\ &\overline{\mu}(a)=\tau(a)+\mu(a),\forall a\in L\end{aligned}$$

It's easy to check that $\overline{\mu}$ is a Lie homomorphism, and $\lambda\overline{\mu} = \pi$. Since $\overline{L}$ is perfect, for any $a \in \overline{L}$, there exist $b, c \in \overline{L}$, such that $[b, c]' = a$. If $\lambda\mu' = \pi = \lambda\overline{\mu}$, where μ' is another homomorphism from $\overline{L}$ to $\hat{L}$, we have $\overline{\mu}(b) - \mu'(b) \in K$ and $\overline{\mu}(c) - \mu'(c) \in K$. So $[\overline{\mu}(b), \overline{\mu}(c)] = [\mu'(b), \mu'(c)]$, $\overline{\mu}(a) = \mu'(a)$. This proves that $\overline{\mu}$ is unique, and also completes the proof the theorem. ■

Corollary 4.11. *The second cohomology $H^2(L, \mathbb{C})$ of the Lie algebra L is 4-dimensional.*

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References

[1] Arbarello, E., C. D. Concini, V. G. Kac, and C. Procesi, *Moduli Spaces of Curves and Representation Theory*, Comm. Math. Phys. **117** (1988), 1–36.

[2] Billig, Y., *Representations of the twisted Heisenberg-Virasoro algebra at level zero*, Canadian Mathematical Bulletin **46** (2003), 529–537.

[3] Berman, S., Y. Gao, and Y. S. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, Journal of Functional Analysis **135** (1996), 339–389.

[4] Chen, Y., M. Xue, W. Lin, and S. Tan, *Structure and automorphism group of a class of derivation Lie algebras over quantum torus*, Chinese Annals of Math. (2005), to appear.

[5] Farnsteiner, R., *Derivations and central extensions of finitely generated graded Lie algebras*, Journal of Algebra **118** (1998), 33–45.

[6] Jiang, C., and D. Meng, *The derivation algebra of the associative algebra* $\mathbb{C}_q[X, Y, X^{-1}, Y^{-1}]$, Commun. in Algebra **26** (1998), 1723–1736.

[7] —, *The automorphism group of the derivation algebra of the virasoro-like algebra*, Advances in Math. (Chinse) **27** (1998), 175–183.

[8] Kassel, C., *Kähler Differential and Covering of Complex Simple Lie Algebras Extended Over a Commutative Algebra*, J. Pure Appl. Algebra **34** (1998), 265–275.

[9] Kirkman, E., C. Procesi, L. A. Small, *q-analog for the Virasoro algebra*, Commun. in Algebra **22** (1994), 3755–3774.

[10] Kac, V., and A. K. Raina, "Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras," World Scientific, Singapore, 1987.

[11] Lin, W., and S. Tan, *Representations of the Lie algebra of derivations for quantum torus*, Journal of Algebra **275** (2004), 250–274.

[12] —, *Central extension and derivations of a class of Lie algebras related to the Quantum Torus*, submitted.

[13] Santharoubane, L. J., *The Second Cohomology Group for Kac-Moody Lie Algebras and Kähler Differentials*, Journal of Algebra **125** (1989), 12–26.

[14] Wilson, R., *Euclidean Lie Algebras are Universal Extensions*, in: Lie Algebras and Related Topics, Lecture Notes in Math. **933** (1982), 210–213.

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