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Reduction Theorems for a Certain Generalization of Contact Metric Manifolds*

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Abstract. We consider a Riemannian manifold with a compatible f-structure which admits a parallelizable kernel. With some additional integrability conditions it is called an (almost) $\mathcal{S}$-manifold and it is a natural generalization of a contact metric and a Sasakian manifold. Then we consider an action of a Lie group preserving the given structures. In such a context we define a momentum map and prove some reduction theorems.
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Introduction

A contact metric manifold may be seen as a Riemannian manifold of dimension $2n+1$ equipped with a compatible f-structure of the rank $2n$ and such that certain integrability conditions are satisfied, cf. [4]. Such manifolds have been intensively studied from the topological and geometrical points of view. A vast set of examples of contact metric manifolds is available too; for a collection of the recent results one may consult an excellent book by D.E. Blair, cf. [5]. In the present paper we consider a generalization of the contact metric manifolds. We consider Riemannian manifolds of dimension $2n+s$ equipped with an f-structure φ of rank $2n$ which is compatible with the metric and such that certain integrability conditions are satisfied; moreover, we assume the kernel bundle of φ is parallelizable. We consider here the so called (almost) $\mathcal{S}$-manifolds which were defined by D.E. Blair, cf. [3]. These structures carries many similarities with the metric contact and (almost) Sasakian manifolds; $\mathcal{S}$-manifolds were studied by various geometers, cf. [21, 11, 18, 15, 13]. An (almost) $\mathcal{S}$-manifold determine naturally the fundamental 2-form, called also the Sasaki form, cf. (2). This form is closed but degenerate.

The aim of this paper is to prove a contact reduction theorem. Such reduction were already profoundly studied in the case of contact, Sasakian and 3-

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Sasakian manifolds. Regular contact reduction of the exact contact manifolds was studied by various authors, cf. [2, 19, 25]; then the generalizations were obtained in [24, 29, 23, 16]. The Sasakian reduction was obtained in [20, 17] and 3-Sasakian reduction was studied in [8, 9, 10].

In Section 2 of our paper we consider an (almost) $\mathcal{S}$-manifold on which acts a Lie group G which preserves the given structures. In this context we define associated momentum and comomentum maps and obtain their characterization and properties. In Section 3 we obtain reduction theorems for (almost) $\mathcal{S}$-manifolds. Our reduction is regular and taken at the zero value of the momentum map. We also prove a version of a symplectic reduction theorem for a symplectic almost Hermitian manifold obtained via the cone construction; we also compare it with the reduction for $\mathcal{S}$-manifold.

All manifolds, maps, distributions considered here are smooth i.e. of the class C^∞; we denote by $\Gamma(-)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

1. Preliminaries

1.1. Actions of Lie groups on manifolds.

In the present subsection we recall basic definitions and properties considering an action of a Lie group on a manifold. These are standard facts and the details may be found in various old and new textbooks, for instance cf. [1, 12, 27].

Let M be an n-dimensional manifold and G a Lie group acting on the left on M by $\psi : G \times M \to M$. We denote by $\mathfrak{g}$ the Lie algebra of G. If $A \in \mathfrak{g}$ then by $\tilde{A}$ we denote the vector field on M determined by A via the action ψ, i.e. if $x \in M$ then $\tilde{A}_x := d_e\psi_x(A)$ where $\psi_x : G \to M$ is such that $\psi_x(a) = \psi(a, x)$ for each $a \in G$; e denotes here the neutral element of G. In such a way there is defined the map $d\psi : \mathfrak{g} \to \Gamma(TM)$ such that $d\psi(A) = \tilde{A}$. The map $d\psi$ is an anti-homomorphism of Lie algebras, i.e. for each $A, B \in \mathfrak{g}$ we have that $[\tilde{A}, \tilde{B}] = -\widetilde{[A, B]}$.

The group G acts by the adjoint representation $Ad : G \to \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}$; for each $a \in G$ and $A \in \mathfrak{g}$ we use the notation $a \cdot A := Ad_a(A)$. Then there is the coadjoint action of $Ad^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ on the real dual space to $\mathfrak{g}$; for each $a \in G$ and $\phi \in \mathfrak{g}^*$ we put $a \cdot \phi := Ad^*_{a^{-1}}(\phi) = \phi \circ Ad_{a^{-1}}$.

There is given a natural pairing $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ such that for each $\phi \in \mathfrak{g}^*$ and $A \in \mathfrak{g}$ we have $\langle \phi, A\rangle := \phi(A)$. We have the following useful property: if $f : M \to \mathfrak{g}^*$ and $A \in \mathfrak{g}$ then $d\langle f, A\rangle = \langle df, A\rangle$.

Suppose that F is a G-invariant 2-form on M. If $\tilde{A} \lrcorner dF = 0$ for all $A \in \mathfrak{g}$ then $\widetilde{[A, B]} \lrcorner F = [\widetilde{B}, \widetilde{A}] \lrcorner F = d(\widetilde{B} \lrcorner (\widetilde{A} \lrcorner F))$ for each $A, B \in \mathfrak{g}$.

1.2. Metric f-manifolds and associated structures. Let M be a $(2n+s)$-dimensional manifold equipped with an f-structure φ, i.e. φ is an endomorphism of TM such that $\varphi^3 + \varphi = 0$. This is a natural generalization of an almost complex structure, cf. [30]. Then $(M, \varphi, \xi_i, \eta^j)$ $(i, j = 1, \ldots, s)$ is said to be an *f-manifold with a parallelizable kernel* (for brevity: *$f.pk$-manifold*) if $\xi_1, \ldots, \xi_s$ are vector fields and $\eta^1, \ldots, \eta^s$ are 1-forms on M such that the following conditions hold

$$\varphi(\xi_i) = 0, \ \eta^i \circ \varphi = 0, \ \varphi^2 = -\mathrm{id} + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \ \eta^i(\xi_j) = \delta^i_j \tag{1}$$

for all $i, j = 1, \ldots, s$. It follows that $\ker(\varphi) = \mathrm{span}\{\xi_1, \ldots, \xi_s\}$. We put $\mathcal{D} := \mathrm{Im}(\varphi)$ and we observe that $TM = \mathcal{D} \oplus \ker(\varphi)$.

Let g be a Riemannian metric on M; then g and φ are said to be *compatible* if for each $X, Y \in TM$ holds $g(\varphi(X), Y) + g(X, \varphi(Y)) = 0$. If g and φ are compatible then it is possible to define also the *Sasaki 2-form* by posing:

$$F(X, Y) := g(X, \varphi(Y)). \tag{2}$$

The triples (M, g, φ) and (M, g, F) determine each other via equation (2). Moreover, the decomposition $TM = \mathcal{D} \oplus \ker(\varphi)$ is orthogonal.

Remark 1.1. If there is given an action of a Lie group G on M such that G preserves g and φ then G preserves also F and the decomposition $\mathcal{D} \oplus \ker(\varphi)$.

The structure $(M, g, \varphi, \xi_i, \eta^j)$, $(i, j = 1, \ldots, s)$, is said to be a *metric f-manifold with a parallelizable kernel* (or for brevity called: *metric $f.pk$-manifold*), if $(M, \varphi, \xi_i, \eta^j)$ is an $f.pk$-manifold and

$$g(X, Y) = g(\varphi(X), \varphi(Y)) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{3}$$

for each $X, Y \in TM$, cf. [18]. A metric g satisfying condition (3) is said to be *adapted* for the $f.pk$-manifold. Such a metric always exists but is not unique. It is easy to observe that an adapted metric is also compatible with φ. Hence there is also given the Sasaki 2-form F.

Summing up, we have the following structure on the manifold M: an f-structure φ, vector fields $\xi_1, \ldots, \xi_s$, 1-forms $\eta^1, \ldots, \eta^s$, an adapted Riemannian metric g and the Sasaki 2-form F. These give $\mathcal{Z} := (M, g, \varphi, \xi_i, \eta^j)$ a metric $f.pk$-manifold.

With the f-structure φ there is naturally associated $\mathcal{N}_\varphi$ the tensor of type $(2,1)$ defined in the following way: $\mathcal{N}_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i$ where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ, cf. [22].

Definition 1.2. It is said that

- $\mathcal{Z}$ is *normal* if $\mathcal{N}_\varphi = 0$
- $\mathcal{Z}$ is an *almost $\mathcal{S}$-manifold* if $d\eta^i = F$ for $i = 1, \ldots, s$
- $\mathcal{Z}$ is an *$\mathcal{S}$-manifold* if $d\eta^i = F$ for $i = 1, \ldots, s$ and $\mathcal{Z}$ is normal.

The above definitions are natural generalizations of the notions of metric contact and Sasakian manifolds, cf. [3].

The $f.pk$-manifolds may be seen from the point of view of the CR-geometry. In fact, given an $f.pk$-manifold $(M, \varphi, \xi_i, \eta^j)$, $(i, j = 1, \ldots, s)$ we may define an almost CR-structure by considering $(M, \mathcal{D}, \varphi|_{\mathcal{D}\varphi})$. Vice versa, given an almost CR-structure (M, H, J) and a parallelization of an H-transversal subbundle of TM we may easily obtain an $f.pk$-manifold. The conditions on $\mathcal{Z}$ for being (almost) $\mathcal{S}$-manifold may be expressed in the language of the CR-geometry. However, we shall not use this language here.

Definition 1.2 may be clarified by considering an s-cone over M. In particular if $\mathcal{Z} = (M, g, \varphi, \xi_i, \eta^j)$, $(i, j = 1, \ldots, s)$, is a metric $fp.k$-manifold then we call an s-cone over M the set $C_s(M) := M \times \mathbb{R}^s_+$ where $\mathbb{R}_+$ denotes the positive real numbers. It is clear that there are the natural global coordinates $r = (r_1, \ldots, r_s)$ on $\mathbb{R}^s_+$. There are also the following injection an projection maps: $u : M \hookrightarrow C_s(M)$ and $p : C_s(M) \to M$ such that $u(x) := (x, 1, \ldots, 1)$ and $p(x, r) := x$. The manifold $C_s(M)$ carries a natural almost Hermitian structure J, h defined as follows: for each $X \in \mathcal{D}$ and each $i \in \{1, \ldots, s\}$

$$J(X) := \varphi(X), \quad J(\xi_i) := r_i \frac{\partial}{\partial r_i}, \quad J(r_i \frac{\partial}{\partial r_i}) := -\xi_i \tag{4}$$

$$h := \sum_{i=1}^{s} (dr_i)^2 + \|r\|^2 p^* g + \sum_{i=1}^{s} (r_i^2 - \|r\|^2) \eta^i \otimes \eta^i. \tag{5}$$

It is straightforward to prove the following proposition, cf. [14].

Proposition 1.3. *If $\mathcal{Z}$ is an almost $\mathcal{S}$-manifold then $(C_s(M), h, J)$ is a symplectic almost Hermitian with the Kähler form*

$$\omega = d\left(\sum_{i=1}^{s} r_i^2 \eta^i\right) = \left(\sum_{i=1}^{s} 2 r_i dr_i \wedge \eta^i\right) + \|r\|^2 F.$$

Moreover, $\mathcal{Z}$ is an $\mathcal{S}$-manifold if and only if $C_s(M)$ is Kähler.

2. A generalization of the momentum maps for almost $\mathcal{S}$ and Kähler manifolds

Throughout all of this section we assume that M is a $(2n+s)$-dimensional manifold and $\mathcal{Z} = (M, g, \varphi, \xi_i, \eta^j)$, $(i, j = 1, \ldots, s)$, is an almost $\mathcal{S}$-manifold with F the associated Sasaki 2-form. By $(C_s(M), h, J, \omega)$ we denote the symplectic almost Hermitian manifold associated with $\mathcal{Z}$. Moreover, we suppose that there is given an action $\psi : G \times M \to M$ of the Lie group G on M; for simplicity we write $a \cdot x := \psi(a, x)$ where $a \in G$ and $x \in M$. We suppose that the action ψ preserves $g, \varphi, \xi_i, \eta^j$ for $i, j = 1, \ldots, s$.

Let $\nu : M \to \mathfrak{g}^* \otimes \mathbb{R}^s$ be the following map:

$$\langle \nu(x), A \rangle := \left(\eta^1(\tilde{A}_x), \ldots, \eta^s(\tilde{A}_x)\right).$$

Definition 2.1. We call the map ν the *momentum map associated with $\mathcal{Z}$*.

Then the map ν satisfies the following properties.

Proposition 2.2. (1). *ν is G-equivariant,* (2). *for each $A \in \mathfrak{g}$ we have that $\langle d\nu, A \rangle = -\tilde{A} \lrcorner (d\eta^1, \ldots, d\eta^s) = -\tilde{A} \lrcorner (F, \ldots, F)$.*

Proof. For each $x \in M$, $a \in G$ and $A \in \mathfrak{g}$ the following holds

$$\begin{aligned}\langle \nu(a\cdot x), A\rangle &= (\eta^1(\tilde{A}_{a\cdot x}), \ldots, \eta^s(\tilde{A}_{a\cdot x}))\\ &= ((L_a^*\eta^1)(\widetilde{Ad_{a^{-1}}(A)}_x), \ldots, (L_a^*\eta^s)(\widetilde{Ad_{a^{-1}}(A)}_x))\\ &= (\eta^1(\widetilde{Ad_{a^{-1}}(A)}_x), \ldots, (\eta^s(\widetilde{Ad_{a^{-1}}(A)}_x))\\ &= \langle \nu(x), Ad_{a^{-1}}(A)\rangle = \langle a\cdot\nu(x), A\rangle;\end{aligned}$$

L_a denotes here the multiplication by a on M. Hence it follows that ν is G-equivariant. On the other hand for each $A \in \mathfrak{g}$

$$\begin{aligned}\langle d\nu, A\rangle &= d\langle \nu, A\rangle = d(\eta^1(\tilde{A}), \ldots, \eta^s(\tilde{A}))\\ &= (d(\tilde{A}\lrcorner\eta^1), \ldots, d(\tilde{A}\lrcorner\eta^s))\\ &= (L_{\tilde{A}}\eta^1 - \tilde{A}\lrcorner d\eta^1, \ldots, L_{\tilde{A}}\eta^s - \tilde{A}\lrcorner d\eta^s)\\ &= -\tilde{A}\lrcorner(d\eta^1, \ldots, d\eta^s) = -\tilde{A}(F, \ldots, F).\end{aligned}$$

Hence *symplectic condition* (2) follows. ■

Let $\mathfrak{h}(\mathcal{Z}) := \{X \in \Gamma(TM) |\ X\lrcorner F \text{ is exact}\}$ and we call it *Hamiltonian vector fields associated with* $\mathcal{Z}$. Then it is easy to observe that $\mathfrak{h}(\mathcal{Z})$ is a vector subspace of $\Gamma(TM)$. Moreover, for each $X, Y \in \mathfrak{h}(\mathcal{Z})$ we have $[X,Y]\lrcorner F = L_X(Y\lrcorner F) - Y\lrcorner(L_X F) = d(X\lrcorner(Y\lrcorner F)) - Y\lrcorner(X\lrcorner dF) = 2d(F(Y,X))$. Hence $[X,Y] \in \mathfrak{h}(\mathcal{Z})$ and then it is a Lie subalgebra of $\Gamma(TM)$.

We put $\mathfrak{sp}(\mathcal{Z}) := \{X \in \Gamma(TM) | X\lrcorner F \text{ is closed}\}$. This is a vector subspace of $\Gamma(TM)$ and we call it the space of *symplectic vector fields associated with* $\mathcal{Z}$. We observe that the property of $X\lrcorner F$ being closed is equivalent to $L_X F = 0$. Moreover, for each $X, Y \in \mathfrak{sp}(\mathcal{Z})$ we have that $L_{[X,Y]}F = L_X(L_Y F) - L_Y(L_X F) = 0$; therefore $\mathfrak{sp}(\mathcal{Z})$ is a Lie subalgebra of $\Gamma(TM)$.

Clearly $\mathfrak{h}(\mathcal{Z})$ is a Lie subalgebra of $\mathfrak{sp}(\mathcal{Z})$. Let $A \in \mathfrak{g}$ then for each $i \in \{1, \ldots, s\}$ we have $\tilde{A}\lrcorner F = \tilde{A}\lrcorner d\eta^i = L_{\tilde{A}}\eta^i - d(\tilde{A}\lrcorner\eta^i) = d(-\eta^i(\tilde{A}))$. This implies that $d\psi$ sends $\mathfrak{g}$ into $\mathfrak{h}(\mathcal{Z})$. Summing up, we get the following inclusions of the Lie algebras:

$$\mathrm{Im}(d\psi) \subset \mathfrak{h}(\mathcal{Z}) \subset \mathfrak{sp}(\mathcal{Z}) \subset \Gamma(TM). \tag{6}$$

Moreover, we have the following exact sequence

$$0 \longrightarrow \mathfrak{h}(\mathcal{Z}) \longrightarrow \mathfrak{sp}(\mathcal{Z}) \xrightarrow{X\mapsto[X\lrcorner F]} H^1_{\mathrm{dR}}(M) \longrightarrow 0.$$

In the case of an action ψ on the almost $\mathcal{S}$-manifold it is possible to define the *comomentum map* $\nu^* : \mathfrak{g} \to C^\infty(M, \mathbb{R}^s)$ as $\nu^*(A) := -(\tilde{A}\lrcorner\eta^1, \ldots, \tilde{A}\lrcorner\eta^s)$. The map ν^* is not a homomorphism since we have not any reasonable Poisson structure on $C^\infty(M, \mathbb{R}^s)$. However the following diagram commutes

$$\begin{array}{ccccccc} 0 \longrightarrow \mathbb{R}^s & \xrightarrow[\text{immersion}]{\text{standard}} & C^\infty(M,\mathbb{R}^s) & \xrightarrow{d} & \Gamma(T^*M\otimes\mathbb{R}^s) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \nu^*} & & \uparrow{\scriptstyle \flat\otimes\mathrm{id}} & & \\ & & \mathfrak{g} & \xrightarrow{d\psi} & \Gamma(TM\otimes\mathbb{R}^s) & & \end{array} \tag{7}$$

where $\flat(X) := (X\lrcorner F)$ for $X \in \Gamma(TM)$. The upper row of the diagram is not exact at $\Gamma(T^*M \otimes \mathbb{R}^s)$. The commutativity of diagram (7) is a description of the fact that ν^* preserves some geometric structures from $\mathfrak{g}$ to $\Gamma(T^*M,\mathbb{R}^s)$. The relationship between ν and ν^* is given by the equality: $\nu^*(A)(x) = -\langle\nu(x), A\rangle$ for each $x\in M$ and $A\in\mathfrak{g}$.

An almost $\mathcal{S}$-manifold M carries a foliation $\mathcal{F}$ defined by the distribution $\ker(\varphi) = \mathrm{span}\{\xi_1,\dots,\xi_s\}$. Then there are defined on M the Lie subalgebra of $\Gamma(TM)$ of the basic vector fields

$$\Gamma_B(TM) := \{X\in\Gamma(TM)|L_XY\in\Gamma(\ker(\varphi)) \text{ for all } Y\in\Gamma(\ker(\varphi))\},$$

cf. [26]. Then $\mathfrak{sp}(\mathcal{Z})$ is contained in $\Gamma_B(TM)$; if fact, for each $i\in\{1,\dots,s\}$ and each $X\in\mathfrak{sp}(\mathcal{Z})$ we have that $[X,\xi_i]\lrcorner F = L_X(\xi_i\lrcorner F) - \xi_i\lrcorner(L_XF) = -\xi_i(X\lrcorner dF) - \xi_i(d(X\lrcorner F)) = 0$. It follows that $[X,\xi_i]\in\Gamma(\ker(\varphi))$ since $[X,\xi_i]$ is an annihilator of F. ¿From inclusions (6) we get that $\mathrm{Im}(d\psi)$ and $\mathfrak{h}(\mathcal{Z})$ are also contained in $\Gamma_B(TM)$.

Then we put $\mathfrak{h}_{\mathcal{D}}(\mathcal{Z}) := \{[X]\in\Gamma(TM/\ker(\varphi))|X\in\mathfrak{h}(\mathcal{Z})\}$ which is a Lie algebra with the bracket inherited from $\mathfrak{h}(\mathcal{Z})$. The canonical isomorphism $D\to TM/\ker(\varphi)$ defines the inclusion of the vector spaces: $\mathfrak{h}_{\mathcal{D}}(\mathcal{Z})\hookrightarrow\Gamma(\mathcal{D})$.

We denote by $C^\infty_B(M) := \{f\in C^\infty(M)|df(\ker(\varphi)) = 0\}$; these are called basic functions on M, cf. [26]. If $f\in C^\infty_B(M)$ then there exist $X_f\in\mathfrak{h}(\mathcal{Z})$ such that $df = X_f\lrcorner F$; this follows from the fact that df annihilates $\ker(\varphi)$ and F is non degenerate on $\mathcal{D}$. Moreover, $X_f+\Gamma(\ker(\varphi))\subset\mathfrak{h}(\mathcal{Z})$. We have the following exact sequence of vector spaces

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty_B(M) \xrightarrow{f\longrightarrow[X_f]} \mathfrak{h}_{\mathcal{D}}(\mathcal{Z}) \longrightarrow 0.$$

On the set $C^\infty_B(M)$ there is defined a bracket $\{\ ,\ \}$ such that

$$\{f_1,f_2\} := -X_{f_2}\lrcorner(X_{f_1}\lrcorner F) \tag{8}$$

where $df_1 = X_{f_1}\lrcorner F$ and $df_2 = X_{f_2}\lrcorner F$. We observe that although the associated Hamiltonian vector fields X_{f_1}, X_{f_2} are not unique but still definition (8) is well posed.

Lemma 2.3. *$(C^\infty_B(M),\{\ ,\ \})$ is a Poisson algebra.*

Proof. These are standard procedures to prove the Jacobi identity and Leibniz rule for $\{\ ,\ \}$ and we omit them here. We only need to prove that $\{f_1,f_2\}\in C^\infty_B(M)$ for each elements $f_1,f_2\in C^\infty_B(M)$. Let X_{f_1}, X_{f_2} be the associated Hamiltonian vector fields. For each $i\in\{1,\dots,s\}$ we have

$$\begin{aligned}\xi_i\lrcorner d(X_{f_2}\lrcorner(X_{f_1}\lrcorner F)) = L_{\xi_i}(X_{f_2}\lrcorner(X_{f_1}\lrcorner F)) = (L_{\xi_i}X_{f_2})\lrcorner(X_{f_1}\lrcorner F)\\ +X_{f_2}\lrcorner((L_{\xi_i}X_{f_1})\lrcorner F) + X_{f_2}\lrcorner(X_{f_1}\lrcorner(\xi_i\lrcorner dF) + X_{f_2}\lrcorner(X_{f_1}\lrcorner(d(\xi_i\lrcorner F)) = 0;\end{aligned} \tag{9}$$

in particular we use here the facts that $L_{\xi_i}X_{f_1}$, $L_{\xi_i}X_{f_2}$ and ξ_i belong to $\Gamma(\ker(\varphi))$ which annihilates F. Then our assertion follows from (9). ∎

The comomentum map ν^* may be written as $\nu^* = (\nu_1^*,\dots,\nu_s^*)$ where $\nu_i^*:\mathfrak{g}\to C^\infty(M)$ is given by $\nu_i^*(A) = -\tilde{A}\lrcorner\eta^i$ for $i\in\{1,\dots,s\}$. It may be easily proved that ν_i^* has its values in $C^\infty_B(M)$; if fact,

$$\xi_i\lrcorner d(\nu_i^*(A)) = -\xi_i\lrcorner(L_{\tilde{A}}\eta^i) + \xi_i\lrcorner(\tilde{A}\lrcorner F) = 0$$

because F is G-invariant and ξ_i annihilates F. We observe that the diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\psi} & \mathfrak{h}(\mathcal{Z}) \\ {\scriptstyle \nu_i^*}\downarrow & & \downarrow {\scriptstyle X \mapsto [X]} \\ C_B^\infty(M) & \xrightarrow{f \mapsto [X_f]} & \mathfrak{h}_{\mathcal{D}}(\mathcal{Z}). \end{array}$$

Moreover, we have the following proposition

Proposition 2.4. *The map ν_i^* is a homomorphism of the Lie algebras for each $i \in \{1, \ldots, s\}$.*

Proof. For given $A, B \in \mathfrak{g}$ we have that the associated Hamiltonian vector fields are the following: $X_{\nu_i^*(A)} = \tilde{A}$, $X_{\nu_i^*(B)} = \tilde{B}$ and $X_{\nu_i^*([A,B])} = \widetilde{[A, B]}$. Then we have

$$\nu_i^*([A, B]) = -\widetilde{[A, B]} \lrcorner \eta^i = -\tilde{B} \lrcorner (\tilde{A} \lrcorner F)) = \{\nu_i^*(A), \nu_i^*(B)\}\,.$$

Hence our assertion follows. ∎

Then the action $\psi : G \times M \to M$ may be extended naturally to the action $\Psi : G \times C_s(M) \to C_s(M)$ such that $\Psi(a, (x, r)) := (\psi(a, x), r)$ where $a \in G$, $x \in M$ and $r = (r_1, \ldots, r_s) \in \mathbb{R}_+^s$. It is an easy observation that Ψ acts by holomorphic and isometric transformations on $C_s(M)$.

Remark 2.5. Let $\tilde{\Psi} : G \times C_s(M) \to C_s(M)$ be an action of G by the holomorphic and isometric transformations preserving $\xi_1, \ldots, \xi_s$. Then it is easy to observe that there exists an action $\tilde{\psi} : G \times M \to M$ preserving $g, \varphi, \xi_1, \ldots, \xi_s$ such that for each $a \in G$ and each $(x, r) \in C_s(M)$ holds $\tilde{\Psi}(a, (x, r)) = (\tilde{\psi}(a, x), r)$. This may be proved by observing that the invariance of $\xi_1, \ldots, \xi_s$ by $\tilde{\Psi}$ implies that

$$\tilde{\Psi}_* \left(r_i \frac{\partial}{\partial r_i} \right) = r_i \frac{\partial}{\partial r_i} \quad \text{for all} \quad i = 1, \ldots, s.$$

These leads to a system of PDEs; the solution of this system forces the action $\tilde{\Psi}$ to leave invariant the $\mathbb{R}_+^s$ component of $C_s(M)$.

Let $\mu : C_s(M) \to \mathfrak{g}^* \otimes \mathbb{R}^s$ be the map defined as follows: if $(x, r) \in C_s(M)$ and $A \in \mathfrak{g}$ then $\langle \mu(x, r), A \rangle := (r_1^2 \eta^1(\tilde{A}_x), \ldots, r_s^2 \eta^s(\tilde{A}_x))$.

Definition 2.6. We call μ the *momentum map for $C_s(M)$ associated with $\mathcal{Z}$*.

Then μ satisfies the following properties.

Proposition 2.7. (1). *μ is G-equivariant,* (2). *for each $A \in \mathfrak{g}$ we have that*

$$\langle d\mu, A \rangle = -\tilde{A} \lrcorner \left(d(r_1^2) \wedge \eta^1 + r_1^2 F, \ldots, d(r_s^2) \wedge \eta^s + r_s^2 F \right),$$

(3). *for each $A \in \mathfrak{g}$ we have $\langle \sum_{i=1}^s d\mu_i, A \rangle = -\tilde{A} \lrcorner \omega$ where $\mu = (\mu_1, \ldots, \mu_s)$,* (4). *$u^* \mu = \nu$.*

Proof. The action Ψ is determined by ψ and then property (1) may be proved in a similar way as that one of Proposition 2.2. On the other hand for each $A \in \mathfrak{g}$ we have

$$\begin{aligned}
\langle d\mu, A\rangle &= \Big(d(r_1^2\tilde{A}\lrcorner\eta^1), \ldots, d(r_s^2\tilde{A}\lrcorner\eta^s)\Big) \\
&= \Big(2(dr_1)\eta^1(\tilde{A}) - r_1^2\tilde{A}\lrcorner d\eta^1, \ldots, 2(dr_s)\eta^s(\tilde{A}) - r_s^2\tilde{A}\lrcorner d\eta^s\Big) \\
&= -\tilde{A}\lrcorner\left(d(r_1^2)\wedge\eta^1 + r_1^2F, \ldots, d(r_s^2)\wedge\eta^s + r_s^2F\right);
\end{aligned}$$

therefore (2) follows. Point (3) follows from (2) and from Proposition 1.3. Point (4) follows from the definitions of ν and μ. ■

We define $\mu^* : \mathfrak{g} \to C^\infty(C_s(M), \mathbb{R}^s)$ by $\mu^*(A) := -\Big(r_1^2\eta^1(\tilde{A}), \ldots, r_s^2\eta^s(\tilde{A})\Big)$ where $A \in \mathfrak{g}$; we call μ^* the *comomentum map of* $C_s(M)$ *associated with* $\mathcal{Z}$. Then we have the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^s \arrow[r, "\text{standard}", "\text{immersion}"'] & C^\infty(C_s(M), \mathbb{R}^s) \arrow[r, "d"] & \Gamma(T^*C_s(M)\otimes\mathbb{R}^s) \arrow[r] & 0 \\
 & & \mathfrak{g} \arrow[u, "\mu^*"] \arrow[r, "d\Psi"] & \Gamma(TC_s(M)\otimes\mathbb{R}^s) \arrow[u, "\flat\otimes\mathrm{id}"'] &
\end{tikzcd}$$

where $\flat(X) := X\lrcorner F$ for each $X \in \Gamma(TC_s(M))$. There is the following relationship between the momentum and comomentum maps: for each $A \in \mathfrak{g}$ and $(x, r) \in C_s(M)$ we have $\mu^*(A)(x, r) = -\langle\mu(x, r), A\rangle$.

3. Reduction theorems

Throughout all of this section we assume that $\mathcal{Z} = (M, g, \varphi, \xi_i, \eta^j)$, $(i, j = 1, \ldots, s)$, is an almost $\mathcal{S}$-manifold and by $(C_s(M), h, J)$ we denote the symplectic almost Hermitian manifold associated with $\mathcal{Z}$, cf. Proposition 1.3; we denote by ω the associated Kähler 2-form on $C_s(M)$. Moreover, we suppose that there is given an action $\psi : G \times M \to M$ of the Lie group G on M; we suppose that the action ψ preserves $g, \varphi, \xi_i, \eta^j$ for $i, j = 1, \ldots, s$. The orbits of the action ψ are denoted by $\mathcal{O}$. Let $\nu : M \to \mathfrak{g}^* \otimes \mathbb{R}^s$ be the momentum map associated with $\mathcal{Z}$ such that $\nu^{-1}(0) \neq \emptyset$. It is easy to observe that $\nu^{-1}(0)$ is G-invariant. We also put for brevity $u_1 : \nu^{-1}(0) \hookrightarrow M$ for the canonical immersion, $\bar{M} := G\backslash\nu^{-1}(0)$ and $\pi : \nu^{-1}(0) \to \bar{M}$ for the canonical projection; since we have the decomposition $TM = \mathcal{D} \oplus \ker(\varphi)$ then we denote by $p_{\mathcal{D}} : TM \to \mathcal{D}$ the projection on the first component of the decomposition. We suppose that the action ψ restricted to $\nu^{-1}(0)$ is free and proper, cf. [6]. Then it follows that the action Ψ on $\mu^{-1}(0)$ is also free and proper.

Theorem 3.1. *If for each $x \in \nu^{-1}(0)$ the map $p_{\mathcal{D}} : T_x\mathcal{O} \to \mathcal{D}_x$ is injective then there exists on $\bar{M}$ the natural structure of an almost $\mathcal{S}$-manifold $\bar{\mathcal{Z}}$. Moreover, $\mathcal{Z}$ is an $\mathcal{S}$-manifold if and only if $\bar{\mathcal{Z}}$ is so.*

Proof. Let $x \in \nu^{-1}(0)$ then

$$\begin{aligned}\ker(d_x\nu) &= \{X \in T_xM | \langle d\nu(X), A\rangle = 0 \text{ for all } A \in \mathfrak{g}\} \\ &= \{X \in T_xM | F(X, \tilde{A}_x) = 0 \text{ for all } A \in \mathfrak{g}\} \\ &= (T_x\mathcal{O})^{\perp_F}\end{aligned}$$

where $(T_x\mathcal{O})^{\perp_F}$ is the orthogonal space to $T_x\mathcal{O}$ with respect to F. We observe that $\dim(\ker(d_x\nu)) = 2n + s - \dim(p_{\mathcal{D}}(T_x\mathcal{O}))$. Since $p_{\mathcal{D}} : T_x\mathcal{O} \to \mathcal{D}$ is injective so $\dim(\ker(d_x\nu)) = 2n + s - d$. The property that the map above is injective is maintained in some open neighbourhood of $\nu^{-1}(0)$. Hence ν has a constant rank equal to d in some open neighbourhood of $\nu^{-1}(0)$. Therefore from the local expression of a map of constant rank, cf. page 41 of [28], it follows that $\nu^{-1}(0)$ is a regular closed submanifold of M of dimension $2n + s - d$.
Since $G \cdot \nu^{-1}(0) \subset \nu^{-1}(0)$ then it follows that for each $A \in \mathfrak{g}$ the vector field $\tilde{A}$ is tangent to $\nu^{-1}(0)$. Therefore, in restriction to $\nu^{-1}(0)$, we have that $F(\tilde{A}, \tilde{B}) = 0$ for each $A, B \in \mathfrak{g}$. Moreover, since for each $i \in \{1, \dots, s\}$ we have that $\xi_i \lrcorner F = 0$ then it follows that ξ_i are tangent to $\nu^{-1}(0)$ too.
Since G acts freely and properly on $\nu^{-1}(0)$ then $\pi : \nu^{-1}(0) \to \bar{M}$ is a left principal fibre bundle with structure group G, cf. [7]. We observe that η^i, F are G-invariant and $\tilde{A}_x \lrcorner \eta^i = \nu^i(x) = 0 = F(\tilde{A}_x, X)$ for each $x \in \nu^{-1}(0)$, $X \in T_x\nu^{-1}(0)$, $A \in \mathfrak{g}$ and $i \in \{1, \dots, s\}$. This means that u_1^*F and $u_1^*\eta^i$ are tensorial forms on the left principal G-bundle $\nu^{-1}(0)$, cf. [22]. Hence there exist the forms $\bar{\eta}^1, \dots, \bar{\eta}^s$ and $\bar{F}$ on $\bar{M}$ such that $\pi^*\bar{\eta}^i = u_1^*\eta^i$ and $\pi^*\bar{F} = u_1^*F$ for $i = 1, \dots, s$. The G-invariant vector fields ξ_i project to the vector fields $\bar{\xi}_i$ on $\bar{M}$ for $i \in \{1, \dots, s\}$. Moreover, $\delta_{ij} = \eta^i(\xi_j) = (\pi^*\bar{\eta}^i)(\xi_j) = \bar{\eta}^i(\bar{\xi}_j)$ and $\pi^*(d\bar{\eta}^i) = d(\pi^*\bar{\eta}^i) = u_1^*F = \pi^*\bar{F}$. Therefore, $d\bar{\eta}^i = \bar{F}$ since π is a submersion.
Let W be the orthogonal subbundle to $p_{\mathcal{D}}(T\mathcal{O})) \oplus \varphi(p_{\mathcal{D}}(T\mathcal{O}))$ in $\mathcal{D}$. We observe that $\varphi(T_x\mathcal{O}) = \varphi(p_{\mathcal{D}}(T_x\mathcal{O}))$ and $T_x\mathcal{O} = \mathrm{Im}(d_x\psi)$ for each $x \in \nu^{-1}(0)$. Then we have the following decomposition

$$\mathcal{D} = \varphi(T\mathcal{O}) \oplus p_{\mathcal{D}}(T\mathcal{O}) \oplus W. \tag{10}$$

Then it is a standard procedure to prove the following properties:
• the decomposition in (10) are G-invariant and orthogonal;
• $\varphi(T\mathcal{O}) \perp T\nu^{-1}(0)$ and the decomposition

$$TM = \varphi(T\mathcal{O}) \oplus \overbrace{T\mathcal{O} \oplus W \oplus \ker(\varphi)}^{=T\nu^{-1}(0)} \tag{11}$$

is G-invariant but usually not orthogonal; a non trivial fact is that the decomposition of $T\nu^{-1}(0)$ is a direct sum of the corresponding components. This follows from the assumption that $p_{\mathcal{D}} : T\mathcal{O} \to \mathcal{D}$ is a monomorphism of vector bundles;
• $\varphi(W) \subset W$ and $F|_{W\times W}$ is non degenerate;
• $d_x\pi : \ker(\varphi_x) \oplus W_x \to T_{\bar{x}}\bar{M}$ is an isomorphism of vector spaces for each $x \in \nu^{-1}(0)$ and $\bar{x} = \pi(x)$;
• $\bar{\eta}^1 \wedge \dots \wedge \bar{\eta}^s \wedge \bar{F}^{2n-2d} \neq 0$.
Then we put $\bar{\mathcal{D}} := d\pi(W)$. The definition of $\bar{\mathcal{D}}$ is well posed since W is G-invariant. Since for each $x \in \nu^{-1}(0)$, $\bar{x} = \pi(x) \in \bar{M}$ we have that $d_x\pi : W_x \oplus \ker(\varphi_x) \to T_{\bar{x}}\bar{M}$ is an isomorphism then we may define the *horizontal lifting*

operator in the following way: if $\bar{X}_{\bar{x}} \in T_{\bar{x}}\bar{M}$ then by $\bar{X}^H_x$ we denote the unique element of $W_x \oplus \ker(\varphi_x)$ such that $d_x\pi(\bar{X}^H_x) = \bar{X}_{\bar{x}}$. Then the lifting may be extended to vector fields: if $\bar{X} \in \Gamma(T\bar{M})$ then there is uniquely defined the horizontal vector field $\bar{X}^H \in \Gamma(T\nu^{-1}(0))$. It is easy to observe that $\bar{\xi}^H_i = \xi_i$ for all $i \in \{1, \ldots, s\}$. Then we define an endomorphism $\bar{\varphi}$ of $T\bar{M}$ such that $\bar{\varphi}(\bar{X}) := d\pi(\varphi(\bar{X}^H))$ for each $\bar{X} \in T\bar{M}$. It follows that $\bar{\varphi}^2 = -\mathrm{id} + \sum_{i=1}^s \bar{\eta}^i \otimes \bar{\xi}_i$. Moreover, we have

$$\pi^*(\bar{g}(\bar{X}, \bar{\varphi}(\bar{Y})) = g(\bar{X}^H, \varphi(\bar{Y}^H)) = F(\bar{X}^H, \bar{Y}^H) = \pi^*(\bar{F}(\bar{X}, \bar{Y}))$$

hence $\bar{g}(\bar{X}, \bar{\varphi}(\bar{Y})) = \bar{F}(\bar{X}, \bar{Y})$ and then $\bar{F}$ is determined by $\bar{g}$ and $\bar{\varphi}$. Wholetogether we get that $\bar{\mathcal{Z}} := (\bar{M}, \bar{g}, \bar{\varphi}, \bar{\xi}_i, \bar{\eta}^j)$ $(i, j = 1, \ldots, s)$ is an almost $\mathcal{S}$-manifold.
On the other hand we have that $[\bar{X}^H, \bar{Y}^H] = [\bar{X}, \bar{Y}]^H +$ terms belonging to $\Gamma(T\mathcal{O}|_{\nu^{-1}(0)})$. Then it follows that $[\varphi, \varphi](\bar{X}^H, \bar{Y}^H) = ([\bar{\varphi}, \bar{\varphi}](\bar{X}, \bar{Y}))^H +$ terms belonging to $\Gamma(T\mathcal{O} \oplus \varphi(T\mathcal{O})|_{\nu^{-1}(0)})$. This implies that

$$\mathcal{N}_\varphi(\bar{X}^H, \bar{Y}^H) = (\mathcal{N}_{\bar{\varphi}}(\bar{X}, \bar{Y}))^H + \text{terms belonging to } \Gamma\left(T\mathcal{O} \oplus \varphi(T\mathcal{O})|_{\nu^{-1}(0)}\right).$$

¿From the direct sum decomposition (11) it follows that the vanishing of $\mathcal{N}_\varphi$ is equivalent to the vanishing of $\mathcal{N}_{\bar{\varphi}}$. This implies that $\mathcal{Z}$ is an $\mathcal{S}$-manifold if and only if $\bar{\mathcal{Z}}$ is so. ■

The momentum map $\mu : C_s(M) \to \mathfrak{g}^* \otimes \mathbb{R}^s$ associated with $\mathcal{Z}$, cf. Definition 2.6, allows us to obtain the following reduction theorem.

Theorem 3.2. *If the map $p_{\mathcal{D}} : T\mathcal{O} \to \mathcal{D}$ is injective then $G\backslash\mu^{-1}(0)$ carries a natural structure of symplectic almost Hermitian manifold. Moreover, if $\mathcal{Z}$ is an $\mathcal{S}$-manifold then $G\backslash\mu^{-1}(0)$ is Kähler.*

Proof. We put, for brevity, $M' := G\backslash\mu^{-1}(0)$. Then we observe that

$$\begin{aligned} \mu^{-1}(0) &= \{(x, r) \in C_s(M) |\ (r_1^2\eta^1(\tilde{A}_x), \ldots, r_s^2\eta^s(\tilde{A}_x)) = 0\} \\ &= \{(x, r) \in C_s(M) |\ \eta^1(\tilde{A}_x) = \ldots \eta^s(\tilde{A}_x) = 0\} \\ &= \{(x, r) \in C_s(M) |\ \nu(x) = 0\} \\ &= \nu^{-1}(0) \times \mathbb{R}^s_+ = C_s(\nu^{-1}(0)). \end{aligned}$$

It follows that $\mu^{-1}(0)$ is a regular submanifold of $C_s(M)$ because $\nu^{-1}(0)$ is a regular submanifold M. The induced action of Ψ on $\mu^{-1}(0)$ is free and proper therefore the canonical projection $\Pi : \mu^{-1}(0) \to M'$ is a left principal fibre bundle with structure group G. The manifold $\mu^{-1}(0)$ carries the Riemannian metric inherited from $C_s(M)$, cf. (5). Then we have the orthogonal and G-invariant decomposition $T\mu^{-1}(0) = T\nu^{-1}(0) \oplus T\mathbb{R}^s$. ¿From (11) follows that $T\mu^{-1}(0) = T\mathcal{O} \oplus W \oplus \ker(\varphi) \oplus T\mathbb{R}^s$ which is G-invariant but usually not orthogonal. The manifold $C_s(M)$ carries also the G-invariant almost complex structure J given explicitly by (4). It is easy to observe that the bundle $W \oplus \ker(\varphi) \oplus T\mathbb{R}^s$ is G- and J-invariant. The following map

$$W \oplus \ker(\varphi) \oplus T\mathbb{R}^s \xrightarrow{d\Pi = d\pi \oplus \mathrm{id}} TM' = T\bar{M} \oplus T\mathbb{R}^s = TC_s(\bar{M}) \qquad (12)$$

is a G-invariant morphism of vector bundles which is an isomorphism when restricted to the fibres. We define an almost Hermitian structure (M', g', J') via the map (12). On the other hand M' may be equipped with the almost Hermitian symplectic structure being an s-cone over an $\mathcal{S}$-manifold $\bar{M}$, cf. Proposition 1.3. It is easy to observe that these two almost Hermitian structures on M' coincide. Then our assertion follows from Proposition 1.3 and Theorem 3.1. ■

The last part of the proof of the above theorem may be simply summed up by saying that the operations of taking an s-cone over and (almost) $\mathcal{S}$-manifold and the reduction commute with each other; it is a generalization of the result of Grantcharov and Ornea, cf. [20].

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