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Wielandt's Results for Algebraic k-Groups

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Abstract. We analyze the relation between subnormality and nilpotence, the subnormal joint property, some criteria of subnormality, the norm and the Wielandt subgroup in the case of algebraic groups defined over an arbitrary field.
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1. Introduction

The fundaments for the theory of subnormal subgroups has been laid down by Wielandt in [15]. His results have inspired many authors, see [7] for more details. The purpose of this note is to carry over the most important results of Wielandt for finite groups into algebraic k-groups. In particular we shall investigate relations between subnormality and nilpotence, the join theorem, some criteria of subnormality, the norm and the Wielandt subgroup. In the case of groups defined over an algebraically closed field some of these questions where analyzed in [18].

K will always denote an algebraically closed field and k will stay for an arbitrary subfield of K. For definition and general facts on algebraic k-groups we refer to [6], [9] and [12]. In the following proposition we collect some known facts about rationality in k-groups that will be needed in the paper.

Proposition 1.1. *Let G be an algebraic k-group.*

(i) *The subgroup generated by a family of connected k-subgroups of G is a connected k-subgroup* ([12], Corollary 2.2.7,(ii)).

(ii) *If A and B are k-subgroups and A is connected then $[A, B]$ is a connected k-subgroup* ([12], Corollary 2.2.8,(ii)).

Now, let G be a connected linear algebraic k-group.

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(iii) *G contains maximal tori defined over k* ([12], Theorem 13.3.6,(i) and Remark 13.3.7)*.*

(iv) *Let G_t be the subgroup of G generated by its maximal tori. Then G_t is the smallest closed, connected, normal subgroup of G whose factor group is unipotent. If k is infinite then G_t is defined over k, generated by the maximal k-tori of G and moreover $G_t(k)$ is dense in G_t* ([12], Proposition 13.3.11)*.*

(v) *G contains a maximal, connected, solvable (resp. unipotent) normal k-subgroup $R_k(G)$ (resp. $R_{u,k}(G)$) called the k-radical (resp. the unipotent k-radical) of G. If $R_k(G)$ (resp. $R_{u,k}(G)$) is reduced to the group formed by the unit, then G is called k-semi-simple (resp. k-reductive). If k is a perfect field, then $R_k(G) = R(G)$ the radical of G and $R_{u,k}(G) = R_u(G)$ the unipotent radical of G. G is k-reductive if and only if $R_u(G)$ does not contain any non-trivial connected k-subgroup* ([12], Proposition 14.4.5 and Lemma 14.4.6)*.*

Let G be an algebraic k-group. For a connected k-subgroup H of G we define the notion of k-subnormality in a natural way: We shall say that H is k-subnormal in G if there exists a non-negative integer m and a series

$$H = H_m \trianglelefteq H_{m-1} \trianglelefteq \cdots \trianglelefteq H_0 = G \tag{1}$$

of connected (except for $H_0 = G$ which may be non-connected) k-subgroups where every subgroup is normal in its predecessor. Then a connected k-subnormal subgroup is subnormal in G also as abstract subgroup but the converse holds too. In fact if H is a connected k-subgroup subnormal as an abstract subgroup in G (i.e. we only require that the terms of the series are subgroups), then we know (see [7], §1.1) that the most rapidly descending series of type (1), the so called normal closure series, can be inductively defined in the following way: We set $H_0 = G$, and we define $H_{i+1} = H^{H_i}$ for all finite $i \geq 0$ where for two abstract subgroups X and Y we define $X^Y = \langle x^y \,|\, x \in X, y \in Y\rangle$. We have

$$H \leq \cdots \trianglelefteq H_{i+1} \trianglelefteq H_i \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G$$

and $H = H_d$ for some integer d, called the defect of H in G. Since $H_i = H[G,_i H]$ (see [7], Proposition 1.1.1 (i)) where $[G,_i H]$ is inductively defined as $[G,_0 H] = G$ and $[G,_i H] = [[G,_{i-1} H], H]$, we have by Proposition 1.1,(i),(ii) that all subgroups H_i are connected k-subgroups for $i \geq 1$, therefore H is k-subnormal in G. Thus, for a connected k-subgroup H of G we will speak simply about its subnormality in an algebraic k-group G using the notation $H \, sn \, G$.

A connected algebraic k-group G is said to be nilpotent if its lower central series defined in the usual way inductively as $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ reaches the group formed by the unit. If H is a connected k-subgroup of G then the normalizer $N_G(H)$ is not in general defined over k, so we define the k-normalizer k-$N_G(H)$ as the subgroup generated by the connected k-subgroups of G which contain H as a normal subgroup. Clearly k-$N_G(H) \leq N_G(H)$ and from Proposition 1.1,(i) we have that k-$N_G(H)$ is a connected k-subgroup.

Finally, we notice that we may reduce ourselves to connected linear algebraic k-groups. In fact, a connected k-subgroup H is subnormal in G if and only if H is subnormal in the identity component G° since $H \leq G^\circ \trianglelefteq G$ (and recall that G° is an algebraic k-group by [9], Section 1). By [9], Corollary 5 to Theorem 16, for a connected algebraic k-group G we have that $G = LD$ where L is the maximal connected linear algebraic subgroup of G and D is the smallest normal algebraic subgroup of G such that G/D is a linear algebraic group. L is k-closed but it is not in general a k-subgroup (see [10], p. 49), while D is a connected central k-subgroup of G. Therefore if H is a connected k-subgroup of G we obtain that H is subnormal in G if and only if HD/D is subnormal in the linear algebraic k-group G/D. We also notice that G is nilpotent if and only if G/D is nilpotent. Using these facts, it will be clear that our results, even when they are stated only for connected linear algebraic k-groups, are valid for arbitrary algebraic k-groups except for Propositions 2.1 and 2.2 which are true only for connected algebraic k-groups.

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2. Nilpotence and subnormality

We prove the following proposition which is the equivalent for connected algebraic k-groups of a well known result for finite groups.

Proposition 2.1. *Let G be a connected linear algebraic k-group. The following statements are equivalent.*

1) *G is nilpotent;*
2) *every connected k-subgroup of G is subnormal;*
3) *for every H connected, proper k-subgroup of G, then $H < k\text{-}N_G(H)$;*
4) *for every H connected, proper k-subgroup of G, then dim $H <$ dim $k\text{-}N_G(H)$;*
5) *every maximal connected k-subgroup of G is normal.*

Proof. To show that the statements from 1) to 5) are equivalent, it will be sufficient to prove that 5) implies 1). Let S be a maximal torus of G defined over k (such a torus exists by Proposition 1.1,(iii)) and let C be its Cartan subgroup which is a nilpotent connected closed subgroup of G, defined over k by [12], Proposition 13.3.1,(ii). If $C \neq G$ then let M be a maximal connected k-subgroup of G such that $C \leq M \neq G$. If g is an arbitrary element of G we have that $T^g \leq M^g = M$ is a maximal torus of M, but then since maximal tori are K-conjugated we have that there exists an element $m \in M$ such that $T^g = T^m$, therefore $gm^{-1} \in N_G(T)$ thus $G = MN_G(T)$ but then G/M would be a connected k-group isomorphic to $N_G(T)/M \cap N_G(T)$ which is finite, therefore $G = M$, a contradiction. ∎

For the group of k-rational points $G(k)$ we can prove the following

Proposition 2.2. *Let G be a connected linear algebraic k-group where k is infinite. If for every connected k-subgroup H we have that $H(k)$ is subnormal in $G(k)$, then G is nilpotent.*

Proof. First of all we prove that $\overline{G(k)}^{\circ}$ is nilpotent. Let T be a maximal torus of G which is defined over k. If T is reduced to the unit, then we have $G = C_G(T)$ the Cartan subgroup associated to T which is nilpotent. So assume $T \neq 1$. By our assumptions we have $T(k)\, sn\, G(k)$. If

$$T(k) = H_n \leq \cdots \trianglelefteq H_{i+1} \trianglelefteq H_i \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G(k)$$

then (see [14], Lemma 5.10)

$$T = \overline{T(k)} = \overline{H_n} \leq \cdots \trianglelefteq \overline{H_{i+1}} \trianglelefteq \overline{H_i} \trianglelefteq \cdots \trianglelefteq \overline{H_1} \trianglelefteq \overline{H_0} = \overline{G(k)}$$

(since k is infinite, then the group of rational points of a k-torus is dense by [12], Proposition 13.2.7,(ii)) therefore $T\, sn\, \overline{G(k)}$ but since T is connected we get $T\, sn\, \overline{G(k)}^{\circ}$. As we have already said in the introduction, we may use the normal closure series in order to obtain a series of connected k-subgroups, each normal in the one above, starting with $\overline{G(k)}^{\circ}$ and having T at the end. We show that T is normal in $\overline{G(k)}^{\circ}$. Set $T \trianglelefteq A \trianglelefteq B$ where A and B are connected k-subgroups. Similarly to the proof of Proposition 2.1 we take $b \in B$ and then T^b is a maximal torus of A, therefore T^b is conjugated to T in A. Hence $T^b = T^a$ for some $a \in A$, thus $ba^{-1} \in N_B(T)$, then $B \leq AN_B(T) = N_B(T)$ i.e. $T \trianglelefteq B$. By induction we obtain $T \trianglelefteq \overline{G(k)}^{\circ}$ and by the rigidity of T we get that $\overline{G(k)}^{\circ}$ is a Cartan subgroup, hence nilpotent.

Now consider the k-subgroup G_t generated by all the maximal tori of G. The same argument shows therefore that $\overline{G_t(k)}^{\circ} = G_t$ is nilpotent (here we use the density of $G_t(k)$ given by Proposition 1.1,(iv)). Then $G_t = T$ is the only maximal k-torus of G and then G is equal to its Cartan subgroup, therefore G is nilpotent. ■

Examples 2.3. The propositions above are not valid in general in the case of non-connected algebraic k-groups as the example $\mathbb{Z}_2 \ltimes K^*$ shows (we suppose $K^2 = K$). Here the cyclic group of order two acts via inversion on the one-dimensional torus. Moreover, Proposition 2.2 does not hold for k finite. Consider the example

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid (a,b) \in K^2, a \neq 0 \right\}$$

where char $K = 2$. G is a connected $\mathbb{F}_2$-group and we have that $G(\mathbb{F}_2)$ is the cyclic group of order two but of course G is not nilpotent.

We may state a result concerning abstract non-nilpotent groups with all subgroups subnormal. Such groups (obviously infinite) exist, as shown for the first time in [5]. Proposition 2.2 implies immediately the following

Corollary 2.4. *There exists no connected algebraic k-group such that its group of k-rational points is a non-nilpotent group with all its subgroups subnormal.*

3. Join Theorem

We now turn to the problem concerning the subnormality of the subgroup generated by two connected subnormal k-subgroups. The respective question for finite groups was answered affirmatively in [15]. As noticed by Wielandt the same proof holds also for groups satisfying the maximal condition on subnormal subgroups. Since algebraic K-groups satisfy the maximal condition on connected subnormal K-subgroups the same proof applies to this case as well. Then the same holds also for the join of subnormal k-subgroups since their join is defined over k by Proposition 1.1,(i). Therefore we have the following

Proposition 3.1. *The subgroup which is generated by two connected subnormal k-subgroups of an algebraic k-group is subnormal.*

Another result of Wielandt (see [7], Lemma 1.3.5 and Theorem 1.3.10) is the following

Theorem 3.2. *Let $\{H_\lambda | \lambda \in \Lambda\}$ be a set of subnormal subgroups of an arbitrary group G and let J be their join. Then J is subnormal in G if and only if the set of subnormal subgroups of G lying in J contains a maximal member.*

Also in this case the proof can be adapted for algebraic k-groups. So we have

Proposition 3.3. *The subgroup generated by a set of connected subnormal k-subgroups of an algebraic k-group is subnormal.*

4. Criteria of subnormality

There are several criteria to determine if a subgroup of a finite group is subnormal. We present two of the most important. Let A be a subgroup of a finite group G. If one of the following is satisfied, then A is subnormal in G.

1. $AA^g = A^gA$ for all elements g of G, see [13] and [17].
2. A is subnormal in $\langle A, A^g\rangle$ for all elements g of G, see [17] and [4].

Before analyzing the respective results for connected linear algebraic k-groups, we mention the following lemma which is immediate. This lemma may also be thought as a subnormality criterion for connected linear algebraic k-groups.

Lemma 4.1. *Let G be a connected linear algebraic k-group where k is infinite. If a connected k-subgroup A is normalized by all elements of the group of k-rational points $G(k)$ then $A \,sn\, G$.*

Proof. Consider the subgroup G_t generated by the maximal tori of G. By Proposition 1.1,(iv) we have that G_t is a normal connected k-subgroup of G, moreover G/G_t is unipotent and $G_t(k)$ is dense in G_t. Now, since $G(k) \leq N_G(A)$ we have $G_t(k) \leq N_G(A)$ and by the density we obtain $G_t \leq N_G(A)$ therefore $A \trianglelefteq AG_t \,sn\, G$. ■

For algebraic k-groups we will follow in principle the analogue proofs for finite groups.

Proposition 4.2. *Let G be a connected linear algebraic k-group where k is an infinite field. If $AA^g = A^gA$ for all elements g of $G(k)$ then $A\,sn\,G$.*

Proof. First of all we show that $A^G = G$ implies $A = G$ using induction on dim $G-$ dim A. If A is normal we have nothing to prove. If all elements of $G(k)$ normalize G then by Lemma 4.1 we obtain $A\,sn\,G$. So let g be an element of $G(k)$ such that $A^g \neq A$ and consider the connected k-subgroup AA^g. Since $\dim G - \dim AA^g < \dim G - \dim A$ and since AA^g satisfies the same hypothesis as A, we have $G = AA^g$ but then we deduce $g \in A$ and $G = A$. The result on the subnormality of A follows by induction on dim G, recalling that the terms of the normal closure series of A in G are connected k-subgroups. ■

Example 4.3. The condition $AA^g = A^gA$ is not necessary for a connected k-subgroup A to be subnormal. In the group $\mathbf{U}(4, K)$ consider

$$A = \left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in K \right\} \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One may verify that $AA^g \neq A^gA$.

For finite groups and other classes of groups the second criterion under examination is a consequence of the so called first maximizer lemma (see [4]). We now present an adapted version for connected linear algebraic k-groups.

Lemma 4.4. *Let G be a connected linear algebraic k-group. Let A be a connected k-subgroup of G and suppose that A is not subnormal in G, but $A\,sn\,H$ for all connected proper k-subgroups H of G containing A. Then*

(i) *A is contained in a unique maximal connected k-subgroup M (we call M the Wielandt k-maximizer of A), and*

(ii) *if $g \in G(k)$, then $A^g \leq M$ implies $M^g = M$.*

Proof. Let $\mathcal{S}$ be the set of all proper k-subgroups of G containg A and which are generated by conjugates of A. We notice that the elements of $\mathcal{S}$ are connected k-groups, and then $\mathcal{S}$ satisfies the maximal condition with respect to inclusion. Moreover, if $H_1 \in \mathcal{S}$, $H_2 \in \mathcal{S}$, and $H_1 \leq H_2$, then $H_1\,sn\,H_2$ as a consequence of Proposition 3.1. We set $L = \langle H \mid H \in \mathcal{S} \rangle$. The proof that $L \in \mathcal{S}$ follows as in [7], Lemma 7.3.15.

Now, set $M = k\text{-}N_G(L)$. Clearly we have $A\,sn\,L \trianglelefteq M$ and then M is a connected proper k-subgroup of G. Let X be a connected proper k-subgroup of G containing A. By assumption we have $A\,sn\,X$. Setting $N = A^X$, we have $N \in \mathcal{S}$ and then $N \leq L$ which implies $N\,sn\,L$. Thus $A \leq N = N^X\,sn\,L^X$ by Proposition 3.1, therefore L^X is a connected proper k-subgroup of G containing A and generated by conjugates of A, thus $L^X \in \mathcal{S}$ hence $L^X \leq L$ but then $L \trianglelefteq \langle L, X \rangle$ which implies $X \leq k\text{-}N_G(L) = M$. We have obtained that M is a maximal connected k-subgroup of G and moreover that M is the only such subgroup which contains A. Let g be an element of $G(k)$ such that $A^g \leq M$. It follows $A \leq M^{g^{-1}}$ therefore $M^g = M$. ■

Proposition 4.5. *Let G be a connected linear algebraic k-group where k is infinite. A connected k-subgroup A of G is subnormal in G if and only if A is subnormal in $\langle A, A^g \rangle$ for all elements g of $G(k)$.*

Proof. Let G be a counterexample of minimal dimension thus A is subnormal in every connected proper k-subgroup H which contains it. By the first maximizer lemma there exists a Wielandt k-maximizer M of A. The assumption $A^g \leq M$ for all $g \in G(k)$ implies that all elements of $G(k)$ normalize M but then $M \, sn \, G$ by Lemma 4.1 hence $A \, sn \, G$, a contradiction. ■

Example 4.6. Lemma 4.1, Propositions 4.2 and 4.5 are not valid in general when the field of definition k is finite. We present an example where $k = \mathbb{F}_2$ which was suggested to me by Th. Weigel. Set K as the algebraic closure of the field $\mathbb{F}_2$. Now consider the special linear group

$$\mathbf{SL}(2, K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}.$$

Take the element $g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Its centralizer in $\mathbf{SL}(2, K)$ is

$$C_{\mathbf{SL}(2,K)}(g) = \left\{ \begin{pmatrix} b + d & b \\ b & d \end{pmatrix} \mid bd + d^2 - b^2 = 1 \right\}.$$

Set $T = C_{\mathbf{SL}(2,K)}(g)$. This is a maximal k-torus of $\mathbf{SL}(2, K)$. Now, the group of k-rational points $\mathbf{SL}(2, K)(\mathbb{F}_2)$ can be generated as follows

$$\mathbf{SL}(2, K)(\mathbb{F}_2) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

therefore one can see that $\mathbf{SL}(2, K)(\mathbb{F}_2)$ is isomorphic to the symmetric group of degree three S_3. It is also easy to verify that T is normalized by the group of k-rational points.

Now, T is a connected $\mathbb{F}_2$-subgroup of the connected linear algebraic $\mathbb{F}_2$-group $\mathbf{SL}(2, K)$ which is normalized by the group of $\mathbb{F}_2$-rational points, so that T verifies the hypothesis of Lemma 4.1, Propositions 4.2 and 4.5 but of course T is not subnormal in $\mathbf{SL}(2, K)$ since the latter is even a $\mathbb{F}_2$-simple group, i.e. $\mathbf{SL}(2, K)$ does not contain any connected normal $\mathbb{F}_2$-subgroup.

5. The norm and the Wielandt subgroup

Let G be an abstract group. The norm of G introduced in [1] is defined as follows:

$$Norm(G) = \bigcap_{H \leq G} N_G(H), \tag{2}$$

thus the norm is constituted by the elements of G which normalize every subgroup of G. $Norm(G)$ is a characteristic subgroup of G which is contained in the second center $Z_2(G)$ of G (see [11]).

For any group G, the Wielandt subgroup $\omega(G)$ was defined in [16] as

$$\omega(G) = \bigcap_{H \trianglelefteq\trianglelefteq G} N_G(H). \tag{3}$$

In the subgroup $\omega(G)$ subnormality is a transitive relation. The Wielandt subgroup may well be trivial (for example when G is the infinite dihedral group), but in [16] it is shown that $\omega(G)$ contains every minimal normal subgroup of G whose subnormal subgroups satisfy the minimal condition. In particular $\omega(G)$ is not trivial when G is a non-trivial finite group.

If G is an algebraic k-group and if we take the intersection in (2) and (3) over the connected k-subgroups of G then such groups are not in general defined over k when k is not perfect.

For example we consider [12], 12.1.6. Assume that k is a non-perfect field of characteristic 2 and take $a \in k \setminus k^2$. Define

$$G = \{(x, y) \in K^2 \mid x^2 - ay^2 \neq 0\},$$

with multiplication

$$(x, y)(x', y') = (xx' + ayy', xy' + x'y).$$

G is a connected linear algebraic k-group. If the k-homomorphism $\phi : G \to \mathbf{G}_m$ is defined by $\phi(x, y) = x^2 - ay^2$ then the kernel of ϕ is not a k-subgroup. Now, let us take the algebraic k-group L with underlying variety $G \times \mathbf{G}_a \times \mathbf{G}_a$ and multiplication defined by

$$(g, x, y)(g', x', y') = (gg', \phi(g')x + x', \phi(g'^{-1})y + y')$$

where $g, g' \in G$ and $x, x', y, y' \in \mathbf{G}_a$. Take the connected subnormal k-subgroup of L formed by the elements $(1, a, a)$ where $a \in \mathbf{G}_a$. Its normalizer is the subgroup Ker $\phi \times \mathbf{G}_a \times \mathbf{G}_a$ of L which is not defined over k. In this case one can verify that this would be the Wielandt subgroup of G while the norm would be Ker $\phi \times \{0\} \times \{0\}$. Both of them are not defined over k.

For this reason we introduce the following definitions:

1. The k-norm of G is the subgroup $k\text{-}Norm(G)$ generated by all connected k-subgroups H of G which satisfies the condition: $L \trianglelefteq \langle H, L\rangle$ for all connected k-subgroups L of G.

2. The k-Wielandt subgroup of G is the subgroup $k\text{-}\omega(G)$ generated by all connected k-subgroups H of G which satisfies the condition: $S \trianglelefteq \langle H, S\rangle$ for all connected subnormal k-subgroups S of G.

$k\text{-}Norm(G)$ and $k\text{-}\omega(G)$ are connected k-subgroups of G by Proposition 1.1,(i). In the example above we have $k\text{-}N(L) = \langle 1\rangle$ and $k\text{-}\omega(L) = \{1\} \times \mathbf{G}_a \times \mathbf{G}_a$.

The properties of the k-norm depend on the characteristic of k. If char $k = 0$ then $k\text{-}Norm(G) = Z(G)^\circ$, which corresponds to a result for local compact topological groups, see [8].

Proposition 5.1. *If G is a connected k-group where k is a field of characteristic 0, then $k\text{-}Norm(G) = Z(G)^\circ$.*

Proof. Assume that G is unipotent, and let $x \in G$. Since $\overline{\langle x \rangle} \simeq K_+$ (see [3]) and Aut $(K_+) \simeq K^*$, we deduce that an element of k-$Norm(G)$ is central. Since k-$Norm(G)$ is connected we get the result. ¿From this it is also immediate to obtain the statement when G is nilpotent. Now, let H be one of the connected k-subgroup that generates k-$Norm(G)$. If T is a maximal torus of G defined over k, then H normalizes T and by the rigidity of T we have that $[H,T]=1$. Therefore k-$Norm(G)$ centralizes every maximal k-torus of G. Moreover k-$Norm(G)$ is contained in every Cartan subgroup. Let C be one. C is a connected nilpotent k-group and since for such groups the statement holds, we have k-$Norm(G) \leq k$-$Norm(C) = Z(C)^\circ$. We get the statement observing that by [12], 13.3.12,(ii) G is generated by the maximal tori of G defined over k together with a Cartan subgroup and by the fact that trivially $Z(G)^\circ \leq k$-$Norm(G)$ because $Z(G)^\circ$ is defined over k since k is perfect, see [12], 12.1.7. ■

When char $k = p > 0$, in general k-$Norm(G) \not\leq Z_2(G)$ as the following example taken from [2] shows.

Example 5.2. Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_0^p & x_1^p & x_2^p \\ 0 & 0 & 1 & x_0^{p^2} & x_1^{p^2} \\ 0 & 0 & 0 & 1 & x_0^{p^3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_0, x_1, x_2.x_3 \in K$ and K is a field of characteristic $p > 0$. G is a group of maximal class, i.e. its nilpotency class equals its dimension, which is 4. Therefore every connected K-subgroup is contained in the commutator subgroup G' of G which is formed by the matrices where $x_0 = 0$. Since in G' every connected K-subgroup is normal, then K-$Norm(G) = G' = Z_3(G)$.

Now we show that the k-Wielandt subgroup of a non-trivial algebraic k-group is not trivial. We may reduce ourselves to connected algebraic k-groups and by Rosenlicht's Theorem we make a further reduction to the connected linear case, because otherwise our group $G = LD$ would contain the central k-subgroup D which of course would be contained in the k-Wielandt subgroup. As a first step, we use [16] to show that k-$\omega(G)$ is not trivial if $G(k)$ is dense in the non-trivial algebraic k-group G.

Proposition 5.3. *Let N be a minimal connected normal k-subgroup of the connected linear algebraic k-group G and assume that $G(k)$ is dense in G. Then N normalizes every connected subnormal k-subgroup of G.*

Proof. Let H be a connected subnormal k-subgroup of G of defect n. If $n \leq 1$, then obviously N normalizes H. We suppose $n > 1$ and we proceed by induction on n. Now, $[H^G, N]$ is a connected normal k-subgroup of N therefore $[H^G, N] = 1$ or $[H^G, N] = N$. In the former case obviously $[H, N] = 1$, hence N normalizes H. If $[H^G, N] = N$ then we have $N \leq H^G$. Let M be a minimal connected normal k-subgroup of H^G such that $M \leq N$. The subgroup

M^g is a minimal connected normal k-subgroup of H^G for any $g \in G(k)$. By induction it follows that M^g normalizes H for any $g \in G(k)$ but then since $M^{G(k)} = M^{\overline{G(k)}} = M^G$ we obtain that $N = M^G$ normalizes H. ∎

The Example 4.6 shows that if the group of k-rational points is not dense then in general $M^{G(k)} \neq M^G$

Now we deal with non-k-reductive groups, see Proposition 1.1,(v). The problem is to look for the right connected k-subgroup which normalizes every connected subnormal k-subgroup of G.

Proposition 5.4. *Let G be a non-k-reductive connected linear algebraic k-group. Then the k-Wielandt subgroup of G is not trivial.*

Proof. Define N to be a minimal normal connected k-subgroup contained in the last non-trivial term of the lower central series of the unipotent k-radical $R_{u,k}(G)$. By Proposition 1.1,(ii) this term is a connected k-subgroup. Notice that N is contained in the center of the unipotent radical of G. Let S be a connected subnormal k-subgroup of G. We prove that $S \trianglelefteq \langle N, S\rangle$. We need to distinguish the solvable from the non-solvable case.

Suppose that S is contained in the radical $R(G)$ of G. $R(G)$ is a connected normal solvable closed subgroup of G. Let T be a maximal torus of $R(G)$. We have $R(G) = T \ltimes R_u(G)$. It is even enough to suppose that S is a connected closed subgroup.

Now, if S is contained in $R_u(G)$, then N trivially normalizes S since N is contained in the center of $R_u(G)$.

If S is not contained in $R_u(G)$, then we may set $S = T_S \ltimes U_S$ where T_S is a non-trivial maximal torus of S and $U_S \leq R_u(G)$ is the unipotent radical of S. The product SN (recall that $N \trianglelefteq G$) is a connected algebraic k-group by Proposition 1.1,(i). S is clearly subnormal in SN and we have $SN = T_S \ltimes U_S N$, in particular T_S is a maximal k-torus of SN.

Set L as the last term of the normal closure series of S in SN different from S, so that $S \trianglelefteq L$ and L is a connected k-subgroup of SN which contains S as a proper subgroup. Clearly T_S is a maximal torus of L as well. If $l \in L$ then the maximal torus S^l is contained in S, but then S^l is a maximal torus of S as well. We know that maximal tori of S are conjugated, therefore there exists an element s of S such that $T^l = T^s$, thus $ls^{-1} \in N_L(T)$ from which we deduce $L \leq SN_L(T_S)$. If C_L is the Cartan subgroup of the maximal torus T in L, by the rigidity of T, we get moreover that $L = SC_L$, so that in particular $L = T_S \ltimes U_S C_1$ where C_1 is a connected closed subgroup contained in the centralizer of T_S (C_1 is in fact the unipotent part of C_L).

One can repeat this argument inductively along the subnormal series which leads S to SN (induction on the defect of S in SN), obtaining at the last step that $SN = T_S \ltimes \langle U_S, C\rangle$ where C is a connected closed subgroup contained in the centralizer of T_S. Recalling that N is central in $R_u(G)$, then U_S is normal in $\langle U_S, C\rangle$, so we get $[S, SN] = [T_S U_S, T_S U_S C] = [U_S, T_S][U_S, U_S][U_S, C] \leq S$ so that $S \trianglelefteq \langle N, S\rangle$.

Now, suppose that our connected subnormal k-subgroup S is not contained in the radical $R(G)$. Therefore $SR(G)/R(G)$ is a connected subnormal

k-subgroup of the connected algebraic k-group $G/R(G)$. We have that $G/R(G)$ is a semi-simple group and its connected K-subnormal subgroups are in fact normal, thus $SR(G)$ is a normal connected k-subgroup of G (see Proposition 1.1,(i)).

Moreover $SR(G)/R(G)$ is a connected K-simple group (i.e. it contains no connected normal K-subgroups) and therefore it is generated by its maximal tori. Since $SR(G)/R(G)$ is naturally isomorphic to the group $S/S \cap R(G)$, we obtain that $S = \langle (S \cap R(G))^\circ, S_t \rangle$ where S_t is the subgroup generated by the maximal tori of S.

Now, since $(S \cap R(G))^\circ \trianglelefteq S \cap R(G) \trianglelefteq S \, sn \, G$, we have that $(S \cap R(G))^\circ$ is a connected closed subnormal subgroup of G contained in the radical $R(G)$, therefore by the case above we have that such a subgroup is normalized by N (the case above works in general for connected subnormal K-subgroups).

Let T be a maximal torus of S. We shall get the statement if we prove that T is normalized by N. The subgroup S is clearly subnormal in SN. We may suppose $S \cap N = 1$. Let

$$S = S_m \trianglelefteq S_{m-1} \trianglelefteq \cdots \trianglelefteq S_0 = SN$$

be normal closure series associated to S, where m is a non-negative integer. It is clear that we may set $S_i = S \ltimes N_i$ where N_i is a connected closed subgroup of N for all $i \in \{0, \ldots, m\}$. From this we deduce the existence of the series

$$T = T_m \trianglelefteq T_{m-1} \trianglelefteq \cdots \trianglelefteq S_0 = TN$$

where $T_i = T \ltimes N_i$ for all $i \in \{0, \ldots, m\}$. Now, T is a maximal torus of SN (recall that N is unipotent) which is subnormal. By an argument similar to the proof of Proposition 2.2 we obtain that T is normalized (in fact centralized) by N, and this ends the proof. ■

Now let G be a connected linear algebraic k-reductive group, i.e. a group such that $R_{u.k}(G) = 1$. If, in addition, G is reductive, the result we are looking for is straightforward.

Proposition 5.5. *Let G be an algebraic k-group. Assume that G is a reductive k-group and $G \neq \langle 1 \rangle$. Then the k-Wielandt subgroup of G is not trivial.*

Proof. If k is infinite, by [12], Corollary 13.3.9,(ii) we have that the group of k-rational points $G(k)$ is dense in G, therefore the result follows by Proposition 5.3.

Let k be finite. If G is semi-simple, then connected subnormal k-subgroups are in fact normal, therefore in this case the k-Wielandt subgroup coincide with G. If G is not semi-simple, the radical $R(G)$ is a non-trivial central torus, which is defined over k by [12], 12.1.7. Hence $R(G)$ is contained in the k-Wielandt subgroup. ■

Finally, we treat the case of non-reductive k-reductive groups.

Proposition 5.6. *Let G be an algebraic k-group. Assume that G is non-reductive k-reductive group. Then the k-Wielandt subgroup of G is not trivial.*

Proof. We notice that our assumptions can be verified only when k is a non-perfect field by Proposition 1.1,(v). By hypothesis we have that the unipotent radical $R_u(G)$ of G is not trivial. Let S be a connected subnormal k-subgroup of G. Assume that S is contained in $R(G)$. By Proposition 1.1,(v) S is not contained in $R_u(G)$, so that we may write $S = T_S \ltimes U_S$ as the usual decomposition of a connected algebraic solvable group, where T_S is a non-trivial maximal k-torus of S (see Proposition 1.1,(iii)) and the unipotent part U_S is not defined over k, if it is not trivial. By Proposition 1.1,(ii), the derived subgroup S' is a connected k-subgroup of S contained in U_S, but then by Proposition 1.1,(v) we get $S' = 1$. Moreover, $S = T_S \times U_S$ is subnormal in the connected algebraic group $SR_u(G)$, and T_S is a maximal k-torus of $SR_u(G)$. With a proof analogue to the first part of Proposition 5.4, we may prove that $R_u(G) = \langle U_S, C\rangle$ where C is a connected k-subgroup which centralizes T_S, therefore T_S is centralized by $R_u(G)$. Let T be the subgroup generated by the maximal k-tori of connected subnormal k-subgroups of G contained in $R(G)$. We have proved that T is a connected central k-subgroup of $R(G)$. In fact T is also a normal k-torus of G.

Assume that T is not trivial. We prove that T is in the k-Wielandt subgroup of G. Of course T normalizes every connected subnormal k-subgroup of G contained in $R(G)$. On the contrary, let S be a connected subnormal k-subgroup of G not contained in $R(G)$. Recall the second part of the proof of Proposition 5.4, where we considered a connected subnormal k-subgroup not contained in the radical of G. The facts stated there over the field K are obviously still valid, so we have $S = \langle (S \cap R(G))^\circ, S_t\rangle$ where S_t is generate by the maximal k-tori of S. The subgroup $(S \cap R(G))^\circ$ may be not defined over k but it is normalized by T since T is central in $R(G)$. With an argument similar to the last part of the proof of Proposition 5.4 we may show that T normalizes S_t as well. Therefore, T is contained in the k-Wielandt subgroup of G, and we get the statement.

Finally, assume that T is trivial, i.e. there is no connected subnormal k-subgroup of G contained in $R(G)$. In particular, $R(G)$ is generated by its maximal tori, hence also G is generated by its maximal tori, so that, using notation and result of Proposition 1.1,(iv) we get $G = G_t$, and the group of k-rational points of G is dense in G. Now, Proposition 5.3 implies that the k-Wielandt subgroup is not trivial in this case as well. ∎

In view of the reduction to the connected linear case and of Propositions 5.4, 5.5 and 5.6, we obtain the following

Theorem 5.7. *Let G be an algebraic k-group, $G \neq \langle 1\rangle$. Then the k-Wielandt subgroup of G is not trivial.*

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