Journal of Lie Theory
Volume **16** (2006) 93–95

On a Special Class of Complex Tori

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Communicated by F. Knop

Abstract. We investigate which complex tori admits complex Lie subgroups whose closure is not complex.
Keywords and phrases: Complex tori, Lie subgroups.
Mathematics Subject Index 2000: 22E10.

It is mentioned in an article of J. Moser ([2]) that for the torus $T = (\mathbb{C}/\mathbb{Z}[i])^2$ every complex connected Lie subgroup of T has a closure which is a complex subtorus of T. In general, i.e., if T is an arbitrary compact complex torus, the closure of a complex Lie subgroup in a compact complex torus is a compact *real* subtorus which need not be complex. For instance, let

$$T = (\mathbb{C}/\mathbb{Z}[i]) \times (\mathbb{C}/\mathbb{Z}[\sqrt{2}i])$$

and let H be the connected complex Lie subgroup which is the image in T of the diagonal line $\{(z,z) : z \in \mathbb{C}\}$ in $\mathbb{C}^2$. Then the preimage of H in $\mathbb{C}^2$ can be described as

$$\pi^{-1}(H) = \{(z+n+mi, z+p+q\sqrt{2}i) : z \in \mathbb{C}; n,m,p,q \in \mathbb{Z}\}$$

whose closure is $\{(z,w) \in \mathbb{C}^2 : \Re(z-w) \in \mathbb{Z}\}$ and therefore of real dimension three. Thus $\bar{H}$ is a real subtorus of T of real codimension one.

Our goal is to determine precisely the class of those compact complex tori for which this phenomenon may occur.

Theorem. *Let T be a compact complex torus of dimension at least two. Then the following two conditions are equivalent:*

(i) *For every connected complex Lie subgroup H of T the closure $\bar{H}$ in T is a* complex *subtorus of T.*

(ii) *There exists an elliptic curve E with complex multiplication such that T is isogenous to E^n (with $n = \dim(T)$).*

We recall that an "elliptic curve" is a compact complex torus of dimension one and that such an elliptic curve is said to have "complex multiplication" if $End_{\mathbb{Q}}(E)$ is larger then $\mathbb{Q}$.

ISSN 0949–5932 / \$2.50

We recall that an endomorphism f of an elliptic curve E is a holomorphic Lie group homomorphism from E to itself. Since E is a commutative group, the set $End(E)$ of all such endomorphisms is a $\mathbb{Z}$-module in a natural way. Then $End_{\mathbb{Q}}(E)$ is defined as $End(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since every endomorphism g of E lifts to an endomorphism of the universal covering $(\mathbb{C}, +)$, there is a natural homomorphism from $End_{\mathbb{Q}}(E)$ to $\mathbb{C}$. Thus $End_{\mathbb{Q}}(E)$ may be regarded as a $\mathbb{Q}$-sub algebra of $\mathbb{C}$.

For a given elliptic curve E there are two possibilities: Either $End_{\mathbb{Q}}(E) = \mathbb{Q}$ or $End_{\mathbb{Q}}(E)$ is larger than $\mathbb{Q}$. In the latter case $End_{\mathbb{Q}}(E)$ (regarded as subalgebra of $\mathbb{C}$) must contain a complex number λ which is not real. For this reason an elliptic curve is said to have "complex multiplication" if $End_{\mathbb{Q}}(E) \neq \mathbb{Q}$.

See e.g. [1] for more information about elliptic curves with complex multiplication.

Proof. First let us assume property *(ii)*. Let $\lambda_0 \in End_{\mathbb{Q}}(E)$ with $\lambda_0 \notin \mathbb{R}$. Then $\lambda_0 m \in End(E)$ for some $m \in \mathbb{N}$. Define $\lambda = \lambda_0 m$. Let $E = \mathbb{C}/\Lambda$. Then we can realize T as a quotient $T = \mathbb{C}^n/\Gamma$ where Γ is commensurable with Λ^n. For every connected complex Lie subgroup $H \subset T$ we consider its preimage under the projection $\pi : \mathbb{C}^n \to T$. Then $\pi^{-1}(H) = V + \Gamma$ for some complex vector subspace V of $\mathbb{C}^n$. Now $\lambda \cdot (V + \Gamma) = V + \lambda \cdot \Gamma$ and $V + \Gamma$ are commensurable. Therefore the connected components of their respective closures in $\mathbb{C}^n$ agree. It follows that the connected component W of the closure of $V + \Gamma$ in $\mathbb{C}^n$ is a real vector subspace which is invariant under multiplication with λ. Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, it follows that W is a complex vector subspace. Consequently $\bar{H} = \pi(W)$ is a *complex* Lie subgroup of T, i.e. a complex subtorus.

Now let us deal with the opposite direction. As a preparation let us discuss real subtori of codimension one. If S is a real subtorus of T of real codimension one, it corresponds to a real hyperplane H in $\mathbb{C}^n$. Then $H \cap iH$ is a complex hyperplane in $\mathbb{C}^n$ which projects onto a connected complex Lie subgroup A of T. By construction either this complex Lie subgroup A is already closed (i.e. a complex subtorus) or its closure equals S.

Let $U = \mathbb{C}^n$ and let $\mathbb{P}^*_{\mathbb{C}}(U)$ resp. $\mathbb{P}^*_{\mathbb{R}}(U)$ denote the spaces parametrizing the complex resp. real hyperplanes in U. Then $H \mapsto H \cap iH$ defines a surjective continuous map from $\mathbb{P}^*_{\mathbb{R}}(U)$ to $\mathbb{P}^*_{\mathbb{C}}(U)$. Let $\mathbb{P}^*_{\Gamma}(U)$ denote the subset of those real hyperplanes which are generated by their intersection with Γ. Observe that a $\mathbb{R}$-linear change of coordinates takes $\mathbb{P}^*_{\Gamma}(U)$ to $\mathbb{P}_{2n-1}(\mathbb{Q})$ and $\mathbb{P}^*_{\mathbb{R}}(U)$ to $\mathbb{P}_{2n-1}(\mathbb{R})$. Therefore $\mathbb{P}^*_{\Gamma}(U)$ is dense in $\mathbb{P}^*_{\mathbb{R}}(U)$ and furthermore projects onto a dense subset of $\mathbb{P}^*_{\mathbb{C}}(U)$.

Let us now assume condition (i). Then for every real subtorus S of codimension one the connected complex Lie subgroup A of codimension one constructed above can not have S as closure and therefore must be complex compact subtorus. We thus obtain the following fact:

Let $\mathbb{P}'$ denote the set of all complex hyperplanes in $\mathbb{P}^*_{\mathbb{C}}(U)$ which project onto compact complex subtori of T. Then $\mathbb{P}'$ is dense in $\mathbb{P}^*_{\mathbb{C}}(U)$.

As a consequence, there are compact complex subtori $(C_i)_{i=1..n}$ of codimension one such that the intersection $\cap_i C_i$ is discrete. It follows that there is a surjective homomorphism of complex tori with finite kernel

$$\psi : T \to \Pi_{i=1}^n \left(T/C_i\right).$$

Thus T is isogenous to a product of elliptic curves.

Let us now discuss two-dimensional quotient tori of B. If $\tau : T \to B$ is a projection onto a two-dimensional torus and $L \subset B$ is a one-dimensional complex Lie subgroup, then $\overline{\tau^{-1}(L)} = \tau^{-1}(\bar{L})$. Thus assuming condition (i) for T implies the same condition for B. If we now define $\mathbb{P}'_B$ as the subset of those complex lines in $\mathbb{C}^2$ whose image in B are complex subtori, then we obtain that $\mathbb{P}'_B$ must be dense in $\mathbb{P}_1(\mathbb{C})$.

Thus let us discuss $\mathbb{P}'_B$ for $B = E' \times E''$ where E' and E'' are elliptic curves. If E' is not isogenous to E'', then $E' \times \{e\}$ and $\{e\} \times E''$ are the only subtori of B and $\mathbb{P}'_B$ can not be dense. Thus we may assume that E' is isogenous to E''.

If E' does not have complex multiplication, then $\mathbb{P}'_N \simeq \mathbb{P}_1(\mathbb{Q})$ whose closure is $\mathbb{P}_1(\mathbb{R})$ and which therefore is not dense.

This leaves the case where E' is isogenous to E'' and has complex multiplication.

If this is to hold for any two-dimensional quotient torus of T, it requires that all the E_i are isogenous to each other and have complex multiplication. ■

References

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Received October 3, 2004
and in final form Mai 5, 2005