On a Special Class of Complex Tori

Jörg Winkelmann

Communicated by F. Knop

Abstract. We investigate which complex tori admits complex Lie subgroups whose closure is not complex.

Keywords and phrases: Complex tori, Lie subgroups.

Mathematics Subject Index 2000: 22E10.

It is mentioned in an article of J. Moser ([2]) that for the torus $T = (\mathbb{C}/\mathbb{Z}[i])^2$ every complex connected Lie subgroup of $T$ has a closure which is a complex subtorus of $T$. In general, i.e., if $T$ is an arbitrary compact complex torus, the closure of a complex Lie subgroup in a compact complex torus is a compact real subtorus which need not be complex. For instance, let

$$T = (\mathbb{C}/\mathbb{Z}[i]) \times (\mathbb{C}/\mathbb{Z}[\sqrt{2}i])$$

and let $H$ be the connected complex Lie subgroup which is the image in $T$ of the diagonal line $\{(z, z) : z \in \mathbb{C}\}$ in $\mathbb{C}^2$. Then the preimage of $H$ in $\mathbb{C}^2$ can be described as

$$\pi^{-1}(H) = \{(z + n + mi, z + p + q\sqrt{2}i) : z \in \mathbb{C}; n, m, p, q \in \mathbb{Z}\}$$

whose closure is $\{(z, w) \in \mathbb{C}^2 : \Re(z - w) \in \mathbb{Z}\}$ and therefore of real dimension three. Thus $\bar{H}$ is a real subtorus of $T$ of real codimension one.

Our goal is to determine precisely the class of those compact complex tori for which this phenomenon may occur.

Theorem. Let $T$ be a compact complex torus of dimension at least two.

Then the following two conditions are equivalent:

(i) For every connected complex Lie subgroup $H$ of $T$ the closure $\bar{H}$ in $T$ is a complex subtorus of $T$.

(ii) There exists an elliptic curve $E$ with complex multiplication such that $T$ is isogenous to $E^n$ (with $n = \dim(T)$).

We recall that an “elliptic curve” is a compact complex torus of dimension one and that such an elliptic curve is said to have “complex multiplication” if $\text{End}_\mathbb{Q}(E)$ is larger than $\mathbb{Q}$. 

ISSN 0949–5932 / $2.50 © Heldermann Verlag
We recall that an endomorphism \( f \) of an elliptic curve \( E \) is a holomorphic Lie group homomorphism from \( E \) to itself. Since \( E \) is a commutative group, the set \( \text{End}(E) \) of all such endomorphisms is a \( \mathbb{Z} \)-module in a natural way. Then \( \text{End}_\mathbb{Q}(E) \) is defined as \( \text{End}(E) \otimes \mathbb{Z} \mathbb{Q} \). Since every endomorphism \( g \) of \( E \) lifts to an endomorphism of the universal covering \( (\mathbb{C},+) \), there is a natural homomorphism from \( \text{End}_\mathbb{Q}(E) \) to \( \mathbb{C} \). Thus \( \text{End}_\mathbb{Q}(E) \) may be regarded as a \( \mathbb{Q} \)-subalgebra of \( \mathbb{C} \).

For a given elliptic curve \( E \) there are two possibilities: Either \( \text{End}_\mathbb{Q}(E) = \mathbb{Q} \) or \( \text{End}_\mathbb{Q}(E) \) is larger than \( \mathbb{Q} \). In the latter case \( \text{End}_\mathbb{Q}(E) \) (regarded as subalgebra of \( \mathbb{C} \)) must contain a complex number \( \lambda \) which is not real. For this reason an elliptic curve is said to have “complex multiplication” if \( \text{End}_\mathbb{Q}(E) \neq \mathbb{Q} \).

See e.g. [1] for more information about elliptic curves with complex multiplication.

**Proof.** First let us assume property \((ii)\). Let \( \lambda_0 \in \text{End}_\mathbb{Q}(E) \) with \( \lambda_0 \not\in \mathbb{R} \). Then \( \lambda_0 m \in \text{End}(E) \) for some \( m \in \mathbb{N} \). Define \( \lambda = \lambda_0 m \). Let \( E = \mathbb{C}/\Lambda \). Then we can realize \( T \) as a quotient \( T = \mathbb{C}^n/\Gamma \) where \( \Gamma \) is commensurable with \( \Lambda^n \). For every connected complex Lie subgroup \( H \subset T \) we consider its preimage under the projection \( \pi : \mathbb{C}^n \to T \). Then \( \pi^{-1}(H) = V + \Gamma \) for some complex vector subspace \( V \) of \( \mathbb{C}^n \). Now \( \lambda \cdot (V + \Gamma) = V + \lambda \cdot \Gamma \) and \( V + \Gamma \) are commensurable. Therefore the connected components of their respective closures in \( \mathbb{C}^n \) agree. It follows that the connected component \( W \) of the closure of \( V + \Gamma \) in \( \mathbb{C}^n \) is a real vector subspace which is invariant under multiplication with \( \lambda \). Since \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), it follows that \( W \) is a complex vector subspace. Consequently \( \bar{H} = \pi(W) \) is a complex Lie subgroup of \( T \), i.e. a complex subtorus.

Now let us deal with the opposite direction. As a preparation let us discuss real subtori of codimension one. If \( S \) is a real subtorus of \( T \) of real codimension one, it corresponds to a real hyperplane \( H \) in \( \mathbb{C}^n \). Then \( H \cap iH \) is a complex hyperplane in \( \mathbb{C}^n \) which projects onto a connected complex Lie subgroup \( A \) of \( T \). By construction either this complex Lie subgroup \( A \) is already closed (i.e. a complex subtorus) or its closure equals \( S \).

Let \( U = \mathbb{C}^n \) and let \( \mathbb{P}^*_\mathbb{C}(U) \) resp. \( \mathbb{P}^*_\mathbb{R}(U) \) denote the spaces parametrizing the complex resp. real hyperplanes in \( U \). Then \( H \mapsto H \cap iH \) defines a surjective continuous map from \( \mathbb{P}^*_\mathbb{R}(U) \) to \( \mathbb{P}^*_\mathbb{C}(U) \). Let \( \mathbb{P}_\mathbb{C}(U) \) denote the subset of those real hyperplanes which are generated by their intersection with \( \Gamma \). Observe that a \( \mathbb{R} \)-linear change of coordinates takes \( \mathbb{P}_\mathbb{C}(U) \) to \( \mathbb{P}_{2n-1}(\mathbb{Q}) \) and \( \mathbb{P}_\mathbb{R}(U) \) to \( \mathbb{P}_{2n-1}(\mathbb{R}) \). Therefore \( \mathbb{P}_\mathbb{C}(U) \) is dense in \( \mathbb{P}^*_\mathbb{R}(U) \) and furthermore projects onto a dense subset of \( \mathbb{P}^*_\mathbb{C}(U) \).

Let us now assume condition \((i)\). Then for every real subtorus \( S \) of codimension one the connected complex Lie subgroup \( A \) of codimension one constructed above can not have \( S \) as closure and therefore must be complex compact subtorus. We thus obtain the following fact:

Let \( \mathbb{P}' \) denote the set of all complex hyperplanes in \( \mathbb{P}^*_\mathbb{C}(U) \) which project onto compact complex subtori of \( T \). Then \( \mathbb{P}' \) is dense in \( \mathbb{P}^*_\mathbb{C}(U) \).

As a consequence, there are compact complex subtori \( (C_i)_{i=1,n} \) of codimension one such that the intersection \( \bigcap C_i \) is discrete. It follows that there is a surjective homomorphism of complex tori with finite kernel

\[ \psi : T \to \prod_{i=1}^n (T/C_i). \]
Thus $T$ is isogenous to a product of elliptic curves.

Let us now discuss two-dimensional quotient tori of $B$. If $\tau : T \to B$ is a projection onto a two-dimensional torus and $L \subset B$ is a one-dimensional complex Lie subgroup, then $\tau^{-1}(\overline{L}) = \tau^{-1}(\overline{L})$. Thus assuming condition (i) for $T$ implies the same condition for $B$. If we now define $\mathbb{P}_B'$ as the subset of those complex lines in $\mathbb{C}^2$ whose image in $B$ are complex subtori, then we obtain that $\mathbb{P}_B'$ must be dense in $\mathbb{P}_1(\mathbb{C})$.

Thus let us discuss $\mathbb{P}_B'$ for $B = E' \times E''$ where $E'$ and $E''$ are elliptic curves. If $E'$ is not isogenous to $E''$, then $E' \times \{e\}$ and $\{e\} \times E''$ are the only subtori of $B$ and $\mathbb{P}_B'$ can not be dense. Thus we may assume that $E'$ is isogenous to $E''$.

If $E'$ does not have complex multiplication, then $\mathbb{P}_N \simeq \mathbb{P}_1(\mathbb{Q})$ whose closure is $\mathbb{P}_1(\mathbb{R})$ and which therefore is not dense.

This leaves the case where $E'$ is isogenous to $E''$ and has complex multiplication.

If this is to hold for any two-dimensional quotient torus of $T$, it requires that all the $E_i$ are isogenous to each other and have complex multiplication. ■

References


Jörg Winkelmann
Institut Elie Cartan (Mathématiques)
Université Henri Poincaré Nancy 1
B.P. 239
F-54506 Vandœuvre-les-Nancy Cedex
France
jwinkel@member.ams.org

Received October 3, 2004
and in final form Mai 5, 2005