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Effective Integration of Lie Algebras

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Abstract. For integrable Banach-Lie algebras the corresponding simply connected Banach-Lie groups are constructed in an explicite way. A concept of discriminant subgroup of a normed Lie algebra is introduced and it is shown that a Banach-Lie algebra is integrable if and only if its discriminant subgroup is discrete. Relation of discriminant subgroup with the Ado and Malcev theorems is also discussed.
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1. Introduction

While every finite dimensional Lie algebra arises as the Lie algebra of some Lie group (E. Cartan [3],[4] 1930), there exist Banach-Lie algebras which are not enlargeable, i.e. which do not integrate to a global Banach-Lie group. This was discovered by van Est and Korthagen ([7], 1964). Using cohomological methods they associated to each Banach-Lie algebra $\mathfrak{g}$ an additive subgroup $\Pi(\mathfrak{g})$ (the 'period group') of the center $z_{\mathfrak{g}}$ of $\mathfrak{g}$ in such a way that $\mathfrak{g}$ is enlargeable if and only if $\Pi(\mathfrak{g})$ is discrete, and they found a Banach-Lie algebra $\mathfrak{g}$ with nondiscrete $\Pi(\mathfrak{g})$.

In 1970 an important contribution to the integration problem was provided by a result of Świerczkowski ([16]). He showed that for each Banach-Lie algebra $\mathfrak{g}$ the Banach-Lie algebra of paths $\sigma(\mathfrak{g}) := \{l \in C([0,1],\mathfrak{g}) : l(0) = e\}$ (with pointwise operations) is enlargeable. This implies the existence of a topological extension

$$0 \longrightarrow \omega(\mathfrak{g}) \xrightarrow{i} \sigma(\mathfrak{g}) \xrightarrow{\exp} \mathfrak{g} \longrightarrow 0,$$

where $\exp : \sigma(\mathfrak{g}) \ni l \to l(1) \in \mathfrak{g}$, and $\omega(\mathfrak{g}) := \ker(\exp)$.

Let $\Sigma(\mathfrak{g})$ denote the simply connected Banach-Lie group corresponding to $\sigma(\mathfrak{g})$ and $\Omega(\mathfrak{g})$ be the normal Banach-Lie subgroup of $\Sigma(\mathfrak{g})$ corresponding to the ideal $\omega(\mathfrak{g})$. As a consequence of Świerczkowski's result one gets the following criterion: a Banach-Lie algebra $\mathfrak{g}$ is enlargeable if and only if $\Omega(\mathfrak{g})$ is closed in $\Sigma(\mathfrak{g})$.

Recently, Świerczkowski's approach was refined by J.J. Duistermaat and J.A Kolk [5]. Applying ordinary differential equations in Banach spaces and using the fact

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that $H^2(G, \mathbf{R}) = 0$ for each finite dimensional simply connected Lie group G, they showed that $\Omega(\mathfrak{g})$ is closed for finite dimensional $\mathfrak{g}$, thus getting a new proof of the Cartan's theorem.

In this paper we present yet another approach to the integration problem. Similarly to Świerczkowski and Duistermaat-Kolk we construct the group as the Hausdorff quotient $G = \exp g_1^{\mathbf{N}} / \exp g_{1,0}^{\mathbf{N}}$ of two Banach-Lie groups. There are, however, major differences:

In place of differential equations applied in [5] we are using the Baker-Campbell-Hausdorff formula. This makes the construction more algebraic and less relying on the completeness of $\mathfrak{g}$. Our arguments being technically simpler than those of Duistermaat and Kolk, we are able to describe intrinsically obstructions to integration in terms of a *discriminant subgroup* $\Gamma_{\mathfrak{g}}$ of $\mathfrak{g}$. This enables us to obtain a relatively simple proof of Cartan's theorem with the simply connected Lie group G corresponding to $\mathfrak{g}$ explicitely described.

In our approach to the integration problem the group $\Gamma_{\mathfrak{g}}$ plays an anologous rôle to $\Pi(\mathfrak{g})$ in [7] : Theorem 4.2 below asserts that a Banach-Lie algebra $\mathfrak{g}$ is enlargeable iff $\Gamma_{\mathfrak{g}}$ is discrete. It seems that $\Gamma_{\mathfrak{g}}$ also may be useful when approaching the problem of injective continuous representability of $\mathfrak{g}$ in an associative Banach algebra. A necessary condition applying $\Gamma_{\mathfrak{g}}$ is given in Theorem 4.7 below.

To make the paper easier to read, in the first part we present the main ideas of the construction, leaving technical details to the appendix. They may be of interest in its own by revealing an interplay between algebraic properties of the Baker-Campbell-Hausdorff formula and local (norm) estimates.

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2. The groups $\exp \mathfrak{g}^{\mathbf{N}}$, $\exp \mathfrak{g}_1^{\mathbf{N}}$ and $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$

The Baker-Campbell-Hausdorff series (abbreviated to the B-C-H series) is a real power series in non-commutative formal variables x, y which is obtained as the composition $\Theta = W \circ Z$ where

$$W(z) = \log(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

and

$$Z(x,y) = e^x e^y - 1 = \sum_{j+k \geq 1} \frac{x^j}{j!} \frac{y^k}{k!}.$$

Gathering together terms of a given order we obtain

$$\Theta(x,y) = \sum_{m=1}^{\infty} \Theta_m(x,y) \tag{2.1}$$

where $\Theta_m(x,y)$ is the (finite) sum of all homogeneous terms of order m.

One of the cornerstones of Lie theory is the observation (cf. [2], [17]) that Θ_m for $m = 1, 2, \dots$ may be expressed as a finite linear combination of $(m-1)$-fold commutators of x and y. In particular, $\Theta_1(x,y) = x + y$ and $\Theta_2(x,y) = \frac{1}{2}(xy - yx)$.

If $\mathfrak{g}$ is a Lie algebra then substituting for the formal variables x and y in (2.1) arbitrary elements of $\mathfrak{g}$ and replacing the commutators by Lie brackets in $\mathfrak{g}$ we obtain the evaluated series with the terms $\Theta_m(x,y)$ in $\mathfrak{g}$.

It is known [2],[6], that if $\mathfrak{g}$ is normed in such a way that the condition (2.3) below is satisfied, then the series (2.1) is absolutely convergent for $(x,y) \in Q$, where

$$Q = \{(x,y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| \leq \ln 2\}$$

Moreover, if $\mathfrak{g}$ is a Banach-Lie algebra the function $Q \ni (x,y) \to \Theta(x,y) \in \mathfrak{g}$ is jointly continuous and defines a local group structure on Q. This local group will be denoted $\exp_0(\mathfrak{g})$ and called the *local B-C-H group of* $\mathfrak{g}$.

For a Lie algebra $\mathfrak{g}$ let $\mathfrak{g}^{\mathbf{N}}$ be be the set of all formal power series $f(t) = \sum_{n=1}^{\infty} a_n t^n$ with coefficients in $\mathfrak{g}$. If $\mathfrak{g}^{\mathbf{N}}$ is equipped with the Cauchy-Lie bracket, i.e. for $f(t) = \sum_{n=1}^{\infty} a_n t^n$ and $g(t) = \sum_{n=1}^{\infty} b_n t^n$,

$$[f,g](t) = \sum_{n=1}^{\infty} c_n t^n \quad \text{where} \quad c_n = \sum_{k+j=n} [a_j, b_k],$$

then $\mathfrak{g}^{\mathbf{N}}$ is a Lie algebra and

$$\mathfrak{g}^{\mathbf{N}} = \prod_{j=1}^{\infty} M_j$$

where M_j for $j = 1,2,3,\ldots$ is the linear space composed of all series with only the j-th coefficient nonvanishing. Moreover, the Lie bracket in $\mathfrak{g}^{\mathbf{N}}$ satisfies the conditions

$$[M_j, M_k] \subset M_{j+k}. \tag{2.2}$$

The property of the B-C-H series that we strongly rely on in this paper is its graded structure: since the n-th coefficient of the series is a linear combination of $(n-1)$-fold Lie brackets, (2.2) implies that the coordinates of this coefficient with indices less than n vanish. Thus (cf. [2], Chapter III) for each pair of elements of $\mathfrak{g}^{\mathbf{N}}$ their B-C-H series is coordinatewise convergent. The function

$$\mathfrak{g}^{\mathbf{N}} \times \mathfrak{g}^{\mathbf{N}} \ni (f,g) \longrightarrow f \circ g := \Theta(f,g) \in \mathfrak{g}^{\mathbf{N}}$$

is jointly continuous and defines a group structure on $\mathfrak{g}^{\mathbf{N}}$.

To see this one may consider all the continuous Lie algebra homomorphisms $\Phi_n : \mathfrak{g}^{\mathbf{N}} \to \mathfrak{g}^{\mathbf{N}}/t^n\mathfrak{g}^{\mathbf{N}}$ into the nilpotent Banach-Lie algebras $\mathfrak{g}^{\mathbf{N}}/t^n\mathfrak{g}^{\mathbf{N}} =: q_n$. It is well known that the BCH-multiplication on each q_n makes the latter a group, and since the mappings Φ_n separate points on $\mathfrak{g}^{\mathbf{N}}$, we deduce that also there the BCH-multiplication is a group multiplication.

Definition 2.1. Given a Lie algebra $\mathfrak{g}$, the above group will be called *the B-C-H group of* $\mathfrak{g}^{\mathbf{N}}$ in the sequel, and it will be denoted by $\exp \mathfrak{g}^{\mathbf{N}}$.

Let $x \otimes t$ stand for the series $f(t) \in g^{\mathbf{N}}$ with the first coefficient equal to x and the remaining ones equal to 0. We also denote

$$\mathfrak{g} \otimes t = \{x \otimes t : x \in \mathfrak{g}\}.$$

The mapping $i : \mathfrak{g} \ni x \to x \otimes t \in \exp \mathfrak{g}^{\mathbf{N}}$ will be called the canonical embedding.

Throughout the rest of this section we will consider $\mathfrak{g}$ to be a Banach-Lie algebra equipped with a norm satisfying

$$\|[x, y]\| \leq \|x\| \cdot \|y\|. \tag{2.3}$$

By $\circ$ we will denote the B-C-H product (for various Lie algebras). We follow the convention that $\exp X$ for $X \subset \mathfrak{g}^{\mathbf{N}}$ denotes the subgroup of $\exp \mathfrak{g}^{\mathbf{N}}$ generated by X. We also abbreviate the B-C-H product $f_1 \circ f_2 \circ \ldots \circ f_m$ to $\circ \prod_{i=1}^{m} f_i$.

Definition 2.2. For $f = \sum_{n=1}^{\infty} a_n t^n \in \mathfrak{g}^{\mathbf{N}}$ let

$$\|f\|_1 = \sum_{n=1}^{\infty} \|a_n\| \tag{2.4}$$

(we set $\| f \|_1 = \infty$ if the series diverges).

Let

$$\mathfrak{g}_1^{\mathbf{N}} = \left\{ f \in \mathfrak{g}^{\mathbf{N}} : \| f \|_1 < \infty \right\}, \tag{2.5}$$

$$\mathfrak{g}_{1,0}^{\mathbf{N}} = \{ f \in \mathfrak{g}_1^{\mathbf{N}} : \sum_{n=1}^{\infty} a_n = 0 \}, \tag{2.6}$$

$$\alpha : \mathfrak{g}_1^{\mathbf{N}} \ni f = \sum_{n=1}^{\infty} x_n t^n \to \sum_{n=1}^{\infty} x_n \in \mathfrak{g}. \tag{2.7}$$

Proposition 2.3.

(a) $\mathfrak{g}_1^{\mathbf{N}}$ *is a Banach-Lie subalgebra of* $\mathfrak{g}^{\mathbf{N}}$ *with the norm* $\| \cdot \|_1$ *which satisfies* (2.3).

(b) $\mathfrak{g}_{1,0}^{\mathbf{N}}$ *is a norm-closed Lie ideal of* $\mathfrak{g}_1^{\mathbf{N}}$.

(c) α *is a continuous Lie algebra homomorphism.*

Proof. (a) The situation is parallel (with the Lie product substituting the associative product) to the one considered in the theory of convolution l_1 (semigroup) Banach algebras. l_1 (semigroup) Banach algebras. We omit the proof.

(b) and (c) Observe that $\| \alpha(f) \| \leq \| f \|_1$ for each $f \in \mathfrak{g}_1^{\mathbf{N}}$, α is a Lie algebra homomorphism and $\ker \alpha = \mathfrak{g}_{1,0}^{\mathbf{N}}$. ■

Definition 2.4.

(a) Let

$$\exp \mathfrak{g}_1^{\mathbf{N}} = \{ f \in \exp \mathfrak{g}^{\mathbf{N}} : f = \circ \prod_{i=1}^{m} h_i, \quad h_i \in \mathfrak{g}_1^{\mathbf{N}}, \; m \in \mathbf{N} \}. \tag{2.8}$$

$$\exp \mathfrak{g}_{1,0}^{\mathbf{N}} = \{f \in \exp \mathfrak{g}^{\mathbf{N}} : f = \circ \prod_{i=1}^{m} h_i, \quad h_i \in \mathfrak{g}_1^{\mathbf{N}}, \ m \in \mathbf{N}\}. \tag{2.9}$$

(b) For $f \in \exp \mathfrak{g}_1^{\mathbf{N}}$ let

$$\|f\|_{(1)} = \inf \left\{ \sum_{i=1}^{m} \|h_i\|_1 : f = \circ \prod_{i=1}^{m} h_i, \quad h_i \in \mathfrak{g}_1^{\mathbf{N}}, \quad m \in \mathbf{N} \right\}. \tag{2.10}$$

For $f \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *let*

$$\| f \|_{(1,0)} = \inf\{\sum_{i=1}^{m} \| h_i \|_1 : f = \circ \prod_{i=1}^{m} h_i, \quad h_i \in \mathfrak{g}_{1,0}^{\mathbf{N}}, \ m \in \mathbf{N}\}. \tag{2.11}$$

(c) *For* $f_1, f_2 \in \exp \mathfrak{g}_1^{\mathbf{N}}$ *let*

$$\rho_{(1)}(f_1, f_2) = \| f_1^{-1} \circ f_2 \|_{(1)} . \tag{2.12}$$

For $f_1, f_2 \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *let*

$$\rho_{(1,0)}(f_1, f_2) = \| f_1^{-1} \circ f_2 \|_{(1,0)} . \tag{2.13}$$

Proposition 2.5.

(a) *The functions* $\rho_{(1)}$ *and* $\rho_{(1,0)}$ *are metrics. The groups* $\exp \mathfrak{g}_1^{\mathbf{N}}$ *and* $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *equipped with the metrics* $\rho_{(1)}$ *and* $\rho_{(o,1)}$ *respectively are metric groups.*

(b) *The group* $\exp \mathfrak{g}_1^{\mathbf{N}} \mathfrak{g}_1^{\mathbf{N}}$ *is simply connected Banach-Lie group corresponding to* $\mathfrak{g}_1^{\mathbf{N}}$ *and compatible with the metric* $\rho_{(1)}$.

(c) *The group* $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *is a connected Banach-Lie group corresponding to* $\mathfrak{g}_{1,0}^{\mathbf{N}}$ *and compatible with the metric* $\rho_{(1,0)}$.

(d) *The group* $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *is the normal Lie subgroup of* $\exp \mathfrak{g}_1^{\mathbf{N}}$ *corresponding to the closed Lie ideal* $\mathfrak{g}_{1,0}^{\mathbf{N}}$.

Proof. (a) This is a consequence of Proposition 6.7 (a) (b) and Proposition 6.5 (see Appendix).

(b) Let r be a number introduced in Proposition 6.2 and let α be as in Lemma 6.8 (c). For $\delta > 0$ let $B_\delta = \{f \in \exp \mathfrak{g}_1^{\mathbf{N}} : \| f \|_{(1)} < \delta\}$ and observe that by Lemma 6.8(a)(c) $B_{\frac{r}{2}}$ is homeomorphic to a suitable neighbourhood of 0 in $\mathfrak{g}_1^{\mathbf{N}}$. Moreover the local group of $\exp \mathfrak{g}_1^{\mathbf{N}}$ restricted to $B_{\frac{\alpha}{2}}$ coincides with the local B-C-H group of $\mathfrak{g}_1^{\mathbf{N}}$ restricted to this neighbourhood. The conclusion follows.

Clearly $\exp \mathfrak{g}_1^{\mathbf{N}}$ as well as $\exp \mathfrak{g}_1^{\mathbf{N}}$ are connected. To show that $\exp \mathfrak{g}_1^{\mathbf{N}}$ is simply connected let $\mathbf{R}_{[0,1]}$ denote the multiplicative semigroup of the real numbers with the underlying set $\{\lambda \in \mathbf{R} : 0 \leq \lambda \leq 1\}$ and let us note that $\exp \mathfrak{g}_1^{\mathbf{N}}$ is closed with respect to the action of $\mathbf{R}_{[0,1]}$ given by $\mathbf{R}_{(0,1]} \times \exp \mathfrak{g}^{\mathbf{N}} \ni (s, f) \to s * f \in \exp \mathfrak{g}^{\mathbf{N}}$ where $s * f := \sum_{n=1}^{\infty} (s^n a_n) t^n$ for $f = \sum_{n=1}^{\infty} a_n t^n$. Indeed, for $s \in \mathbf{R}_{(0,1]}$ and $f_1, f_2 \in \exp \mathfrak{g}\mathbf{N}$ one has $s * (f_1 \circ f_2) = (s * f_1) \circ (s * f_2)$, moreover for $f \in \mathfrak{g}_1^{\mathbf{N}}$ also $s * f \in \mathfrak{g}_1^{\mathbf{N}}$.

Hint: *To show continuity of the map* $\mathbf{R}_{[0,1]} \times \exp \mathfrak{g}_1^{\mathbf{N}} \ni (s,f) \to s*f \in \exp \mathfrak{g}_1^{\mathbf{N}}$, *observe that (2.12), and (6.2.1)(b),(c) together with Proposition 6.7(a) imply for* $f_1, f_2 \in \exp \mathfrak{g}_1^{\mathbf{N}}$ *and* $s_1, s_2 \in \mathbf{R}_{[0,1]}$ *the inequality*

$$\rho((s_1 * f_1),(s_2 * f_2)) \leq \rho((s_1 * f_1),(s_2 * f_1)) + \rho((s_2 * f_1),(s_2 * f_2)).$$

This reduces the problem to showing that for a fixed $f \in \exp \mathfrak{g}_1^{\mathbf{N}}$ *the function* $\mathbf{R}_{[0,1]} \to s*f \in \exp \mathfrak{g}_1^{\mathbf{N}}$ *is continuous and also that for a fixed* $s \in \mathbf{R}_{[0,1]}$ *the function* $\exp : \mathfrak{g}_1^{\mathbf{N}} \ni f \to s*f \in \exp \mathfrak{g}_1^{\mathbf{N}}$ *is continuous. Next, the definition (2.11) and properties (6.2.1)(b), (6.2.4) and (6.2.7) of the norm* $\| \cdot \|_{(1)}$ *enable further reduction to the analogous statements with* $\mathfrak{g}_1^{\mathbf{N}}$ *substituting* $\exp \mathfrak{g}_1^{\mathbf{N}}$.

(c) The proof is similar to the first part of the proof of (b).

(d) results from (c) and the observation (cf. the proof of Lemma 6.8 (d)) that $f \circ \phi \circ f^{-1} \in \mathfrak{g}_{1,0}^{\mathbf{N}}$ for $f \in \mathfrak{g}_1^{\mathbf{N}}$ and $\phi \in \mathfrak{g}_{1,0}^{\mathbf{N}}$. ■

3. The discriminant subgroup of a Banach-Lie algebra.

Let $\mathfrak{g}$ be a Banach-Lie algebra with center $z_{\mathfrak{g}}$ and central descending series $\left(\mathfrak{g}^j\right)_{j\geq 1}$ (i.e. the sequence of ideals $\mathfrak{g}^1 = g$ and $\mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}]$ for $j \geq 2$, (cf. [2], Chap.I, Section 1, p.5). We will also consider *the closed central descending series* $\left(\overline{\mathfrak{g}^j}\right)_{j\geq 1}$ which is the sequence of the closures of the ideals in the central descending series. Define

$$\overline{\mathfrak{g}^{\infty}} = \bigcap_{j=1}^{\infty} \overline{\mathfrak{g}^j}.$$

In connection with the problems of integration and of continuous embedding of a given Banach-Lie algebra into a Banach algebra, a role is played by

$$E\mathfrak{g} = \{x \in \mathfrak{g} : x \otimes t \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}\}.$$

Define

$$\Gamma_{\mathfrak{g}} = E_{\mathfrak{g}} \cap z_{\mathfrak{g}}.$$

Let us observe that for $x, y \in \Gamma_{\mathfrak{g}}$, the elements $x \otimes t$, $y \otimes t$ are central in $\exp \mathfrak{g}^{\mathbf{N}}$, hence $(x \otimes t) \circ (y \otimes t) = (x+y) \otimes t \in \exp g_{1,0}^{\mathbf{N}} \cap g \otimes t$, i.e. $x + y \in \Gamma_g$. It follows that $\Gamma_{\mathfrak{g}}$ is an additive subgroup of $z_{\mathfrak{g}}$.

Definition 3.1. $\Gamma_{\mathfrak{g}}$ will be called the discriminant subgroup of $\mathfrak{g}$.

Proposition 3.2.

(a) *If* $\| x \| < 2\pi$ *then* $x \in E_{\mathfrak{g}}$ *implies* $x \in z_{\mathfrak{g}}$.

(b) *If* $\mathfrak{g}$ *is the Lie algebra of a connected Banach-Lie group* G, Z_G *is the center of* G *and* $\exp : g \to G$ *is the exponential map, then*

$$E_{\mathfrak{g}} \subset \{x \in \mathfrak{g} : \exp x \in Z_G\}.$$

(c) $E_{\mathfrak{g}} \subset \bar{\mathfrak{g}}^{\infty}$.

Proof. (a) Let $x \otimes t \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$. Considering $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ with its Banach manifold topology, i.e. the topology implemented by the metric (2.13), we see (cf. Lemma 6.8(b)) that $B_\beta = \{f \in g_{1,0}^{\mathbf{N}} :\| f \|_1 < \beta\}$ is a neighbourhood of the unit element in $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ for small $\beta > 0$. Since $(-y \otimes t) \circ (x \otimes t) \circ (y \otimes t) = e^{\mathrm{ad}_{(y\otimes t)}}(x \otimes t)$ using (6.2.6) we infer that $(-y \otimes t) \circ (x \otimes t) \circ (y \otimes t) \in (x \otimes t) \circ B_\beta$, for sufficiently small y i.e. that

$$(-y \otimes t) \circ (x \otimes t) \circ (y \otimes t) \circ (-x \otimes t) \in \mathfrak{g}_{1,0}^{\mathbf{N}}. \tag{3.1}$$

Note that $A_x(f) = (x\otimes t)\circ f \circ(-x\otimes t) = e^{ad_{(x\otimes t)}}(f)$ is a bounded linear operator on $\mathfrak{g}_1^{\mathbf{N}}$, and thus the mapping $y \to (-y \otimes t) \circ A_x(y \otimes t)$ is defined and differentiable in some neighbourhood of zero in $\mathfrak{g}$ and by (3.1) it maps this neighbourhood into $\mathfrak{g}_{1,0}^{\mathbf{N}}$. e same applies to its differential at 0 and we infer that for each $y \in \mathfrak{g}$,

$$e^{ad_{(x\otimes t)}}(y \otimes t) - (y \otimes t) \in g_{1,0}^{\mathbf{N}}.$$

Applying the homomorphism α we get $e^{ad_x}(y) - y = 0$ for each $y \in \mathfrak{g}$, i.e.

$$e^{ad_x} = I. \tag{3.2}$$

If $\| x \| < 2\pi$ then also $\| ad_x \| < 2\pi$ and (3.2) implies $ad_x = 0$, i.e. $x \in z$.

(b) It is well known that (3.2) is equivalent to the condition

$$\exp(x) \exp y(\exp(x))^{-1} = \exp(y),$$

i.e. $\exp(x) \in Z_G$.

(c) Let m be a positive integer. To show that $E_{\mathfrak{g}} \subset \bar{\mathfrak{g}}^m$ let $x \otimes t = \circ \prod_{j=1}^k f_j$, where $f_j \in \mathfrak{g}_{1,0}^{\mathbf{N}}$ for $j = 1, \ldots, k$. Decompose this coordinatewise convergent iterated B-C-H series into two parts: $x \otimes t = W + V$, where $W = \sum_{i=1}^{m-1} a_i$ and $V = \sum_{i=m}^{\infty} a_i$, where a_i for $i = 1, 2, \ldots,$ is the i-th homogeneous part. Due to the graded structure of $\mathfrak{g}^{\mathbf{N}}$, the $m-1$ initial coordinates of V vanish and its $i-th$ coordinate v_i belongs to $\mathfrak{g}^m$ for $i = m, m+1, \ldots$. For $W = \sum_{j=1}^{\infty} w_j t^j$ the equality $x \otimes t = W + V$ implies that $w_1 = x$ and $w_j = 0$ for $1 < j \leq m-1$. Also $w_j = -v_j$ for $j \geq m$. Since $f_j \in g_{1,0}^{\mathbf{N}}$ for $j = 1 \ldots k$, each a_j is in $\mathfrak{g}_{1,0}^{\mathbf{N}}$ and thus $W \in \mathfrak{g}_{1,0}^{\mathbf{N}}$. It follows that $x = \sum_{i=1}^{m-1} w_i = -\sum_{i=m}^{\infty} w_i = \sum_{i=m}^{\infty} v_i \in \bar{\mathfrak{g}}^m$. ■

Proposition 3.3. *The following are equivalent:*

(a) $\Gamma_{\mathfrak{g}}$ *is closed.*

(b) $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ *is closed in* $\exp \mathfrak{g}_1^{\mathbf{N}}$

Proof. We claim that there exist a neighbourhood U of 0 in $\mathfrak{g}_1^{\mathbf{N}}$, a neighbourhood V_1 of 0 in $\mathfrak{g}$ and a neighbourhood V_2 of 0 in $\mathfrak{g}_{1,0}^{\mathbf{N}}$ such that the mapping

$$\sigma : V_1 \times V_2 \ni (k, \phi) \to k \circ \phi \in U$$

is a homeomorphism identifying k with $k \otimes t$. Indeed σ is defined on the product of suitable neighbourhoods $\tilde{V_1}$ of 0 in $\mathfrak{g}$ and $\tilde{V_2}$ of 0 in $\mathfrak{g}_{1,0}^{\mathbf{N}}$, it is differentiable and $d_{(0,0)}\sigma(x, k) = x \otimes t + k$, in particular $d_{(0,0)}\sigma$ is an isomorphism. The claim follows.

Let V_1 and V_2 be such neighbourhoods. Observe that the assertion of the proposition follows from the equality

$$U \cap \exp \mathfrak{g}^{\mathbf{N}}_{1,0} = (\Gamma_{\mathfrak{g}} \cap V_1) \circ V_2 \qquad (3.3).$$

The inclusion $(\Gamma_{\mathfrak{g}} \cap V_1) \circ V_2 \subset U \cap \exp \mathfrak{g}^{\mathbf{N}}_{1,0}$ is obvious.

To show the reverse inclusion, let $f = (x \otimes t) \circ h \in \exp \mathfrak{g}^{\mathbf{N}}_{1,0} \cap U$ where $x \in V_1$ and $h \in V_2$. Then $x \otimes t \in \exp \mathfrak{g}^{\mathbf{N}}_{1,0}$, i.e. $x \in \Gamma_{\mathfrak{g}} \cap V_1$. ■

Proposition 3.4.

(a) *If $\mathfrak{g}$ is separable then $\Gamma_{\mathfrak{g}}$ is at most countable.*

(b) *If $\mathfrak{g}$ is separable and $\Gamma_{\mathfrak{g}}$ is closed then $\Gamma_{\mathfrak{g}}$ is discrete.*

Proof. (a) By Lemma 6.8 (b), $B_r \cap \mathfrak{g}^{\mathbf{N}}_{1,0}$ is a neighbourhood of the unit in $\exp \mathfrak{g}^{\mathbf{N}}_{1,0}$. Let $x \in \Gamma_{\mathfrak{g}}$. Since x is in the center of $\mathfrak{g}$ hence $(x \otimes t) + (B_r \cap \mathfrak{g}^{\mathbf{N}}_{1,0}) = (x \otimes t) \circ (B_r \cap \mathfrak{g}^{\mathbf{N}}_{1,0})$ thus $(x \otimes t) + (B_r \cap \mathfrak{g}^{\mathbf{N}}_{1,0})$ is a neighbourhood of $x \otimes t$ in $\exp \mathfrak{g}^{\mathbf{N}}_{1,0}$. Observe that for $x_1 \neq x_2$ the corresponding neighbourhoods are disjoint and moreover $(y \otimes t) \in (x \otimes t) + (B_r \cap \mathfrak{g}^{\mathbf{N}}_{1,0})$ implies $x = y$. Since for $\mathfrak{g}$ separable, $\exp \mathfrak{g}^{\mathbf{N}}_{1,0}$ is also separable, this yields the statement.

(b) If $\Gamma_{\mathfrak{g}}$ is closed but not discrete then being a complete and perfect metric space it cannot be countable, contrary to (a). ■

Proposition 3.5. *Let $\mathfrak{g}$ be a finite dimensional real or complex Lie algebra. Then* $\Gamma_{\mathfrak{g}} = \{0\}$.

Proof. Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of the real normed Lie algebra $\mathfrak{g}$. From Definition 3.1 one deduces that $\Gamma_{\mathfrak{g}} \subset \Gamma_{\mathfrak{g}_{\mathbf{C}}}$. It follows that we may restrict attention to the complex case.

Let $\mathfrak{g} = S + R$ be a Levi decomposition of $\mathfrak{g}$, i.e. S is a semisimple (Levi) subalgebra of $\mathfrak{g}$ and R is the radical of $\mathfrak{g}$. Let $x \in \Gamma_{\mathfrak{g}}$ so that $x \otimes t = \circ \prod_{j=1}^{k} f_j$, where $f_j \in \mathfrak{g}^{\mathbf{N}}_{1,0}$ for $j = 1, \ldots, k$. With no loss of generality we may assume that each f_j is as small as we wish. Decomposing $f_j = s_j + r_j$ for $j = 1, 2, \ldots, k$, where $s_j \in S^{\mathbf{N}}_{1,0}$ and $r_j \in R^{\mathbf{N}}_{1,0}$, and applying the inverse mapping theorem to the function $F : S^{\mathbf{N}}_{1,0} \times R^{\mathbf{N}}_{1,0} \ni (s, r) \to s \circ r \in \mathfrak{g}^{\mathbf{N}}_{1,0}$ (cf. the proof of Proposition 3.3), we may write each f_j as the B-C-H product $f_j = s'_j \circ r'_j$ where $s'_j \in S^{\mathbf{N}}_{1,0}$ and $r'_j \in R^{\mathbf{N}}_{1,0}$. Changing the order and applying the fact that $\exp R^{\mathbf{N}}_{1,0}$ is a normal subgroup of $\exp \mathfrak{g}^{\mathbf{N}}_{1,0}$, we get

$$x \otimes t = \circ \prod_{j=1}^{k} (s'_j \circ r'_j) = s_0 \circ \prod_{j=1}^{k} r''_j, \qquad (3.4)$$

where $r''_j \in R^{\mathbf{N}}_{1,0}$ for $j = 1, \ldots, k$ and $s_0 = \circ \prod_{j=1}^{k} s'_j \in \exp S^{\mathbf{N}}_{1,0}$ has its coordinates in S.

Thus each coordinate of $x \otimes t$ is the sum of the corresponding coordinate of s_0 which belongs to S and an element of R. It follows that $s_0 = 0$, i.e. $x \otimes t \in \exp R^{\mathbf{N}}_{1,0}$.

Let $R = A + N$ be the Iwasawa decomposition of R, where A is an abelian subalgebra and N is a nilpotent ideal. Arguments as before with A substituting S and N substituting R yield $x \otimes t \in \exp N^{\mathbf{N}}_{1,0}$. But then Proposition 3.2(c) implies that $x \in \bar{N}^{\infty} = \{0\}$. ■

4. Results

For a Banach-Lie algebra $\mathfrak{g}$ with discriminant subgroup $\Gamma_{\mathfrak{g}}$ and $\exp \mathfrak{g}_1^{\mathbf{N}}$ equipped with the metric (2.11) define

$$G = \exp \mathfrak{g}_1^{\mathbf{N}} / \exp \mathfrak{g}_{1,0}^{\mathbf{N}}. \tag{4.1}$$

Let h be the Banach-Lie algebra corresponding to a Banach-Lie group H. Let $I : \mathfrak{g} \to h$ be a continuous homomorphism. Then $\delta : \mathfrak{g}_1^{\mathbf{N}} \ni f \to I(\alpha(f)) \in h$ is a Lie algebra homomorphism which may be also treated as a continuous homomorphism of the corresponding local B-C-H groups. Since these local groups may be identified respectively with a local group of $\exp \mathfrak{g}_1^{\mathbf{N}}$ and a local group of H (the first by Proposition 2.5(b)), and moreover the group $\exp \mathfrak{g}_1^{\mathbf{N}}$ is simply connected, the local homomorphism δ may be extended to a global continuous homomorphism $\tilde{\delta} : \exp \mathfrak{g}_1^{\mathbf{N}} \to H$.

Lemma 4.1.

(a) *If I is injective then $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp \mathfrak{g}_1^{\mathbf{N}}$, and G is a Banach-Lie group corresponding to $\mathfrak{g}$. Moreover, $\Gamma_{\mathfrak{g}}$ is discrete.*

(b) *If I is an isomorphism and H is simply connected, then $\ker \tilde{\delta} = \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ and $H = G$.*

Proof. (a) Since by Lemma 6.8(a), $U \subset \mathfrak{g}_1^{\mathbf{N}}$ for a sufficiently small neighbourhood U of e in $\exp \mathfrak{g}_1^{\mathbf{N}}$ and moreover $\tilde{\delta}(f) = I(\alpha(f))$ for $f \in U$, we get $\ker \tilde{\delta} \cap U = \mathfrak{g}_{1,0}^{\mathbf{N}} \cap U$ and the inclusion $\exp \mathfrak{g}_{1,0}^{\mathbf{N}} \subset \ker \tilde{\delta}$ implies $\mathfrak{g}_{1,0}^{\mathbf{N}} \cap U = \exp \mathfrak{g}_{1,0}^{\mathbf{N}} \cap U$. It follows that $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp g_1^{\mathbf{N}}$ and that $\exp g_{1,0}^{\mathbf{N}}$ is the connected component of the identity in $\ker \tilde{\delta}$.

Considering if necessary a smaller neighbourhood V of e in $\exp \mathfrak{g}_1^{\mathbf{N}}$ we observe that for $x_1, x_2 \in V$ their cosets mod $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ are equal iff $\alpha(x_1) = \alpha(x_2)$. Thus α maps a neighbourhood of unity in G to a neighbourhood of 0 in $\mathfrak{g}$ transferring the quotient group multiplication to the B-C-H multiplication in $\mathfrak{g}$.

We also observe that $U \cap (\mathfrak{g} \otimes t) \cap \exp \mathfrak{g}_{1,0}^{\mathbf{N}} = \{0\}$, hence $\Gamma_{\mathfrak{g}}$ is discrete.

(b) If I is an isomorphism, then δ is open and hence also $\tilde{\delta}$ is open. It follows that $H = \exp \mathfrak{g}_1^{\mathbf{N}} / \ker \tilde{\delta}$ topologically. Thus the inclusion $\exp \mathfrak{g}_{1,0}^{\mathbf{N}} \subset \ker \tilde{\delta}$ induces a continuous homomorphism of G onto H. Since both groups have the same Lie algebra, are connected and H is simply connected this homomorphism has to be injective, i.e. $\ker \tilde{\delta} = \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$. ■

Theorem 4.2.

(a) *If $\mathfrak{g}$ is the Lie algebra of a simply connected Banach-Lie group H then $\Gamma_{\mathfrak{g}}$ is discrete, $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp \mathfrak{g}_1^{\mathbf{N}}$, and $H \cong G$. Moreover, the exponential map $\exp : \mathfrak{g} \to G$, may be interpreted as the restriction to $\mathfrak{g} \otimes t$ of the quotient homomorphism $\pi : \exp \mathfrak{g}_1^{\mathbf{N}} \to G$.*

(b) *If $\Gamma_{\mathfrak{g}}$ is discrete, then $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp \mathfrak{g}_1^{\mathbf{N}}$ and G is the simply connected Banach-Lie group corresponding to $\mathfrak{g}$.*

Proof. (a) The first part results from Lemma 4.1(a) and (b). For the second part, given $x \in \mathfrak{g}$ the function $\phi_x : \mathbf{R} \ni s \to \pi(sx \otimes t) \in G$ is a continuous one-parameter subgroup of G, and identifying a local group of G with the appropriate local B-C-H group of $\mathfrak{g}$ we infer that the groups ϕ_x for $x \in \mathfrak{g}$ are the only continuous one-parameter subgroups of G. It follows that $\exp(x) = \phi_x(1) = \pi(x \otimes t)$.

(b) By Proposition 3.3, $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$. For G defined by (4.1) consider the map $\phi : \mathfrak{g} \ni x \to \pi(x \otimes t) \in G$ where $\pi : \exp \mathfrak{g}_{1,0}^{\mathbf{N}} \to G$ is the quotient homomorphism. Applying σ defined in the proof of Proposition 3.3 (a) and Lemma 6.8 (a) we see that restriction of ϕ to a sufficiently small neighbourhood of 0 is an open map.

Observe that for discrete $\Gamma_{\mathfrak{g}}$ and suitably small $x, y \in \mathfrak{g}$ equality $\phi(x) = \phi(y)$ is equivalent to $\pi((x \otimes t)(y \otimes t)^{-1}) \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ and this by (3.3) implies $((x \otimes t) \circ (y \otimes t)^{-1}) \in \mathfrak{g}_{1,0}^{\mathbf{N}}$ Hence $0 = x \otimes (-y) = \alpha((x \otimes t) \circ (y \otimes t)^{-1})$ i.e. $-x = -y$ .It follows that ϕ restricted to a suitably small neighbourhood of 0 maps it homeomorphically onto a neighbourhood of the unit in G.

Note also that for a suitably small neighbourhood U of 0 in $\mathfrak{g}$ (3.3) yields

$$(\phi|_U)^{-1} = \alpha|_{\phi(u)}. \tag{4.2}$$

This shows that a local group of G is isomorphic to the appropriate local B-C-H group of $\mathfrak{g}$. Thus G is a Lie group corresponding to $\mathfrak{g}$. Part (a) now entails that G is simply connected. ∎

Proposition 3.2 (b) may now be formulated in a more precise way:

Corollary 4.3. *If $\mathfrak{g}$ is the Lie algebra of a simply connected Banach-Lie group G, and $\exp : \mathfrak{g} \to G$ is the exponential map, then*

$$E_{\mathfrak{g}} = \{x \in \mathfrak{g} : \exp x = e\}. \tag{4.3}$$

Proof. By Theorem 4.2(a),

$$E_{\mathfrak{g}} = \{x \in \mathfrak{g} : x \otimes t \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}\} = \{x \in \mathfrak{g} : \pi(x \otimes t) = e\} = \{x \in \mathfrak{g} : \exp(x) = e\}. \quad ∎$$

Example 4.4. Let $\mathfrak{g}$ be the Lie algebra of all bounded linear operators on an infinite dimensional Hilbert space H. Then $\Gamma_{\mathfrak{g}}$ consists of all $2k\pi i$-multiples of the identity operator I where k is an integer.

Indeed, $\mathfrak{g}$ is the Lie algebra of the group of all bounded and invertible endomorphisms of H, which by Kuiper's theorem [13] is simply connected. Thus Corollary 4.3 implies the conclusion.

Theorem 4.5. *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then $\Gamma_{\mathfrak{g}} = \{0\}$, $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is closed in $\exp \mathfrak{g}_1^{\mathbf{N}}$ and $G = \exp \mathfrak{g}_1^{\mathbf{N}} / \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is the simply connected Lie group of $\mathfrak{g}$.*

Proof. Since $\Gamma_{\mathfrak{g}} = \{0\}$ by Proposition 3.5, we can apply Theorem 4.2(b). ∎

Definition 4.6. Let $[\mathfrak{g}, \mathfrak{g}]_0 = \{[x, y] : x, y \in \mathfrak{g}\}$ and let $\Gamma_{\mathfrak{g}}^0 = \Gamma_{\mathfrak{g}} \cap [\mathfrak{g}, \mathfrak{g}]_0$.

Theorem 4.7. *For $\mathfrak{g}$ a Banach-Lie algebra let $\rho : \mathfrak{g} \to A$ be a continuous homomorphism into an associative Banach algebra A. Then $\rho(\Gamma^0_{\mathfrak{g}}) = 0$.*

Proof. With no loss of generality, we may assume that A is complex, unital and that $\rho(\mathfrak{g})$ generates A. Let us apply Lemma 4.1 with $h = A$, and $H = \tilde{A}$, where $\tilde{A}$ is the group of units of A. We get a local homomorphism δ of the form $\delta(f) = \exp(\rho(\alpha f))$ for small $f \in \mathfrak{g}^{\mathbf{N}}_{1,0}$, where $\exp : A \to \tilde{A}$.

Let us observe that for $x \in \mathfrak{g}$ and $\tilde{\delta}$ as in Lemma 4.1 one has

$$\tilde{\delta}(x \otimes t) = \exp(\rho(x)). \tag{4.4}$$

Indeed, (4.4) holds for small x by the definition of δ. Since $x \otimes t = (\frac{x}{n} \otimes t)^n$ we get

$$\tilde{\delta}(x \otimes t) = (\tilde{\delta}(\frac{x}{n} \otimes t))^n = (\exp \rho(\frac{x}{n}))^n = \exp(\rho(x)).$$

Let $x_0 \in \Gamma^0_{\mathfrak{g}}$. Suppose that $\rho(x_0) \neq 0$. Then the one-parameter subgroup $\mathbf{R} \ni s \to \phi(s) = \tilde{\delta}(sx_0 \otimes t)$ is central, non-trivial and 1-periodic. Consider the induced representation of the circle group $T = \mathbf{R}/\mathbf{Z}$ and the resulting representation $C(T) \ni f \to a_f \in A$ of the convolution algebra $C(T)$ of all continuous complex functions on T, where

$$a_f = \int_T f(s)\phi(s)ds \qquad (4.5).$$

(integration with respect to the normalized Lebesgue measure.) The condition $\rho(x_0) \neq 0$ implies that there exists a non-zero $b = a_f$ with $f(s) = e^{2k\pi is}$ for a certain integer k. Observe that b is central and $b = b^2$. Thus we get a decomposition of A into the direct sum of ideals:

$$A = bA \oplus (e - b)A.$$

Consider the representation induced by ρ in bA. Invariance of the measure ds yields $\phi(s) \cdot b = e^{2k\pi is}b$ for $s \in T$, hence the operators of multiplication from the left by $\phi(s)$ when restricted to bA are multiples of the identity. This contradicts the Wielands theorem (canonical commutation relations cannot be realized in a Banach algebra) (cf. also [12], Problem 182). ■

As observed by the referee, another shorter proof of Theorem 4.7 may be obtained by reduction via Corollary 4.3 to the following Lemma of van Est and Świerczkowski ([8] p.54)

Lemma . *Let G be a simply connected Banach-Lie group such that its Lie algebra $\mathfrak{g}$ contains two elements p, q with the properties*

(i) $0 \neq [p, q] \in$ *centre of* $\mathfrak{g}$.

(ii) *the one-parameter subgroup tangent to* $[p, q]$ *is a circle.*

Then $\mathfrak{g}$ is not faithfully representable.

The first step of reduction is the observation that Theorem 4.7 may be equivalently formulated in the following way:

Theorem 4.7'. *For $\mathfrak{g}'$ a Banach-Lie algebra let $\rho' : \mathfrak{g}' \to A$ be a continuous injective homomorphism into an associative Banach algebra A. Then $(\Gamma_{\mathfrak{g}'})^0 = \{0\}$.*

Proof. To show the equivalence of theorems 4.7 and 4.7' note that Theorem 4.7' is a special case of Theorem 4.7.

Assume that Theorem 4.7' holds true and let $\mathfrak{g}, \rho, A$ be as in assumption of the Theorem 4.7. For $\mathfrak{g}' = \mathfrak{g}/\ker\rho$ consider the induced representation $\rho' : \mathfrak{g}' \to A$. Note that for a continuous surjective homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}'$ one has

$$\pi(\Gamma_{\mathfrak{g}}) \subset \Gamma_{\mathfrak{g}'}. \tag{4.5}$$

Substituting here for π the quotient homomorphism, we see that the condition $(\Gamma_{\mathfrak{g}'})^0 = \{0\}$ implies the assertion of Theorem 4.7. ∎

Proof of the Theorem 4.7' Let z' be the center of $\mathfrak{g}'$. Since $\mathfrak{g}'$ admits a continuous injective representation into an associative Banach algebra, $\mathfrak{g}'$ is enlargeable. Let G' be its simply connected Banach-Lie group and let $\exp \mathfrak{g}' \to G'$ be the exponential map. By Corollary 4.3 $\Gamma_{\mathfrak{g}'} = \ker\exp \cap z'$. Thus each nonzero element of $(\Gamma_{\mathfrak{g}'})^0$ would confirm with the assumptions of the Lemma what contradicts the injectivity of ρ'. ∎

5. Remarks

In this paper, we describe intrinsic obstructions to integration of a Lie algebra $\mathfrak{g}$ in terms of the discriminant subgroup $\Gamma_{\mathfrak{g}}$. We do not know whether it is equal to the period group $\Pi(\mathfrak{g})$ of van Est and Korthagen. The following was observed by the referee of the paper:

Proposition 5.1. *The following are equivalent*

(a) $\Gamma_{\mathfrak{g}}$ *is discrete.*

(b) $\Pi(\mathfrak{g})$ *is discrete.*

(c) $\mathfrak{g}$ *is enlargeable.*

Proof. The conditions (b) and (c) are equivalent by [7], the conditions (a) and (c) are equivalent by Theorem 4.2. ∎

It was also observed by the referee that each of the conditions (a),(b),(c) implies $\Pi(\mathfrak{g}) = \Gamma_{\mathfrak{g}}$. In fact, (c) implies by Corollary 4.3 that $\Gamma_{\mathfrak{g}} = \ker\exp \cap z$,where z is the center of $\mathfrak{g}$. The same characterization holds for $\Pi(g)$, namely Proposition III.8 in Glöckner-Neeb [10] asserts that (c) implies $\Pi(\mathfrak{g}) = \ker\exp \cap z$.

It was noted by the referee of the earlier version of the paper that the statement (a) of Proposition 3.4 which was observed earlier implies (b) of the same Proposition. This raises the question whether for non-separable $\mathfrak{g}$ the situation when $\Gamma_{\mathfrak{g}}$ is closed but not discrete may really occur. In such a case our method of integration would provide a new class of topological groups.

Representing a simply connected Lie group G in the form (4.1), we get a new insight into finite dimensional Lie group theory. For instance, Corollary 4.3 and Proposition 3.2(c) imply the statement:

If $x \in \mathfrak{g}$ generates a periodic one-parameter subgroup in the associated simply connected group G then $x \in \mathfrak{g}^\infty$.

Nonenlargeability of a Banach-Lie algebra $\mathfrak{g}$ may be equivalently formulated by stating that the local B-C-H group of $\mathfrak{g}$ does not embed in any global group. Let us note (cf. [14]) that a necessary and sufficient condition for an abstract local group H to embed in a global group is the following *generalized associativity law:*

for each $m \in \mathbf{N}$, each m-tuple $A = (a_1, a_2, \ldots, a_m)$ of elements of H and each distribution of parentheses (indicating how to evaluate A in H), the evaluated elements (if they exist) do not depend on this distribution.

It follows that a Banach-Lie algebra $\mathfrak{g}$ is not integrable to a Banach-Lie group iff for each neighbourhood V of zero in $\mathfrak{g}$ there exists a positive integer m and an m-tuple $A = (x_1, x_2, \ldots, x_m)$ of elements of V such that for two distributions of parentheses the resulting iterated B-C-H products (say w_1 and w_2) do exist in V but are different.

We claim that in this situation $w_1 \circ w_2^{-1} \in \Gamma_{\mathfrak{g}}$ povided this product can be defined. For this, given $m \in \mathbf{N}$, an m-tuple $A = (x_1, x_2, \ldots, x_m)$ of elements of $\mathfrak{g}$ and a distribution of parentheses, simultaneously with $w \in \exp_0 \mathfrak{g}$ resulting as the iterated B-C-H product we consider $\tilde{w} \in \exp \mathfrak{g}_1^{\mathbf{N}}$ obtained similarly to w but starting from $x_i \otimes t$ instead of x_i $(i = 1, \ldots, m)$ and using the B-C-H product in $\exp \mathfrak{g}_1^{\mathbf{N}}$ instead of that in $\exp_0(\mathfrak{g})$. Since $\exp \mathfrak{g}_1^{\mathbf{N}}$ is a full group, all the elements $\tilde{w}$ obtained for a fixed m-tuple A and any distributions of parentheses are equal. Elements $w \in \mathfrak{g}$ and $\tilde{w} \in \exp \mathfrak{g}_1^{\mathbf{N}}$ resulting from a common A and common choice of parentheses will be called similar. The least m such that an m-tuple A with some choice of parentheses produces $\tilde{w}$ will be called the length of $\tilde{w}$.

Our claim may be deduced from the following statement:

Lemma 5.2. *Let $w \in \mathfrak{g}$ and $\tilde{w} \in \exp \mathfrak{g}_1^{\mathbf{N}}$ be a pair of similar elements. Then*

$$(w \otimes t) \circ \tilde{w}^{-1} \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}. \tag{5.1}$$

Before proving the lemma we indicate how the claim may be deduced from it. For a fixed A let w_1, w_2 be obtained for certain choices of parentheses. Let $\tilde{w_1}, \tilde{w_2}$ be the similar elements to w_1, w_2 respectively. Then $\tilde{w_1} = \tilde{w_2}$ and by (5.1),

$$w_1 \circ w_2^{-1} \otimes t = (w_1 \circ w_2^{-1} \otimes t) \circ (\tilde{w_1} \circ \tilde{w_2}^{-1})^{-1} \in \exp g_{1,0}^{\mathbf{N}}.$$

Proof. We proceed by induction on the length of $\tilde{w}$. (5.1) is obvious for length 1.

Observe that for sufficiently small elements $x, y \in \mathfrak{g}$ we get

$$v = ((x \circ y) \otimes t) \circ ((x \otimes t) \circ (y \otimes t))^{-1}) \in \mathfrak{g}_1^{\mathbf{N}}$$

by Proposition 6.1. Moreover $\alpha : \mathfrak{g}_{1,0}^{\mathbf{N}} \to \mathfrak{g}$ being a homomorphism of local B-C-H groups, $\alpha(v) = x \circ y - (\alpha(x \otimes t) \circ \alpha(y \otimes t)) = 0$, i.e.

$$((x \circ y) \otimes t) \circ ((x \otimes t) \circ (y \otimes t))^{-1} \in \mathfrak{g}_{1,0}^{\mathbf{N}}. \tag{5.2}$$

Suppose that (5.1) holds for each pair $(z, \tilde{z})$ with the length of $\tilde{z}$ less than n and consider a pair $(w, \tilde{w})$ with the length of $\tilde{w}$ equal to n. Let $w = w_1 \circ w_2$ and $\tilde{w} = \tilde{w_1} \circ \tilde{w_2}$ with the length of $\tilde{w_j}$ for $j = 1, 2$ less than n. Applying (5.2) we get the following equalities modulo $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ which hold provided w_1 and w_2 are sufficiently small :

$$\begin{aligned}(w \otimes t) \circ \tilde{w}^{-1} &= (w_1 \circ w_2 \otimes t) \circ ((w_1 \otimes t) \circ (w_2 \otimes t))^{-1} \circ ((w_1 \otimes t) \circ (w_2 \otimes t)) \circ (\tilde{w_1} \circ \tilde{w_2})^{-1} = \\ &= (w_1 \otimes t) \circ ((w_2 \otimes t)) \circ \tilde{w_2}^{-1}) \circ (w_1 \otimes t)^{-1} \circ (w_1 \otimes t) \circ \tilde{w_1}^{-1}.\end{aligned}$$

Since $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$ is a normal subgroup of $\exp \mathfrak{g}_1^{\mathbf{N}}$, the induction hypothesis gives (5.1). ■

Question 5.3 *Given a neighbourhood V of 0 in $\mathfrak{g}$, let $\Gamma_{\mathfrak{g},V}$ be the additive subgroup of $\mathfrak{g}$ generated by all the elements $w_1 \circ w_2^{-1}$ with $w_1, w_2, w_1 \circ w_2^{-1} \in V$ and w_1, w_2 having a common similar element. What is the relation of $\Gamma_{\mathfrak{g}}$ and $\Gamma_{\mathfrak{g},V}$?*

We believe that the present approach is not limited to the class of normed Lie algebras which are complete. Since adjustments to meet a more general situation would increase the volume, we leave it to a future publication.

6. Appendix

In this section we present some technical details omitted in the main text for its transparency. We organize it in two subsections corresponding to the main subjects treated. These are: the norm inequalities in local B-C-H groups and the 'group norms' and resulting Banach manifold structure on $\exp \mathfrak{g}_1^{\mathbf{N}}$ and on $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$.

6.1. Norm inequalities in local Banach-Lie groups

Let $\mathfrak{g}$ be a Banach-Lie algebra equipped with a norm satisfying condition (2.3). For $y_1, y_2 \in \mathfrak{g}$ consider the series $\Theta(y_1, y_2)$ (cf. (2.2)). If this series converges we denote its sum by $y_1 \circ y_2$ and we say that $y_1 \circ y_2$ is correctly defined. Proceeding by induction assume that $y_1 \circ \ldots \circ y_{m-1}$ is correctly defined and consider the series $\Theta(y_1 \circ \ldots \circ y_{m-1}, y_m)$. If it converges we denote its sum by $y_1 \circ \ldots \circ y_m$ and we say that $y_1 \circ \ldots \circ y_m$ is correctly defined.

Proposition 6.1. *There exists $r > 0$ such that for each integer $m \geq 2$ the inequality*

$$\sum_{i=1}^{m} \| y_i \| < r$$

implies that $y_1 \circ \ldots \circ y_m$ is correctly defined.

Proof. Using the Taylor expansion at 0 of the function $\Theta(x, y)$ we infer (compare Remark 6.3 below) that there exists a positive constant R (which may and will be assumed to be less than $\frac{1}{2}\ln 2$) and a positive constant M not depending on the particular choice of $\mathfrak{g}$, such that for $x, y \in \mathfrak{g}$ with $\| x \| \leq R$, $\| y \| \leq R$ one has

$$\|x \circ y - (x + y)\| \leq M\|x\| \cdot \|y\|. \tag{6.1.1}$$

Note that (6.1.1) implies

$$1 + M\|x \circ y\| \leq (1 + M\|x\| \cdot (1 + M\|y\|). \tag{6.1.2}$$

Let $r = \frac{1}{M}\ln(1 + MR)$. We claim that for each integer $m \geq 2$ and $y_1, \ldots, y_m \in g$ the inequality $\sum_{i=1}^{m} \|y_i\| < r$ implies that

$$y_1 \circ \ldots \circ y_m \text{ is correctly defined,} \tag{a}$$

$$1+M\|y_1\circ\ldots\circ y_m\|\le(1+M\|y_1\|)\cdot\ldots\cdot(1+M\|y_m\|), \qquad (b) \qquad (6.1.3)$$

$$\|y_1\circ\ldots\circ y_m\|\le R. \qquad (c)$$

We prove (6.1.3) by induction. For $m=2$ observe that $r<R\le\frac{1}{2}\ln 2$ and thus (6.1.3)(a) holds. Also (6.1.3)(b) reduces to (6.1.2). To prove (6.1.3)(c) note that (6.1.2) implies

$$\begin{aligned}\ln(1+M\|y_1\circ y_2\|)&\le\ln(1+M\|y_1\|)+\ln(1+M\|y_2\|)\\ &\le M(\|y_1\|+\|y_2\|)\le\ln(1+MR)\end{aligned}$$

and so $\|y_1\circ y_2\|\le R$.

Passing to the general case assume that $\sum_{i=1}^m\|y_i\|<r$. Then $\sum_{i=1}^{m-1}\|y_i\|<r$ so by the induction hypothesis $x=y_1\circ\ldots\circ y_{m-1}$ is correctly defined and has norm at most R. Moreover, since $\|y_m\|\le R$ and $R\le\frac{1}{2}\ln 2$ also $y_1\circ\ldots\circ y_m$ is correctly defined.

For (6.1.3)(b), applying (6.1.2) and the induction hypothesis (6.1.3)(b) with respect to x one gets

$$1+M\|y_1\circ\ldots\circ y_m\|=1+M\|x\circ y_m\|$$

$$\le(1+M\|x\|)\cdot(1+M\|y_m\|)\le(1+M\|y_1\|)\cdot\ldots\cdot(1+M\|y_m\|).$$

To prove (6.1.3)(c) note that (6.1.3)(b) implies

$$\ln(1+M\|y_1\circ\ldots\circ y_m\|)\le\sum_{i=1}^m\ln(1+M\|y_i\|)\le M\sum_{i=1}^m\|y_i\|\le\ln(1+MR)$$

and so $\|y_1\circ\ldots\circ y_m\|\le R$. This completes the proof. ∎

For $\mathbf{N}$ the set of positive integers and $x\in\mathfrak{g}$ define

$$\|x\|_\circ=\inf\ \{\sum_{i=1}^m\|y_i\|:\quad x=y_1\circ\ldots\circ y_m,\quad m\in\mathbf{N}\}. \qquad (6.1.4)$$

Observe that $\|x\|_\circ$ is well defined, since each $x\in\mathfrak{g}$ has an obvious representation with $m=1$ and $y_1=x$. Moreover

$$\|x\|_\circ\le\|x\|. \qquad (6.1.5)$$

Proposition 6.2. *There exist $r>0$ and $L\ge1$ such that $\|x\|_\circ<r$ implies*

$$\|x\|\le L\|x\|_\circ. \qquad (6.1.6)$$

Proof. Let r,R and M be as in the proof of Proposition 6.1. Let K be such that for $0\le s\le R$,

$$Ks\le\ln(1+Ms),$$

and let

$$w=y_1\circ\ldots\circ y_m \qquad \text{with} \quad \sum_{i=1}^m\|\,y_i\,\|<r. \qquad (6.1.7)$$

As in the proof of (6.1.3)(c) we get

$$K \parallel w \parallel \leq \ln(1 + M \parallel w \parallel) \leq \ln(\prod_{i=1}^{m}(1 + M \parallel y_i \parallel)) \leq M \sum_{i=1}^{m} \parallel y_i \parallel$$

and letting $L = \frac{M}{K}$ gives

$$\parallel w \parallel \leq L \sum_{i=1}^{m} \parallel y_i \parallel$$

for w satisfying the condition (6.1.7). This gives the required conclusion. ■

Remark 6.3. The constants $r < \frac{\ln 2}{2}$ and $L > 1$ in Proposition 6.2 may be chosen universally for all Banach-Lie algebras, provided the norm satisfies the condition (2.3).

Since we do not need the Remark and moreover our proof, based on modified Dynkin estimates (like the proof of Lemma 6.8(c) below) is rather long we decided not to present it here.

6.2. Group norms

Throughout the rest of this section $r > 0$ and $L \geq 1$ are the constants introduced in Proposition 6.2.

Definition 6.4. Let P be a group. Put $\{f, g\} = fgf^{-1}g^{-1}$ for $f, g \in P$. A function $P \ni f \to \parallel f \parallel \in \mathbf{R}^+$ satisfying

$$\|f\| = 0 \Leftrightarrow f = e, \qquad (a)$$
$$\|f \cdot g\| \leq \|f\| + \|g\|, \qquad (b)$$
$$\|f^{-1}\| = \|f\|, \qquad (c) \qquad (6.2.1)$$
$$\exists_{\epsilon>0} \ \exists_{C>0}(\parallel f \parallel < \epsilon, \parallel g \parallel < \epsilon) \Rightarrow (\parallel \{f, g\} \parallel \leq C \parallel f \parallel \cdot \parallel g \parallel)$$
$$\text{and} \ \ \{g \in G : \parallel g \parallel < \epsilon\} \ \ \text{generates} \ \ G \qquad (d)$$

is said to be a group norm on P.

The following two propositions correspond to well-known facts in the theory of normed linear spaces. We omit the proofs.

Proposition 6.5. *Let $\|\cdot\|$ be a group norm on P. Then*

(a) *The function*

$$P \times P \ni (f, g) \to \rho_l(f, g) = \|f^{-1} \cdot g\| \in \mathbf{R}^+ \qquad (6.2.2)$$

is a left-invariant metric providing P with a topological group structure.

(b) *The function*

$$P \times P \ni (f, g) \to \rho_r(f, g) = \|f \cdot g^{-1}\| \in \mathbf{R}^+ \qquad (6.2.3)$$

is a right-invariant metric providing P with a topological group structure.

(c) *The metrics* (6.2.2) *and* (6.2.3) *are equivalent and define the same topology on P.*

A topological group equipped with a group norm which induces the topology will be called a *normed group.* The topology induced by a group norm will be referred to as a *norm topology.*

Proposition 6.6. *Let P be a normed group with a closed normal subgroup H and let $Q = P/H$. Then*

$$Q \ni fH \to \| fH \|_{P/H} = \inf_{h \in H} \| f \cdot h \| = \inf_{h \in H} \| h \cdot f \| \in \mathbf{R}^+$$

is a group norm inducing the quotient topology of Q.

Proposition 6.7. *Let $\mathfrak{g}$ be a Banach-Lie algebra with norm $\| \cdot \|$. Then*

(a) *The function* (2.10) *is a group norm on* $\exp \mathfrak{g}_1^{\mathbf{N}}$.

(b) *The function* (2.11) *is a group norm on* $\exp \mathfrak{g}_{1,0}^{\mathbf{N}}$.

To prove this proposition we will need the following properties of the functions (2.10) and (2.11).

Lemma 6.8.

(a) *If $f \in \exp \mathfrak{g}_1^{\mathbf{N}}$ and $\| f \|_{(1)} < r$, then $f \in \mathfrak{g}_1^{\mathbf{N}}$ and*

$$\| f \|_{(1)} \leq \| f \|_1 \leq L \| f \|_{(1)} . \tag{6.2.4}$$

(b) *If $f \in \exp g_{1,0}^{\mathbf{N}}$ and $\| f \|_{(1,0)} < r$ then $f \in \mathfrak{g}_{1,0}^{\mathbf{N}}$ and*

$$\| f \|_{(1,0)} \leq \| f \|_1 \leq L \| f \|_{(1,0)} . \tag{6.2.5}$$

(c) *There exist constants $C_1, C_2 > 0$ and $0 < \alpha \leq r$ such that for $f_1, f_2 \in \mathfrak{g}_1^{\mathbf{N}}$ satisfying $\| f_1 \|_1 + \| f_2 \|_1 \leq \alpha$,*

$$C_1 \| f_1 - f_2 \|_1 \leq \| f_1 \circ f_2^{-1} \|_1 \leq C_2 \| f_1 - f_2 \|_1 . \tag{6.2.6}$$

(d) *If $f_1, f_2 \in \exp \mathfrak{g}_1^{\mathbf{N}}$, then*

$$\|f_1 \circ f_2 \circ f_1^{-1}\|_{(1)} \leq \|f_2\|_{(1)} e^{\|f_1\|_{(1)}}. \tag{6.2.7}$$

If $f_1 \in \exp \mathfrak{g}_1^{\mathbf{N}}$ and $f_2 \in \exp \mathfrak{g}_{1,0}^{\mathbf{N}}$, then

$$\|f_1 \circ f_2 \circ f_1^{-1}\|_{(1,0)} \leq \|f_2\|_{(1,0)} e^{\|f_1\|_{(1)}}. \tag{6.2.7$'$}$$

(e) *For $x \in \mathfrak{g}$,*

$$\|x \otimes t\|_{(1)} = \|x\|. \tag{6.2.8}$$

Proof. (a) Let $f = \circ \prod_{i=1}^m h_i$ where $h_i \in \mathfrak{g}_1^{\mathbf{N}}$ and $\sum_{i=1}^m \|h_i\|_1 < r$. From Proposition 6.2 (applied to the Banach-Lie algebra $\mathfrak{g}_1^{\mathbf{N}}$) the product $h_1 \circ h_2 \circ \ldots \circ h_m$ is correctly defined and $f \in \mathfrak{g}_1^{\mathbf{N}}$. Also $\|f\|_1 \leq L\|f\|_{(1)}$.

(b) The proof of (6.2.5) is similar to that of (6.2.4).

(c) To show the right hand inequality of (6.2.6) we follow Dynkin's method (cf. [6]) yielding the upper bound of the sum of the norms of all Lie monomials in the B-C-H series.

We get $f_1 \circ (-f_2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} W_n$ where W_n has associative form

$$(e^x e^y - 1)^n = \sum_{S_n} \frac{f_1^{k_1} (-f_2)^{j_1} \ldots f_1^{k_n} (-f_2)^{j_n}}{k_1! j_1! \ldots k_n! j_n!},$$

where S_n is the set of all $2n$-multiindices $(k_1, k_2, \dots, k_n, j_1, j_2, \dots, j_n)$ with $k_i, j_i \in \mathbf{N}$ and $k_i + j_i \geq 1$ for $i = 1, 2, \dots, n$. The Lie form of W_n is (cf. [2] Chapter 2, Section 3, p. 2)

$$W_n = \sum_{S_n} \frac{1}{k_1 + j_1 + \ldots + k_n + j_n}$$

$$\cdot \frac{[\underbrace{f_1, [\ldots [f_1,}_{k_1 times} \underbrace{[(-f_2), [\ldots [(-f_2),}_{j_1 times} \ldots \underbrace{[f_1, [\ldots [f_1,}_{k_n times} \underbrace{[(-f_2), [\ldots [(-f_2), (-f_2)]}_{j_n times} \ldots]}{k_1! j_1! \ldots k_n! j_n!}.$$

Let us observe that each nonvanishing homogeneous Lie monomial in W_n terminates either with $[f_1, (-f_2)]$ or $[(-f_2), f_1]$. Since $[f_1, (-f_2)] = [f_1, f_1 - f_2]$ and also $[f_1, (-f_2)] = [f_1 - f_2, -f_2]$, applying (2.3) we get $\|[f_1, -f_2]\|_1 \leq \|f_1\|_1 \|f_2\|_1 \cdot 2 \frac{\|f_1 - f_2\|_1}{\|f_1 + f_2\|_1}$ Following Dynkin this leads to the estimate

$$\|W_n\|_1 \leq (e^{\|f_1\|_1 + \|f_2\|_1} - 1)^n \cdot 2 \frac{\|f_1 - f_2\|_1}{\|f_1 + f_2\|_1}.$$

Since $\sum_{n=1}^{\infty} \frac{a^n}{n} = -\log(1-a)$ for $0 \leq a < 1$ we get

$$\|f_1 \circ f_2^{-1}\|_1 = \|\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} W_n\|_1 \leq \sum_{n=1}^{\infty} \frac{\|W_n\|_1}{n} \leq \|f_1 - f_2\|_1 \cdot 2 \frac{\log(2 - e^{\|f_1\|_1 + \|f_2\|_1})^{-1}}{(\|f_1\|_1 + \|f_2\|_1)}$$

and the right hand inequality of (6.2.6) follows with $C_2 = -2 \frac{\log(2 - e^r)}{r}$.

As remarked by the referee the inequalities 6.2.6 may be obtained much simpler but with constants depending on the Lie algebra:

The first order Taylor expansion of the smooth map $(f, g) \rightarrow f \circ (f + g)^{-1}$ *around* $(f, 0)$ *reads* $f \circ (f + g)^{-1} = -g + A(f) \cdot g + o(\| g \|)$*, where* $A(f)$ *is a continuous linear map depending continuously on* f*, with* $A(0) = 0$ *(and where the remainder is* $o(\| g \|)$ *uniformly in* f*). Thus* $f_1 \circ f_2^{-1} = f_1 \circ (f_1 + (f_2 - f_1))^{-1} = (A(f_1) - id) \cdot (f_2 - f_1) + o(\| f_2 - f_1 \|)$*, from which the assertion is apparent.*

To show the left hand inequality of (6.2.6) with the constant C_1 not depending of the Lie algebra, observe that by Lemma 6.8(a) we may assume that for α sufficiently small $f_2 = h \circ f_1$ where $h \in \mathfrak{g}_1^{\mathbf{N}}$. Then $\|f_1 - f_2\|_1 = \|h + \Theta_1(f_1, h)\|_1$ for f_1 and h siutably small where $\|\Theta_1(f_1, h)\|_1 \leq D \|f_1\| \|h\|$ for some constant D.

The conclusion follows.

(d) To show (6.2.7) observe that for $f_1, f_2 \in \exp \mathfrak{g}^{\mathbf{N}}$,

$$f_1 \circ f_2 \circ f_1^{-1} = e^{ad_{f_1}}(f_2) = \sum_{n=1}^{\infty} \frac{\mathrm{ad}_{f_1}^n (f_2)}{n!},$$

where the series is coordinatewise convergent due to the graded structure of $\mathfrak{g}^{\mathbf{N}}$.

The above formula may be justified by arguments similar to those explaining the group properties and joint continuity of the function

$$\mathfrak{g}^{\mathbf{N}} \times \mathfrak{g}^{\mathbf{N}} \ni (f, g) \longrightarrow f \circ g := \Theta(f, g) \in \mathfrak{g}^{\mathbf{N}}$$

in the Section 2.

Let us note that if $f_1, f_2 \in \mathfrak{g}_1^{\mathbf{N}}$ then by (2.3) $\| \operatorname{ad}^n_{f_1}(f_2) \|_1 \leq \| f_1 \|_1^n \| f_2 \|_1$, and the above series converges absolutely in $\mathfrak{g}_1^{\mathbf{N}}$ so that in particular $f_1 \circ f_2 \circ f_1^{-1} \in \mathfrak{g}_1^{\mathbf{N}}$ and

$$\| f_1 \circ f_2 \circ f_1^{-1} \|_1 \leq \| f_2 \|_1 e^{\|f_1\|_1}. \tag{6.2.9}$$

Given $\epsilon > 0$ let $f_1 = \circ \prod_{i=1}^m h_i$ and $f_2 = \circ \prod_{j=1}^n k_j$ where $h_i, k_j \in g_1^{\mathbf{N}}$ and suppose $\sum_{i=1}^m \| h_i \|_1 \leq \| f_1 \|_{(1)} + \epsilon$ and $\sum_{j=1}^n \| k_j \|_1 \leq \| f_2 \|_{(1)} + \epsilon$. Then

$$f_1 \circ f_2 \circ f_1^{-1} = \circ \prod_{j=1}^n (h_1 \circ (h_2 (\circ \ldots (h_m \circ k_j \circ h_m^{-1}) \circ \ldots) \circ h_2^{-1}) \circ h_1^{-1}),$$

thus,

$$\|f_1 \circ f_2 \circ f_1^{-1}\|_{(1)} \leq \sum_{j=1}^n (\prod_{i=1}^m e^{\|h_i\|_1}) \|k_j\|_1 \leq e^{\|f_1\|_{(1)}+\epsilon} (\|f_2\|_{(1)} + \epsilon).$$

The conclusion follows.

The proof of (6.2.7') is similar.

(e) Since $\|x \otimes t\|_1 = \|x\|$ and $\|x \otimes t\|_{(1)} \leq \|x \otimes t\|_1$, we get $\|x \otimes t\|_{(1)} \leq \|x\|$. Given $\epsilon > 0$ let $x \otimes t = \circ \prod_{i=1}^m h_i$ with $h_i \in g_1^{\mathbf{N}}$ and $\|(x \otimes t)\|_{(1)} + \epsilon \geq \sum_{i=1}^m \|h_i\|_1$. Let $a_{1,i}$ denote the first coordinate of h_i, $i = 1, 2, \ldots, m$. Then $x = \sum_{i=1}^m a_{1,i}$ and

$$\|x\| \leq \sum_{i=1}^m \|a_{1,i}\| \leq \sum_{i=1}^m \|h_i\|_1 \leq \|x \otimes t\|_{(1)} + \epsilon.$$

The conclusion follows. ∎

Proof of Proposition 6.7. We prove (a) only. The argument for (b) is similar. The properties (b) and (c) of (6.2.1) and the implication $\Leftarrow$ in (a) are straightforward and we omit their proofs (for (c) note that $(\circ \prod_{j=1}^p h_j)^{-1} = \circ \prod_{j=1}^p -h_{p-j}$).

For the implication $\Rightarrow$ in (6.2.1)(a), pick $f = \sum_{i=1}^\infty a_i t^i \in \exp \mathfrak{g}_1^{\mathbf{N}}$ with $\|f\|_{(1)} = 0$. By Lemma 6.8(a), $f \in \mathfrak{g}_1^{\mathbf{N}}$ and $\|f\|_1 = 0$. Thus $f = 0$. For (6.2.1)(d), let $\|f\|_1$ and $\|g\|_1$ be sufficiently small. Then, by Lemma 6.8(c),

$$\|f \circ g \circ f^{-1} \circ g^{-1}\|_1 \leq C_2 \|f \circ g \circ f^{-1} - g\|_1$$

$$= C_2 \|e^{\operatorname{ad}_f}(g) - g\|_1 \leq C_3 (e^{\|f\|_1} - 1) \|g\|_1 \leq C_4 \|f\|_1 \|g\|_1$$

where C_2, C_3, C_4 are appropriate constants. Applying Lemma 6.8(a) we get (6.2.1)(d). ∎

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