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A Generalization of Helling-Kim-Mennicke Groups and Manifolds

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Abstract. We construct an infinite family of closed connected orientable 3–manifolds by pairwise identifications of faces in the boundary of certain polyhedral 3–cells. We determine geometric presentations (that is, induced by Heegaard diagrams of the constructed manifolds) of the fundamental group, and study the split extension of it. Then we prove that these manifolds are n–fold cyclic coverings of the 3–sphere branched over some pretzel links. Our results generalize those of Helling, Kim and Mennicke [Comm. in Algebra **23** (1995), 5169–5206] and Cavicchioli and Paoluzzi [Manuscripta Math. **101** (2000), 457–494] on cyclic branched coverings of the Whitehead link, and their fundamental groups.
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1. Introduction

In 1933 Seifert and Weber constructed a nice example of a compact orientable hyperbolic 3–manifold by pairwise identification of faces of the $2\pi/5$–regular dodecahedron in the hyperbolic 3–space (see, for example, [11] and [14]). It is well-known that this manifold is the 5–fold cyclic branched covering of the 3–sphere branched over the Whitehead link (with covering isometry corresponding to the cyclic symmetry of the dodecahedron). In [6] Helling, Kim and Mennicke generalized the Seifert-Weber polyhedral scheme to construct a 2–parameter family of closed orientable 3–manifolds $M_{n,k}$ which are n–fold strongly cyclic branched coverings of the Whitehead link. According to [16], *strongly cyclic branched covering* for $M_{n,k}$ means that n and k are coprime, so the branching indices on the components of the Whitehead link are equal to n. For $(n,k) = (5,2)$ we get the Seifert-Weber hyperbolic 3–manifold. For $(n,k) = (5,1)$ the manifold $M_{n,k}$ was also considered in [1]. A polyhedral description of the whole family of (possibly, nonstrongly) cyclic branched coverings of the Whitehead link was given by Cavicchioli and Paoluzzi [3]. In the present paper we generalize polyhedral schemata

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from [3] and [6] to define (see Theorem 2.1) an infinite 3–parameter family of closed connected orientable 3–manifolds $M_{n,k,\ell}$ for $n \geq 2$, $1 \leq k \leq n-1$ and $\ell \geq 1$. Here $M_{n,k} = M_{n,k,\ell}$ for $\ell = 1$. By the construction, $M_{n,k,\ell}$ admits a cyclic symmetry of order n. In Theorem 2.5 we prove that this symmetry induces a cyclic covering of the 3–sphere branched over the 2–bridge link $\mathcal{W}_\ell$ determined by the rational parameter $(6\ell+2)/(2\ell+1)$, that is, the pretzel link $P(2,1,2\ell)$ (for the theory of knots and links see, for example, [2], [10], and [12]). In particular, $\mathcal{W}_1$ is the Whitehead link and $M_{5,2,1}$ is the Seifert-Weber dodecahedral hyperbolic 3–manifold. The crucial step in the proof is to apply methods from [3] to modify Heegaard diagrams of closed 3–orbifolds (i.e., Heegaard diagrams of their underlying manifolds plus the curves of the singular sets) by simplifications along closed curves and cancellations of handles. This extends to the case of orbifolds some techniques described in [5] for link complements. In Theorem 2.2 we determine geometric presentations (that is, induced by Heegaard diagrams of the constructed manifolds) of the fundamental group, and study the split extension of it (see Theorem 2.4). Finally, we discuss the classification problem for our manifolds up to homeomorphism (resp. isometry).

2. A generalization of the manifolds $M_{n,k}$

For every $n \geq 2$, $\ell \geq 1$ and $1 \leq k \leq n-1$, let us consider the combinatorial 3–cell $P_{n,k,\ell}$ whose 2–sphere boundary consists of two n–gons F and F' in the northern and southern hemispheres, respectively, and $2n$ $(3\ell+2)$–gons, labelled by F_i and F'_i, $i = 1, 2, \ldots, n$, in the equatorial zone. Then $\partial P_{n,k,\ell}$ has exactly $2n+2$ faces, $3n(\ell+1)$ edges and $n(3\ell+1)$ vertices. The side pairing of index k is again determined by identifying the pairs of faces (F_i, F'_i) and (F, F'), and k is the shift to do before gluing F_i with F'_i (see Figure 1 for $P_{n,k,\ell} = P_{4,2,3}$, where $d = (n,k) = (4,2) = 2$). The resulting identification space $M_{n,k,\ell}$ has d vertices, $n+d$ 1–cells, $n+1$ 2–cells, and one 3–cell, hence it is a closed 3–manifold. Let $G_{n,k,\ell}$ denote the fundamental group of $M_{n,k,\ell}$, and a_i $(i = 1, \ldots, n)$ and b the affine transformations which identify the pairs of faces (F_i, F'_i) and (F, F'), respectively. Following the cycles of equivalent edges we get for the label x_i ($\equiv i$ in the figure), $i = 1, \ldots, n$, the relation

$$a_i(a_{i-1}^{-1}a_{i-k}^{-1})a_{i-(k+1)}(a_{i-(k+1)-1}^{-1}a_{i-2(k+1)+1}^{-1})\cdots$$

$$\cdots a_{i-(\ell-1)(k+1)}(a_{i-(\ell-1)(k+1)-1}^{-1}a_{i-\ell(k+1)+1}^{-1})a_{i-\ell(k+1)}b^{-1} = 1$$

and for the label y_i, $i = 1, \ldots, d$, $d = (n,k)$, the relation

$$a_i a_{i+k} a_{i+2k} \ldots a_{i+k(n-1)} = 1.$$

Therefore we have the following result

Theorem 2.1. *The polyhedral* 3*–cell* $P_{n,k,\ell}$ *with identifications,* $n \geq 2$, $\ell \geq 1$, $1 \leq k \leq n-1$, *constructed above, defines a closed connected orientable* 3*–manifold* $M_{n,k,\ell}$ *which has a spine modeled on the finite presentation* $G_{n,k,\ell} \cong$
$\langle a_1, \ldots, a_n, b : b = a_i(a_{i-1}^{-1}a_{i-k}^{-1})a_{i-(k+1)}(a_{i-(k+1)-1}^{-1}a_{i-2(k+1)+1}^{-1})\cdots$
$a_{i-(\ell-1)(k+1)}(a_{i-(\ell-1)(k+1)-1}^{-1}a_{i-\ell(k+1)+1}^{-1})a_{i-\ell(k+1)}, \quad (i = 1, \ldots, n),$
$a_i a_{i+k} a_{i+2k} \ldots a_{i+k(n-1)} = 1, \quad (i = 1, \ldots, d)\rangle$
where the indices are taken mod n, *and* $d = (n,k)$.

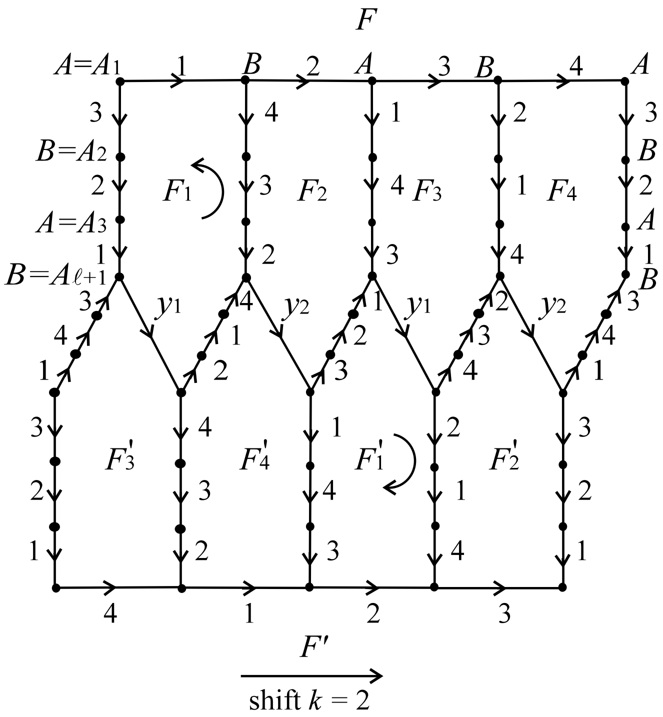

Figure 1. The polyhedral scheme $P_{n,k,\ell} = P_{4,2,3}$ with identifications

Of course, if $\ell = 1$, then the manifolds $M_{n,k,\ell}$ are exactly the manifolds $M_{n,k}$ considered in [3] and [6]. Suppose now that n and k are coprime, i.e., $d = (n, k) = 1$. Then the quotient complex $M_{n,k,\ell}$ has exactly one vertex, so we can obtain a further presentation for $G_{n,k,\ell} = \pi_1(M_{n,k,\ell})$ with generators $x_1, \dots x_n, y$ and relators arising from the boundaries of the 2–cells $F_1, \dots, F_n, F$ of the polyhedral scheme. For each $i = 1, \dots, n$, the boundary cycle of the polygons F_i and F'_i is

$$y = (x^{-1}_{i+\ell(k+1)-1} x^{-1}_{i+(\ell-1)(k+1)-1} \cdots x^{-1}_{i+2(k+1)-1} x^{-1}_{i+k})$$
$$x_i (x_{i+k+1} x_{i+2(k+1)} \cdots x_{i+(\ell-1)(k+1)} x_{i+\ell(k+1)})$$
$$(x^{-1}_{i+(\ell-1)(k+1)+1} x^{-1}_{i+(\ell-2)(k+1)+1} \cdots x^{-1}_{i+(k+1)+1} x^{-1}_{i+1})$$

where the indices are taken mod n, and the boundary cycle of the polygons F and F' is

$$x_1 x_2 \dots x_n = 1.$$

Then we have the following result.

Theorem 2.2. *If n and k are coprime, then the fundamental group $G_{n,k,\ell}$ of the manifold $M_{n,k,\ell}$, $n \geq 2$, $\ell \geq 1$, $1 \leq k \leq n-1$, admits the finite presentation*

$$G_{n,k,\ell} \cong \langle x_1, \dots, x_n, y : \; x_1 x_2 \dots x_n = 1,$$

$$y = (x^{-1}_{i+\ell(k+1)-1} x^{-1}_{i+(\ell-1)(k+1)-1} \cdots x^{-1}_{i+2(k+1)-1} x^{-1}_{i+k})$$
$$x_i (x_{i+k+1} x_{i+2(k+1)} \cdots x_{i+(\ell-1)(k+1)} x_{i+\ell(k+1)})$$
$$(x^{-1}_{i+(\ell-1)(k+1)+1} x^{-1}_{i+(\ell-2)(k+1)+1} \cdots x^{-1}_{i+(k+1)+1} x^{-1}_{i+1})$$

$$(i = 1, \dots, n;\ \textit{indices mod } n)\ \rangle$$

which corresponds to a spine (or, equivalently, to a Heegaard diagram) of the manifold (hence it is geometric).

In particular, if $k = n-1$, then we get the following result.

Corollary 2.3. *The fundamental group $G_{n,n-1,\ell}$ of the closed 3–manifold $M_{n,n-1,\ell}$, $n \geq 2$, $\ell \geq 1$, admits the following geometric balanced presentations*

$$\begin{aligned} G_{n,n-1,\ell} &\cong \langle a_1, \dots, a_n,\ b :\ a_n a_{n-1} \dots a_2 a_1 = 1, \\ &\qquad\qquad b = [a_i(a_{i-1}^{-1} a_{i+1}^{-1})]^{\ell} a_i \\ &\qquad\qquad\qquad (i = 1, \dots n)\ \rangle \\ &\cong \langle x_1, \dots, x_n,\ y :\ x_1 x_2 \dots x_n = 1, \\ &\qquad\qquad y = x_{i-1}^{-\ell} x_i^{\ell+1} x_{i+1}^{-\ell} \\ &\qquad\qquad\qquad (i = 1, \dots, n)\ \rangle \end{aligned}$$

where the indices are taken mod n.

Now we study the split extension group of $G_{n,k,\ell}$ by the cyclic automorphism corresponding to the presentation given in Theorem 2.1. More precisely, we have the following theorem.

Theorem 2.4. *Let $H_{n,k,\ell}$ be the split extension group of $G_{n,k,\ell}$, where $n \geq 2$, $\ell \geq 1$ and $1 \leq k \leq n-1$. Then $H_{n,k,\ell}$ is isomorphic to the fundamental group of the orbifold $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{W}_\ell)$ whose underlying space is the 3–sphere $\mathbb{S}^3$ and whose singular set is the 2–bridge link $\mathcal{W}_\ell = \frac{6\ell+2}{2\ell+1}$, i.e., the pretzel link $P(2,1,2\ell)$ (if $\ell = 1$, then $\mathcal{W}_\ell = \frac{8}{3}$ is the Whitehead link) with branching indices n and n/d on its components, where $d = (n,k)$ (see Figure 2).*

Proof. Let us consider the finite presentation of $G_{n,k,\ell}$ given in the statement of Theorem 2.1. Let ρ be the automorphism of $G_{n,k,\ell}$ defined by $\rho(a_i) = a_{i+1}$ (indices mod n), and $\rho(b) = b$. The split extension group $H_{n,k,\ell}$ of $G_{n,k,\ell}$ has the finite presentation

$$\begin{aligned} H_{n,k,\ell} = \langle\, a, b, \rho :\ & \rho^n = 1,\ \rho b = b\rho, \\ & a\rho^{-k} a \rho^{k} \rho^{-2k} a \rho^{2k} \dots \rho^{-k(n-1)} a \rho^{k(n-1)} = 1, \\ & b = a(\rho^{-(n-1)} a^{-1} \rho^{n-1+k} a^{-1} \rho^{-k}) \rho^{k+1} a \rho^{-(k+1)} \\ & (\rho^{(k+1)+1} a^{-1} \rho^{-(k+1)-1} \rho^{2(k+1)-1} a^{-1} \rho^{-2(k+1)+1}) \dots \\ & \dots \rho^{(\ell-1)(k+1)} a \rho^{-(\ell-1)(k+1)} (\rho^{(\ell-1)(k+1)+1} a^{-1} \rho^{-(\ell-1)(k+1)-1} \\ & \rho^{\ell(k+1)-1} a^{-1} \rho^{-\ell(k+1)+1}) \rho^{\ell(k+1)} a \rho^{-\ell(k+1)}\,\rangle \\ \cong \langle\, a, b, \rho :\ & \rho^n = 1,\ \rho b = b\rho,\ (a\rho^{-k})^{\frac{n}{d}} = 1, \\ & b = (a\rho a^{-1} \rho^{k-1} a^{-1} \rho)^{\ell} a \rho^{-\ell(k+1)}\,\rangle \end{aligned}$$

where $a_1 = a$, $a_{i+1} = \rho^{-i} a \rho^{i}$ and $b = \rho^{-1} b \rho$. Setting $\tau = a\rho^{-k}$ and eliminating $a = \tau\rho^{k}$ and

$$b = (\tau\rho\tau^{-1}\rho^{-1}\tau^{-1}\rho)^{\ell}\tau\rho^{-\ell(k+1)+k}$$

yields the finite presentation

$$H_{n,k,\ell} = \langle\, \tau, \rho \,:\, \rho^{n} = 1,\ \tau^{\frac{n}{d}} = 1,\ w_{\ell}\rho = \rho w_{\ell} \,\rangle$$

where

$$w_{\ell} = (\tau\rho\tau^{-1}\rho^{-1}\tau^{-1}\rho)^{\ell}\tau.$$

Because

$$w_{\ell} = \tau^{\epsilon_1}\rho^{\epsilon_2}\tau^{\epsilon_3}\ldots\tau^{\epsilon_{6\ell-1}}\rho^{\epsilon_{6\ell}}\tau^{\epsilon_{6\ell+1}}$$

where ϵ_i is the sign (± 1) of $(2\ell+1)i$ reduced mod $2(6\ell+2)$ in the interval $(-(6\ell+2), 6\ell+2)$, the word w_{ℓ} corresponds to the 2–bridge link $\mathcal{W}_{\ell} = \frac{6\ell+2}{2\ell+1}$. In particular, the finite presentation $\langle\, \rho, \tau \,:\, w_{\ell}\rho = \rho w_{\ell} \,\rangle$ defines the link group of $\mathcal{W}_{\ell}$, where ρ and τ are meridians around its components. Therefore, $H_{n,k,\ell}$ is the fundamental group of the orbifold $\mathcal{O}_{n,\frac{n}{d}}(\mathcal{W}_{\ell})$ defined in the statement of the theorem. ∎

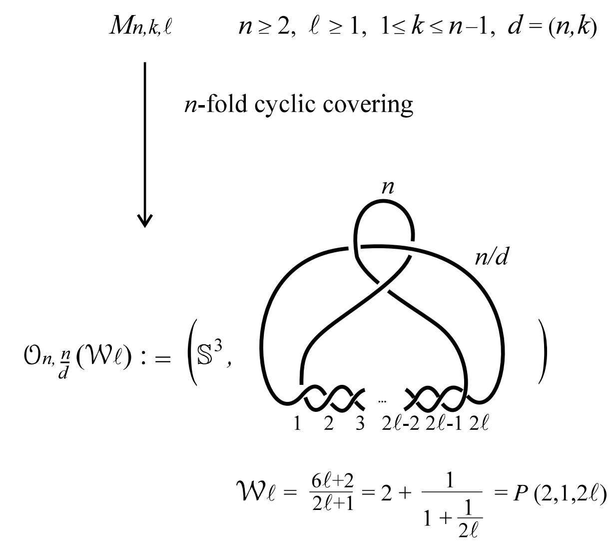

Figure 2. Representing the manifolds $M_{n,k,\ell}$ as branched coverings

Theorem 2.5. *The closed connected orientable 3–manifolds $M_{n,k,\ell}$, $n \geq 2$, $\ell \geq 1$, $1 \leq k \leq n-1$, are cyclic branched n–fold coverings of the 2–bridge link $\mathcal{W}_{\ell} = \frac{6\ell+2}{2\ell+1}$ in the 3–sphere, where the branching indices of its components are n and $\frac{n}{d}$, respectively, where $d = (n,k)$.*

Proof. Let us consider the automorphism ρ of $G_{n,k,\ell}$ defined in the proof of Theorem 2.4, and denote the corresponding homeomorphism of $M_{n,k,\ell}$, also by ρ. Since ρ corresponds to the n–rotational symmetry of the polyhedron $P_{n,k,\ell}$, it follows that the $\frac{1}{n}$–piece $\Pi_{n,k,\ell}$ of $P_{n,k,\ell}$, pictured in Figure 3, is the fundamental polyhedron for the quotient space $M_{n,k,\ell}/{<\rho>}$.

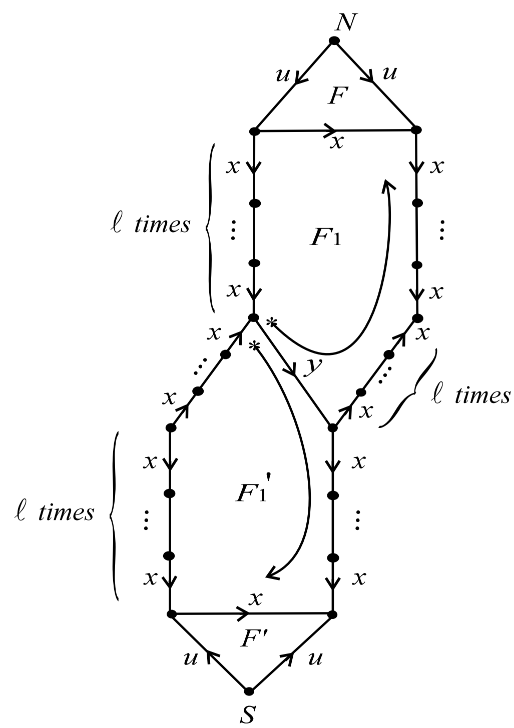

Figure 3. The $\frac{1}{n}$–piece $\Pi_{n,k,\ell}$

Moreover, the fundamental group of the orbifold $M_{n,k,\ell}/< \rho >$ is isomorphic to $H_{n,k,\ell}$. The faces of the complex $\Pi_{n,k,\ell}$ are pairwise equivalent under the identifications $F \equiv F'$ and $F_1 \equiv F_1'$ induced by the side pairings of $P_{n,k,\ell}$. The faces F_1 and F_1' are to be paired so that the index stars in Figure 3 match up. Note that the topological space underlying the orbifold $M_{n,k,\ell}/< \rho >$ is just the closed 3–manifold $M_{1,0,\ell}$. The fundamental group of $M_{1,0,\ell}$ is isomorphic to the group presented by $\langle\, x,\, y \colon\, y^{-1}x^{-\ell}x^{\ell+1}x^{-\ell} = 1,\ x = 1\,\rangle$, and so it is trivial. A Heegaard diagram of $M_{1,0,\ell}$ (plus the arc from N to S representing the symmetry axis of the rotation ρ) is depicted in Figure 4. Since $M_{1,0,\ell}$ is a simply–connected closed 3–manifold of Heegaard genus ≤ 2, it is a genuine 3–sphere by [9]. In other words, the space underlying the quotient orbifold $M_{n,k,\ell}/< \rho >$ is topologically homeomorphic to $\mathbb{S}^3$. Furthermore, the singular set of $M_{n,k,\ell}/< \rho >$ is the image of the rotational axis of ρ, consisting of points of order n, plus the image of the edges of type x_i, consisting of points of order n/d, where $d = (n, k)$. The cellular decomposition of the $\frac{1}{n}$–piece $\Pi_{n,k,\ell}$ in Figure 3 defines in a natural way a decomposition of $M_{1,0,\ell}$ ($\cong \mathbb{S}^3$) into handles. The 3–handle is a neighbourhood in $M_{1,0,\ell}$ of the images of the vertices of $\Pi_{n,k,\ell}$. The 2–handles are neighbourhoods of the images of the edges of $\Pi_{n,k,\ell}$. The 1–handles are neighbourhoods of the images of the faces of $\Pi_{n,k,\ell}$ which are identified in pairs. There is a unique 0–handle obtained after cutting off all other handles; it is the image of a 3–ball inside the polyhedron $\Pi_{n,k,\ell}$. All the information concerning the side pairings of the boundary faces of $\Pi_{n,k,\ell}$ are stored in the planar graph shown in Figure 4.

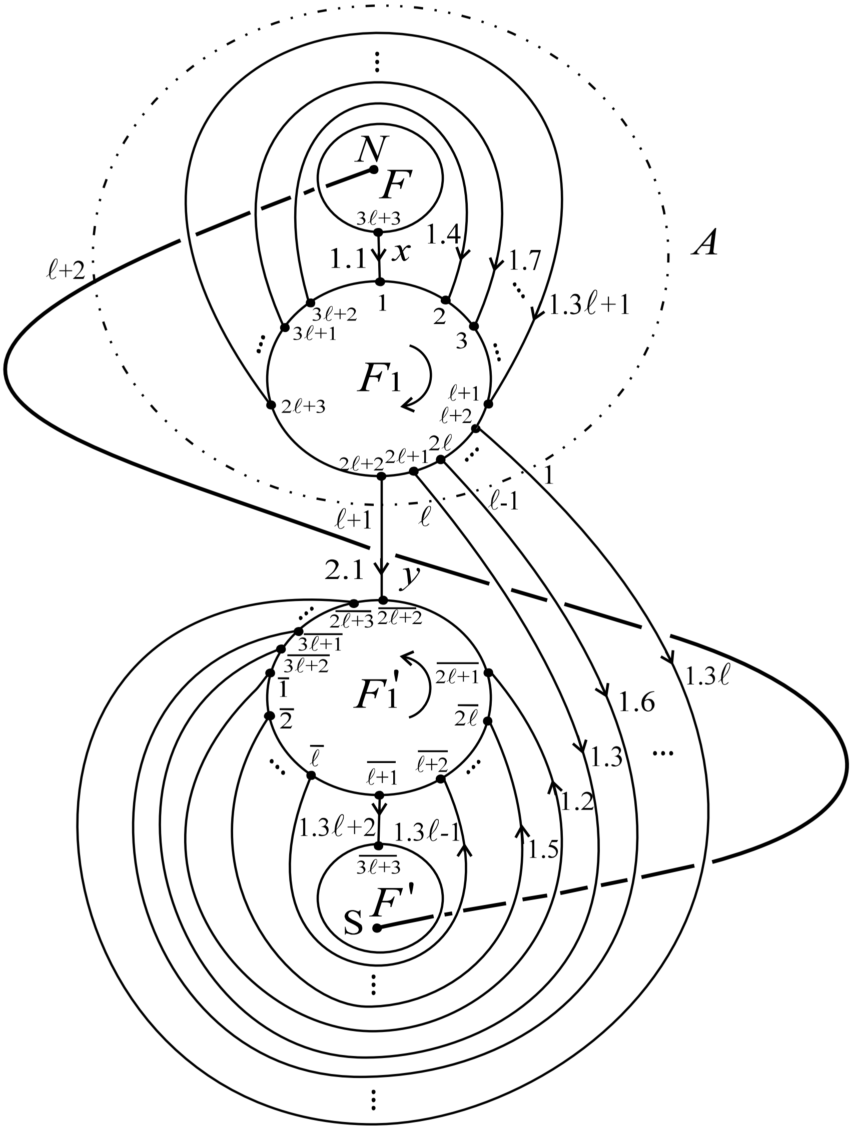

Figure Figure 4. A Heegaard diagram of the quotient space ($\cong \mathbb{S}^3$) obtained from $\Pi_{n,k,\ell}$ plus the arc from N to S representing the symmetry axis of the rotation

Let us consider the oriented arc in Figure 4 with endpoints labelled $3\ell+3$ (in ∂F) and 1 (in ∂F_1). Denote it by 1.1. The endpoint 1 of this arc is glued onto another point (labelled by $\bar{1}$ in $\partial F_1'$) by the identification of the faces F_1 and F_1'. Denote by 1.2 the oriented arc starting at $\bar{1}$ (in $\partial F_1'$) which ends to the point labelled $\overline{2\ell+1}$ (in $\partial F_1'$). Go on like this until you return to the initial arc 1.1. We obtain the cycle x which represents a 2–handle in $M_{1,0,\ell}$. There remains only one arc in the graph, and we denote it by 2.1. It connects the point labelled $2\ell+2$ (in ∂F_1) with the point labelled $\overline{2\ell+2}$ (in $\partial F_1'$), and represents the 2–handle along y in $M_{1,0,\ell}$. By construction, we have exactly four 2–discs F, F', F_1, F_1' (i.e., two 1–handles in $M_{1,0,\ell}$) and two cycle relations (i.e., two 2–handles in $M_{1,0,\ell}$) in our graph. For every planar graph (as that in Figure 4) we define the *complexity* of the graph (see [5]) to be the pair (g,n), where g is the number of 1–handles (i.e., half the number of faces in the polyhedral schemata) and n is the number of arcs of the graph (i.e., the number of edges in the polyhedral schemata). In our case, we have $(g,n) = (2, 3\ell+2)$. We write $(g,n) > (g',n')$ if either $g > g'$ or $g = g'$ and $n > n'$.

Now we apply techniques from [3] (see also [5]) to modify the graph (without changing the represented manifold $M_{1,0,\ell} \cong \mathbb{S}^3$) in order to diminish the complexity of it. We have already pointed out that the singular set of our orbifold $M_{n,k,\ell}/< \rho >$ consists of the symmetry axis (with branching index n) and the image of the edges of type x_i in the polyhedron $P_{n,k,\ell}$ (with branching index n/d). The axis is represented in Figure 4 as a marked line inside the semispace, starting and ending in two points N and S which are equivalent in the side pairing. While diminishing complexity we must take care to check what happens to the axis. It will be cut into pieces and rearranged inside the semispace. Let A denote a Jordan curve in the plane (the dotted circle in Figure 4) which intersects the graph only along its arcs and transversally. The curve A separates the pairs of identified 2–discs (F, F') and (F_1, F_1') such that the number of points along their boundaries is larger than the number of intersections of A with the graph. Figure 5 is obtained from Figure 4 by a simplification along the closed simple curve A which surrounds the "holes" F and F_1. The resulting graph is again a Heegaard diagram for $M_{1,0,\ell}$ ($\cong \mathbb{S}^3$). The graphical move is realized as follows. Dig a 2–ball along the curve A and glue it back along the pair of 2–discs (F_1, F_1') which disappear. The marked line representing the symmetry axis is modified consequently as done in Figure 5. The complexity of the new planar graph is $(g', n') = (2, \ell+3) < (g, n) = (2, 3\ell+2)$. To diminish "$g$" we have to cancel a 1–handle with a 2–handle. To do this we need to check that the sequence of arcs defining the glueing diagram for the 2–handle intersects the two discs defining the 1–handle in exactly one point each one. Also in this case, we must see how the remaining arcs of the graph are deformed. The procedure gives at the end a Heegaard diagram of lower genus for the quotient space. Figure 6 is obtained from Figure 5 by cancelling the 2–handle along y between the "holes" A^+ and A^- defining a 1–handle. The result is again a Heegaard diagram (of genus one) for the quotient space $M_{1,0,\ell} \cong \mathbb{S}^3$ underlying the orbifold $M_{n,k,\ell}/ < \rho > \cong \Pi_{n,k,\ell}/ \sim$. The marked line in Figure 6 represents the symmetry axis, modified according to the above move. We get a reduction of the complexity of the graph; in fact, we now have $(g'', n'') = (1, 1)$. The graphical move is realized as follows. Push the 1–handle with one of its "feet", say A^+, along the attaching sphere y of the 2–handle until we come to A^-. Of course, in doing so, we have to pull along with A^+ all the other cocores of the 2–handles running over the A–handle. In our case, we have to pull along with A^+ the cocore x and the singular arcs representing the symmetry axis (see Figure 5). This explains how the remaining edges of the graph adjacent to A^+ and A^- must be changed when (A^+, A^-) and the arc 2.1 (representing the cocore y) are erased (see Figure 6). We note that the singular set consists of the axis of symmetry (with branching index n) together with the cocore of the 2–handle along x (with branching index n/d). Cancelling the 2–handle along x between the "holes" F and F' we get the singular set of the branched covering which is the link pictured in Figure 7. It is equivalent to the 2–bridge link $\mathcal{W}_\ell$ determined by the rational parameter $\frac{6\ell+2}{2\ell+1}$ with branching indices n and n/d on its components, where $\ell \geq 1$ and $d = (n, k)$. ∎

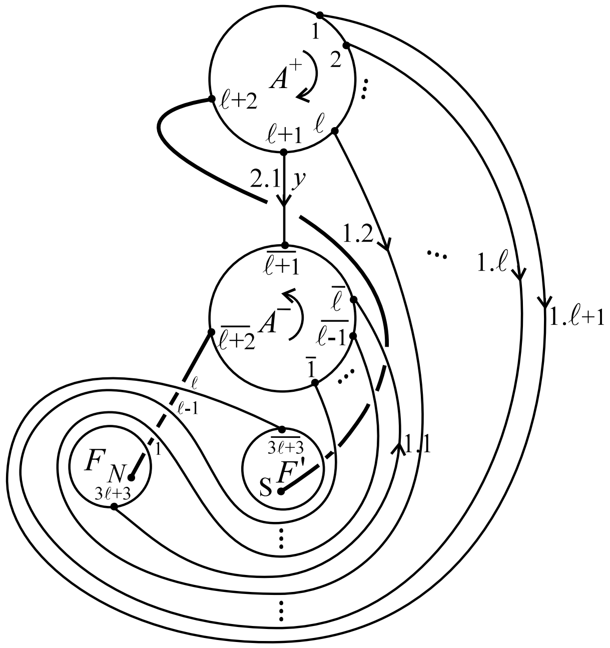

Figure 5. Simplification along the closed simple curve A

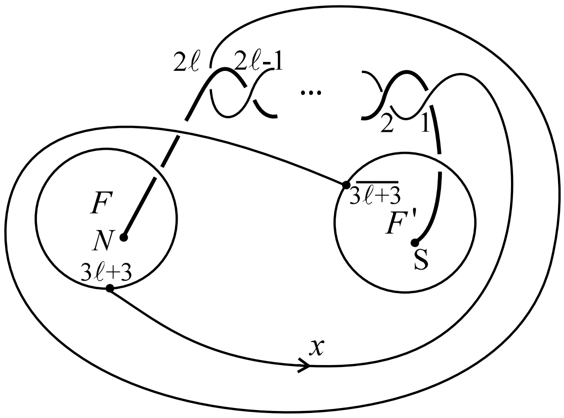

Figure 6. Cancelling the handle along y between A^+ and A^-

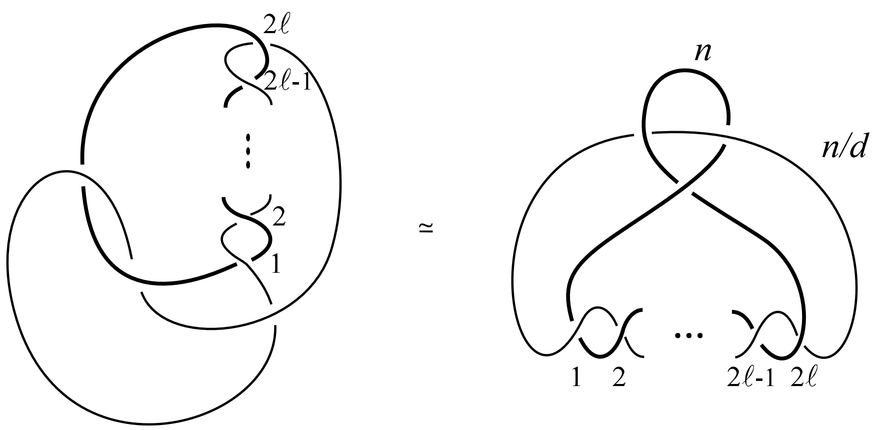

Figure 7. The 2–bridge link $\mathcal{W}_\ell = \frac{6\ell+2}{2\ell+1}$ as singular set

As it is well-known the n–fold strongly cyclic covering of $\mathbb{S}^3$ branched over a link L has the same geometric structure of the orbifold which has $\mathbb{S}^3$ as its underlying space and L as its singular set (see for example [4]). By Theorem 3.1 of [8] the orbifold $\mathcal{O}_{n,n}(\mathcal{W}_\ell)$ is hyperbolic for every $n \geq 3$ and spherical for $n = 1, 2$. This implies immediately the following consequence about the geometric structures of our manifold.

Theorem 2.6. *If n and k are coprime, then the manifolds $M_{n,k,\ell}$, $n \geq 1$, $\ell \geq 1$, $0 \leq k \leq n-1$, has a hyperbolic structure for every $n \geq 3$, and a spherical structure for $n = 1, 2$ (in particular, $M_{1,0,\ell} \cong \mathbb{S}^3$ and $M_{2,1,\ell}$ is homeomorphic to the lens space $L(6\ell + 2, 2\ell + 1)$).*

By Theorem 4.1 of [13] the symmetry group of the 2–bridge link $\mathcal{W}_\ell = \frac{6\ell+2}{2\ell+1}$ is isomorphic to either D_4 or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see also [7]). So we can apply Theorem 1 of [15] for the case $d = (n, k) = 1$ to get the topological classification of manifolds $M_{n,k,\ell}$.

Theorem 2.7. *If $n \geq 3$, ℓ, $\ell' > 1$ and $d = (n, k) = (n, k') = 1$, then $M_{n,k,\ell}$ is isometric (homeomorphic) to $M_{n,k',\ell'}$ if and only if $\ell = \ell'$ and $k' \equiv k^{\pm 1}$ (mod n).*

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