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L^p-Boundedness of Bergman Projections in the Tube Domain over Vinberg's Cone

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Abstract. In this paper, we determine a range of p for which there is L^p-continuity of the Bergman projector in the tube domain over Vinberg's cone. This is the simplest example of a homogeneous non-symmetric cone. Our main tool is the existence of a suitable isometry between the cone and its dual.
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1. Introduction

Let V be the Euclidean vector space of 3×3 matrices with real entries of the form

$$x = \begin{pmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{pmatrix},$$

where the inner product is defined as follows

$$(x|y) = \mathrm{Trace}(xy) = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4y_4 + 2x_5y_5.$$

Let

$$\Omega = \{x \in V : \ Q_j(x) > 0, \ j = 1, 2, 3\},$$

where

$$Q_1(x) = x_1, \quad Q_2(x) = x_2 - \frac{x_4^2}{x_1}, \quad Q_3(x) = x_3 - \frac{x_5^2}{x_1},$$

be the Vinberg's cone. Then Ω is a proper open convex cone and its dual cone is given by

$$\Omega^* = \{\xi \in V : \ Q_j^*(\xi) > 0, \quad j = 1, 2, 3\}$$

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where

$$Q_1^*(\xi) = \xi_1 - \frac{\xi_4^2}{\xi_2} - \frac{\xi_5^2}{\xi_3}, \qquad Q_2^*(\xi) = \xi_2, \qquad Q_3^*(\xi) = \xi_3.$$

In [4], Casalis proved that the cone Ω is non self-dual for any positive symmetric bilinear form defined on V.

In the sequel, we will use these notations: for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $x \in \Omega$ and $\xi \in \Omega^*$

$$Q^\alpha(x) = \prod_{j=1}^{3} Q_j^{\alpha_j}(x) \text{ and } (Q^*)^\alpha(\xi) = \prod_{j=1}^{3} (Q_j^*)^{\alpha_j}(\xi).$$

Let $T_\Omega = V + i\Omega$ be the associated tube domain. We put $\tau = (2, \frac{3}{2}, \frac{3}{2})$. For $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ we denote by $L^p_\nu(T_\Omega)$, $1 \leq p \leq \infty$, the Lebesgue space $L^p(T_\Omega, Q^{\nu-\tau}(\Im m\, w)dv(w))$ where $Q^{-2\tau}(\Im m\, w)dv(w)$ is the invariant measure with respect to the group of automorphisms of Ω. Here dv is the Lebesgue measure defined on $\mathbb{C}^n$.

The *weighted Bergman space* $A^p_\nu(T_\Omega)$ is the closed subspace of $L^p_\nu(T_\Omega)$ consisting of holomorphic functions. In order to have a non-trivial subspace, we take $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that $\nu_1 > 1$, $\nu_2 > \frac{1}{2}$ and $\nu_3 > \frac{1}{2}$.

The orthogonal projection of the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ is the *weighted Bergman projection* P_ν. It is well known that P_ν is defined by the integral

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z, w) f(w) Q^{\nu-\tau}(\Im m\, w)dv(w)$$

where

$$B_\nu(z,\, w) = d_\nu Q^{-\nu-\tau}\left(\frac{z - \overline{w}}{i}\right)$$

is the *weighted Bergman kernel*, that is the reproducing kernel on $A^2_\nu(T_\Omega)$.

The L^p-boundedness of Bergman projections on tube domains over cones have been studied by many authors these last years. In [1], D. Békollé and A. Bonami considered the forward light cone; they obtained some sufficient conditions using Schur's Lemma. They have proved that this condition is necessary and sufficient for the positive Bergman operator; that is the Bergman projector with positive Bergman kernel. This result has been improved in [3]. There are values of p for which the Bergman projection is bounded whenever the positive Bergman operator is unbounded.

In the Lecture Notes [2] of the Workshop "PDE, Classical Analysis and Applications" held in Yaoundé in December 2001, the authors generalized to the case of general symmetric cones the results of [3]. Moreover, D. Debertol in his thesis [6] obtained the sufficient conditions above with general weighted measures.

On the other hand, A. Temgoua in his thesis [12] gave sufficient conditions in the case of Siegel domains of type II, symmetric or not; he applied Schur's Lemma to the positive Bergman operator.

It is important for us to mention that all these sufficient conditions are far from being necessary when the rank of the cone is greater than 1.

The aim of our work is the generalization of D. Debertol's result in the setting that A. Temgoua has considered, that is, tubular domains over homogeneous cones.

The purpose of this paper is to make sure that all results known in the case of tubular domains over general symmetric cones are also valid if the homogeneous cone considered is not necessarily self-dual. This justifies the fact that we focus our attention in the particular case of Vinberg's cone which is the simplest example of a non self-dual homogeneous cone. In a joint work with B. Trojan, the second author is looking for the generalization of results announced here in the case of general homogeneous cones. This cannot be done without the use of deep tools of algebra, this is why we have first considered this typical example, where everything can be written directly.

In order to present our results, let us consider the positive Bergman operator

$$P_\nu^+ f(z) = \int_{T_\Omega} |B_\nu(z,w)| f(w) Q^{\nu-\tau}(\Im m\, w) dv(w).$$

We have the following:

Theorem 1.1. *The operator P_ν^+ is bounded on $L_\nu^p(T_\Omega)$ if and only if $\frac{\nu_1+1}{\nu_1} < p < \nu_1 + 1$. Hence P_ν is bounded for this range of p.*

As we mentioned above, the "if" part of this Theorem has been proved by A. Temgoua [12] and we prove here that the "only if" part is also valid. For the case of tube domains over symmetric cones, authors of [2] established that there are values of p for which P_ν is bounded, but P_ν^+ is unbounded. We extend this result to the tube domain over Vinberg's cone. We have the following theorem which is the main result of this paper.

Theorem 1.2. *The Bergman projector P_ν extends to a bounded operator on $L_\nu^p(T_\Omega)$ for $\frac{\nu_1+2}{\nu_1+1} < p < \nu_1 + 2$.*

We describe the main ideas in the proof of this result. We must take advantage of the oscillations of the Bergman kernel. Hence, we are induced to use the Fourier transform in the x variables and consequently to focus on L^2-norms in these variables. For this reason, we introduce mixed norm spaces. For $1 \le p, q \le \infty$, let $L_\nu^{p,q}(T_\Omega) = L^q(\Omega, Q^{\nu-\tau}(y)dy; L^p(V, dx))$ be the space of functions f on T_Ω such that

$$\|f\|_{L_\nu^{p,q}(T_\Omega)} := \left(\int_\Omega \left(\int_V |f(x+iy)|^p dx \right)^{\frac{q}{p}} Q^{\nu-\tau}(y) dy \right)^{\frac{1}{q}}$$

is finite (with obvious modification if $p, q = \infty$). As before, we call $A_\nu^{p,q}(T_\Omega)$ the closed subspace of $L_\nu^{p,q}(T_\Omega)$ consisting of holomorphic functions.

For $p = 2$, we obtain that P_ν is bounded on $L_\nu^{2,q}(T_\Omega)$ if and only if $\frac{2\nu_1+2}{2\nu_1+1} < q < 2\nu_1 + 2$. We observe that, as in the case of symmetric cones, we have this necessary and sufficient condition for the Bergman projection. Then Theorem 1.2 follows by interpolation with Theorem 1.1.

This paper is divided in 7 sections, the first one is the introduction above. In the second section, we give an isometry between the open convex homogeneous cone and its dual. The third section describes a solvable group that acts simply transitively on the Vinberg's cone and gives its Whitney decomposition. In section 4, we compute some important integrals. Section 5 deals with Bergman spaces and the last two sections state the proofs of results announced in the first section.

2. An isometry between an open convex homogeneous cone and its dual

In this section, we describe the main tool that we use to obtain our result; we show that there is an isometry between an open convex homogeneous cone and its dual. We begin by recalling some definitions.

2.1. Preliminary definitions.

A subset C of $\mathbb{R}^n$ is said to be a *convex cone* if for $x, y \in C$ and $\lambda,\ \mu > 0$ we have $\lambda x \in C$ and $\lambda x + \mu y \in C$.

We define its dual cone by

$$C^* = \{\xi \in \mathbb{R}^n : (\xi|x) > 0,\ \forall x \in \overline{C} \setminus \{0\}\}.$$

Let C be an open cone. We say that C is *self-dual* if $C = C^*$. The cone C is said to be a *proper* cone if $\overline{C} \cap (-\overline{C}) = \{0\}$.

The cone C is said to be *homogeneous* if the group $G(C) = \{g \in Gl(\mathbb{R}^n) : gC = C\}$ acts transtively on it i.e for $x,\ y \in C$ there is an element $g \in G(C)$ such that $y = gx$.

A homogeneous cone that is self-dual is said to be a *symmetric* cone. Let us remark that $G(C^*) = (G(C))^*$ and $C^{**} = C$.

2.2. The Riemannian structure of a proper open convex cone. Let C be a proper open convex cone. We denote by φ (resp. φ_*) the *characteristic function* of the cone C (resp. C^*); then for $x \in C$ and $\xi \in C^*$,

$$\varphi(x) = \int_{C^*} e^{-(x|\xi)} d\xi \quad \text{and} \quad \varphi_*(\xi) = \int_C e^{-(\xi|x)} dx.$$

From a change of variables (see also [13], Chapter I, formula (2)), we have

$$\forall g \in G(C),\ \varphi(gx) = |\mathrm{Det}\, g|^{-1} \varphi(x) \quad \text{and} \quad \varphi_*(g^*\xi) = |\mathrm{Det}\, g|^{-1} \varphi_*(\xi).$$

We have then the following

Lemma 2.1. *Let C be a proper open convex cone. The measure m (resp. m_*) defined on C (resp. C^*) by*

$$dm(x) = \varphi(x)dx \quad (\text{resp.} \quad dm_*(\xi) = \varphi_*(\xi)d\xi)$$

is invariant under the action of the group $G(C)$ (resp. $G(C^*)$).

The gradient of a differentiable function f at the point $x \in \mathbb{R}^n$ is defined by

$$(\nabla f(x)|u) = D_u f(x) = \frac{d}{dt} f(x + tu)\Big|_{t=0}$$

for all $u \in \mathbb{R}^n$.

For $x \in C$ we define $x^* \in C^*$ by

$$x^* = -\nabla \log \varphi(x).$$

Similarly, for $\xi \in C^*$ we define

$$\xi^* = -\nabla \log \varphi_*(\xi).$$

Since the function $\log \varphi$ is strictly convex (cf. Proposition I.3.3 of [7]), the symmetric bilinear form on $\mathbb{R}^n$

$$G_x(u,v) = D_u D_v \log \varphi(x) \quad (\text{resp.} \quad G^*_\xi(u,v) = D_u D_v \log \varphi_*(\xi))$$

where $u,\, v \in \mathbb{R}^n$ defines on C (resp. C^*) a structure of Riemannian manifold. The corresponding Riemannian distances are given by

$$d(x,y) = \inf_{\gamma} \left\{ \int_0^1 \sqrt{G_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt \right\}$$

and

$$d_*(\xi,\eta) = \inf_{\gamma^*} \left\{ \int_0^1 \sqrt{G^*_{\gamma^*(t)}(\dot{\gamma^*}(t), \dot{\gamma^*}(t))}dt \right\}$$

where the infimum is taken on the smooth path $\gamma : [0,1] \to C$ (resp. $\gamma^* : [0,1] \to C^*$) such that $\gamma(0) = x,\ \gamma(1) = y$ (resp. $\gamma^*(0) = \xi,\ \gamma^*(1) = \eta$).

Lemma 2.2. *The Riemannian distances d and d_* are invariant under the action of $G(C)$ and $G(C^*)$ respectively i.e*

$$\forall x,\, y \in C, \forall g \in G(C),\ d(gx, gy) = d(x,y)$$

and

$$\forall \xi,\, \eta \in C^*, \forall g \in G(C),\ d_*(g^*\xi, g^*\eta) = d_*(\xi,\eta).$$

Proof. For the proof see [7], pages 15-16. ■

Proposition 2.3. *Let C be a homogeneous cone. For $x \in C$ and $g \in G(C)$, we have the following:*

$$(gx)^* = g^{*-1}x^*;$$
$$\varphi(x)\varphi_*(x^*) \text{ is constant;}$$
$$(x^*)^* = x.$$

Proof. See [13], Chapter 1, Section 4. ■

The first equality of the Proposition 2.3 shows that whenever the cone C is homogeneous, so is its dual C^*.

For $x \in C$ and $\xi \in C^*$, we denote by $A(x)$ and $A_*(\xi)$ the selfadjoint endomorphisms defined on $\mathbb{R}^n$ by

$$(A(x)u|v) = G_x(u,v) \quad \text{and} \quad (A_*(\xi)u|v) = G^*_\xi(u,v)$$

where $u, v \in T_xC \cong \mathbb{R}^n$. By the $G(C)$-invariance of G_x (resp. the $G(C^*)$-invariance of G^*_ξ), one has

$$A(gx) = g^{*-1}A(x)g^{-1} \quad \text{and} \quad A_*(g^*\xi) = g^{-1}A_*(\xi)g^{*-1} \tag{1}$$

for all $g \in G(C)$. Moreover, the following equalities hold

$$x^* = A(x)x \quad \text{and} \quad (x|x^*) = n$$

for all $x \in C$.

We have the following result which is the most important of this section.

Theorem 2.4. *Let C be an open convex homogeneous cone. The map $\sigma : x \mapsto x^*$ between the Riemannian manifolds C and C^* is an isometry; that is*

$$d_*(x^*, y^*) = d(x,y).$$

Proof. It suffices to establish that

$$G^*_{\sigma(x)}(D_u\sigma(x), D_v\sigma(x)) = G_x(u,v),$$

for all $u, v \in T_xC \cong \mathbb{R}^n$. The map $\sigma_* : \xi \mapsto \xi^*$ defined from C^* to C is the inverse of the bijection $\sigma : x \mapsto x^*$ defined from C to C^* since $C^{**} = C$. Moreover, for all $\xi \in C^*$, one has (cf. [7], page 17)

$$D_u\sigma_*(\xi) = -A_*(\xi)u.$$

We deduce that for all $x \in C$,

$$\begin{aligned} u = D_u(\sigma_* \circ \sigma)(x) &= D_{D_u\sigma(x)}\sigma_*(\sigma(x)) \\ &= -A_*(\sigma(x))D_u\sigma(x) = -A_*(x^*)(-A(x)u) = A_*(x^*)A(x)u; \end{aligned}$$

thus

$$A_*(x^*)A(x) = Id_{\mathbb{R}^n}$$

where $Id_{\mathbb{R}^n}$ denotes the identity map defined on $\mathbb{R}^n$. Hence,

$$\begin{aligned} G^*_{\sigma(x)}(D_u\sigma(x), D_v\sigma(x)) &= G^*_{\sigma(x)}(-A(x)u, -A(x)v) \\ &= G^*_{x^*}(A(x)u, A(x)v) \\ &= (A_*(x^*)A(x)u|A(x)v) \\ &= (u|A(x)v) = (A(x)u|v) \\ &= G_x(u,v). \end{aligned}$$

∎

Remark 2.5. It is important to mention that the map $\sigma_* : \xi \mapsto \xi^*$ is not necessarily equal to the inverse of $\sigma : x \mapsto x^*$ for a non-homogeneous cone. (For instance, see [10] for a counterexample where the cone is even proper). On the other hand, according to Lemma 8.3 of [5], in a general proper open convex cone, one can define a metric on the dual cone as a second derivative of a smooth function so that σ is an isometry.

3. Geometric properties of the Vinberg's cone

It is well known that the Vinberg's cone is homogeneous (cf. [13]). In this section, we give a description of a solvable group that acts simply transitively on the cone Ω. For $x \in V$, we define

$$pr_1(x) = \begin{pmatrix} x_1 & x_4 \\ x_4 & x_2 \end{pmatrix}, \quad pr_2(x) = \begin{pmatrix} x_1 & x_5 \\ x_5 & x_3 \end{pmatrix}.$$

If

$$g_i = \begin{pmatrix} \alpha & \gamma_i \\ 0 & \delta_i \end{pmatrix}, \quad \alpha\delta_i \neq 0, \quad i = 1, 2$$

then one defines an element g of $Gl(V)$ by

$$pr_i(gx) = g_i^{'} pr_i(x) g_i, \quad i = 1, 2 \tag{2}$$

where $g_i^{'}$ denote the transpose of g_i. Let us remark that when $x \in \Omega$, then $pr_1(x)$ and $pr_2(x)$ belong to the cone of 2×2 positive definite symmetric matrices. The endomorphism g is then well-defined by (2) and we have

$$gx = \begin{pmatrix} \alpha^2 x_1 & \alpha\gamma_1 x_1 + \alpha\delta_1 x_4 & \alpha\gamma_2 x_1 + \alpha\delta_2 x_5 \\ \alpha\gamma_1 x_1 + \alpha\delta_1 x_4 & \gamma_1^2 x_1 + \delta_1^2 x_2 + 2\delta_1\gamma_1 x_4 & 0 \\ \alpha\gamma_2 x_1 + \alpha\delta_2 x_5 & 0 & \gamma_2^2 x_1 + \delta_2^2 x_3 + 2\delta_2\gamma_2 x_5 \end{pmatrix}.$$

We denote by $\underline{\mathbf{e}}$ the identity matrix of V. The following identities are consequences of the same properties in the case of 2×2 positive definite symmetric matrices (see for instance [3], section 2):

$$\begin{aligned} Q_1(gx) &= \alpha^2 Q_1(x) = Q_1(g\underline{\mathbf{e}}) Q_1(x) && (3) \\ Q_2(gx) &= \delta_1^2 Q_2(x) = Q_2(g\underline{\mathbf{e}}) Q_2(x) && (4) \\ Q_3(gx) &= \delta_2^2 Q_3(x) = Q_3(g\underline{\mathbf{e}}) Q_3(x) && (5) \end{aligned}$$

where $\underline{\mathbf{e}}$ is the identity matrix of V. It follows that $g \in G(\Omega)$ and

$$Q^\tau(gx) = |\mathrm{Det}\, g|\, Q^\tau(x) \tag{6}$$

since

$$\mathrm{Det}\, g = \alpha^4 \delta_1^3 \delta_2^3 = Q^\tau(g\underline{\mathbf{e}}). \tag{7}$$

3.1. Description of a solvable group that acts simply transitively on Ω. Let us now consider the solvable subgroup H of $G(\Omega)$ consisting of elements h defined as in (2) by

$$h_i = \begin{pmatrix} \alpha & \gamma_i \\ 0 & \delta_i \end{pmatrix}, \quad \alpha > 0,\, \delta_i > 0, \quad i = 1, 2.$$

For all $y \in \Omega$, the equation $y = h\underline{\mathbf{e}}$ with $h \in H$ has a unique solution $h(y)$ defined as follows

$$h_1(y) = \begin{pmatrix} \sqrt{y_1} & \frac{y_4}{\sqrt{y_1}} \\ 0 & \sqrt{y_2 - \frac{{y_4}^2}{y_1}} \end{pmatrix} \quad \text{and} \quad h_2(y) = \begin{pmatrix} \sqrt{y_1} & \frac{y_5}{\sqrt{y_1}} \\ 0 & \sqrt{y_3 - \frac{{y_5}^2}{y_1}} \end{pmatrix}. \tag{8}$$

This shows that the group H acts simply transitively on the cone Ω. Therefore we have this identification $\Omega \equiv H\underline{\mathbf{e}}$.

For all $x \in \Omega$, $\xi \in \Omega^*$ and $g \in G$, $(gx|\xi) = (x|g^*\xi)$ where the adjoint endomorphism $g^* : V \to V$ is defined by

$$g^*\xi = \begin{pmatrix} \alpha^2\xi_1+\gamma_1^2\xi_2+\gamma_2^2\xi_3+2\alpha\gamma_1\xi_4+2\alpha\gamma_2\xi_5 & \gamma_1\delta_1\xi_2+\alpha\delta_1\xi_4 & \gamma_2\delta_2\xi_3+\alpha\delta_2\xi_5 \\ \gamma_1\delta_1\xi_2 + \alpha\delta_1\xi_4 & \delta_1^2\xi_2 & 0 \\ \gamma_2\delta_2\xi_3 + \alpha\delta_2\xi_5 & 0 & \delta_2^2\xi_3 \end{pmatrix}.$$

Let $\xi \in \Omega^*$. If we put

$$\pi_1(\xi) = \begin{pmatrix} \sqrt{\xi_1 - \frac{\xi_4^2}{\xi_2} - \frac{\xi_5^2}{\xi_3}} & \frac{\xi_4}{\sqrt{\xi_2}} \\ 0 & \sqrt{\xi_2} \end{pmatrix}, \qquad \pi_2(\xi) = \begin{pmatrix} \sqrt{\xi_1 - \frac{\xi_4^2}{\xi_2} - \frac{\xi_5^2}{\xi_3}} & \frac{\xi_5}{\sqrt{\xi_3}} \\ 0 & \sqrt{\xi_3} \end{pmatrix}; \tag{9}$$

then $\pi(\xi)$ is the unique element of H that satisfies

$$\xi = \pi(\xi)^*\underline{\mathbf{e}}.$$

This gives the identification $\Omega^* \equiv H^*\underline{\mathbf{e}}$.

3.2. Invariant measures on Ω and Ω^*. Let $y \in \Omega$ and $\xi \in \Omega^*$. From (6), the measure

$$dm(y) = Q^{-\tau}(y)dy \quad (\text{resp. } dm_*(\xi) = (Q^*)^{-\tau}(\xi)d\xi)$$

is $G(\Omega)$-invariant on Ω (resp. $G(\Omega^*)$-invariant on Ω^*).

The element $\underline{\mathbf{e}}^*$ is given by

$$\underline{\mathbf{e}}^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

We have

$$Q_1^*(\underline{\mathbf{e}}^*)Q_1(\underline{\mathbf{e}}) = 2, \qquad Q_2^*(\underline{\mathbf{e}}^*)Q_2(\underline{\mathbf{e}}) = \frac{3}{2}, \qquad Q_3^*(\underline{\mathbf{e}}^*)Q_3(\underline{\mathbf{e}}^*) = \frac{3}{2}. \tag{10}$$

It follows by homogeneity that if $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ then

$$Q^\alpha(x)(Q^*)^\alpha(x^*) = 2^{\alpha_1 - \alpha_2 - \alpha_3}\ 3^{\alpha_2 + \alpha_3} \tag{11}$$

for all $x \in \Omega$.

3.3. Some comments about the metrics.

We have established in the second section that the map $\sigma : x \mapsto x^*$ is an isometry from Ω to Ω^* with the metrics

$$G_x(u,v) = D_u D_v \log \varphi(x) \quad \text{and} \quad G_\xi^*(u,v) = D_u D_v \log \varphi_*(\xi)$$

where u, $v \in V$.

Now, we want to show that if we change our invariant metrics, this result can be false. Given a $G(\Omega)$-invariant metric $\mathcal{H}$ and a $G(\Omega^*)$-invariant metric $\mathcal{H}^*$, they realise an isometry between Ω and Ω^* if and only if

$$\mathcal{H}^*_{\underline{\mathbf{e}}^*}(A(\underline{\mathbf{e}})u, A(\underline{\mathbf{e}})v) = \mathcal{H}_{\underline{\mathbf{e}}}(u,v).$$

According to the identifications $\Omega = H\underline{\mathbf{e}}$ and $\Omega^* = H^*\underline{\mathbf{e}}$, we consider the following metrics on Ω and Ω^* defined as in [2]: for $x \in \Omega$, $\xi \in \Omega^*$, we put

$$\mathcal{G}_x(u,v) = (h^{-1}u | h^{-1}v)$$

and

$$\mathcal{G}_\xi^*(u,v) = (g^{*-1}u | g^{*-1}v)$$

where $x = h\underline{\mathbf{e}}$, $\xi = g^*\underline{\mathbf{e}}$ with u, $v \in V$ and $h, g \in H$. We have the following proposition:

Proposition 3.1. *i) The metric $\mathcal{G}_x$ (resp. $\mathcal{G}_\xi^*$) is an H-invariant metric (resp. H^*-invariant metric).*

ii) The metric $\mathcal{G}_x$ (resp. $\mathcal{G}_\xi^*$) is different from the H-invariant metric G_x (resp. H^*-invariant metric G_ξ^*) defined by the characteristic function of the cone Ω (resp. Ω^*).

iii) We have

$$\mathcal{G}^*_{\underline{\mathbf{e}}^*}(A(\underline{\mathbf{e}})u, A(\underline{\mathbf{e}})v) \neq \mathcal{G}_{\underline{\mathbf{e}}}(u,v).$$

Proof. i) Let $u, v \in V$ and $g \in G$, we want to show that

$$\mathcal{G}_{gx}(gu, gv) = \mathcal{G}_x(u,v).$$

The result is obvious by the definition of the metric.

ii) Since metrics G_x and $\mathcal{G}_x$ are H-invariant, it suffices to take $x = \underline{\mathbf{e}}$. We have

$$G_{\underline{\mathbf{e}}}(u,v) = (A(\underline{\mathbf{e}})u|v) \neq (u|v) = \mathcal{G}_{\underline{\mathbf{e}}}(u,v)$$

since $A(\underline{\mathbf{e}}) \neq Id_V$. In fact $A(\underline{\mathbf{e}})\underline{\mathbf{e}} = \underline{\mathbf{e}}^* \neq \underline{\mathbf{e}}$.

iii) Since $\Omega^* \equiv H^*\underline{\mathbf{e}}$, we have $\underline{\mathbf{e}}^* = h^*\underline{\mathbf{e}}$ where h is defined by $h_1 = h_2 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}$ according to (9). Since $\underline{\mathbf{e}}^* = A(\underline{\mathbf{e}})\underline{\mathbf{e}}$, we obtain

$$\mathcal{G}^*_{\underline{\mathbf{e}}^*}(A(\underline{\mathbf{e}})u, A(\underline{\mathbf{e}})v) = (h^{*-1}A(\underline{\mathbf{e}})u|h^{*-1}A(\underline{\mathbf{e}})v).$$

But $h^{*-1}A(\underline{\mathbf{e}})$ cannot be the identity because $A(\underline{\mathbf{e}})$ is not in $G(\Omega)$. ∎

Remark 3.2. For the case of symmetric cones, $x^* = \frac{n}{r}x^{-1}$ (Proposition III.4.3 of [7]) and we obtain that $G_x = \frac{n}{r}\mathcal{G}_x$. Thus, the map $\sigma : C \to C$ defined by $\sigma(x) = x^* = \frac{n}{r}x^{-1}$ is an isometry for the metric $\mathcal{G}_x$. (See [2], Theorem 2.50).

For $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, all the following identities hold

$$\begin{aligned} Q^\mu(h(y)t) &= Q^\mu(y)Q^\mu(t) && (12)\\ Q^\mu(\pi(\xi)t) &= (Q^*)^\mu(\xi)Q^\mu(t) && (13)\\ (Q^*)^\mu(h(y)^*\xi) &= Q^\mu(y)(Q^*)^\mu(\xi) && (14)\\ (Q^*)^\mu(\pi(\xi)^*\eta) &= (Q^*)^\mu(\xi)(Q^*)^\mu(\eta) && (15) \end{aligned}$$

for all $y, t \in \Omega$ and $\eta, \xi \in \Omega^*$.

3.4. The Whitney decomposition of the Vinberg's cone. In this section, we will prove some lemmas and propositions which are useful in the sequel; the most important of them is the Whitney decomposition of the cone Ω.

Lemma 3.3. *Given $\lambda > 0$, there is a constant $C = C(\lambda) > 0$ such that:*

i) if $d(y, t) \leq \lambda$ then $\frac{1}{C} \leq \frac{Q_j(y)}{Q_j(t)} \leq C$ for all $j = 1, 2, 3$ and $x, y \in \Omega$;

ii) if $d_*(\xi, \eta) \leq \lambda$ then $\frac{1}{C} \leq \frac{Q_j^*(\xi)}{Q_j^*(\eta)} \leq C$ for all $j = 1, 2, 3$ and $\xi, \eta \in \Omega^*$.

Proof. Since $\frac{Q_j(gy)}{Q_j(gt)} = \frac{Q_j(y)}{Q_j(t)}$ (resp. $\frac{Q_j^*(g^*\xi)}{Q_j^*(g^*\eta)} = \frac{Q_j^*(\xi)}{Q_j^*(\eta)}$) for all $g \in G(\Omega)$ defined by (2), it is sufficient to take $t = \underline{\mathbf{e}}$ (resp. $\eta = \underline{\mathbf{e}}$). The conclusion then follows from the continuity of the functions Q_j (resp. Q_j^*) and a compactness argument. ∎

Let $\lambda > 0$, $y \in \Omega$ and d the $G(\Omega)$-invariant distance defined in Ω. We denote by

$$B_\lambda(y) = \{t \in \Omega : d(y, t) < \lambda\}$$

the d-ball centered at the point y with the radius λ.

Lemma 3.4. *Let $0 < \lambda < 1$. Then*

$$m(B_\lambda(y)) \sim \lambda^5 \quad \text{and} \quad m_*(B_\lambda(\xi)) \sim \lambda^5.$$

Proof. By the $G(\Omega)$-invariance of the distance, we have for all $g \in G(\Omega)$, $B_\lambda(\underline{\mathbf{e}}) = gB_\lambda(\underline{\mathbf{e}})$, so $m(B_\lambda(y)) = m(B_\lambda(\underline{\mathbf{e}}))$ for all $y \in \Omega$. It follows that

$$m(B_\lambda(\underline{\mathbf{e}})) = \int_{B_\lambda(\underline{\mathbf{e}})} dm(y) = \int_{B_\lambda(\underline{\mathbf{e}})} Q^{-\tau}(y)dy \sim \int_{B_\lambda(\underline{\mathbf{e}})} dy.$$

It is well known that the distance d is equivalent to the Euclidean distance on compact subsets of V (cf. [11]); hence there is two positive constants c_1 and c_2 depending on Ω such that

$$\{y \in \Omega : |y - \underline{\mathbf{e}}| \leq c_1\lambda\} \subset B_\lambda(\underline{\mathbf{e}}) \subset \{y \in \Omega : |y - \underline{\mathbf{e}}| \leq c_2\lambda\}$$

and the result follows. ∎

We give now the Whitney decomposition of the cone Ω.

Lemma 3.5. *Given $0 < \lambda < 1$, there exists a sequence $\{y_j\}_j$ of points of Ω such that the following three properties hold:*

i) the balls $B_{\frac{\lambda}{2}}(y_j)$ are pairwise disjoint;

ii) the balls $B_\lambda(y_j)$ form a covering of Ω;

iii) there is an integer $N = N(\Omega)$ such that every $y \in \Omega$ belongs to at most N balls $B_\lambda(y_j)$.

Proof. Take $E = \{y_j\}$ a maximal subset of Ω (under inclusion) among those with the property that their elements are distant at least λ one from the other. Clearly the balls $B_{\frac{\lambda}{2}}(y_j)$ are pairwise disjoint. If the balls $B_\lambda(y_j)$ were not a covering of Ω, this would contradict the maximality of E.

Let us prove iii). Let $y \in \Omega$ such that $y \in \cap_{j=1}^N B_\lambda(y_j)$, then $\cup_{j=1}^N B_{\frac{\lambda}{2}}(y_j) \subset B_{\frac{3\lambda}{2}}(y)$. Hence $m(\cup_{j=1}^N B_{\frac{\lambda}{2}}(y_j)) \leq m(B_{\frac{3\lambda}{2}}(y)) = m(B_{\frac{3\lambda}{2}}(\underline{\mathbf{e}}))$ i.e $\sum_{j=1}^N m(B_{\frac{\lambda}{2}}(y_j)) \leq m(B_{\frac{3\lambda}{2}}(\underline{\mathbf{e}}))$ i.e $Nm(B_{\frac{\lambda}{2}}(\underline{\mathbf{e}})) \leq m(B_{\frac{3\lambda}{2}}(\underline{\mathbf{e}}))$ and N is finite. ∎

Remark 3.6. This lemma is also valid for the dual cone Ω^*.

Definition 3.7. Sequences $\{y_j\}_j$ (resp. $\{\xi_j\}_j$) of points of Ω (resp. Ω^*) that satisfy properties of Lemma 3.5 are called λ-*lattices* of Ω (resp. Ω^*.).

The family $\{B_\lambda(y_j)\}_j$ (resp. $\{B^*_\lambda(\xi_j)\}_j$) is called a *Whitney decomposition* of the cone Ω (resp. Ω^*).

Proposition 3.8. *The sequence $\{y_j\}_j$ is a λ-lattice of Ω if and only if $\{y^*_j\}_j$ is a λ-lattice in Ω^*. The sequence $\{y^*_j\}_j$ is called the dual lattice of the λ-lattice $\{y_j\}_j$.*

Proof. The proof follows immediately from the definitions and Theorem 2.4. ∎

Lemma 3.9. *Let $(y_0, \xi_0) \in \Omega \times \Omega^*$; then*

$$|B_\lambda(y_0)| = C_\lambda Q^\tau(y_0) \quad \text{and} \quad |B^*_\lambda(\xi_0)| = C_\lambda (Q^*)^\tau(\xi_0). \tag{16}$$

Proof. We know that $y_0 = h(y_0)\underline{\mathbf{e}}$ with $h(y_0) \in H$; if we use the change of variables $y = h(y_0)t$, $dy = Q^\tau(y_0)dt$, and, since the distance d is $G(\Omega)$-invariant, $d(y, y_0) = d(h(y_0)t, h(y_0)\underline{\mathbf{e}}) = d(t, \underline{\mathbf{e}})$. Hence,

$$|B_\lambda(y_0)| = Q^\tau(y_0) \int_{B_\lambda(\underline{\mathbf{e}})} dy = C_\lambda Q^\tau(y_0).$$

The same argument holds for $B^*_\lambda(\xi_0)$. ∎

In the sequel, we will consider the following disjoint covering of the cone Ω^*

$$E_1^* = B_1^*, \ \ E_j^* = B_j^* \setminus \bigcup_{k=1}^{j-1} B_k^*, \ \ j = 2, \ldots .$$

where $B_j^* = B_\lambda^*(y_j^*)$. We have

$$|E_j^*| \sim |B_j^*| \sim (Q^*)^\tau (y_j^*).$$

Proposition 3.10. *Let $y \in \Omega$ (resp. $\xi \in \Omega^*$.) There is a constant $\gamma = \gamma(\Omega, \Omega^*) \geq 1$ such that*

$$\frac{1}{\gamma} < \frac{(y|\xi)}{(y|\xi_0)} < \gamma \qquad \left(\text{resp.} \quad \frac{1}{\gamma} < \frac{(y|\xi)}{(y_0|\xi)} < \gamma\right)$$

whenever $\xi \in B_\lambda^*(\xi_0)$ (resp. $y \in B_\lambda(y_0)$).

Proof. Suppose first that $\xi_0 = \underline{\mathbf{e}}$. The inner product $(y, \xi) \mapsto (y|\xi)$ is continuous from $\overline{\Omega} \setminus \{0\} \times \Omega^*$ to $\mathbb{R}_+^*$; thus, restricted to the compact set $\{t \in \overline{\Omega} : |t| = 1\} \times \overline{B_\lambda^*(\underline{\mathbf{e}})}$, this function of two variables is between two constants $C_1 > 0$ and $C_2 > 0$. Replacing y by $\frac{y}{|y|}$ in $\frac{(y|\xi)}{(y|\underline{\mathbf{e}})}$, we establish the proof for $\xi_0 = \underline{\mathbf{e}}$.

For the general case, just write $\xi_0 = \pi(\xi_0)^* \underline{\mathbf{e}}$ with $\pi(\xi_0) \in H$ and notice that

$$\frac{(y|\xi)}{(y|\xi_0)} = \frac{(\pi(\xi_0) y | \pi(\xi_0)^{*-1} \xi)}{(\pi(\xi_0) y | \underline{\mathbf{e}})}.$$

Then one concludes using the first case. ■

Corollary 3.11. *Let $(y_0, \xi_0) \in \Omega \times \Omega^*$. There is a constant $\gamma > 0$ such that*

$$\frac{5}{\gamma} \leq (y|\xi) \leq 5\gamma$$

for all $(y, \xi) \in B_\lambda(y_0) \times B_\lambda^*(\xi_0)$.

Proof. We recall that $(y_0|y_0^*) = 5$. Using Proposition 3.10, there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{1}{c_1} < \frac{(y|\xi)}{(y|y_0^*)} < c_1 \quad \text{and} \quad \frac{1}{c_2} < \frac{(y|y_0^*)}{(y_0|y_0^*)} < c_2$$

whenever $\xi \in B_\lambda^*(\xi_0)$ and $y \in B_\lambda(y_0)$. The result follows from the fact that

$$\frac{(y|\xi)}{(y_0|y_0^*)} = \frac{(y|\xi)}{(y|y_0^*)} \times \frac{(y|y_0^*)}{(y_0|y_0^*)}.$$

Take $\gamma = c_1 c_2$. ■

4. Some integral formulas

We recall that

$$T_\Omega = V + i\Omega \quad (\text{resp.} \quad T_{\Omega^*} = V + i\Omega^* \quad).$$

We denote respectively by

$$\mathbb{H}_1 = \{z = x + iy \in \mathbb{C} : y > 0\} \quad \text{and} \quad \mathbb{H}_2 = \{z = x + iy \in \mathbb{C}^3 : y \in \Lambda_3\}$$

the upper half plane of $\mathbb{C}$ and the tube domain over the spherical cone Λ_3. For $z \in T_\Omega$ (resp. $\zeta \in T_{\Omega^*}$), we write

$$Q_1(z) = z_1, \quad Q_2(z) = z_2 - \frac{{z_4}^2}{z_1}, \quad Q_3(z) = z_3 - \frac{{z_5}^2}{z_1};$$
$$Q_1{}^*(\zeta) = \zeta_1 - \frac{{\zeta_4}^2}{\zeta_2} - \frac{{\zeta_5}^2}{\zeta_3}, \quad Q_2{}^*(\zeta) = \zeta_2, \quad Q_3{}^*(\zeta) = \zeta_3.$$

Proposition 4.1. *For all $z = x + iy \in T_\Omega$ (resp. $\zeta = \eta + i\xi \in T_{\Omega^*}$), we have*

$$\Re e\, Q_j\left(\frac{z}{i}\right) > 0 \qquad (\textit{resp.} \qquad \Re e\, Q_j{}^*\left(\frac{\zeta}{i}\right) > 0), \qquad j = 1, 2, 3.$$

Moreover, for all $y \in \Omega$ (resp. $\xi \in \Omega^*$),

$$\left|Q_j\left(\frac{x+iy}{i}\right)\right| \geq Q_j(y) \qquad (\textit{resp.} \qquad \left|Q_j{}^*\left(\frac{\eta+i\xi}{i}\right)\right| \geq Q_j{}^*(\xi)), \qquad j = 1, 2, 3.$$

Proof. Let $z = x+iy \in T_\Omega$. Then $z_1 = x_1+iy_1 \in \mathbb{H}_1$, which allows to conclude directly. In the same way, for $k = 2,\ 3$, one has $(z_1,\ z_k,\ z_{k+2}) \in \mathbb{H}_2$, which allows to conclude as in [3], Lemma 3.1.

Now let $\zeta = \eta + i\xi \in T_{\Omega^*}$. Assume first that $\xi = \underline{\mathbf{e}}$. Then, $\Im m\, Q_j^*(\zeta) = 1 > 0$, $j = 2, 3$ and

$$\Im m\, Q_1^*(\zeta) = 1 + \frac{\eta_4^2}{\eta_2^2 + 1} + \frac{\eta_5^2}{\eta_3^2 + 1} > 0.$$

So, for $j = 1, 2, 3$, we have $Q_j^*(\zeta) \in \mathbb{H}_1$. More generally, if $\xi \in \Omega^*$, then $\xi = \pi(\xi)^* \underline{\mathbf{e}}$; for $\eta = \pi(\xi)^*\beta$ with $\beta \in V$, we obtain $\Im m\, Q_j^*(\zeta) = Q_j^*(\xi) > 0$, $j = 2, 3$ and

$$\Im m\, Q_1^*(\zeta) = Q_1^*(\xi)\left(1 + \frac{\beta_4^2}{\beta_2^2 + 1} + \frac{\beta_5^2}{\beta_3^2 + 1}\right) > 0.$$

This implies that for $j = 1, 2, 3$, $\quad Q_j^*(\zeta) \in \mathbb{H}_1$. It follows that

$$\Re e\, Q_j^*\left(\frac{\zeta}{i}\right) = \Im m\, Q_j^*(\zeta) > 0$$

and

$$\left|Q_j{}^*\left(\frac{\eta+i\xi}{i}\right)\right| = |Q_j^*(\eta + i\xi)| \geq \Im m\, Q_j{}^*(\eta + i\xi) \geq Q_j^*(\xi) \qquad j = 1, 2, 3.$$

$\blacksquare$

Notation

For $\nu = (\nu_1, \nu_2, \nu_3)$ and j=1,2,3, we shall denote by

$$Q_j^{\nu_j}\left(\frac{z}{i}\right) \qquad (z \in T_\Omega)$$

(resp.

$$(Q_j^*)^{\nu_j}\left(\frac{\zeta}{i}\right) \qquad (\zeta \in T_{\Omega^*}\ ))$$

the determination of the ν_j-th power that corresponds to the holomorphic determination of the logarithm of $Q_j^{\nu_j}(\frac{z}{i})$ (resp. $(Q_j^*)^{\nu_j}(\frac{\zeta}{i})$) which is real and positive on $i\Omega$ (resp. $i\Omega^*$).

We recall that

$$\tau = (2, 3/2, 3/2).$$

Lemma 4.2. *Let $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$. The integral*

$$\int_\Omega e^{-(\xi|y)} Q^{\nu-\tau}(y)dy \qquad (resp. \quad \int_{\Omega^*} e^{-(y|\xi)} (Q^*)^{\nu-\tau}(\xi)d\xi \quad)$$

is finite for all $\xi \in \Omega^*$ (resp. $y \in \Omega$) if and only if

$$\nu_1 > 0, \qquad \nu_2 > \frac{1}{2}, \qquad \nu_3 > \frac{1}{2} \qquad (resp. \quad \nu_1 > 1, \quad \nu_2 > 0, \quad \nu_3 > 0 \quad).$$

For these values of ν and for all $\zeta = \eta + i\xi \in T_{\Omega^*}$ (resp. $z = x + iy \in T_\Omega$),

$$\int_\Omega e^{i(\zeta|y)} Q^{\nu-\tau}(y)dy = \pi\Gamma(\nu_1)\Gamma(\nu_2 - \frac{1}{2})\Gamma(\nu_3 - \frac{1}{2})(Q^*)^{-\nu}\left(\frac{\zeta}{i}\right) \tag{17}$$

(resp.

$$\int_{\Omega^*} e^{i(z|\xi)} (Q^*)^{\nu-\tau}(\xi)d\xi = \pi\Gamma(\nu_1 - 1)\Gamma(\nu_2)\Gamma(\nu_3)Q^{-\nu}\left(\frac{z}{i}\right) \quad). \tag{18}$$

Proof. To prove (17), by homogeneity, it suffices to compute the integral

$$\int_\Omega e^{-(\underline{\mathbf{e}}|y)} Q^{\nu-\tau}(y)dy.$$

For $i = 1, 2,$ we put

$$(s)_i = \begin{pmatrix} s_1 & s_{i+3} \\ 0 & s_{i+1} \end{pmatrix}$$

with $s_1 > 0$, $s_2 > 0$, $s_3 > 0$. Let $y \in \Omega$, we use this change of variable

$$pr_i(y) = (s)_i^{'}(s)_i, \quad i = 1,\ 2;$$

then

$$Q(y) = s_1^2 s_2^2 s_3^2, \ \ dy = 2^3 s_1^3 s_2 s_3 ds, \ \ (\underline{\mathbf{e}}|y) = \sum_{k=1}^{5} s_k^2.$$

We obtain

$$\int_\Omega e^{-(\underline{\mathbf{e}}|y)} Q^{\nu-\tau}(y)dy \;=\; 2^3\pi \left(\int_0^{+\infty} e^{-s_1^2} s_1^{2\nu_1 - 1} ds_1 \right) \prod_{j=2}^{3} \left(\int_0^{+\infty} e^{-s_j^2} s_j^{2\nu_j - 2} ds_j \right);$$

the integrals on the right hand side are convergent if and only if $\nu_1 > 0$, $\nu_2 > \frac{1}{2}$ and $\nu_3 > \frac{1}{2}$.

The proof of (18) follows directly from the properties of the Gamma function of the cone. (See [9]). ∎

Lemma 4.3. *Let $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$. For all $y \in \Omega$, the integral*

$$J_{\mu\lambda}(y) = \int_\Omega Q^\mu(y+v) Q^\lambda(v) dv$$

is finite if and only if

$$\lambda_1 > -2,\ \lambda_2 > -1,\ \lambda_3 > -1,\ \mu_1 + \lambda_1 < -3,\ \mu_2 + \lambda_2 < -\frac{3}{2},\ \mu_3 + \lambda_3 < -\frac{3}{2}.$$

In this case,

$$J_{\mu\lambda}(y) = M_{\mu\lambda} Q^{\mu+\lambda+\tau}(y).$$

Proof. One can observe that the convergence of the integral $J_{\lambda\mu}(y)$ is established for $y = \underline{\mathbf{e}}$; the rest follows by the identification $\Omega \equiv H\underline{\mathbf{e}}$ and the fact that $Q(h(y)t) = Q(y)Q(t)$ where $y = h(y)\underline{\mathbf{e}}$, with $h(y) \in H$ (here, $Q \equiv Q^{(1,1,1)}$). By (18), we write

$$Q^\mu(\underline{\mathbf{e}} + v) = c(\mu) \int_{\Omega^*} e^{-(\underline{\mathbf{e}}+v|\xi)} (Q^*)^{-\mu-\tau}(\xi) d\xi$$

if and only if $\mu_1 < -1$, $\mu_2 < 0$ and $\mu_3 < 0$. According to Fubini's Theorem, (17) and (18), we obtain

$$\begin{aligned} J_{\mu\lambda}(\underline{\mathbf{e}}) &= c(\mu) \int_{\Omega^*} e^{-(\underline{\mathbf{e}}|\xi)} (Q^*)^{-\mu-\tau}(\xi) \left(\int_\Omega e^{-(v|\xi)} Q^\lambda(v) dv \right) d\xi \\ &= c(\mu)c'(\lambda) \int_{\Omega^*} e^{-(\underline{\mathbf{e}}|\xi)} (Q^*)^{-\mu-\lambda-2\tau}(\xi) d\xi := M_{\lambda\mu} < +\infty \end{aligned}$$

if and only if

$$\lambda_1 > -2,\ \lambda_2 > -1,\ \lambda_3 > -1,\ \mu_1 + \lambda_1 < -3,\ \mu_2 + \lambda_2 < -\frac{3}{2},\ \mu_3 + \lambda_3 < -\frac{3}{2}.$$

∎

Lemma 4.4. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$.*

i) The integral

$$J_\alpha(y) = \int_V \left| Q^{-\alpha}\left(\frac{x+iy}{i} \right) \right| dx \qquad (y \in \Omega) \tag{19}$$

converges if and only if $\alpha_1 > 3,\ \alpha_2 > \frac{3}{2},\ \alpha_3 > \frac{3}{2}$. *If this case,*

$$J_\alpha(y) = m_\alpha Q^{-\alpha+\tau}(y).$$

ii) *The function*

$$F(z) = Q^{-\alpha}\left(\frac{z+it}{i}\right) \qquad (z \in T_\Omega)$$

with $t \in \Omega$, *belongs to* $A^{p,q}_\nu(T_\Omega)$ *if and only if*

$$\alpha_1 > max\left\{\frac{3}{p}, \frac{\nu_1+1}{q} + \frac{2}{p}\right\}, \quad \alpha_j > \frac{\nu_j}{q} + \frac{3}{2p}, \quad j = 2,3.$$

Proof. For fixed $y \in \Omega$, interpret (19) as the L^2-norm in dx of $Q^{\frac{-\alpha}{2}}\left(\frac{x+iy}{i}\right)$. By (18) and Plancherel's formula, the integral in (19) is finite if and only if the integral

$$\int_{\Omega^*} e^{-2(y|\xi)}(Q^*)^{\alpha-2\tau}(\xi)d\xi$$

is finite. This proves (i).

The rest follows by Lemma 4.3.

■

Lemma 4.5. *Let* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ *and* $0 < \lambda < \frac{1}{4}$. *There is a constant* C_α *such that for all* $y \in \Omega,\ |y| < \lambda$,

$$\int_{\{x\in V:|x|<1\}} \left|Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx \geq C_\alpha Q^{-\alpha+\tau}(y).$$

Proof. We set $x = h(y)x'$ where $h(y)$ is given by (8), and we use the fact that $dx = Q^\tau(y)dx'$. Then

$$\int_{\{x\in V:\ |x|<1\}} \left|Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx = Q^{-\alpha+\tau}(y)\int_{\{x'\in V:\ |h(y)x'|<1\}} \left|Q^{-\alpha}\left(\frac{x'+i\mathbf{\underline{e}}}{i}\right)\right| dx'$$

$\geq C_\alpha r Q^{-\alpha+\tau}(y)$ with

$$C_\alpha = \int_{\{x'\in V:\ |x'|<1\}} \left|Q^{-\alpha}\left(\frac{x'+i\mathbf{\underline{e}}}{i}\right)\right| dx'.$$

In fact, by our assumption, $|y| < \frac{1}{4}$; this implies that $\|h(y)\| < 1$. It follows that set $\{x' \in V :\ |h(y)x'| < 1\}$ contains the set $\{x' \in V :\ |x'| < 1\}$.

■

The characteristic function of Ω and Ω^*. We deduce from (18) and (17) that

$$\varphi(x) = \int_{\Omega^*} e^{-(x|\xi)}d\xi = \int_{\Omega^*} e^{-(x|\xi)}(Q^*)^{\tau-\tau}(\xi)d\xi = \frac{\pi^2}{4}Q^{-\tau}(x) \tag{20}$$

and

$$\varphi_*(\xi) = \int_\Omega e^{-(\xi|y)}dy = \int_\Omega e^{-(\xi|y)}Q^{\tau-\tau}(y)dy = \pi(Q^*)^{-\tau}(\xi). \tag{21}$$

5. The Bergman spaces

Here, we recall some basic facts about Bergman spaces. All these results are basically the same as those obtained in the paper [3]. The reader can look at this paper to have more details of proofs omitted here. For $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that $\nu_1 > 1$, $\nu_2 > 1/2$ and $\nu_3 > 1/2$, we shall denote $L^2_{(\nu)}(\Omega^*) = L^2(\Omega^*, (Q^*)^\nu(\xi)d\xi)$ and we define by

$$\mathcal{L}g(z) = (2\pi)^{-\frac{5}{2}} \int_{\Omega^*} e^{i(z|\xi)} g(\xi) d\xi$$

the Laplace transform of a locally integrable function g. We have this Paley-Wiener type theorem

Theorem 5.1. *Let $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ with $\nu_1 > 1$, $\nu_j > 1/2$, $j = 2, 3$. A function F belongs to $A^2_\nu(T_\Omega)$ if and only if $F = \mathcal{L}g$, with $g \in L^2_{(-\nu)}(\Omega^*)$. Moreover*

$$\|F\|^2_{A^2_\nu(T_\Omega)} = e_\nu \|g\|^2_{L^2_{(-\nu)}(\Omega^*)} \tag{22}$$

where

$$e_\nu = \pi 2^{-|\nu|+|\tau|} \Gamma(\nu_1 + 2)\Gamma(\nu_2 - 1/2)\Gamma(\nu_3 - 1/2).$$

Proof. Cf. [7], Proposition IX.3.3. ■

We denote by $\langle , \rangle_\nu$ the Hermitian form induced by the $A^2_\nu(T_\Omega)$-norm. It follows that for $F \in A^2_\nu(T_\Omega)$, since the Bergman kernel is a reproducing kernel of $A^2_\nu(T_\Omega)$, by polarization of (22),

$$F(w) = \langle F, B_\nu(\cdot, w)\rangle_\nu = e_\nu \langle g, g_w\rangle_{L^2_{(-\nu)}(\Omega^*)} = \int_{\Omega^*} g(\xi) e_\nu \overline{g_w(\xi)} (Q^*)^{-\nu}(\xi) d\xi.$$

Since $F = \mathcal{L}g$, one has

$$g_w(\xi) = (2\pi)^{-\frac{5}{2}} e_\nu^{-1} e^{-i(\overline{w}|\xi)} (Q^*)^\nu(\xi).$$

Hence, by (18),

$$B_\nu(z, w) = (2\pi)^{-\frac{5}{2}} \mathcal{L}g_w(z) = d_\nu Q^{-\nu-\tau}\left(\frac{z - \overline{w}}{i}\right).$$

The operator

$$P_\nu f(z) = \int_{T_\Omega)} B_\nu(z, w) f(w) Q^{\nu-\tau}(\Im m\, w) dv(w)$$

is the identity of $A^2_\nu(T_\Omega)$; it provides the orthogonal projection of $L^2_\nu(T_\Omega)$ onto $A^2_\nu(T_\Omega)$ i.e it is the Bergman projection.

Lemma 5.2. *Let $F \in A_\nu^{p,q}(T_\Omega)$. The following assertions hold:*

i) There is a constant $C = C(p,q,\nu) > 0$ such that for all $z = x+iy \in T_\Omega$,

$$|F(x+iy)| \leq CQ^{-\frac{\nu}{q}-\frac{\tau}{2p}}(y)\|F\|_{A_\nu^{p,q}(T_\Omega)}. \tag{23}$$

ii) There is a constant $C = C(p,q,\nu) > 0$ such that for all $y \in \Omega$,

$$\|F(\cdot+iy)\|_p \leq CQ^{-\frac{\nu}{q}}(y)\|F\|_{A_\nu^{p,q}(T_\Omega)}. \tag{24}$$

iii) There is a constant $C = C(p,q,\nu) > 0$ such that for all $y \in \Omega$ and all $s > p$,

$$\|F(\cdot+iy)\|_s \leq CQ^{-\frac{\nu}{q}-\frac{\tau}{2}\left(\frac{1}{p}-\frac{1}{s}\right)}(y)\|F\|_{A_\nu^{p,q}(T_\Omega)}. \tag{25}$$

Proof. The proof is the same as in [3], Lemma 4.2. ■

Corollary 5.3. *The Bergman space $A_\nu^{p,q}(T_\Omega)$ is a Banach space.*

Proof. Taking $s = \infty$ in Lemma 5.2, we see that convergence in $A_\nu^{p,q}(T_\Omega)$ implies convergence over compact subsets of T_Ω. So $A_\nu^{p,q}(T_\Omega)$ is a closed subspace of $L_\nu^{p,q}(T_\Omega)$. This one is known to be a Banach space. ■

Remark 5.4. Let $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3$ such that $\alpha_1 > 1,\ \alpha_j > 0,\ j = 2,3$; for all $(x,y,t) \in V \times \Omega \times \Omega$,

$$\left|Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| \leq Q^{-\alpha}(y) \tag{26}$$

$$Q^{-\alpha}(y+t) < Q^{-\alpha}(y). \tag{27}$$

The inequality (26) is a direct application of Proposition 4.1 and inequality (27) follows from Lemma 4.2.

Corollary 5.5. *Let $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{R}^3$ such that $\nu_1 > 1$, $\nu_j > 1/2$, $j = 2,3$ and $F \in A_\nu^{p,q}(T_\Omega)$;*

i) for every $t \in \Omega$, the function $F_t(z) = F(z+it)$ belongs to the Hardy space $\mathsf{H}^s(T_\Omega)$ for $s \geq p$;

ii) for $y,t \in \Omega$

$$\|F(\cdot+i(y+t))\|_s \leq \|F(\cdot+iy)\|_s;$$

iii) for $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3$ such that $\alpha_1 > 1$, $\alpha_j > 0$, $j = 2,3$ and $\varepsilon > 0$, let

$$F_{\varepsilon,\alpha}(z) = F(z+i\varepsilon\underline{\mathbf{e}})Q^{-\alpha}\left(\frac{\varepsilon z+i\underline{\mathbf{e}}}{i}\right).$$

Then $F_{\varepsilon,\alpha} \in A_\nu^{p,q}(T_\Omega)$ and we have

$$\lim_{\varepsilon\to 0}\|F - F_{\varepsilon,\alpha}\|_{A_\nu^{p,q}(T_\Omega)} = 0.$$

Proof. The proof is the same as in Corollary 4.4 of [3]. ■

Corollary 5.6. *Let $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ and $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ such that $\nu_1 > 1$, $\mu_1 > 1$ and $\nu_j > 1/2$, $\mu_j > 1/2$, $j = 2, 3$. The subspace $A_\nu^{p,q}(T_\Omega) \cap A_\mu^{s,r}(T_\Omega)$ of the Bergman spaces $A_\nu^{p,q}(T_\Omega)$ and $A_\mu^{s,r}(T_\Omega)$ is dense in each of them.*

Proof. The proof is the same as in Corollary 4.5 of [3]. ■

6. Proof of Theorem 1.1

In order to prove Theorem 1.1, we will state that the $L_\nu^{p,q}(T_\Omega)$-boundedness of the operator P_ν^+ is related to the $L^q(\Omega, Q^{\nu-\tau}(y)dy)$-boundedness of a positive integral operator on the cone Ω.

6.1. Positive integral operator on the cone Ω. Consider the positive integral operator S defined on Ω by

$$Sg(y) = \int_\Omega Q^{-\nu}(y+v)g(v)Q^{\nu-\tau}(v)dv. \tag{28}$$

It is easy to verify that S is a self-adjoint operator. We put

$$q_\nu = \nu_1 + 1.$$

Theorem 6.1. *Let $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that $\nu_1 > 1$, $\nu_2 > 1/2$, $\nu_3 > 1/2$. The operator S is bounded on $L^q(\Omega, Q^{\nu-\tau}(v)dv)$ if and only if $q'_\nu < q < q_\nu$.*

Proof. *The sufficiency.*

We will use Schur's Lemma (See [8]). The kernel of the operator S relative to the measure $Q^{\nu-\tau}(v)dv$ is given by

$$N(y, v) = Q^{-\nu}(y+v)$$

and it is positive. By Schur's Lemma, it is sufficient to find a positive and measurable function φ defined on Ω such that

$$\int_\Omega N(y, v)\varphi(v)^{q'}Q^{\nu-\tau}(v)dv \leq C\varphi(y)^{q'} \tag{29}$$

and

$$\int_\Omega N(y, v)\varphi(y)^{q}Q^{\nu-\tau}(y)dy \leq C\varphi(v)^{q}. \tag{30}$$

We take as test functions $\varphi(v) = Q^\gamma(v)$ where $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$ has to be determined. An application of Lemma 4.3 gives that (29) holds whenever

$$\frac{-\nu_1}{q'} < \gamma_1 < \frac{-1}{q'} \quad \text{and} \quad \frac{1/2 - \nu_j}{q'} < \gamma_j < 0, \quad j = 2, 3$$

and (30) holds when

$$\frac{-\nu_1}{q} < \gamma_1 < \frac{-1}{q} \quad \text{and} \quad \frac{1/2 - \nu_j}{q} < \gamma_j < 0, \quad j = 2, 3.$$

Assume $q < 2 < q'$ then $\frac{1/2-\nu_j}{q} < \frac{1/2-\nu_j}{q'}$, $j = 1, 2, 3$. In this case the inequalities (29) and (30) are simultaneously satisfied if the following conditions hold

$$\gamma_j \in \left]\frac{1/2-\nu_j}{q}, 0\right[, \quad j = 2, 3 \tag{31}$$

and

$$\gamma_1 \in \left]\frac{-\nu_1}{q'}, \frac{-1}{q'}\right[\bigcap \left]\frac{-\nu_1}{q}, \frac{-1}{q}\right[. \tag{32}$$

The interval in (31) is non-empty since $\nu_j > 1/2$, for $j = 2, 3$ and the intersection in (32) is not empty if $\frac{-\nu_1}{q'} < \frac{-1}{q}$, i.e if $q > \frac{\nu_1+1}{\nu_1} = q'_\nu$.

The case $q' < 2 < q$ is obtained accordingly since the operator is self-adjoint; we obtain the dual condition $q' > q'_\nu$ i.e $q < q_\nu$.

The necessity.

If we take for g the characteristic function of the invariant ball $B_\delta(\underline{\mathbf{e}})$, $0 < \delta < 1$; from Lemma 3.3 we know that the functions $v \mapsto Q(v)$ is almost constant on $B_\delta(\underline{\mathbf{e}})$. Moreover, take δ so small that there are two positive constants c_1 and c_2 such that $\{t \in V : |t - \underline{\mathbf{e}}| \leq c_1\} \subset B_\delta(\underline{\mathbf{e}}) \subset \{t \in V : |t - \underline{\mathbf{e}}| \leq c_2\}$. For a fixed $y \in \Omega$, the function $v \in \Omega \mapsto Q(y+v)$ is continuous on the compact subset $\{t \in \Omega : |t - \underline{\mathbf{e}}| \leq c_2\}$ of V contained in Ω; then there exists $b \in \{t \in \Omega : |t - \underline{\mathbf{e}}| \leq c_2\}$ such that $Q(y+v) \leq Q(y+b)$ for all $v \in \{t \in \Omega : |t - \underline{\mathbf{e}}| \leq c_2\} \supset B_\delta(\underline{\mathbf{e}})$. It follows that

$$\begin{aligned}
\|Sg\|_{L^q(\Omega, Q^{\nu-\tau}(v)dv)} &= \left(\int_\Omega \left(\int_{B_\delta(\underline{\mathbf{e}})} Q^{-\nu}(y+v) Q^{\nu-\tau}(v)dv\right)^q Q^{\nu-\tau}(y)dy\right)^{\frac{1}{q}} \\
&\geq C\left(\int_\Omega \left(\int_{B_\delta(\underline{\mathbf{e}})} Q^{-\nu}(y+b) Q^{\nu-\tau}(\underline{\mathbf{e}})dv\right)^q Q^{\nu-\tau}(y)dy\right)^{\frac{1}{q}} \\
&\geq C\left(\int_\Omega Q^{-\nu q}(y+b) Q^{\nu-\tau}(y)dy\right)^{\frac{1}{q}} \int_{B_\delta(\underline{\mathbf{e}})} dv
\end{aligned}$$

i.e

$$\|Sg\|_{L^q(\Omega, Q^{\nu-\tau}(v)dv)} \geq C_{b\delta} \|Q^{-\nu}(\cdot + b)\|_{L^q(\Omega, Q^{\nu-\tau}(v)dv)}.$$

So if S is bounded on $L^q(\Omega, Q^{\nu-\tau}(v)dv)$, the function $y \mapsto Q^{-\nu}(y+b)$ belongs to $L^q(\Omega, Q^{\nu-\tau}(y)dy)$. Using Lemma 4.3, we get the necessary condition $q > q'_\nu$. The dual condition $q < q_\nu$ follows from the self-adjointness of S. ■

6.2. Proof of Theorem 1.1.

We prove the following:

Theorem 6.2. *Let $1 \leq p \leq +\infty$ and $1 \leq q < +\infty$. The operator P^+_ν is bounded on $L^{p,q}_\nu(T_\Omega)$ if and only if $q'_\nu < q < q_\nu$. Moreover, the weighted Bergman projector P_ν is bounded from $L^{p,q}_\nu(T_\Omega)$ to $A^{p,q}_\nu(T_\Omega)$ whenever $q'_\nu < q < q_\nu$.*

Proof. *The sufficiency*

Let $f \in L^{p,q}_\nu(T_\Omega)$; we use this notation $f_y(x) = f(x+iy)$. Then

$$P^+_\nu f(x+iy) \;=\; b_\nu \int_\Omega (|Q^{-\nu-\tau}_{y+v}| * f_v)(x) Q^{\nu-\tau}(v)dv;$$

by Minkowski's inequality and the Young's inequality, we obtain that

$$\begin{aligned}\left(\int_V |P^+_\nu f(x+iy)|^p dx\right)^{\frac{1}{p}} &\le b_\nu \left(\int_V \left(\int_\Omega ||Q^{-\nu-\tau}_{y+v}| * f_v(x)| Q^{\nu-\tau}(v)dv\right)^p dx\right)^{\frac{1}{p}} \\ &\le b_\nu \int_\Omega \left(\int_V ||Q^{-\nu-\tau}_{y+v}| * f_v(x)|^p dx\right)^{\frac{1}{p}} Q^{\nu-\tau}(v)dv \\ &\le b_\nu \int_\Omega \||Q^{-\nu-\tau}_{y+v}| * f_v\|_p Q^{\nu-\tau}(v)dv \\ &\le b_\nu \int_\Omega \||Q^{-\nu-\tau}_{y+v}|\|_1 \|f_v\|_p Q^{\nu-\tau}(v)dv.\end{aligned}$$

Moreover, by Lemma 4.4,

$$\||Q^{-\nu-\tau}_{y+v}|\|_1 = CQ^{-\nu}(y+v)$$

hence,

$$\left(\int_V |P^+_\nu f(x+iy)|^p dx\right)^{\frac{1}{p}} \le Cb_\nu Sg(y)$$

where S is the positive integral operator defined in (28) and $g(v) = \|f_v\|_p$. The function g belongs to $L^q(\Omega, Q^{\nu-\tau}(v)dv)$ since $f \in L^{p,q}_\nu(T_\Omega)$. It follows from Theorem 6.1 that for $q'_\nu < q < q_\nu$, one has

$$\|P^+_\nu f\|_{L^{p,q}_\nu(T_\Omega)} \le Cb_\nu \|Sg\|_{L^q(\Omega, Q^{\nu-\tau}(v)dv)} \le Cb_\nu \|g\|_{L^q(\Omega, Q^{\nu-\tau}(v)dv)} = C'b_\nu \|f\|_{L^{p,q}_\nu(T_\Omega)}.$$

This finishes the proof of the sufficiency part.

The necessity

Conversely, we shall prove that P^+_ν is not bounded on $L^{p,q}_\nu(T_\Omega)$ when $q \ge q_\nu$.

Let $\lambda \in \mathbb{R}$ such that $0 < \lambda < \frac{1}{4}$ and g the function defined on T_Ω by $g(x+iy) = \chi_{\{|x|<1\}}(x)k(y)$ where $k \in L^q(\Omega, Q^{\nu-\tau}(v)dv)$ is positive with support in $\{y \in \Omega : |y| < \frac{\lambda}{2}\}$. By Lemma 4.5, there is a constant C such that for all $v, y \in \Omega$ which satify $|v| < \frac{\lambda}{2}$, $|y| < \frac{\lambda}{2}$,

$$P^+_\nu g(x+iy) \ge Cb_\nu \int_{|v|<\frac{\lambda}{2}} Q^{-\nu}(y+v)k(v)Q^{\nu-\tau}(v)dv = Cb_\nu Sk(y).$$

It follows from our hypothesis that

$$\begin{aligned}\int_{|y|<\frac{\lambda}{2}} & (Sk(y))^q Q^{\nu-\tau}(y)dy = c\int_{|y|<\frac{\lambda}{2}} \left(\int_{|x|<1} (Sk(y))^p dx\right)^{\frac{q}{p}} Q^{\nu-\tau}(y)dy \\ & \le c(Cb_\nu)^{-q} \int_{|y|<\frac{\lambda}{2}} \left(\int_{|x|<1} |P^+_\nu g(x+iy)|^p dx\right)^{\frac{q}{p}} Q^{\nu-\tau}(y)dy \\ & \le c(Cb_\nu)^{-q} \|P^+_\nu g\|^q_{L^{p,q}_\nu(T_\Omega)} \le c'(Cb_\nu)^{-q} \|g\|^q_{L^{p,q}_\nu(T_\Omega)}.\end{aligned}$$

Let $N \in \mathbb{N}^*$. By applying the statement above to the Euclidean ball $B(O,N)$, we obtain

$$\int_{|y|<N}\left(\int_{|v|<N} Q(y+v)^{-\nu}k(v)Q^{\nu-\tau}(v)dv\right)^q Q^{\nu-\tau}(y)dy$$
$$\leq c(Cb_\nu)^{-q}\int_{|y|<N}(k(y))^q Q^{\nu-\tau}(y)dy.$$

The continuous functions with compact support are dense in $L^q(\Omega,dz)$, $dz = Q^{\nu-\tau}(y)dy)$, so by letting N tend to infinity, we conclude by the Lebesgue monotone convergence Theorem that

$$\int_\Omega (Sk(y))^q Q^{\nu-\tau}(y)dy \leq C_{\nu q}\|P_\nu^+ g\|^q_{L^{p,q}_\nu(T_\Omega)} \leq C'_{\nu q}\int_\Omega (k(y))^q Q^{\nu-\tau}(y)dy.$$

Thus, if P_ν^+ is bounded on $L^{p,q}_\nu(T_\Omega)$ then S is bounded on $L^q(\Omega, Q^{\nu-\tau}(y)dy)$; we deduce that, if S is not bounded on $L^q(\Omega, Q^{\nu-\tau}(y)dy)$, i.e if $q \geq q_\nu$, then P_ν^+ is not bounded on $L^{p,q}_\nu(T_\Omega)$. This gives the necessary condition.

To obtain the result for the Bergman projector, it is sufficient to remark that for all $f \in L^{p,q}_\nu(T_\Omega)$, one has $\|P_\nu f\|_{L^{p,q}_\nu(T_\Omega)} \leq \|P_\nu^+|f|\|_{L^{p,q}_\nu(T_\Omega)} \leq C_{\nu q}\||f|\|_{L^{p,q}_\nu(T_\Omega)} = C_{\nu q}\|f\|_{L^{p,q}_\nu(T_\Omega)}$ whenever $q'_\nu < q < q_\nu$.

■

7. L^p-estimates of the Bergman projector P_ν.

In this section, we shall find values of p for which the Bergman projector P_ν is bounded whenever the operator P_ν^+ is not bounded. We will use the Paley-Wiener Theorem (Theorem 5.1) to prove that the Laplace transform is an isomorphism between $A^{2,q}_\nu(T_\Omega)$ and the space $b^q_\nu(\Omega^*)$. We conclude then by interpolation. The results here are the same as in the paper [2] which deals with symmetric cones. We will give only statements of the proofs that emphasize some differences.

We recall that $\Omega^* = \bigcup_j E^*_j$ where the sets E^*_j are pairwise disjoint.

Definition 7.1. Let $q \geq 1$, $0 < \lambda < 1$ and $\{\xi_j\}$ a λ-lattice in Ω^*. We denote by $b^q_\nu(\Omega^*)$ the space of all measurable functions g which are locally square integrable and satisfy the estimate

$$\|g\|_{b^q_\nu(\Omega^*)} := \left(\sum_j (Q^*)^{-\nu}(\xi_j)\left(\int_{E^*_j}|g(\xi)|^2 d\xi\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} < +\infty.$$

We say that a sequence $\{\lambda_j\}_j$ belongs to l^q_ν if it satisfies

$$\sum_j |\lambda_j|^q (Q^*)^{-\nu}(\xi_j) < +\infty.$$

Lemma 7.2. *The space $b^q_\nu(\Omega^*)$ is a Banach space.*

Proof. Just remark that $b^q_\nu(\Omega^*) = l^q_\nu(L^2(E^*_j))$. ■

Remark 7.3. Let $\{a_j\}_j$ a positive sequence. Then

$$\left(\sum_j a_j\right)^\delta \leq \sum_j a_j^\delta \qquad \text{if} \qquad 0 < \delta \leq 1 \tag{33}$$

and

$$\sum_j a_j^\delta \leq \left(\sum_j a_j\right)^\delta \qquad \text{if} \qquad \delta \geq 1. \tag{34}$$

7.1. The boundedness of the Bergman projector P_ν on $L^{2,q}_\nu(T_\Omega)$.

We shall show that the Laplace transform $\mathcal{L}$ is isomorphically bounded from $b^q_\nu(\Omega^*)$ onto $A^{2,q}_\nu(T_\Omega)$.

The following proposition proves the statement for $q = 2$.

Proposition 7.4. *There is a constant $C = C(\nu) > 1$ such that for all $F \in A^{2,2}_\nu(T_\Omega)$,*

$$\frac{1}{C}\sum_j (Q^*)^{-\nu}(\xi_j)\int_{E^*_j}|g(\xi)|^2 d\xi \leq \|F\|^2_{A^{2,2}_\nu} \leq C\sum_j (Q^*)^{-\nu}(\xi_j)\int_{E^*_j}|g(\xi)|^2 d\xi;$$

where $F = \mathcal{L}g$ with $g \in L^2_{(-\nu)}(\Omega^*)$.

Proof. According to Theorem 5.1, $F \in A^{2,2}_\nu(T_\Omega)$ if and only if $F = \mathcal{L}g$ with $g \in L^2_{(-\nu)}(\Omega^*)$. By Lemma 3.3,

$$\|F\|^2_{A^{2,2}_\nu} = e_\nu \sum_j \int_{E^*_j}|g(\xi)|^2 (Q^*)^{-\nu}(\xi)d\xi \leq e'_\nu \sum_j (Q^*)^{-\nu}(\xi_j)\int_{E^*_j}|g(\xi)|^2 d\xi.$$

Conversely, by Lemma 3.3 and (iii) of Lemma 3.5,

$$\begin{aligned}\sum_j (Q^*)^{-\nu}(\xi_j)\int_{E^*_j}|g(\xi)|^2 d\xi &\leq c_\nu \int_{\Omega^*}|g(\xi)|^2\left(\sum_j \chi_{B^*_j}(\xi)\right)(Q^*)^{-\nu}(\xi)d\xi \\ &\leq c_\nu N e_\nu^{-1}\|F\|^2_{A^{2,2}_\nu}.\end{aligned}$$

■

Lemma 7.5. *Let $q \geq 1$. There is a constant $C = C(\nu, \tau, q) > 0$ such that for all $g \in b^q_\nu(\Omega^*)$ and all $y \in \Omega$,*

$$\int_{\Omega^*}|g(\xi)|e^{-(y|\xi)}d\xi \leq C\|g\|_{b^q_\nu(\Omega^*)}Q^{-\frac{\nu}{q}-\frac{\tau}{2}}(y).$$

In particular, g is locally integrable on Ω^*.

Proof. See Lemma 3.27 of [2]. ■

Theorem 7.6. *Let $q \geq 1$. For all $F \in A_\nu^{2,q}(T_\Omega)$, there is a unique function $g \in b_\nu^q(\Omega^*)$ such that $F = \mathcal{L}g$ and*

$$\|g\|_{b_\nu^q(\Omega^*)} \leq C\|F\|_{A_\nu^{2,q}(T_\Omega)}.$$

Proof. By density (see Corollary 5.6), take $F \in A_\nu^{2,q}(T_\Omega) \cap A_\nu^{2,2}(T_\Omega)$. By Paley-Wiener Theorem (Theorem 5.1), there exists a function $g \in L^2_{(-\nu)}(\Omega^*)$ such that

$$F(x+iy) = \mathcal{L}g(x+iy) = (2\pi)^{-\frac{5}{2}} \int_{\Omega^*} g(\xi) e^{i(x+iy|\xi)} d\xi.$$

Let $\{y_j\}_j$ be a λ-lattice of Ω and let $\{y_j^*\}_j$ be the dual lattice of the λ-lattice $\{y_j\}_j$. We saw that the map $x \mapsto x^*$ is an isometry from Ω on Ω^* (cf. Theorem 2.4). Thus for $y \in B_j = B_\lambda(y_j)$, one has $y^* \in B_j^* = B_\lambda(y_j^*)$; moreover, by Corollary 3.11, there is a constant γ such that $\frac{1}{\gamma} \leq (y|\xi) \leq \gamma$ whenever $y \in B_j$ and $\xi \in B_j^*$. Then, for $y \in B_j$, according to Corollary 3.11, we have

$$\int_{E_j^*} |g(\xi)|^2 d\xi \leq c_\gamma \int_{\Omega^*} |g(\xi)|^2 e^{-2(y|\xi)} d\xi = C' \int_V |F(x+iy)|^2 dx$$

by Plancherel's formula. It follows from (16) that

$$\left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \leq c_q' Q^{-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx\right)^{\frac{q}{2}} dy.$$

If we denote by $\{\xi_j\}_j$ the dual λ-lattice of $\{y_j\}_j$, then by (11) and $i)$ of Lemma 3.3, $(Q^*)^{-\nu}(\xi_j) \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}}$

$$\begin{aligned}
&\leq c_q (Q^*)^{-\nu}(\xi_j) Q^{-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx\right)^{\frac{q}{2}} dy \\
&\leq c_{\nu q} Q^{\nu-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx\right)^{\frac{q}{2}} dy \\
&\leq c'_{\nu q} \int_{B_j} \left(\int_V |F(x+iy)|^2 dx\right)^{\frac{q}{2}} Q^{\nu-\tau}(y) dy;
\end{aligned}$$

hence

$$\|g\|_{b_\nu^q(\Omega^*)} \leq C_{\nu,q} \|F\|_{A_\nu^{2,q}(T_\Omega)}. \tag{35}$$

This finishes the proof. ■

We prove now the converse of the previous theorem.

Theorem 7.7. *Assume $1 \leq q < 2q_\nu$. Given $g \in b_\nu^q(\Omega^*)$, then $\mathcal{L}g \in A_\nu^{2,q}(T_\Omega)$ and*

$$\|\mathcal{L}g\|_{A_\nu^{2,q}(T_\Omega)} \leq C\|g\|_{b_\nu^q(\Omega^*)}.$$

Proof. Write $F(x+iy) = F_y(x) = \mathcal{L}g(x+iy)$. For every $y \in \Omega$, the function $x \mapsto F_y(x)$ is the inverse Fourier transform of the function $\xi \mapsto \psi_y(\xi) = g(\xi)e^{-(y|\xi)}$. By Plancherel's formula,

$$\|F\|^q_{A^{2,q}_\nu(T_\Omega)} = \int_\Omega \left(\int_{\Omega^*} |g(\xi)|^2 e^{-2(y|\xi)} d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy.$$

By (ii) of Lemma 3.5 and Proposition 3.10, we deduce that

$$\|F\|^q_{A^{2,q}_\nu(T_\Omega)} \leq \int_\Omega \left(\sum_j e^{-2\gamma(y|\xi_j)} \int_{E^*_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy. \tag{36}$$

First assume that $1 \leq q \leq 2$. Since $\frac{q}{2} \leq 1$, we deduce from the inequality (33) and Lemma 4.2 that

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu(T_\Omega)} &\leq \int_\Omega \sum_j e^{-q\gamma(y|\xi_j)} \left(\int_{E^*_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy \\ &\leq \sum_j \left(\int_{B^*_j} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \int_\Omega e^{-q\gamma(y|\xi_j)} Q^{\nu-\tau}(y)dy \leq C_{\nu q\gamma} \|g\|^q_{b^q_\nu(\Omega^*)}.\end{aligned}$$

Assume next that $2 \leq q < 2q_\nu$. Let $\rho = \frac{q}{2}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. By Hölder's inequality,

$$\begin{aligned}\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E^*_j} |g(\xi)|^2 d\xi\right) &\leq \left(\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E^*_j} |g(\xi)|^2 d\xi\right)^\rho (Q^*)^{-\alpha\rho}(\xi_j)\right)^{\frac{1}{\rho}} \\ &\quad \times \left(\sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'}(\xi_j)\right)^{\frac{1}{\rho'}}.\end{aligned}$$

From (36), it follows that

$$\begin{aligned}\|F\|^q_{A^{2,q}_\nu(T_\Omega)} \leq \int_\Omega &\left[\left(\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E^*_j} |g(\xi)|^2 d\xi\right)^\rho (Q^*)^{-\alpha\rho}(\xi_j)\right)\right. \\ &\left.\times \left(\sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'}(\xi_j)\right)^{\frac{\rho}{\rho'}}\right] Q^{\nu-\tau}(y)dy.\end{aligned} \tag{37}$$

From (16), $ii)$ of Lemma 3.3, Proposition 3.10 and $iii)$ of Lemma 3.5 , we have

$$\begin{aligned}\sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'}(\xi_j) &\leq c \sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'-\tau}(\xi_j) \int_{B^*_j} d\xi \\ &\leq CN \int_{\Omega^*} e^{-2(y|\xi)} (Q^*)^{\alpha\rho'-\tau}(\xi) d\xi.\end{aligned}$$

We deduce from (18) that

$$\sum_j e^{-2\gamma(y|\xi_j)}(Q^*)^{\alpha\rho'}(\xi_j) \leq C_{\alpha\rho} Q^{-\alpha\rho'}(y)$$

whenever $\alpha_1\rho' > 1,\ \alpha_j\rho' > 0,\ j = 2, 3.$

So for $\alpha_1\rho' > 1,\ \alpha_j\rho' > 0,\ j = 2, 3$, from inequality (37) we obtain:

$$\|F\|^q_{A^{2,q}_\nu(T_\Omega)} \leq C_{\alpha\rho} \int_\Omega \left(\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E^*_j} |g(\xi)|^2 d\xi \right)^\rho (Q^*)^{-\alpha\rho}(\xi_j) \right) Q^{-\alpha\rho+\nu-\tau}(y) dy$$

$$\leq C_{\alpha\rho} \sum_j \left(\int_{E^*_j} |g(\xi)|^2 d\xi \right)^\rho (Q^*)^{-\alpha\rho}(\xi_j) \int_\Omega e^{-2\gamma(y|\xi_j)} Q^{-\alpha\rho+\nu-\tau}(y) dy.$$

Moreover, if $-\alpha_1\rho + \nu_1 > 0,\ -\alpha_j\rho + \nu_j > \frac{1}{2}$, by (17), we have

$$\int_\Omega e^{-2\gamma(y|\xi_j)} Q^{-\alpha\rho+\nu-\tau}(y) dy = c_{\alpha\nu\rho} (Q^*)^{\alpha\rho-\nu}(\gamma\xi_j);$$

it follows that,

$$\|F\|^q_{A^{2,q}_\nu(T_\Omega)} \leq C_{\alpha\nu\rho} \|g\|^q_{b^q_\nu(\Omega^*)}.$$

Therefore, the conclusion follows if we choose $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1\rho' > 1, \quad \alpha_2\rho' > 0, \quad \alpha_3\rho' > 0$$

and

$$-\alpha_1\rho + \nu_1 > 0, \quad -\alpha_2\rho + \nu_2 > \frac{1}{2}, \quad -\alpha_3\rho + \nu_3 > \frac{1}{2}.$$

The parameters α_2 and α_3 can be suitably chosen since $0 < \alpha_j < \frac{\nu_j - 1/2}{\rho},\ j = 2, 3,$ and α_1 must lie in $\left]\frac{1}{\rho'}, \frac{\nu_1}{\rho}\right[$ which is a non-empty interval. ∎

We have proved that the Laplace transform $\mathcal{L}$ maps $b^q_\nu(\Omega^*)$ isomorphically onto $A^{2,q}_\nu(T_\Omega)$ whenever $1 \leq q < 2q_\nu$.

Lemma 7.8. *For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$, the integral*

$$I_{\alpha\beta} = \int_{\Omega^*} (Q^*)^\alpha(\xi) \left(1 + |\log Q^*_1(\xi)|\right)^\beta e^{-(\xi|\mathbf{e})} d\xi$$

is finite if and only if one of the following two conditions is satisfied:

i) $\alpha_1 > -1$ and $\alpha_j > -\frac{3}{2},\ j = 2, 3;$

ii) $\alpha_1 = -1,\ \alpha_j > -\frac{3}{2},\ j = 2, 3$ and $\beta < -1.$

Proof. We use this change of variable: $\xi = uu^*$ where $u = \begin{pmatrix} u_1 & u_4 & u_5 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}$ with $u_j > 0,\ j = 1, 2, 3$. We have $d\xi = 2^3 u_1 u_2^2 u_3^2 du$ and $Q^*(\xi) = u_1^2 u_2^2 u_3^2$. It follows that

$$\begin{aligned} I_{\alpha\beta} &= 4\pi \left(\int_0^{+\infty} u_1^{2\alpha_1+1} (1 + |\log u_1^2|)^\beta e^{-u_1^2} du_1 \right) \times \prod_{j=2}^{3} \left(\int_0^{+\infty} u_j^{2\alpha_j+2} e^{-u_j^2} du_j \right) \\ &= 4\pi J_\alpha K_\alpha \end{aligned}$$

where

$$J_\alpha = \int_0^{+\infty} u_1^{2\alpha_1+1} (1 + |\log u_1^2|)^\beta e^{-u_1^2} du_1 \quad \text{and} \quad K_\alpha = \prod_{j=2}^{3} \left(\int_0^{+\infty} u_j^{2\alpha_j+2} e^{-u_j^2} du_j \right).$$

We observe that K_α is finite if and only if $\alpha_j > -\frac{3}{2},\ j = 2, 3$ while J_α is finite if and only if either $\alpha_1 > -1$ or both $\alpha_1 = -1$ and $\beta < -1$.

■

Theorem 7.9. *For $q \geq 2\nu_1 + 2$, there is a function $g \in b_\nu^q(\Omega^*)$ such that $\mathcal{L}g$ does not belong to $L^{2,q}(T_\Omega)$.*

Proof. Let $q = 2\nu_1 + 2$; we will find a positive function g on Ω^* such that $\|g\|_{b_\nu^q(\Omega^*)} < +\infty$ but $I(y) = \int_{\Omega^*} |g(\xi)|^2 e^{-2(y|\xi)} d\xi = \infty$ for all $y \in \Omega$.

Take

$$g(\xi) = e^{-(\xi|\underline{\mathbf{e}})} (Q^*)^\alpha(\xi) (1 + |\log Q_1^*(\xi)|)^{-\frac{1}{2}}$$

with $\alpha = (-1/2, \alpha_2, \alpha_3)$ such that $\alpha_j q > \nu_j - \frac{3}{4}q,\ j = 2, 3$.

By Plancherel formula, we have the following

$$\|\mathcal{L}g\|_{L_\nu^{2,q}(T_\Omega)} = \int_\Omega I(y)^{\frac{q}{2}} Q^{\nu-\tau}(y) dy;$$

in particular, $I(\underline{\mathbf{e}}) = \int_{\Omega^*} (Q^*)^{2\alpha}(\xi) \left(1 + |\log Q_1^*(\xi)|\right)^{-1} e^{-2(\xi|\underline{\mathbf{e}})} d\xi$. According to Lemma 7.8, this integral is not finite. This shows that $\mathcal{L}g \notin L_\nu^{2,q}(T_\Omega)$. Moreover, from Lemma 3.3, Hölder inequality and (iii) of Lemma 3.5,

$$\begin{aligned} \|g\|_{b_\nu^q(\Omega^*)}^q &= \sum_j (Q^*)^{-\nu}(\xi_j) \left(\int_{E_j^*} |g(\xi)|^2 d\xi \right)^{\frac{q}{2}} \\ &= \sum_j (Q^*)^{-\nu}(\xi_j) \left(\int_{E_j^*} e^{-2(\xi|\underline{\mathbf{e}})} (Q^*)^{2\alpha}(\xi) (1 + |\log Q_1^*(\xi)|)^{-1} d\xi \right)^{\frac{q}{2}} \\ &\leq C \sum_j \left(\int_{E_j^*} e^{-2(\xi|\underline{\mathbf{e}})} (Q^*)^{2\alpha - \frac{2}{q}\nu + \tau}(\xi) (1 + |\log Q_1^*(\xi)|)^{-1} \frac{d\xi}{(Q^*)^\tau(\xi)} \right)^{\frac{q}{2}} \\ &\leq C_q \sum_j \int_{E_j^*} e^{-q(\xi|\underline{\mathbf{e}})} (Q^*)^{(2\alpha+\tau)\frac{q}{2} - \nu}(\xi) (1 + |\log Q_1^*(\xi)|)^{-\frac{q}{2}} \frac{d\xi}{(Q^*)^\tau(\xi)} \\ &\leq C_q N \int_{\Omega^*} e^{-q(\xi|\underline{\mathbf{e}})} (Q^*)^{(2\alpha+\tau)\frac{q}{2} - \nu - \tau}(\xi) (1 + |\log Q_1^*(\xi)|)^{-\frac{q}{2}} d\xi. \end{aligned}$$

According to Lemma 7.8, the last integral above is convergent; so the function $g \in b^q_\nu(\Omega^*)$ whenever its Laplace transform does not belong to $L^{2,q}_\nu(T_\Omega)$ for $q = 2\nu_1 + 2$. ■

Let us now consider the operator $R = \mathcal{L}^{-1}P_\nu$. The operator R is surjective. We will now show that the operator R is bounded from $L^{2,q}_\nu(T_\Omega)$ to $b^q_\nu(\Omega^*)$.

Let $\phi \in L^2_\nu(T_\Omega)$; by Paley-Wiener Theorem, $F \in A^2_\nu(T_\Omega)$ if and only if $F = \mathcal{L}g$ with $g \in L^2_{(-\nu)}(\Omega^*)$. The self-adjointness of P_ν implies

$$\langle P_\nu\phi, F\rangle_{A^2_\nu(T_\Omega)} = \langle \phi, F\rangle_{L^2_\nu(T_\Omega)} = \langle \phi, \mathcal{L}g\rangle_{L^2_\nu(T_\Omega)}.$$

Now, by the Plancherel formula and Fubini's Theorem

$$\begin{aligned}
\langle \phi, \mathcal{L}g\rangle_{L^2_\nu(T_\Omega)} &= \int_\Omega \left(\int_{\mathbb{R}^5} \phi_y(x)\overline{\mathcal{F}^{-1}(g(\xi)e^{-(y|\xi)})(x)}dx\right) Q^{\nu-\tau}(y)dy \\
&= \int_\Omega \left(\int_{\Omega^*} \mathcal{F}(\phi_y)(\xi)\overline{g(\xi)}e^{-(y|\xi)}d\xi\right) Q^{\nu-\tau}(y)dy \\
&= \int_{\Omega^*} \Big\{\left((Q^*)^\nu(\xi)\int_\Omega \mathcal{F}(\phi_y)(\xi)e^{-(y|\xi)}Q^{\nu-\tau}(y)dy\right) \\
&\qquad \overline{g(\xi)}(Q^*)^{-\nu}(\xi)d\xi\Big\}
\end{aligned} \tag{38}$$

where $\mathcal{F}$ is the Fourier transform. Therefore, for $g \in L^2_{(-\nu)}(\Omega^*)$, equality (38) and the polarization of isometry (22) in the Paley-Wiener Theorem imply that

$$\begin{aligned}
\langle \phi, \mathcal{L}g\rangle_{L^2_\nu(T_\Omega)} &= \langle P_\nu\phi, F\rangle_{A^2_\nu(T_\Omega)} \\
&= e_\nu\langle \mathcal{L}^{-1}P_\nu, g\rangle_{L^2_{(-\nu)}(\Omega^*)} = e_\nu\langle R\phi, g\rangle_{L^2_{(-\nu)}(\Omega^*)}.
\end{aligned} \tag{39}$$

Comparing (38) and (39) then gives

$$R\phi(\xi) = e_\nu^{-1}(Q^*)^\nu(\xi)\left(\int_\Omega \mathcal{F}\phi_y(\xi)e^{-(y|\xi)}Q^{\nu-\tau}(y)dy\right).$$

We shall need the following lemma.

Lemma 7.10. *If $q \geq 2$, then for all $\phi \in L^{2,q}_\nu(T_\Omega)$, $R\phi \in b^q_\nu(\Omega^*)$ and*

$$\|R\phi\|_{b^q_\nu(\Omega^*)} \leq C\|\phi\|_{L^{2,q}_\nu(T_\Omega)}.$$

Proof. See Lemma 4.21 of [2]. ■

Let

$$Q_\nu = 2q_\nu = 2\nu_1 + 2;$$

we can prove now the following result

Corollary 7.11. *The Bergman projector P_ν extends to a bounded operator from $L^{2,q}_\nu(T_\Omega)$ to $A^{2,q}_\nu(T_\Omega)$ if and only if $Q'_\nu < q < Q_\nu$.*

Proof. Assume that $2 \leq q < Q_\nu$. By Lemma 7.10, the operator R is bounded from $L_\nu^{2,q}(T_\Omega)$ to $b_\nu^q(\Omega^*)$ and according to the Theorem 7.7, the Laplace transform $\mathcal{L}$ is bounded from $b_\nu^q(\Omega^*)$ to $A_\nu^{2,q}(T_\Omega)$. We conclude that the Bergman projector $P_\nu = \mathcal{L} \circ R$ is bounded from $L_\nu^{2,q}(T_\Omega)$ to $A_\nu^{2,q}(T_\Omega)$. We obtain the other part by self-adjointness of P_ν. This proves the sufficiency.

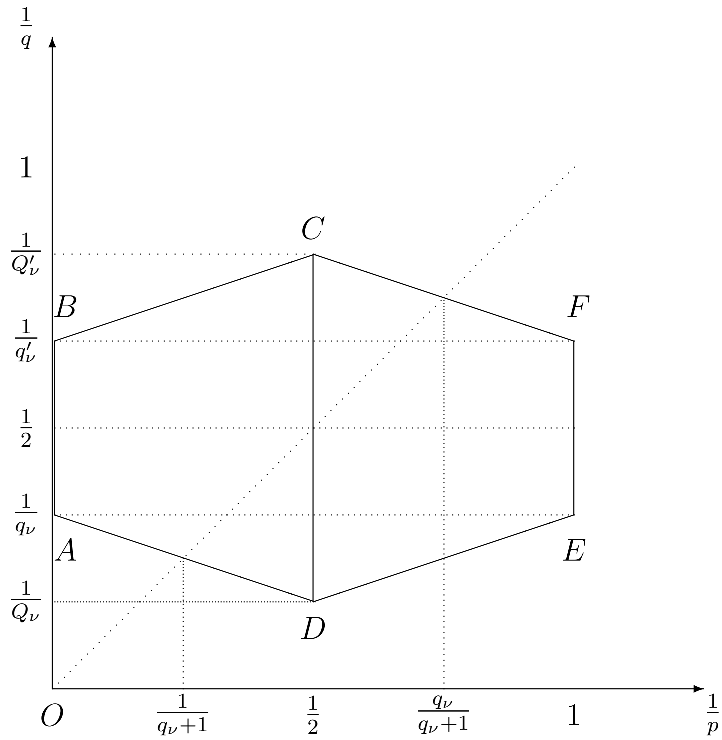
The necessary condition follows from Theorem 7.9. ∎

7.2. Proof of Theorem 1.2.

Theorem 7.12. *The Bergman projector P_ν extends to a bounded operator from $L_\nu^{p,q}(T_\Omega)$ to $A_\nu^{p,q}(T_\Omega)$ if*

$$\begin{cases} 0 \leq \frac{1}{p} \leq \frac{1}{2} \\ \frac{1}{q_\nu p'} < \frac{1}{q} < 1 - \frac{1}{q_\nu p'} \end{cases} \qquad \text{or} \qquad \begin{cases} \frac{1}{2} \leq \frac{1}{p} \leq 1 \\ \frac{1}{q_\nu p} < \frac{1}{q} < 1 - \frac{1}{q_\nu p}. \end{cases}$$

Proof. For a fixed $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ that satisfies $\nu_1 > 1$, $\nu_2 > 1/2$ and $\nu_3 > 1/2$, let us consider the following picture



Picture.

By interpolation, P_ν is bounded on $L_\nu^{p,q}(T_\Omega)$ for $\left(\frac{1}{p}, \frac{1}{q}\right)$ in the interior of the hexagon of vertices

$$A\left(0, \frac{1}{\nu_1 + 1}\right), \quad D\left(\frac{1}{2}, \frac{1}{2\nu_1 + 2}\right), \quad E\left(1, \frac{1}{\nu_1 + 1}\right)$$

and their symmetric points with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$. ∎

Theorem 1.2 is the particular case $p = q$ of Theorem 7.12. As we remark, our condition depends only on ν_1. It is then important to say that, for the dual cone Ω^*, we obtain

$$q_\nu = 1 + \min\{2\nu_2, 2\nu_3\};$$

this result depends on ν_2 and ν_3 and is different from the one we obtained in this paper. This shows that when the homogeneous cone is non-symmetric, there are different indices when one considers the cone or its dual.

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