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Centralizers of Lie Algebras Associated to Descending Central Series of Certain Poly-Free Groups

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Abstract. Poly-free groups are constructed as iterated semidirect products of free groups. The class of poly-free groups includes the classical pure braid groups, fundamental groups of fiber-type hyperplane arrangements, and certain subgroups of the automorphism groups of free groups. The purpose of this article is to compute centralizers of certain natural Lie subalgebras of the Lie algebra obtained from the descending central series of poly-free groups Γ including some of the geometrically interesting classes of groups mentioned above. The main results here extend the result in Cohen, F. R., and S. Prassidis: On injective homomorphisms for pure braid groups, and associated Lie algebras, J. Algebra 298 (2006), 363–370, for such groups. These results imply that a homomorphism $f : \Gamma \to G$ is faithful, essentially, if it is faithful when restricted to the level of Lie algebras obtained from the descending central series for the product $F_T \times Z$, where F_T is the "top" free group in the semidirect products of free groups and Z is the center of Γ. The arguments use a mixture of homological, and Lie algebraic methods applied to certain choices of extensions. The limitations of these methods are illustrated using the "poison groups" of Formanek and Procesi Formanek, E., and C. Procesi: The automorphism group of a free group is not linear, J. Algebra 149 (1992), 494–499, poly-free groups whose Lie algebras do not have certain properties considered here.
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1. Introduction

A group Γ is *poly-free* if there is a sequence

$$\{1\} = \Gamma_0 \leq \Gamma_1 \leq \cdots \leq \Gamma_n = \Gamma$$

such that Γ_{i+1} is a normal subgroup of Γ_i and the quotient Γ_i/Γ_{i+1} is isomorphic to a free group. Examples include the Artin pure braid group, the fundamental

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group of the configuration space of n ordered points in $\mathbb{C}$, as well as fundamental groups of certain orbit configuration spaces [5, 8]. There are also natural subgroups of the automorphism group of a free group which are poly-free. One such example considered here is the "upper triangular" version of McCool's subgroup of basis-conjugating automorphisms [9]. Homological properties of such groups have been investigated in [7] and [19]. Geometric properties of poly-free groups were investigated in [1] and [17].

The main objective of this paper is to determine the centralizers of certain Lie subalgebras of the Lie algebras which arise from the classical descending central series of the families of discrete groups given above. These Lie algebras have been used in contexts ranging from Vassiliev invariants of pure braids [21, 22] to structures of simplicial groups and the loop space of the 2-sphere [2, 11]. Other applications of these Lie algebras arise in the study of homotopy groups of higher dimensional knot spaces, see [28].

For nonempty subsets U and V of a (discrete) group G, let $[U, V]$ denote the subgroup of G generated by all commutators $[u, v] = uvu^{-1}v^{-1}$ of elements $u \in U$ and $v \in V$. Let G_n be the n-th stage of the descending central series of G, defined inductively by $G_1 = G$, and $G_{n+1} = [G_n, G]$ for $n \geq 1$, and let $\mathrm{gr}_n(G) = G_n/G_{n+1}$ be the n-th associated quotient. Let $\mathrm{gr}_*(G) = \bigoplus_{n\geq 1} \mathrm{gr}_n(G)$. There is a bilinear homomorphism

$$[-,-] : \mathrm{gr}_p(G) \otimes_{\mathbb{Z}} \mathrm{gr}_q(G) \longrightarrow \mathrm{gr}_{p+q}(G)$$

induced by the commutator map

$$c : G \times G \longrightarrow G$$

(which is itself not in general a homomorphism).

The construction of $\mathrm{gr}_*(G)$ is a functor from the category of discrete groups to the category of Lie algebras over $\mathbb{Z}$. This construction has potential applications to questions concerning representations. Specifically, if G is *residually nilpotent* and a group homomorphism f out of G induces a Lie algebra monomorphism on the corresponding Lie algebras, then f is itself a monomorphism ([11]). Based on this property, conditions which insure that a representation of G is faithful are recorded in Corollary 1.2 below. It should be noted, however, that these methods have not yet succeeded in producing faithful representations of the poly-free groups considered here.

Summary of the Results. Let B be a subset of the Lie algebra A. Then *the centralizer of B in A* is defined by

$$\mathcal{C}_A(B) = \{a \in A : \ [a, b] = 0, \text{ for all } b \in B\}.$$

Abbreviate $\mathcal{C}_A(A)$ by $\mathcal{C}(A)$. The main result of this paper gives centralizers of Lie subalgebras of $\mathrm{gr}_*(G)$ for the following groups G:

I. The "upper triangular McCool groups", subgroups $P\Sigma_n^+$ of the automorphism group of the free group $F_n = F[x_1, \dots, x_n]$ generated by automorphisms $\beta_{i,j}$, $1 \leq i < j \leq n$, defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k, & \text{if } k \neq j, \\ x_i^{-1}x_jx_i, & \text{if } k = j. \end{cases}$$

II. Fundamental groups $P_G(n)$ of orbit configuration spaces associated to surface groups G acting freely, and properly discontinuously on the upper 1/2-plane $\mathbb{H}$. In this case, $\mathrm{gr}_*(P_G(n))$ is generated by $B^{\sigma}_{i,j}$ with $\sigma \in G$, $1 \le i < j \le n$.

III. Fundamental groups $P(r,n)$ of orbit configuration spaces associated to finite cyclic groups, of order r, acting freely on $\mathbb{C}^*$, by rotation. The generators of the Lie algebra $\mathrm{gr}_*(P(r,n))$ are given by $B^{(p)}_{i,j}$ with $1 \le p \le r$, $1 \le i < j \le n$, and Z_k for $1 \le k \le n$.

The groups Γ_n given by $P\Sigma_n^+$, $P_G(n)$, or $P(r,n)$ all have the property that there is an extension

$$1 \longrightarrow F_{\alpha(n)} \xrightarrow{j} \Gamma_n \xrightarrow{p} \Gamma_{n-1} \longrightarrow 1 \tag{1}$$

which satisfies the following properties.

Γ_0 is a free group; (2a)

$F_{\alpha(n)}$ is a free group on a countable set $\alpha(n)$ of cardinality at least two; (2b)

the map $p : \Gamma_n \to \Gamma_{n-1}$ is a split epimorphism; and (2c)

the action of Γ_{n-1} on $H_*(F_{\alpha(n)})$ is trivial. (2d)

By work in [20, 14, 29], for such a group Γ_n, there is a split extension of Lie algebras

$$0 \longrightarrow \mathrm{gr}_*(F_{\alpha(n)}) \xrightarrow{\mathrm{gr}_*(j)} \mathrm{gr}_*(\Gamma_n) \xrightarrow{\mathrm{gr}_*(p)} \mathrm{gr}_*(\Gamma_{n-1}) \longrightarrow 0. \tag{3}$$

In particular the map $j : F_{\alpha(n)} \to \Gamma_n$ induces an injection on the level of Lie algebras

$$\mathrm{gr}_*(j) : L[V_{\alpha(n)}] = \mathrm{gr}_*(F_{\alpha(n)}) \longrightarrow \mathrm{gr}_*(\Gamma_n),$$

where $L[V_{\alpha(n)}]$ denotes the free Lie algebra generated by the set $\alpha(n)$. Groups Γ_n which admit this structure are the main focus of this paper.

Theorem 1.1. *Let Γ_n be one of the groups described above. The centralizer of the free Lie algebra $L[V_{\alpha(n)}]$ in $\mathrm{gr}_*(\Gamma_n)$ is given as follows:*

I. *If $\Gamma_n = P\Sigma_n^+$ is the upper triangular McCool group, then*

$$\mathcal{C}_{\mathrm{gr}_*(\Gamma_n)}(L[V_{\alpha(n)}]) = L[B_n],$$

where $B_{i,j}$ is the image of $\beta_{i,j}$ in $\mathrm{gr}_*(P\Sigma_n^+)$, and $B_n = \sum_{j=2}^n B_{1,j}$.

II. *If $\Gamma_n = P_G(n)$ is the fundamental group of a surface group orbit configuration space, then*

$$\mathcal{C}_{\mathrm{gr}_*(\Gamma_n)}(L[V_{\alpha(n)}]) = 0.$$

III. *If $\Gamma_n = P(r,n)$ is the fundamental group of a cyclic group orbit configuration space, then*

$$\mathcal{C}_{\mathrm{gr}_*(\Gamma_n)}(L[V_{\alpha(n)}]) = L[\Delta(r,n)],$$

where

$$\Delta(r,n) = \sum_{k=1}^{n} Z_k + \sum_{p=1}^{r} \sum_{1 \le i < j \le n} B^{(p)}_{i,j}.$$

For a poly-free group Γ_n satisfying conditions (2), the center of the Lie algebra $\mathrm{gr}_*(\Gamma_n)$ is determined in section 2. Namely, it is shown in Proposition 2.1 that the center of $\mathrm{gr}_*(\Gamma_n)$ is either infinite cyclic or trivial. Furthermore, if the free group Γ_0 is of rank greater than one (as is the case for the groups $F_{\alpha(k)}$ by (2b)), then both the center of $\mathrm{gr}_*(\Gamma_n)$ and the center of Γ_n itself are trivial. This property need not be satisfied for an arbitrary poly-free group, as illustrated by the direct product $F_1 \times F_1$, where F_1 is free of rank one.

Recall the classical adjoint representation

$$\mathrm{ad} : L \longrightarrow \mathrm{Der}(L)$$

of a graded Lie algebra L, where $\mathrm{Der}(L)$ denotes the graded Lie algebra of graded derivations of L. The map ad is defined by the equation $\mathrm{ad}(X)(Y) = [X, Y]$ for X and Y in L. The center of L is precisely the kernel of $\mathrm{ad} : L \to \mathrm{Der}(L)$. If I is a Lie ideal of L, then the natural restriction map is denoted by

$$\mathrm{ad}\,|_I : L \longrightarrow \mathrm{Der}(I).$$

Let Γ_n be one of the groups considered above. Regard $\mathrm{gr}_*(\Gamma_n)$ as a graded Lie algebra by the convention that $\mathrm{gr}_q(\Gamma_n)$ has degree $2q$ (as the axioms for a graded Lie algebra are not satisfied without this convention). In these cases, $L[V_{\alpha(n)}]$ is a Lie ideal. Consider the two adjoint representations:

$$\mathrm{ad} : \mathrm{gr}_*(\Gamma_n) \longrightarrow \mathrm{Der}(\mathrm{gr}_*(\Gamma_n)) \quad \text{and} \quad \mathrm{ad}|_{L[V_{\alpha(n)}]} : \mathrm{gr}_*(\Gamma_n) \longrightarrow \mathrm{Der}(L[V_{\alpha(n)}]).$$

Corollary 1.1. *Let Γ_n be one of the above groups.*

I. *If $\Gamma_n = P\Sigma_n^+$, then* $\ker(\mathrm{ad}) = \ker(\mathrm{ad}|_{L[V_{\alpha(n)}]})$ *is the cyclic group generated by B_n in $\mathrm{gr}_1(P\Sigma_n^+)$. Thus there is a short exact sequence of Lie algebras*

$$0 \longrightarrow L[B_n] \longrightarrow \mathrm{gr}_*(P\Sigma_n^+) \xrightarrow{\mathrm{ad}\,|_{L[V_n]}} \mathrm{Image}(\mathrm{ad}\,|_{L[V_n]}) \longrightarrow 0.$$

II. *If $\Gamma_n = P_G(n)$, then* $\ker(\mathrm{ad}) = \ker(\mathrm{ad}|_{L[V_{\alpha(n)}]})$ *is trivial. Thus there is an isomorphism*

$$\mathrm{gr}_*(P_G(n)) \xrightarrow{\mathrm{ad}\,|_{L[V_n]}} \mathrm{Image}(\mathrm{ad}\,|_{L[V_n]}).$$

III. *If $\Gamma_n = P(r, n)$, then* $\ker(\mathrm{ad}) = \ker(\mathrm{ad}|_{L[V_{\alpha(n)}]})$ *is the cyclic group generated by $\Delta(r, n)$ in $\mathrm{gr}_1(P(r, n))$. Thus there is a short exact sequence of Lie algebras*

$$0 \longrightarrow L[\Delta(r, n)] \longrightarrow \mathrm{gr}_*(P(r, n)) \xrightarrow{\mathrm{ad}\,|_{L[V_n]}} \mathrm{Image}(\mathrm{ad}\,|_{L[V_n]}) \longrightarrow 0.$$

Recent work on the Isomorphism Conjecture [15, 16] has renewed interest in detecting monomorphisms on discrete groups, especially when the target space is a finite dimensional linear group and the image is a discrete subgroup. The linearity of the classical Artin braid groups, and hence the Artin pure braid groups, was established by Bigelow [4] and Krammer [23]. Subsequently, Digne [12] showed that Artin groups of crystallographic type are linear. This paper may be viewed as an attempt to develop methods for detecting faithful finite dimensional representations of poly-free groups, as illustrated by the following consequence of Theorem 1.1 and the results of [10].

Corollary 1.2. *Let Γ_n be one of the poly-free groups $P\Sigma_n^+$, $P_G(n)$, or $P(r,n)$, and $f:\Gamma_n \to G$ a homomorphism of groups. If the morphism of Lie algebras*

$$\mathrm{gr}_*(f)|_{L[V_{\alpha(n)}]} : L[V_{\alpha(n)}] \longrightarrow \mathrm{gr}_*(G)$$

is a monomorphism and

I. $\mathrm{gr}_*(f)|_{L[B_n]} : L[B_n] \to \mathrm{gr}_*(G)$ *is a monomorphism when* $\Gamma_n = P\Sigma_n^+$*;*

II. *no further conditions when* $\Gamma_n = P_G(n)$*;*

III. $\mathrm{gr}_*(f)|_{L[\Delta(r,n)]} : L[\Delta(r,n)] \to \mathrm{gr}_*(G)$ *is a monomorphism when* $\Gamma_n = P(r,n)$*,*

then f is a monomorphism. In addition, the following two statements are equivalent:

(i) The map $f:\Gamma_n \to G$ is one-to-one.

(ii) The maps of Lie algebras

$$\mathrm{gr}_*(f)|_{L[V_{\alpha(n)}]} : L[V_{\alpha(n)}] \longrightarrow \mathrm{gr}_*(f(\Gamma_n))$$

is a monomorphism and

I. $\mathrm{gr}_*(f)|_{L[B_n]} : L[B_n] \to \mathrm{gr}_*(f(P\Sigma_n^+))$ *is a monomorphism when* $\Gamma_n = P\Sigma_n^+$*;*

II. *no further conditions when* $\Gamma_n = P_G(n)$*;*

III. $\mathrm{gr}_*(f)|_{L[\Delta(r,n)]} : L[\Delta(r,n)] \to \mathrm{gr}_*(f(P(r,n))$ *is a monomorphism when* $\Gamma_n = P(r,n)$*,*

where $f(-)$ denotes the image of f.

The Artin pure braid groups, and the groups $P(r,n)$, may be realized as fundamental groups of complements of fiber-type hyperplane arrangements. For any such arrangement, the fundamental group of the complement is poly-free, and satisfies the conditions (2), see Falk and Randell [14]. Thus, there are corresponding split, short exact sequences of descending central series Lie algebras as in (3). Furthermore, using the linearity of the pure braid group [4, 23] and topological properties of fiber-type arrangements [5, 6], one can show that the fundamental group of the complement of any fiber-type arrangement is linear, see Theorem 6.4. Consequently, many poly-free groups which fit into exact sequences of the form (1) admit faithful finite dimensional linear representations.

There is, however, a dichotomy as follows. The "poison group" H of Formanek and Procesi admits no faithful, finite dimensional linear representation [18]. While this group is poly-free, and fits into a split exact sequence

$$1 \longrightarrow F_3 \xrightarrow{\ j\ } H \xrightarrow{\ p\ } F_2 \longrightarrow 1,$$

where F_k is a rank k free group, information is lost upon passage to the descending central series Lie algebra. Specifically, there are homomorphisms out of H which have non-trivial kernels, but induce monomorphisms on the level of descending central series Lie algebras. The natural map

$$p \times \alpha : H \longrightarrow F_2 \times H_1(H),$$

given by the projection $p : H \to F_2$ and the abelianization map $\alpha : H \to H_1(H)$, is one such homomorphism. The map $p \times \alpha$ has non-trivial kernel but induces a monomorphism on the Lie algebras. Thus, the Lie algebraic methods here fail to inform on representations of the group H. This failure arises directly from the fact that the local coefficient system in homology is non-trivial, that is, the action of F_2 on $H_*(F_3)$ is non-trivial. For details, see Proposition 7.1.

The above discussion suggests the following.

Question 1. *Let Γ be a group that fits into a split exact sequence*

$$1 \longrightarrow F \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

with F a finitely generated free group and G a group that admits a finite dimensional faithful linear representation. If G acts on F by basis-conjugating automorphisms, does Γ then admit a finite dimensional faithful linear representation? More generally, if G acts trivially on the homology of F, does Γ then admit a finite dimensional faithful linear representation?

2. Centers

The purpose of this section is to identify the center of the descending central series Lie algebra $\mathrm{gr}_*(\Gamma_n)$ in the case where Γ_n belongs to class of poly-free groups which satisfy the conditions (2). Recall that such groups are given inductively by extensions

$$1 \longrightarrow F_{\alpha(n)} \xrightarrow{j} \Gamma_n \xrightarrow{p} \Gamma_{n-1} \longrightarrow 1,$$

where Γ_0 is a free group; $F_{\alpha(n)}$ is a free group of rank at least 2; the map $p : \Gamma_n \to \Gamma_{n-1}$ is a split epimorphism; and the action of Γ_{n-1} on $H_*(F_{\alpha(n)})$ is trivial. Denote the center of a Lie algebra $\mathfrak{g}$ by $\mathcal{C}(\mathfrak{g})$, and write $F_{\alpha(0)} = \Gamma_0$.

Proposition 2.1. *Let Γ_n be a poly-free group satisfying conditions* (2). *Then the center of* $\mathrm{gr}_*(\Gamma_n)$ *is cyclic. Furthermore, if $F_{\alpha(k)}$ is of rank at least two for every k, $0 \le k \le n$, then both the center of* $\mathrm{gr}_*(\Gamma_n)$ *and the center of Γ_n are trivial.*

Proof. By [14], there is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{gr}_*(F_{\alpha(n)}) \xrightarrow{\mathrm{gr}_*(j)} \mathrm{gr}_*(\Gamma_n) \xrightarrow{\mathrm{gr}_*(p)} \mathrm{gr}_*(\Gamma_{n-1}) \longrightarrow 0,$$

where $\mathrm{gr}_*(j)$ and $\mathrm{gr}_*(p)$ are the maps induced by j and p, and the first homology of Γ_k is free abelian for each k. Since Γ_0 is free of rank at least one, the center of $\mathrm{gr}_*(\Gamma_0)$ is either infinite cyclic if $\Gamma_0 = \mathbb{Z}$, or is trivial if the rank of Γ_0 is at least two [10].

Assume inductively that the center of $\mathrm{gr}_*(\Gamma_{n-1})$ is concentrated in degree 1 and is cyclic. Observe that $\mathrm{gr}_*(p)(\mathcal{C}(\mathrm{gr}_*(\Gamma_n)))$, the image of the center of $\mathrm{gr}_*(\Gamma_n)$

in $\mathrm{gr}_*(\Gamma_{n-1})$, is also cyclic. In addition, it may be assumed inductively that the center $\mathcal{C}(\mathrm{gr}_*(\Gamma_{n-1}))$ is trivial in case the rank of $F_{\alpha(k)}$ is at least two for all k, $0 \le k \le n-1$.

Choose an element Δ_1 in $\mathcal{C}(\mathrm{gr}_*(\Gamma_n))$ which projects to a generator of $\mathrm{gr}_*(p)(\mathcal{C}(\mathrm{gr}_*(\Gamma_n)))$. Let Δ_2 be an arbitrary element in the center $\mathcal{C}(\mathrm{gr}_*(\Gamma_n))$. If Δ_2 is concentrated in degree greater than 1, then $\mathrm{gr}_*(p)(\Delta_2) = 0$. Consequently, there is an element X in $\mathrm{gr}_*(F_{\alpha(n)})$ such that $\mathrm{gr}_*(j)(X) = \Delta_2$. Since the center of $\mathrm{gr}_*(F_{\alpha(n)})$ is trivial whenever the cardinality of $\alpha(n)$ is at least two, $\Delta_2 = 0$. Thus it suffices to assume that Δ_2 is concentrated in degree 1 in case the cardinality of $\alpha(n)$ is at least two.

It follows that there is an integer q such that $q\Delta_1$ and Δ_2 project to the same element in $\mathcal{C}(\mathrm{gr}_*(\Gamma_{n-1}))$. Then $\Delta = q\Delta_1 - \Delta_2$ is an element of $\mathcal{C}(\mathrm{gr}_*(\Gamma_n))$, and is in the image of $\mathrm{gr}_*(j)$, say $\Delta = \mathrm{gr}_*(j)(\delta)$, where $\delta \in \mathcal{C}(\mathrm{gr}_*(F_{\alpha(n)}))$. However, the center of $\mathrm{gr}_*(F_{\alpha(n)})$ is trivial whenever the cardinality of $\alpha(n)$ is at least two and so $q\Delta_1 - \Delta_2 = 0$. Thus the center $\mathcal{C}(\mathrm{gr}_*(\Gamma_n))$ is cyclic, and is trivial in the case where the rank of $F_{\alpha(k)}$ is at least 2 for all k, $0 \le k \le n$.

To finish, assume that the rank of $F_{\alpha(k)}$ is at least 2 for all k, $0 \le k \le n$. Since the center of $F_{\alpha(k)}$ is trivial for $0 \le k \le n$, the center of Γ_n is trivial by the natural induction on n. ■

The previous proposition suggests the following.

Question 2. *Let Γ be a group which fits into an exact sequence*

$$1 \longrightarrow F \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

where F is a free group of rank at least two and the center of $\mathrm{gr}_*(G)$ is cyclic. Is then the center of $\mathrm{gr}_*(\Gamma)$ cyclic? Note that no assumptions are made regarding the existence of a splitting, or the action of G on the homology of F.

3. Upper Triangular McCool Groups

Let F_n be the free group on the set $\{x_1, x_2, \cdots, x_n\}$. Let IA_n denote the kernel of the natural map

$$\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$$

induced by the map of F_n to its abelianization. As shown by Nielsen, and Magnus, the group IA_n is generated by automorphisms $\beta_{i,j}$, $1 \le i, j \le n$, $i \ne j$, and $\Theta_{j;[s,t]}$, where $1 \le j, s, t \le n$ and j, s, t are distinct, see [24]. These automorphisms are given by

$$\beta_{i,j}(x_k) = \begin{cases} x_k, & \text{if } k \ne j, \\ x_i^{-1} x_j x_i, & \text{if } k = j, \end{cases} \quad \text{and} \quad \Theta_{j;[s,t]}(x_k) = \begin{cases} x_k, & \text{if } k \ne j, \\ x_j [x_s, x_t], & \text{if } k = j. \end{cases}$$

Let $P\Sigma_n$ be the group of *basis-conjugating automorphisms*, the subgroup of IA_n generated by the automorphisms $\beta_{i,j}$. McCool [25] showed that the group $P\Sigma_n$ admits a presentation with these generators and relations

$$\left\{ \begin{array}{ll} [\beta_{i,j}, \beta_{k,l}] & \text{for } i,j,k,l \text{ distinct} \\ [\beta_{i,j}, \beta_{i,k}] & \text{for } i,j,k \text{ distinct} \\ [\beta_{j,k}, \beta_{i,j}\beta_{i,k}] & \text{for } i,j,k \text{ distinct} \end{array} \right\}. \tag{4}$$

Let $P\Sigma_n^+$ be the *upper triangular McCool group*, the subgroup of $P\Sigma_n$ generated by the automorphisms $\beta_{i,j}$ with $i < j$. The (relevant) relations (4) may be used to show that $P\Sigma_n^+$ is a semidirect product, $P\Sigma_n^+ = F_{n-1} \rtimes P\Sigma_{n-1}^+$, where F_{n-1} is the free group on the set

$$\{\beta_{i,n} : \ i = 1, 2, \dots n-1\}.$$

In other words, there is a split exact sequence

$$1 \longrightarrow F_{n-1} \longrightarrow P\Sigma_n^+ \longrightarrow P\Sigma_{n-1}^+ \longrightarrow 1,$$

see [9] (where the automorphisms $\beta_{i,j}$ are denoted by $\chi(j,i)$). It is readily checked that the action of $P\Sigma_{n-1}^+$ on $H_*(F_{n-1})$ is trivial.

The structure of the Lie algebra $\mathrm{gr}_*(P\Sigma_n^+)$ is determined in [9]. Denote the image of $\beta_{i,j}$ in $\mathrm{gr}_*(P\Sigma_n^+)$ by $B_{i,j}$. Then $\mathrm{gr}_*(P\Sigma_n^+)$ has generators $B_{i,j}$, $1 \le i < j \le n$, and relations

$$[B_{i,j}, B_{k,l}] = 0, \quad \text{for } i, j, k, l \text{ distinct}, \tag{5a}$$
$$[B_{i,j}, B_{i,k}] = 0, \quad \text{for } i, j, k \text{ distinct}, \tag{5b}$$
$$[B_{j,k}, B_{i,j} + B_{i,k}] = 0, \quad \text{for } i, j, k \text{ distinct}. \tag{5c}$$

Note that the last of these is one of the classical infinitesimal braid relations. It is not, however, the case that all of the infinitesimal braid relations are satisfied. For example, if $i < j < k$, the element $[B_{i,k}, B_{i,j} + B_{j,k}]$ need not vanish in $\mathrm{gr}_*(P\Sigma_n^+)$.

For $2 \le k \le n$, let $V_k = \{B_{i,k} : \ 1 \le i \le k-1\}$, and let $L[V_k]$ be the corresponding free Lie algebra. There is an additive isomorphism

$$\mathrm{gr}_*(P\Sigma_n^+) \cong \bigoplus_{2 \le k \le n} L[V_k].$$

Moreover, it is clear from the relations (5) that the Lie algebra $\mathrm{gr}_*(F_{n-1}) = L[V_n]$ is a Lie ideal of $\mathrm{gr}_*(P\Sigma_n^+)$.

The next result is the portion of the Theorem 1.1 pertaining to Case **I**, where $\Gamma_n = P\Sigma_n^+$.

Theorem 3.1. *The centralizer of $L[V_n]$ in $\mathrm{gr}_*(P\Sigma_n^+)$ is the linear span of the element*

$$B_n = \sum_{j=2}^{n} B_{1,j}.$$

Proof. Proposition 2.1 implies that an element in the centralizer of $L[V_n]$ must be of weight 1. Let

$$X = \sum_{1 \le i < j \le n} a_{i,j} B_{i,j}$$

be an element that centralizes $L[V_n]$. Since X centralizes $L[V_n]$,

$$[X, B_{1,n}] = 0.$$

Relation (5a) implies that

$$\sum_{j=2}^{n-1} a_{1,j}[B_{1,j}, B_{1,n}] + \sum_{i=2}^{n-1} a_{i,n}[B_{i,n}, B_{1,n}] = 0.$$

Relation (5b) implies that each term in the first sum vanishes. Therefore,

$$\sum_{i=2}^{n-1} a_{i,n}[B_{i,n}, B_{1,n}] = 0.$$

But the last equation takes place in the free Lie algebra $L[V_n]$. Thus, $a_{i,n} = 0$ for $1 < i < n$, and the element X has the form

$$X = \sum_{1 \le i < j < n} a_{i,j} B_{i,j} + a_{1,n} B_{1,n}.$$

Fix j, $1 < j < n$. Start with the identity

$$[X, B_{j,n}] = 0.$$

Relation (5a) implies that

$$\sum_{i=1}^{j-1} a_{i,j}[B_{i,j}, B_{j,n}] + \sum_{k=j+1}^{n-1} a_{j,k}[B_{j,k}, B_{j,n}] + a_{1,n}[B_{1,n}, B_{j,n}] = 0.$$

Relation (5b) implies that each term in the middle sum vanishes. Thus,

$$\sum_{i=1}^{j-1} a_{i,j}[B_{i,j}, B_{j,n}] + a_{1,n}[B_{1,n}, B_{j,n}] = 0.$$

Using the relation (5c), notice that the next relation follows

$$-\sum_{i=1}^{j-1} a_{i,j}[B_{i,n}, B_{j,n}] + a_{1,n}[B_{1,n}, B_{j,n}] = 0$$

which implies that

$$-\sum_{i=2}^{j-1} a_{i,j}[B_{i,n}, B_{j,n}] + (-a_{1,j} + a_{1,n})[B_{1,n}, B_{j,n}] = 0.$$

Since all the commutators are linearly independent, $a_{i,j} = 0$ for $1 < i < j$, and $a_{1,j} = a_{1,n}$. Since j was arbitrary, these relations hold for all j with $1 < j < n$. The result follows. ∎

4. Surface Group Orbit Configuration Spaces

Let M be a manifold without boundary, and let G be a discrete group which acts freely on M. The *orbit configuration space* consists of all ordered n-tuples of points in M which lie in distinct orbits:

$$\mathrm{Conf}^G(M,n) = \{(x_1,\dots,x_n) \in M^n :\ G\cdot x_i \cap G\cdot x_j = \emptyset \text{ if } i \neq j\}.$$

In [29], Xicoténcatl proves that, for $\ell \leq n$, projection onto the first ℓ coordinates,

$$p_G : \mathrm{Conf}^G(M,n) \to \mathrm{Conf}^G(M,\ell),$$

is a locally trivial bundle, with fiber $\mathrm{Conf}^G(M\setminus Q^G_\ell, n-\ell)$, where Q^G_ℓ denote the union of ℓ distinct orbits, $G\cdot x_1,\dots,G\cdot x_n$, in M. This result generalizes the Fadell-Neuwirth theorem [13].

Let G be a discrete subgroup of $PSL(2,\mathbb{R})$ acting freely and properly discontinuously on the upper–half plane $\mathbb{H}^2$ by fractional linear transformations. Let $\mathrm{Conf}^G(\mathbb{H}^2,n)$ be the corresponding orbit configuration space and $P_G(n)$ its fundamental group. Then there is a fiber bundle

$$\mathbb{H}^2\backslash Q^G_{n-1} \to \mathrm{Conf}^G(\mathbb{H}^2,n) \to \mathrm{Conf}^G(\mathbb{H}^2,n-1).$$

This induces a split exact sequence:

$$1 \longrightarrow F \longrightarrow P_G(n) \longrightarrow P_G(n-1) \longrightarrow 1,$$

where F is the free group on Q^G_{n-1}. Notice that in this case F is an infinitely generated free group.

The structure of $\mathrm{gr}_*(P_G(n))$ is determined in [8]. The generators are $B^\sigma_{i,j}$, where $1\leq i<j\leq n$ and $\sigma\in G$. In this case, the infinitesimal braid relations are:

$$\begin{aligned}
[B^\sigma_{i,j}, B^\tau_{s,t}] &= 0, \quad \text{if } \{i,j\}\cap\{s,t\} = \emptyset,\\
[B^\tau_{i,j}, B^{\tau\sigma^{-1}}_{s,j} + B^\sigma_{i,s}] &= 0, \quad \text{if } 1\leq i<s<j\leq n,\\
[B^\sigma_{i,s}, B^\tau_{i,j} + B^{\tau\sigma^{-1}}_{s,j}] &= 0, \quad \text{if } 1\leq i<s<j\leq n,\\
[B^{\tau\sigma^{-1}}_{s,j}, B^\tau_{i,j} + B^\sigma_{i,s}] &= 0, \quad \text{if } 1\leq i<s<j\leq n.
\end{aligned} \tag{6}$$

The last equation follows from the previous two equations. Note that, for $0<i<k<n$, this last equation implies that

$$[B^{\tau\sigma^{-1}}_{k,n}, B^\tau_{i,n} + B^\sigma_{i,k}] = 0. \tag{7}$$

Let V_j be the linear span of the set $\{B^\sigma_{i,j} :\ 1\leq i<j,\ \sigma\in G\}$. Then, there is a splitting

$$\mathrm{gr}_i(P_G(n)) = \bigoplus_{j=2}^n \mathrm{gr}_i(L[V_j]),$$

as abelian groups.

The next result is the portion of the Theorem 1.1 pertaining to Case **II**, where $\Gamma_n = P_G(n)$.

Theorem 4.1. *The centralizer of $L[V_n]$ in $\mathrm{gr}_*(P_G(n))$ is zero:*

$$\mathcal{C}_{\mathrm{gr}_*(P_G(n))}(L[V_n]) = 0.$$

Proof. Proposition 2.1 implies that elements in the centralizer are of weight 1. Let

$$x = \sum_{k=1}^{\ell} a_k B_{i_k,j_k}^{\sigma_k} + \sum_{m=1}^{p} b_m B_{m,n}^{\tau_m}$$

be such an element. Since G is infinite, there exists $\tau \in G$ such that

$$\tau\sigma_k^{-1} \neq \tau_m, \quad \text{for all } k = 1, 2, \ldots, \ell,\ m = 1, 2, \ldots, p.$$

Since x centralizes $L[V_n]$,

$$0 = [x, B_{1,n}^{\tau}] = \sum_{k=1}^{\ell} a_k [B_{i_k,j_k}^{\sigma_k}, B_{1,n}^{\tau}] + \sum_{m=1}^{p} b_m [B_{m,N}^{\tau_m}, B_{1,n}^{\tau}].$$

The infinitesimal braid relations (6) imply that

$$[B_{i_k,j_k}^{\sigma_k}, B_{1,n}^{\tau}] = 0, \quad \text{if } 1 \neq i_k,$$

and that

$$[B_{1,j_k}^{\sigma_k}, B_{1,n}^{\tau}] = -[B_{j_k,n}^{\tau\sigma_k^{-1}}, B_{j_k,n}^{\tau}].$$

Thus, for eack $1 \leq i_k < n$,

$$[B_{i_k,j_k}^{\sigma_k}, B_{1,n}^{\tau}] = -[B_{j_k,n}^{\tau\sigma_k^{-1}}, B_{1,n}^{\tau}],$$

and

$$-\sum_{k=1}^{\ell} a_k [B_{j_k,n}^{\tau\sigma_k^{-1}}, B_{1,n}^{\tau}] + \sum_{m=1}^{p} b_m [B_{m,n}^{\tau_m}, B_{1,n}^{\tau}] = 0.$$

Since the commutators are all different and they are linearly independent, all the coefficients are equal to 0 and thus $x = 0$. ∎

5. Cyclic Group Orbit Configuration Spaces

Let $G = \mathbb{Z}/r\mathbb{Z}$ be a finite cyclic group. The group G acts freely on the manifold $M = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by multiplication by the primitive r-th root of unity $\zeta = \exp(2\pi\sqrt{-1}/r)$. The corresponding orbit configuration space is given by

$$\mathrm{Conf}^G(\mathbb{C}^*, n) = \{(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \mid x_j \neq \zeta^p x_i \text{ for } i \neq j \text{ and } 1 \leq p \leq r\}.$$

Denote the fundamental group of $\mathrm{Conf}^G(\mathbb{C}^*, n)$ by $P(r, n)$.

Let $\ell_n = rn + 1$, and define a map $g_n : \mathrm{Conf}^G(\mathbb{C}^*, n) \to \mathrm{Conf}(\mathbb{C}, \ell_n)$ from the orbit configuration space to the classical configuration space by sending a point to its orbits (together with 0). Explicitly, if $(x_1, \ldots, x_n) \in \mathrm{Conf}^G(\mathbb{C}^*, n)$, define

$$g_n(x_1, \ldots, x_n) = (0, \zeta x_1, \ldots, \zeta^r x_1, \zeta x_2, \ldots, \zeta^r x_2, \ldots\ldots, \zeta x_n, \ldots, \zeta^r x_n)$$

in $\mathrm{Conf}(\mathbb{C}, \ell_n)$. Then, one has the following result (see [5, Thm. 2.1.3] and [6, §3]).

Theorem 5.1. *The orbit configuration space bundle* $p_G : \mathrm{Conf}^G(\mathbb{C}^*, n+1) \to \mathrm{Conf}^G(\mathbb{C}^*, n)$ *is equivalent to the pullback of the configuration space bundle* $p : \mathrm{Conf}(\mathbb{C}, \ell_n + 1) \to \mathrm{Conf}(\mathbb{C}, \ell_n)$ *along the map* g_n*.*

Passing to fundamental groups, there is an induced commutative diagram with split rows,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_{\ell_n} & \longrightarrow & P(r, n+1) & \longrightarrow & P(r,n) & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle (g_n)_*} & & \\ 1 & \longrightarrow & F_{\ell_n} & \longrightarrow & P_{\ell_n+1} & \longrightarrow & P_{\ell_n} & \longrightarrow & 1 \end{array} \tag{8}$$

where F_N is a free group on N generators. Passing further to descending central series Lie algebras, there is a commutative diagram, again with split rows (see [6, §4]).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L[V_{\ell_n}] & \longrightarrow & \mathrm{gr}_*(P(r, n+1)) & \longrightarrow & \mathrm{gr}_*(P(r,n)) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{gr}_*(g_n)} & & \\ 0 & \longrightarrow & L[V_{\ell_n}] & \longrightarrow & \mathrm{gr}_*(P_{\ell_n+1}) & \longrightarrow & \mathrm{gr}_*(P_{\ell_n}) & \longrightarrow & 0 \end{array} \tag{9}$$

This realizes the Lie algebra $\mathrm{gr}_*(P(r, n+1))$ as the semidirect product of $\mathrm{gr}_*(P(r,n))$ by $L[V_{\ell_n}]$ determined by the homomorphism

$$\theta_{\ell_n} \circ \mathrm{gr}_*(g_n) : \mathrm{gr}_*(P(r,n)) \to \mathrm{Der}(L[V_{\ell_n}]),$$

where $\theta_N : \mathrm{gr}_*(P_N) \to \mathrm{Der}(L[V_N])$ is given by $\theta_N(B_{i,j}) = \mathrm{ad}(B_{i,j})$, see [6, Thm. 4.4]. More explicitly, the structure of $\mathrm{gr}_*(P(r,n))$ is given in the following theorem, proved in [5].

Theorem 5.2. *Let* $\mathrm{gr}_*(P(r,n))$ *be the Lie algebra associated to the descending central series of the group* $P(r,n) = \pi_1(\mathrm{Conf}^G(\mathbb{C}^*, n))$*, where* $G = \mathbb{Z}/r\mathbb{Z}$*. Then,*

$$\mathrm{gr}_*(P(r,n)) \cong \bigoplus_{j=0}^{n-1} L(rj+1)$$

as abelian groups, where $L(rj+1)$ *is generated by* Z_{j+1} *and* $B_{i,j+1}^{(p)}$*,* $1 \le i \le j$*,* $1 \le p \le r$*. The Lie bracket relations in* $\mathrm{gr}_*(P(r,n))$ *are given by*

$$\begin{aligned} \big[Z_j + Z_l + \sum_{q=1}^{r} B_{j,l}^{(q)}, Y\big] &= 0 \quad \text{for } Y = Z_l,\ Y = B_{j,l}^{(p)},\ 1 \le p \le r, \\ [B_{i,j}^{(p)} + B_{i,k}^{(q)} + B_{j,k}^{(m)}, Y] &= 0 \quad \text{for } Y = B_{i,k}^{(q)}, B_{j,k}^{(m)},\ q \equiv p + m \mod r, \\ [B_{i,j}^{(p)}, B_{k,l}^{(q)}] &= 0 \quad \text{for } \{i,j\} \cap \{k,l\} = \emptyset,\ 1 \le p, q \le r, \text{ and} \\ [Z_k, B_{i,j}^{(p)}] &= 0 \quad \text{for } k \ne i, j \text{ and } 1 \le p \le r. \end{aligned}$$

Proposition 5.3. *The map* $g_n : \mathrm{Conf}^G(\mathbb{C}^*, n) \to \mathrm{Conf}(\mathbb{C}, \ell_n)$ *induces injections on fundamental groups and descending central series Lie algebras. More precisely, the maps*

$$(g_n)_* : P(r,n) \longrightarrow P_{\ell_n} \quad \textit{and} \quad \mathrm{gr}_*(g_n) : \mathrm{gr}_*(P(r,n)) \longrightarrow \mathrm{gr}_*(P_{\ell_n})$$

induced by g_n*, are monomorphisms.*

Proof. The proof is by induction on n. In the case $n = 1$, notice that $\mathrm{Conf}^G(\mathbb{C}^*, 1) = \mathbb{C}^*$, and for $x \in \mathbb{C}^*$, $g_1(x) = (0, \zeta x, \dots, \zeta^r x) \in \mathrm{Conf}(\mathbb{C}, r+1)$. Let $\gamma \in \pi_1(\mathbb{C}^*)$ be the standard generator, and check that $(g_1)_*(\gamma) = \Delta(r+1)$ generates the center of P_{r+1}. It follows that both $(g_1)_* : P(r,1) \to P_{\ell_1}$ and $\mathrm{gr}_*(g_1) : \mathrm{gr}_*(P(r,1)) \to \mathrm{gr}_*(P_{\ell_1})$ are injective.

Assume inductively that $(g_n)_* : P(r,n) \to P_{\ell_n}$ and $\mathrm{gr}_*(g_n) : \mathrm{gr}_*(P(r,n)) \to \mathrm{gr}_*(P_{\ell_n})$ are injective. It must be shown that $(g_{n+1})_* : P(r, n+1) \to P_{\ell_{n+1}}$ and $\mathrm{gr}_*(g_{n+1}) : \mathrm{gr}_*(P(r, n+1)) \to \mathrm{gr}_*(P_{\ell_{n+1}})$ are also injective, where $\ell_n = rn + 1$ and $\ell_{n+1} = r(n+1) + 1$. Let $\tilde{g}_n : \mathrm{Conf}^G(\mathbb{C}^*, n+1) \to \mathrm{Conf}(\mathbb{C}, \ell_n + 1)$ denote the map on the pullback induced by g_n. Note that

$$\tilde{g}_n(x_1, \dots, x_n, z) = (0, \zeta x_1, \dots, \zeta^r x_1, \dots\dots, x_n, \zeta x_n, \dots, \zeta^r x_n, z).$$

The map $\tilde{g}_n$ may be factored as follows. Let $p_{m,k} : \mathrm{Conf}(\mathbb{C}, m) \to \mathrm{Conf}(\mathbb{C}, k)$ be the projection which forgets the last $m - k$ points. Then $\tilde{g}_n = p_{m,k} \circ g_{n+1}$, where $m = \ell_{n+1}$ and $k = \ell_n + 1$.

Since $(g_n)_*$ and $\mathrm{gr}_*(g_n)$ are injective by induction, it follows from (8) and (9) that $(\tilde{g}_n)_*$ and $\mathrm{gr}_*(\tilde{g}_n)$ are also injective. This, together with the fact that $\tilde{g}_n = p_{m,k} \circ g_{n+1}$, implies that $(g_{n+1})_*$ and $\mathrm{gr}_*(g_{n+1})$ are injective. ∎

Corollary 5.4. *The group $P(r,n)$ is linear.*

Proof. The group $P(r,n)$ embeds in the Artin pure braid group, which is linear [4, 23]. ∎

The next result is the portion of the Theorem 1.1 pertaining to Case **III**. It is notationally convenient to state the result for the group $\Gamma_{n+1} = P(r, n+1)$.

Theorem 5.5. *The centralizer of $L[V_{\ell_n}]$ in $\mathrm{gr}_*(P(r, n+1))$ is the linear span of the element*

$$\Delta(r, n+1) = \sum_{k=1}^{n+1} Z_k + \sum_{p=1}^{r} \sum_{1 \le i < j \le n+1} B_{i,j}^{(p)}.$$

Proof. Denote the generators of $L[V_{\ell_n}]$ by Z_{n+1} and $B_{i,n+1}^{(p)}$, $1 \le i \le n$, $1 \le p \le r$, and let

$$\mathcal{B}_{\ell_n} = Z_{n+1} + \sum_{p=1}^{r} \sum_{i=1}^{n} B_{i,n+1}^{(p)}.$$

Let $x \in \mathrm{gr}_*(P(r, n+1))$, and assume that $[x, B] = 0$ for all $B \in L[V_{\ell_n}]$. Write $x = u + v$, where $u \in \mathrm{gr}_*(P(r,n))$ and $v \in L[V_{\ell_n}]$. Then, for all $B \in L[V_{\ell_n}]$, it follows that $\mathrm{gr}_*(\tilde{g}_n)[x, B] = [\mathrm{gr}_*(g_n)(u) + v, B] = 0$ in $\mathrm{gr}_*(P_{\ell_n+1})$. So $\mathrm{gr}_*(g_n)(u) + v = k \cdot \Delta(\ell_n + 1)$ for some constant k. Consequently, $\mathrm{gr}_*(g_n)(u) = k \cdot \Delta(\ell_n)$ and $v = k \cdot \mathcal{B}_{\ell_n}$. Since $\mathrm{gr}_*(g_n) : \mathrm{gr}_*(P(r,n)) \to \mathrm{gr}_*(P_{\ell_n})$ is injective (as is $\mathrm{gr}_*(\tilde{g}_n)$), it follows that the centralizer of $L[V_{\ell_n}]$ in $\mathrm{gr}_*(P(r, n+1))$ is the linear span of the element

$$\mathrm{gr}_*(\tilde{g}_n)^{-1}(\Delta(\ell_n + 1)) = \mathrm{gr}_*(g_n)^{-1}(\Delta(\ell_n)) + \mathcal{B}_{\ell_n}.$$

So it suffices to show that

$$\mathrm{gr}_*(g_n)^{-1}(\Delta(\ell_n)) = \sum_{k=1}^{n} Z_k + \sum_{p=1}^{r} \sum_{1\le i<j\le n} B_{i,j}^{(p)}.$$

The map $g_n : \mathrm{Conf}^G(\mathbb{C}^*, n) \to \mathrm{Conf}(\mathbb{C}, \ell_n)$ is the restriction of the affine transformation $g_n : \mathbb{C}^n \to \mathbb{C}^{\ell_n}$, defined by the same formula, and, abusing notation, denoted by the same symbol. The orbit configuration space $\mathrm{Conf}^G(\mathbb{C}^*, n)$ may be realized as the complement of the hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^n$ with hyperplanes $H_i = \{x_i = 0\}$, $1 \le i \le n$, and $H_{i,j}^{(p)} = \{x_i = \zeta^p x_j\}$, $1 \le i < j \le n$, $1 \le p \le r$. The generators of the Lie algebra $\mathrm{gr}_*(P(r,n))$ are in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. If $B_H \in \mathrm{gr}_*(P(r,n))$ denotes the generator corresponding to $H \in \mathcal{A}$, it follows from [6, Prop. 3.4] that

$$\mathrm{gr}_*(g_n)(B_H) = \sum_{g_n(H)\subset H_{r,s}} B_{r,s},$$

where $H_{r,s} = \{x_r = x_s\} \subset \mathbb{C}^{\ell_n}$, and $B_{r,s} \in \mathrm{gr}_*(P_{\ell_n})$ is the corresponding generator of the descending central series Lie algebra of the pure braid group. Let $S_i = \{H_{r,s} \mid g_n(H_i) \subset H_{r,s}\}$ and $S_{i,j}^{(p)} = \{H_{r,s} \mid g_n(H_{i,j}^{(p)}) \subset H_{r,s}\}$. Then, one can check that the sets S_i, $1 \le i \le n$, and $S_{i,j}^{(p)}$, $1 \le i < j \le n$, $1 \le p \le r$, form a partition of the (entire) set of $\binom{\ell_n}{2}$ hyperplanes $H_{r,s}$ in $\mathbb{C}^{\ell_n}$. It follows that

$$\mathrm{gr}_*(g_n)^{-1}(\Delta(\ell_n)) = \mathrm{gr}_*(g_n)^{-1}\Big(\sum_{1\le r<s\le \ell_n} B_{r,s}\Big) = \sum_{k=1}^{n} Z_k + \sum_{p=1}^{r} \sum_{1\le i<j\le n} B_{i,j}^{(p)},$$

completing the proof. ∎

6. Fiber-Type Arrangements

In light of Corollary 5.4, it is natural to speculate that the fundamental group of the complement of an arbitrary fiber-type hyperplane arrangement is linear. The purpose of this section is to show that this is indeed the case.

A hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one affine subspaces of Euclidean space $\mathbb{C}^n$. See Orlik and Terao [26] as a general reference on arrangements. The complement of an arrangement $\mathcal{A}$ is the manifold $X = X(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$. Denote the fundamental group of the complement by $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$.

Definition 6.1. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^{n+1}$ is *strictly linearly fibered* if there is a choice of coordinates $(\mathbf{x}, z) = (x_1, \dots, x_n, z)$ on $\mathbb{C}^{n+1}$ so that the restriction, p, of the projection $\mathbb{C}^{n+1} \to \mathbb{C}^n$, $(\mathbf{x}, z) \mapsto \mathbf{x}$, to the complement $X(\mathcal{A})$ is a fiber bundle projection, with base $p(X(\mathcal{A})) = X(\mathcal{B})$, the complement of an arrangement $\mathcal{B}$ in $\mathbb{C}^n$, and fiber the complement of finitely many points in $\mathbb{C}$. In this case $\mathcal{A}$ is said to be strictly linearly fibered over $\mathcal{B}$.

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^{n+1}$, strictly linearly fibered over $\mathcal{B} \subset \mathbb{C}^n$. For each hyperplane H of $\mathcal{A}$, let f_H be a linear polynomial with $H = \ker f_H$.

Then $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H$ is a defining polynomial for $\mathcal{A}$. From the definition, there is a choice of coordinates for which a defining polynomial for $\mathcal{A}$ factors as $Q(\mathcal{A}) = Q(\mathcal{B}) \cdot \phi(\mathbf{x}, z)$, where $Q(\mathcal{B}) = Q(\mathcal{B})(\mathbf{x})$ is a defining polynomial for $\mathcal{B}$, and $\phi(\mathbf{x}, z)$ is a product

$$\phi(\mathbf{x}, z) = (z - g_1(\mathbf{x}))(z - g_2(\mathbf{x})) \cdots (z - g_m(\mathbf{x})),$$

with $g_j(\mathbf{x})$ linear. Define $g : \mathbb{C}^n \to \mathbb{C}^m$ by

$$g(\mathbf{x}) = \big(g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_m(\mathbf{x})\big), \tag{10}$$

Since $\phi(\mathbf{x}, z)$ necessarily has distinct roots for any $\mathbf{x} \in X(\mathcal{B})$, the restriction of g to $X(\mathcal{B})$ takes values in the configuration space $\mathrm{Conf}(\mathbb{C}, m)$. The next result, proved in [5], generalizes Theorem 5.1.

Theorem 6.1. *Let $\mathcal{B}$ be an arrangement of m hyperplanes, and let $\mathcal{A}$ be an arrangement of $m+n$ hyperplanes which is strictly linearly fibered over $\mathcal{B}$. Then the bundle $p : X(\mathcal{A}) \to X(\mathcal{B})$ is equivalent to the pullback of the bundle of configuration spaces $p_{n+1} : \mathrm{Conf}(\mathbb{C}, n+1) \to \mathrm{Conf}(\mathbb{C}, n)$ along the map g. Consequently, the bundle $p : X(\mathcal{A}) \to X(\mathcal{B})$ admits a cross-section, and has trivial local coefficients in homology.*

Passing to fundamental groups, there is an induced commutative diagram with split rows.

$$\begin{tikzcd}
1 \arrow[r] & F_m \arrow[r] \arrow[d, "\mathrm{id}"] & G(\mathcal{A}) \arrow[r] \arrow[d] & G(\mathcal{B}) \arrow[r] \arrow[d, "g_*"] & 1 \\
1 \arrow[r] & F_m \arrow[r] & P_{m+1} \arrow[r] & P_m \arrow[r] & 1
\end{tikzcd} \tag{11}$$

realizing $G(\mathcal{A})$ as a pullback.

Lemma 6.2. *Given a pullback of groups*

$$\begin{tikzcd}
 & & P \arrow[r] \arrow[d] & H \arrow[d] & \\
1 \arrow[r] & K \arrow[r] & G \arrow[r] & Q \arrow[r] & 1
\end{tikzcd}$$

if G and H are linear, then the pullback P is also linear.

Proof. The pullback P is a subgroup of $G \times H$, which is linear if G and H are. ∎

Corollary 6.3. *If $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$, and $G(\mathcal{B})$ is linear, then $G(\mathcal{A})$ is also linear.*

Definition 6.2. An arrangement $\mathcal{A} = \mathcal{A}_1$ of finitely many points in $\mathbb{C}^1$ is *fiber-type*. An arrangement $\mathcal{A} = \mathcal{A}_n$ of hyperplanes in $\mathbb{C}^n$ is *fiber-type* if $\mathcal{A}$ is strictly linearly fibered over a fiber-type arrangement $\mathcal{A}_{n-1}$ in $\mathbb{C}^{n-1}$.

Examples include the braid arrangement with defining polynomial $Q(\mathcal{A}) = \prod_{i<j}(y_i - y_j)$, and complement $X(\mathcal{A}) = \mathrm{Conf}(\mathbb{C}, n)$, and the full monomial arrangement with defining polynomial $Q(\mathcal{A}) = x_1 \cdots x_n \prod_{i<j}(x_i^r - x_j^r)$, and complement $X(\mathcal{A}) = \mathrm{Conf}^G(\mathbb{C}^*, n)$, where $G = \mathbb{Z}/r\mathbb{Z}$.

Theorem 6.4. *The fundamental group of the complement of a fiber-type hyperplane arrangement is linear.*

Proof. Let $\mathcal{A} = \mathcal{A}_n$ be a fiber-type arrangement in $\mathbb{C}^n$, with complement $X(\mathcal{A}_n)$. The proof is by induction on n.

In the case $n = 1$, denote the cardinality of $\mathcal{A}_1$ by d. If $d = 0$ ($\mathcal{A}_1$ is the empty arrangement), then $X(\mathcal{A}_1) = \mathbb{C}$ and $G(\mathcal{A}_1)$ is the trivial group. If $d > 0$, then $X(\mathcal{A}_1)$ has the homotopy type of a bouquet of d circles and $G(\mathcal{A}_1) = F_d$ is a free group on d generators. So $G(\mathcal{A}_1)$ is linear.

Assume the result holds for any fiber-type arrangement $\mathcal{A}_n$ in $\mathbb{C}^n$, and let $\mathcal{A}_{n+1}$ be a fiber-type arrangement in $\mathbb{C}^{n+1}$. Then $\mathcal{A} = \mathcal{A}_{n+1}$ is strictly linearly fibered over $\mathcal{B} = \mathcal{A}_n$, a fiber-type arrangement in $\mathbb{C}^n$. By induction, the fundamental group $G(\mathcal{B}) = \pi_1(X(\mathcal{B}))$ is linear. Hence, $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$ is also linear by Corollary 6.3. ∎

7. The Poison Group

One group which does not admit a faithful finite dimensional linear representation is the so-called poison group. This group appears in the work of Formanek and Procesi [18], where it is realized as a subgroup of $\mathrm{Aut}(F_n)$ for $n \geq 3$, proving that the latter admits no faithful finite dimensional linear representation. The poison group may also be realized as a subgroup of IA_n for $n \geq 5$, see Pettet [27]. On the other hand, Brendle and Hamidi-Tehrani [3] have shown that the mapping class group of a genus g surface with one fixed point has no subgroup isomorphic to the poison group.

The purpose of this section to show how the Lie algebraic criteria in this article fail in a strong way for the poison group H, a group given by a split extension

$$1 \longrightarrow F_3 \xrightarrow{j} H \xrightarrow{p} F_2 \longrightarrow 1.$$

The group H admits the following presentation:

$$H = \langle a_1, a_2, a_3, \phi_1, \phi_2 \mid \phi_i a_j \phi_i^{-1} = a_j, \phi_i a_3 \phi_i^{-1} = a_3 a_i, i, j = 1, 2\rangle. \tag{12}$$

From this presentation, it is clear that $H = F_3 \rtimes F_2$ is a semidirect product, where F_3 is generated by $\{a_1, a_2, a_3\}$ and F_2 by $\{\phi_1, \phi_2\}$. Thus, H is poly-free, and it is natural to consider how the structure of the descending central series Lie algebra fails to inform on representations for this group. For a group G, let $\alpha : G \to H_1(G)$ denote the abelianization map.

Proposition 7.1. *There is a split exact sequence of Lie algebras*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{gr}_*(H) \xrightarrow{\mathrm{gr}_*(p)} \mathrm{gr}_*(F_2) \longrightarrow 0$$

with the center, $\mathcal{C}(\mathrm{gr}_*(H))$, given by $\mathbb{Z}$, generated the class of a_3.

The induced map

$$\mathrm{gr}_*(F_3) \xrightarrow{\mathrm{gr}_*(j)} \mathrm{gr}_*(H)$$

factors through the center $\mathcal{C}(\mathrm{gr}_*(H)) = \mathbb{Z}$, and

$$\mathrm{gr}_*(H)/\mathcal{C}(\mathrm{gr}_*(H)) \xrightarrow{\mathrm{gr}_*(j)} \mathrm{gr}_*(F_2)$$

is an isomorphism of Lie algebras.

Furthermore, the natural map

$$p \times \alpha : H \longrightarrow F_2 \times H_1(H)$$

has non-trivial kernel, but induces a monomorphism

$$\mathrm{gr}_*(p \times \alpha) : \mathrm{gr}_*(H) \longrightarrow \mathrm{gr}_*(F_2 \times H_1(H))$$

on the level of descending central series Lie algebras.

Consequently, the Lie algebra obtained from the descending central series of H provides little information about embeddings as the subgroup $F_3 = \langle a_1, a_2, a_3 \rangle$ has image which factors through $\mathbb{Z}$ on the level of Lie algebras.

Proof. It follows from the presentation (12) that, for $1 \le i, j \le 2$, the relations $[\phi_i, a_j] = 1$ and $[\phi_i, a_3] = a_3 a_i a_3^{-1}$ hold in H. Denote the images of the generators ϕ_i and a_j in $\mathrm{gr}_*(H)$ by the same symbols. Then, for $1 \le i \le 2$, $a_i = 0$ in $\mathrm{gr}_1(H)$ since a_i is conjugate to a commutator in H. For any element $X \in \mathrm{gr}_*(H)$, it follows that $[a_i, X] = 0$ in $\mathrm{gr}_*(H)$ if $1 \le i \le 2$. Also, since $[\phi_i, a_3] = [a_3, a_i] \cdot a_i$ in H, $[\phi_i, a_3] = 0$ in $\mathrm{gr}_1(H)$. It follows that the map $\mathrm{gr}_*(j) : \mathrm{gr}_*(F_3) \to \mathrm{gr}_*(H)$ factors through the Lie subalgebra generated by a_3:

$$\mathrm{gr}_*(j) : \mathrm{gr}_*(F[a_1, a_2, a_3]) \longrightarrow L[a_3] \longrightarrow \mathrm{gr}_*(H)$$

Since $[\phi_i, a_3]$ is zero in $\mathrm{gr}_*(H)$, the class of a_3 centralizes $\mathrm{gr}_*(H)$. It follows that the Lie algebra kernel of $\mathrm{gr}_*(p) : \mathrm{gr}_*(H) \to \mathrm{gr}_*(F_2)$ is exactly $\mathcal{C}(\mathrm{gr}_*(H))$, a copy of the integers generated by a_3.

Note that $H_1(H)$ is free abelian of rank 3, generated by the classes of ϕ_1, ϕ_2, and a_3. It follows that the natural map $p \times \alpha : H \to F_2 \times H_1(H)$ induces a monomorphism

$$\mathrm{gr}_*(p \times \alpha) : \mathrm{gr}_*(H) \longrightarrow \mathrm{gr}_*(F_2 \times H_1(H))$$

on the level of Lie algebras, which completes the proof. ■

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