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Reduction Theorems for Manifolds with Degenerate 2–form

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Abstract. We consider a manifold with a 2–form and an action of a Lie group on the manifold which preserves the form. We define a momentum map and study its properties in this context. In particular we obtain a reduction theorem. Then we apply our reduction theorem to a certain generalization of the contact metric manifolds.
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Introduction

In recent years we observe a very rapid development of the symplectic geometry. In particular, interesting results were obtained using the symplectic reduction theorem. This theorem has various generalizations for symplectic manifolds enriched with supplementary structures. There were obtained the reductions of Kählerian, Sasakian, contact, hyper–Kähler, 3-Sasakian and many other structures, cf. [33]. These generalizations of the classical symplectic reduction theorem lead to the construction of new examples of various important structures on manifolds.

A contact metric manifold may be seen as a Riemannian manifold of dimension $2n+1$ equipped with an f-structure φ, i.e. $\varphi^3+\varphi=0$, such that the Riemannian metric is compatible with φ and certain integrability conditions are satisfied, cf. [6]. Such manifolds have been intensively studied from different points of view: topological and geometrical. A vast set of examples of contact metric manifolds is available too; for a rich collection of results one may consult an excellent book by D. E. Blair, cf. [7].

In the present paper we consider a generalization of the contact metric manifolds. We consider Riemannian manifolds of dimension $2n+s$ equipped with an f–structure φ of rank $2n$ which is compatible with the metric and such that certain integrability conditions are satisfied; moreover, we assume that the kernel bundle of φ is parallelizable. The $f-$structure and the Riemannian metric determine naturally the fundamental 2–form, called also the Sasaki form, cf. (1.2).

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This form is usually not closed neither non-degenerate. There is a wide possibility to study such manifolds under different assumptions. We consider here the so called (almost) $\mathcal{K}, \mathcal{C}, \mathcal{S}$–manifolds which were defined by D.E. Blair, cf. [5]. These structures carry many similarities with the (almost) Sasakian and cosympletic structures and have been studied by various geometers, cf. [24, 13, 18, 17].

In section 2. of our paper we consider a manifold on which acts a Lie group G. There is also given a 2–form which is invariant with respect to the action of G. Then we define a momentum map for such a structure and obtain its various characterizations and properties. In section 3. we consider a manifold with a 2–form such that its rank may vary from point to point. On such a manifold acts a Lie group leaving invariant the form. Then we prove a reduction theorem for such manifolds. In section 4. we prove various reduction theorems for the (almost) $\mathcal{K}, \mathcal{C}, \mathcal{S}$-manifolds. In section 5. we present applications of our reduction theorems as well as methods of constructing examples.

All manifolds, maps, distributions considered here are smooth i.e. of the class C^∞; we denote by $\Gamma(-)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

1. Preliminaries

1.1. Actions of Lie groups on manifolds. In the present subsection we recall basic definitions and properties considering the action of a Lie group on manifolds in relation with the symplectic geometry. We extend these properties for a manifold with any invariant 2-form. We use these definitions and properties later on in our paper. There is a vast very good classical bibliography about the subject, cf. [2, 1, 29, 23]; there are also some new brilliant textbooks too, cf. [12, 4, 14, 31, 33]. One can find an extensive and updated bibliography about the subject in [33].

Let M be an n–dimensional manifold and G a Lie group acting on the left on M by $\psi : G \times M \to M$. We denote by $\mathfrak{g}$ the Lie algebra of G. If $A \in \mathfrak{g}$ then by $\tilde{A}$ we denote the vector field on M determined by A via the action ψ, i.e. if $x \in M$ then $\tilde{A}_x := d_e\psi_x(A)$ where $\psi_x : G \to M$ is such that $\psi_x(a) = \psi(a, x)$ for each $a \in G$; e denotes here the neutral element of G. In such a way there is defined the map $d\psi : \mathfrak{g} \to \Gamma(TM)$ such that $d\psi(A) = \tilde{A}$. The map $d\psi$ is an anti–homomorphism of Lie algebras, i.e. for each $A, B \in \mathfrak{g}$ we have that $[\tilde{A}, \tilde{B}] = -\widetilde{[A, B]}$.

The group G acts by the adjoint representation $Ad : G \to Aut(\mathfrak{g})$ on $\mathfrak{g}$; for each $a \in G$ and $A \in \mathfrak{g}$ we denote $a \cdot A := Ad_a(A)$. Then there is the coadjoint action $Ad^* : G \to Aut(\mathfrak{g}^*)$ on the real dual space to $\mathfrak{g}$; for each $a \in G$ and $\phi \in \mathfrak{g}^*$ we put $a \cdot \phi := Ad^*_{a^{-1}}(\phi) = \phi \circ Ad_{a^{-1}}$. Then the action of G may be extended for the tensorial and wedge products of the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$. In particular, if $a \in G$, $\phi \in \wedge^2\mathfrak{g}^*$ and $A, B \in \mathfrak{g}$ then $(a \cdot \phi)(A, B) = \phi(a \cdot A, a \cdot B)$. The adjoint action of $\mathfrak{g}$ on itself $ad : \mathfrak{g} \to End(\mathfrak{g})$ determines the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, i.e. for each $A \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$ we have $A \cdot \phi := \phi \circ ad_A$. This action extends also to the higher tensorial powers of $\mathfrak{g}$ and $\mathfrak{g}^*$.

There is given a natural pairing $\langle \ , \ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ such that for each $\phi \in \mathfrak{g}^*$ and $A \in \mathfrak{g}$ we have $\langle \phi, A \rangle := \phi(A)$. This pairing extends to the pairs of external products of Lie algebras, in particular we have $\langle\ ,\ \rangle : \mathfrak{g}^* \wedge \mathfrak{g}^* \times \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$ such that for each $\phi \in \mathfrak{g}^* \wedge \mathfrak{g}^*$ and $A, B \in \mathfrak{g}$ we have $\langle \phi, A \wedge B\rangle := \phi(A, B)$. We have the following useful property: if $f : M \to \mathfrak{g}^*$ and $A \in \mathfrak{g}$ then $d\langle f, A\rangle = \langle df, A\rangle$. The same property is valid for smooth maps from M to $\wedge^2\mathfrak{g}^*$.

There is defined the *Chevalley cocomplex* $0 \xrightarrow{\delta} \mathfrak{g}^* \xrightarrow{\delta} \wedge^2\mathfrak{g}^* \xrightarrow{\delta} \wedge^3\mathfrak{g}^* \xrightarrow{\delta} \dots$ with the coboundary operator δ. We recall the explicit formula for the first cocycle space

$$C^1(\mathfrak{g}, \mathbb{R}) = \{\phi \in \mathfrak{g}^* |\ \text{ for each } A, B \in \mathfrak{g},\ \phi([A, B]) = 0\} = ([\mathfrak{g}, \mathfrak{g}])^0 \tag{1.1}$$

where $([\mathfrak{g}, \mathfrak{g}])^0$ denotes the subspace of $\mathfrak{g}^*$ consisting of the annihilators of $[\mathfrak{g}, \mathfrak{g}]$. Then there are defined the cohomology spaces $H^\bullet(\mathfrak{g}, \mathbb{R})$ which play important role in describing the properties of the Lie algebra $\mathfrak{g}$. We would like to remark that the action of G and that one of $\mathfrak{g}$ commute with δ. Hence $H^\bullet(\mathfrak{g}, \mathbb{R})$ is also invariant by the action of G.

Observation 1.1. *Suppose that F is a G-invariant 2-form on M. If $\tilde{A} \lrcorner dF = 0$ for all $A \in \mathfrak{g}$ then $[\widetilde{B}, \widetilde{A}] \lrcorner F = 2d(F(\widetilde{A}, \widetilde{B}))$ for each $A, B \in \mathfrak{g}$.*

Proof. If fact, for each $A, B \in \mathfrak{g}$ we have

$$\begin{aligned} 2d(F(\widetilde{A}, \widetilde{B})) &= d(\widetilde{B} \lrcorner (\widetilde{A} \lrcorner F)) = L_{\widetilde{B}}(\widetilde{A} \lrcorner F) - \widetilde{B} \lrcorner (d(\widetilde{A} \lrcorner F)) \\ &= L_{\widetilde{B}}(\widetilde{A} \lrcorner F) - \widetilde{B} \lrcorner (L_{\widetilde{A}} \lrcorner F - \widetilde{A} \lrcorner dF) = L_{\widetilde{B}}(\widetilde{A} \lrcorner F) = [\widetilde{B}, \widetilde{A}] \lrcorner F. \end{aligned}$$

■

1.2. Metric f–manifolds and associated structures. Let M be a m–dimensional manifold equipped with an f–structure φ, i.e. φ is an endomorphism of TM such that $\varphi^3 + \varphi = 0$. This is a natural generalization of an almost complex structure, cf. [37].

A Riemannian metric g and φ are said to be *compatible* if for each $X, Y \in TM$ holds $g(\varphi(X), Y) + g(X, \varphi(Y)) = 0$. If g and φ are compatible then it is possible to define the *Sasaki 2–form* by posing:

$$F(X, Y) := g(X, \varphi(Y)). \tag{1.2}$$

Moreover, we have the following simple observation.

Observation 1.2. *A Riemannian metric g and an f–structure φ are compatible if and only if at each point $x \in M$ there exists an orthonormal basis of T_xM such that the matrix of φ at the point x is given by*

$$\begin{pmatrix} \mathbb{O}^s_s & \mathbb{O}^s_n & \mathbb{O}^s_n \\ \mathbb{O}^n_s & \mathbb{O}^n_n & \mathbb{I}_n \\ \mathbb{O}^n_s & -\mathbb{I}_n & \mathbb{O}^n_n \end{pmatrix} \tag{1.3}$$

where $\mathbb{O}^p_q$ denotes null $p \times q$ matrix and $\mathbb{I}_p$ denotes the $p \times p$ identity matrix. In particular it follows from (1.3) *that* $\operatorname{rank}(\varphi_x) = \operatorname{rank} F_x = 2n$. *Clearly the numbers n, s may vary from point to point, but $2n + s = m$ in each point.*

Let M be equipped with a compatible Riemannian metric g and an f-structure φ. We denote by $N(F)$ the null distribution determined by F, i.e.

$$N(F) = \bigcup_{x\in M} N(F_x) = \{X \in T_xM | X \lrcorner F = 0\}. \tag{1.4}$$

It is clear that $N(F)$ is not a subbundle of TM in the strict sense since its fibres may have different dimensions. We observe that $N(F) = \ker\varphi$. Then we put $\mathcal{D} := \mathrm{Im}\varphi$. We have also the following orthogonal decomposition

$$TM = \mathcal{D} \oplus N(F). \tag{1.5}$$

Remark 1.1. If there is given an action of a Lie group G on M such that G preserves g and φ then G preserves also F and the decomposition (1.5).

(M, g, φ) is said to be an *f–structure with parallelizable kernel* (we write: *f.pk–structure*), if there exist s global vector fields $\xi_1, \dots, \xi_s$ and dual 1–forms $\eta^1, \dots, \eta^s$ on M satisfying the following conditions

$$\varphi(\xi_i) = 0,\ \eta^i \circ \varphi = 0,\ \varphi^2 = -I + \sum_{j=1}^{s} \eta^j \otimes \xi_j,\ \eta^i(\xi_j) = \delta^i_j \tag{1.6}$$

for all $i, j = 1, \dots, s$. On such manifolds, among compatible metrics, there always exists an *adapted* Riemannian metric g, in the sense that for each $X, Y \in \Gamma(TM)$

$$g(X, Y) = g(\varphi(X), \varphi(Y)) + \sum_{j=1}^{s} \eta^j(X)\eta^j(Y). \tag{1.7}$$

Hence there is also given the Sasaki 2–form F as in (1.2).

Reassuming, we have the following structures on the manifold M: an f-structure φ, the vector fields $\xi_1, \dots, \xi_s$, the 1-forms $\eta^1, \dots, \eta^s$, an adapted Riemannian metric g and the Sasaki 2-form F. We put $\mathcal{Z} := (M, g, \varphi, \xi_i, \eta^j)$; it is called *metric f-structure with a parallelizable kernel* or *metric f.pk-structure*, cf. [18]. It is easy to observe that $\mathcal{D}^\perp = \mathrm{span}\{\xi_1, \dots, \xi_s\} = N(F)$. With the f-structure φ there is naturally associated a tensor $\mathcal{N}_\varphi$ of type $(2,1)$ defined in the following way: $\mathcal{N}_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i$ where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ, cf. [28].

Definition 1.1. It is said that

- $\mathcal{Z}$ is *normal* if $\mathcal{N}_\varphi = 0$
- $\mathcal{Z}$ is an *almost $\mathcal{C}$-structure* if $d\eta^i = 0$ for $i = 1, \dots, s$ and $dF = 0$
- $\mathcal{Z}$ is a *$\mathcal{C}$-structure* if $d\eta^i = 0$ for $i = 1, \dots, s$, $dF = 0$ and $\mathcal{Z}$ is normal
- $\mathcal{Z}$ is an *almost $\mathcal{S}$-structure* if $d\eta^i = F$ for $i = 1, \dots, s$
- $\mathcal{Z}$ is a *$\mathcal{S}$-structure* if $d\eta^i = dF$ for $i = 1, \dots, s$ and $\mathcal{Z}$ is normal
- $\mathcal{Z}$ is a *$\mathcal{K}$-structure* if $dF = 0$ and $\mathcal{Z}$ is normal.

The above definitions are natural generalizations of the notions of the metric contact, Sasakian and cosymplectic manifolds, cf. [6].

The $f.pk$–structures may be seen from a different point of view. Namely as a certain type of almost CR–manifolds. In fact, given an $f.pk$–manifold $\mathcal{Z}$, we may define an almost CR–structure by considering $(M, \mathrm{Im}\varphi, \varphi|_{\mathrm{Im}\varphi})$. This structure is usually far from being integrable. However, the conditions on $\mathcal{Z}$ of being (almost) $\mathcal{C}, \mathcal{S}, \mathcal{K}-$ structures may be expressed in the language of the CR–geometry. Vice versa, an almost CR–structure (M, H, J) with a parallelizable transversal bundle determines an $f.pk$–structure. However, we shall not use the language of CR–geometry here.

Suppose that $M' \subset M$ is a submanifold such that $\varphi(TM') \subset TM'$ and for each $i \in \{1, \dots, s\}$, $x \in M'$, $(\xi_i)_x \in T_xM'$; then we put $\xi_i' := \xi_i|_{M'}$, $(\eta')^i := \eta^i|_{M'}$, $\varphi' := \varphi|_{M'}$ and $g' := g|_{M'}$. We also put $\mathcal{Z}' := (M', g', \varphi', \xi_i', (\eta')^j)$.

Proposition 1.1. **([16, 27])** *$\mathcal{Z}'$ is a metric $f.pk$-structure; moreover, if $\mathcal{Z}$ is (almost) $\mathcal{K}-, \mathcal{C}-, \mathcal{S}-$ structure then so is $\mathcal{Z}'$, respectively.*

Suppose that $\pi : (M, g) \to (\bar{M}, \bar{g})$ is a Riemannian submersion such that $\xi_1, \dots, \xi_{s-k}$ are horizontal and projectable onto $\bar{\xi}_1, \dots, \bar{\xi}_{s-k}$ via π; moreover, $\xi_{s-k+1}, \dots, \xi_s$ are vertical. We suppose also that φ is projectable onto $\bar{\varphi}$ via π. Then it is easy to observe that $\eta^1, \dots, \eta^{s-k}$ are projectable onto $\bar{\eta}^1, \dots, \bar{\eta}^{s-k}$ via π. We put $\bar{\mathcal{Z}} := (\bar{M}, \bar{g}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\beta)$, $\alpha, \beta \in \{1, \dots, s-k\}$.

Proposition 1.2. *$\bar{\mathcal{Z}}$ is a metric $f.pk$–structure. Moreover, if $\mathcal{Z}$ is (almost) $\mathcal{K}-, \mathcal{C}-, \mathcal{S}-$ structure then so is $\bar{\mathcal{Z}}$, respectively.*

Proof. Let $\bar{X}, \bar{Y} \in T_{\bar{x}}\bar{M}$ and let $X, Y \in T_xM$ be their horizontal lifts at a point $x \in M$. Then

$$\begin{aligned} \bar{g}(\bar{\varphi}(\bar{X}), \bar{Y}) &= \bar{g}(\pi_*(\varphi(X)), \pi_*(Y)) = (\pi^*\bar{g})(\varphi(X), Y) = g(\varphi(X), Y) \\ &= -g(X, \varphi(Y)) = -\bar{g}(\bar{X}, \bar{\varphi}(\bar{Y})) \end{aligned} \tag{1.8}$$

hence it follows that $\bar{g}, \bar{\varphi}$ are compatible. We denote by $\bar{F}$ the associated Sasaki fundamental 2–form. Using similar equations as (1.8) it is easy to prove that $\pi^*\bar{F} = F$, $\pi^*\bar{\eta}^\alpha = \eta^\alpha$ and $(\mathcal{N}_{\bar{\varphi}})(\bar{X}, \bar{Y}) = \pi_*(\mathcal{N}_\varphi(X, Y))$; therefore the second part of the assertion follows. ∎

2. Momentum maps for manifolds with degenerate 2–forms

Throughout all of this section we suppose that M is an n dimensional manifold with a 2–form F and $\psi : G \times M \to M$ is an action of a Lie group G on M which preserves F. The following definition is a natural generalization of a well known concept from the symplectic geometry.

Definition 2.1. A smooth map $\mu : M \to \mathfrak{g}^*$ is said to be a *momentum map for the action ψ* iff

(a) for each $A \in \mathfrak{g}$ we have $\langle d\mu, A \rangle = \widetilde{A} \lrcorner F$

(b) μ is G–equivariant.

Remark 2.1. We do not assume that F is closed neither non-degenerate 2–form on M. Usual momentum maps are related with the symplectic manifolds. The cases when the 2–form F is degenerate but still closed is considered in [36, 3]. With some regularity conditions about $N(F)$ there are obtained the symplectic structure, the momentum map and the symplectic reduction theorem on the induced quotient manifold. A very general aproach to the momentum map is considered in [21].

If there exists a momentum map for the action ψ then for each $A \in \mathfrak{g}$ we have that $\widetilde{A} \lrcorner dF = 0$. In fact, we have

$$\widetilde{A} \lrcorner dF = L_{\widetilde{A}} F - d(\widetilde{A} \lrcorner F) = -d\langle d\mu, A\rangle = -\langle d^2\mu, A\rangle = 0.$$

Assumption. From now on throughout all of this section we assume that $\widetilde{A} \lrcorner dF = 0$ for all $A \in \mathfrak{g}$. In many cases this condition appears to be sufficient for the existence of a momentum map for ψ.

We consider the following vector subspaces of $\Gamma(TM)$

$$\mathfrak{h}(F) := \{X \in \Gamma(TM) | \exists f \in C^\infty(M)\ df = X \lrcorner F \text{ and } X \lrcorner dF = 0\},$$
$$\mathfrak{sp}(F) := \{X \in \Gamma(TM) |\ L_X F = 0\}.$$

We call the vector fields in $\mathfrak{h}(F)$ ($\mathfrak{sp}(F)$) *Hamiltonian* (respectively: *symplectic*) *vector fields associated with* F. We observe that for each $X, Y \in \mathfrak{h}(F)$ we have $[X, Y] \lrcorner F = (L_X Y) \lrcorner F = L_X(Y \lrcorner F) - Y \lrcorner (L_X F) = 2d(F(Y, X))$. Hence it follows that $[X, Y] \in \mathfrak{h}(F)$ and $\mathfrak{h}(F)$ is a Lie subalgebra of $\Gamma(TM)$. Similarily, $L_{[X,Y]}F$ $= L_X(L_Y F) - L_Y(L_X F) = 0$ and then $\mathfrak{sp}(F)$ is a Lie subalgebra of $\Gamma(TM)$. It is easy to observe that $\mathfrak{h}(F) \subset \mathfrak{sp}(F)$.

Example 2.1. We consider $M = \mathbb{R}^4$ with its standard coordinates x, y, z, t and a 2–form $F = dx \wedge dy$. Then we consider the function $f \in C^\infty(M)$ such that $f(x, y, z, t) = z$. It is easy to observe that there is no vector field X on M such that $X \lrcorner F = df$; this is in contrast with the case of the standard symplectic structure on $\mathbb{R}^4$.

Since for each $A \in \mathfrak{g}$ we have $L_{\widetilde{A}} F = 0$ then the map $d\psi$ sends $\mathfrak{g}$ into $\mathfrak{sp}(F)$. If $X \in \Gamma(TM)$ and $X \lrcorner F = df$ then it does not imply that $X \in \mathfrak{sp}(F)$ unlike in the symplectic case. In fact, there is the following counterexample.

Example 2.2. Suppose that there are given: the manifold $M := \mathbb{R}^3$ with its standard coordinates x, y, z, the 2-form $F := zdx \wedge dy$ and the vector field $X = \frac{\partial}{\partial z}$. It holds $X \lrcorner F = 0 = d(\text{constant})$. On the other hand

$$L_X F = d(X \lrcorner F) + X \lrcorner dF = \frac{\partial}{\partial z} \lrcorner (dx \wedge dy \wedge dz) \neq 0;$$

therefore $X \notin \mathfrak{sp}(F)$.

Since for each $A, B \in \mathfrak{g}$ we have $\widetilde{[A, B]} \lrcorner F = 2d(F(\widetilde{A}, \widetilde{B}))$, cf. Observation 1.1, then $\rho : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to H^1_{\mathrm{dR}}(M)$ is a well defined map where $\rho(A) := [\widetilde{A} \lrcorner F]$. Then we have the following condition for the existence of a momentum map.

Proposition 2.1. *There exists a smooth map $\mu : M \to \mathfrak{g}^*$ satisfying condition* (a). *of Definition* 2.1 *if and only if* $\rho \equiv 0$.

Proof. If there exists μ satisfying 2.1(a) then for each $A \in \mathfrak{g}$ we have $\rho(A) = [\widetilde{A} \lrcorner F] = [d\langle \mu, A\rangle] = 0$. Vice versa, suppose that $\rho \equiv 0$. We consider a basis $A_1, \dots, A_d$ of $\mathfrak{g}$; then $\widetilde{A}_i \lrcorner F = df_i$ for some $f_1, \dots, f_d \in C^\infty(M)$. We put $\mu := \sum_{i=1}^d f_i A_i^*$ where $A_1^*, \dots, A_d^*$ is the dual basis of $\mathfrak{g}^*$. ∎

Hence there exists μ satisfying 2.1(a) in the case when $H^1_{\mathrm{dR}}(M) = 0$ or when $\mathfrak{g}$ is *perfect*, i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. If, in addition, G is compact then we can average such μ to get a momentum map. In fact, we have the following proposition.

Proposition 2.2. *Suppose that G is compact and that there exists a map $\widetilde{\mu} : M \to \mathfrak{g}^*$ satisfying* 2.1(a) *then there exists a momentum map for ψ.*

Proof. We put $\mu_a := (L^*_{a^{-1}}\widetilde{\mu}) \circ Ad_{a^{-1}}$ for each $a \in G$ and we observe that μ_a is smooth and satisfies 2.1(a). Then we define

$$\mu := \int_G \mu_a \nu_G$$

where ν_G is the left invariant volume form on G such that $\int_G \nu_G = 1$. It is easy to observe that the constructed μ is a momentum map for ψ. ∎

Proposition 2.3. *Suppose that there exists a momentum map $\mu : M \to \mathfrak{g}^*$ and that $H^1(\mathfrak{g}, \mathbb{R}) = 0$ then μ is unique.*

Proof. Suppose that $\mu' : M \to \mathfrak{g}^*$ is another momentum map for ψ. Then the map $\mu' - \mu$ takes its values in the cocycle space $C^1(\mathfrak{g}, \mathbb{R})$, cf. (1.1). Since $H^1(\mathfrak{g}, \mathbb{R}) = C^1(\mathfrak{g}, \mathbb{R}) = 0$ then it follows that μ and μ' coincide. ∎

Suppose that $\mu : M \to \mathfrak{g}^*$ is a momentum map for ψ and $\phi \in \mathfrak{g}^*$ then $\mu + \phi$ is a momentum map if and only if ϕ is G-invariant; if G is connected then this is equivalent to $\phi \in ([\mathfrak{g}, \mathfrak{g}])^0$. In fact

$$\begin{aligned} &\langle d(\mu + \phi), A\rangle = \langle d\mu, A\rangle = \widetilde{A} \lrcorner F, \\ &a \cdot (\mu + \phi)(x) = \mu(x) \circ Ad_{a^{-1}} + \phi = (\mu(x) + \phi) \circ Ad_{a^{-1}}. \end{aligned}$$

In the symplectic geometry the existence of the momentum map is strictly related to the existence of the so called comomentum map. If F is degenerate it is not possible to define in the same way comomentum map as in the symplectic context. However, we propose the following definition.

Definition 2.2. A map $\mu^* : \mathfrak{g} \to C^\infty(M)$ is said to be a *comomentum map* iff for each $A, B \in \mathfrak{g}$ we have $\mu^*([A, B]) = F(\widetilde{A}, \widetilde{B})$ and the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mu^*} & C^\infty(M) \\ {\scriptstyle -d\psi}\Big\downarrow & & \Big\downarrow{\scriptstyle d} \\ \Gamma(TM) & \xrightarrow{F^\flat} & \Gamma(T^*M). \end{array} \tag{2.1}$$

Then we have the following proposition.

Proposition 2.4. *If $\mu : M \to \mathfrak{g}^*$ is a momentum map for the action ψ then the map $\mu^* : \mathfrak{g} \to C^\infty(M)$ defined as $\mu^*(A) := -\frac{1}{2}\langle \mu, A\rangle$, $(A \in \mathfrak{g})$ is a comomentum map.*

Proof. In fact, let $A, B \in \mathfrak{g}$ and let a_t denote the 1-parameter subgroup of transformations of G determined by A then

$$\begin{aligned} d\mu^*([A,B]) &= -\frac{1}{2}\langle \mu, [A,B]\rangle = -\frac{1}{2}\langle \mu, \frac{d}{dt}(Ad_{a_{-t}})_*(B)\rangle|_{t=0} \\ &= -\frac{1}{2}\frac{d}{dt}\langle \mu \circ L_{a_t}, B\rangle|_{t=0} = -\frac{1}{2}\langle d\mu(\widetilde{A}), B\rangle = F(\widetilde{A}, \widetilde{B}). \end{aligned}$$

On the other hand for each $A \in \mathfrak{g}$ we have that

$$d(\mu^*(A)) = -\frac{1}{2}d\langle \mu, A\rangle = -\frac{1}{2}\langle d\mu, A\rangle = -\frac{1}{2}\widetilde{A}\lrcorner F = (F^\flat \circ (-d\psi))(A),$$

therefore diagram (2.1) commutes. ∎

The inverse of the above proposition is also true.

Proposition 2.5. *If $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a comomentum map and G is connected then the map $\mu : M \to \mathfrak{g}^*$ defined as $\langle \mu, A\rangle := -2\mu^*(A)$ for each $A \in \mathfrak{g}$ is a momentum map.*

Proof. In fact, for each $A \in \mathfrak{g}$ we have

$$d\langle \mu, A\rangle = -2d\mu^*(A) = 2F^\flat(d\psi(A)) = 2F^\flat(\widetilde{A}) = \widetilde{A}\lrcorner F,$$

therefore condition 2.1(a) holds. Suppose that there are given $A, B \in \mathfrak{g}$ and a_t the 1–parameter subgroup of transformations determined by A; we fix also $x \in M$. Then we consider the function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma(t) := \langle \mu(a_t \cdot x), B\rangle - \langle \mu(x), Ad_{a_{-t}}(B)\rangle$. Then clearly $\gamma(0) = 0$ and we have that

$$\begin{aligned} \gamma'(t) &= \langle d\mu(\widetilde{A}_{a_t\cdot x}), B\rangle - \langle \mu(x), [A, Ad_{a_{-t}}(B)]\rangle \\ &= F_{a_t\cdot x}(\widetilde{B}_{a_t\cdot x}, \widetilde{A}_{a_t\cdot x}) - F_x(\widetilde{Ad_{a_{-t}}(B)}_x, \widetilde{A}_x) \\ &= (L^*_{a_t}F)_x(dL_{a_{-t}}(\widetilde{B}_{a_t\cdot x}), dL_{a_{-t}}(\widetilde{A}_{a_t\cdot x})) - F_x(\widetilde{Ad_{a_{-t}}(B)}_x, \widetilde{A}_x) \\ &= F_x(\widetilde{Ad_{a_{-t}}(B)}_x, \widetilde{A}_x) - F_x(\widetilde{Ad_{a_{-t}}(B)}_x, \widetilde{A}_x) = 0. \end{aligned}$$

It follows that γ is constant equal to zero; this means that μ is G–equivariant. ∎

We would like to refine a little Propositions 2.4 and 2.5. For this purpose we define $C^\infty_{N(F)}(M) := \{f \in C^\infty(M) |\ df(N(F)) = 0\}$ and $\Gamma_{N(F)}(T^*M) := \{\Psi \in T^*M |\ \Psi(N(F)) = 0\}$. It is easy to observe that if there exists a comomentum map $\mu^* : \mathfrak{g} \to C^\infty(M)$ then μ^* takes actually its values in the space $C^\infty_{N(F)}(M)$ and the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mu^*} & C^\infty_{N(F)}(M) \\ {\scriptstyle -d\psi}\downarrow & & \downarrow{\scriptstyle d} \\ \Gamma(TM) & \xrightarrow{F^\flat} & \Gamma_{N(F)}(T^*M). \end{array}$$

In our case of F possibly degenerate we may define the Chu momentum map. We consider the map $\Phi : M \to \wedge^2\mathfrak{g}^*$ such that for each $A, B \in \mathfrak{g}$ we have $\Phi(x)(A, B) := F(\widetilde{A}_x, \widetilde{B}_x)$. It is clear that for each $x \in M$ the map $\Phi(x)$ is an element of $\wedge^2\mathfrak{g}^*$. Moreover for each $A, B, C \in \mathfrak{g}$ the following holds

$$0 \overset{\text{(i)}}{=} 3dF(\widetilde{A}_x, \widetilde{B}_x, \widetilde{C}_x) \overset{\text{(ii)}}{=} -\delta(\Phi(x))(A, B, C).$$

The essential fact in proving equalities (i) and (ii) is that for each $A \in \mathfrak{g}$ we have $\widetilde{A} \lrcorner dF = 0$ which is the light motive assumption of this paper. Therefore we have that Φ has its values in the cocycle space $C^2(\mathfrak{g}, \mathbb{R})$. The map $\Phi : M \to C^2(\mathfrak{g}, \mathbb{R})$ is called *Chu momentum map*, cf. [14, 33]. The Chu momentum map enjoys many symmetry properties as stated in the following theorem.

Theorem 2.1. *The Chu momentum map $\Phi : M \to C^2(\mathfrak{g}, \mathbb{R})$ satisfies the following properties:*

(1) *Φ is G–equivariant*

(2) *for each $A, B \in \mathfrak{g}$ we have $\langle d\Phi, (A, B)\rangle = \widetilde{[A, B]} \lrcorner F$*

(3) *for each $A, B, C \in \mathfrak{g}$ we have $\langle d\Phi(\widetilde{C}), (A, B)\rangle = -\langle C \cdot (d\Phi), (A, B)\rangle$*

(4) $\ker d\Phi = \{X \in TM | \ \ \forall A, B \in \mathfrak{g} \ \ F(\widetilde{[A, B]}, X) = 0\}$.

In Theorem 2.1 there is considered the extension of the adjoint action of G on the cocycle space $C^2(\mathfrak{g}, \mathbb{R})$ as well as the extension of the representation of $\mathfrak{g}$, cf. Subsection 1.. Theorem 2.1 may be proved in a similar way as the corresponding theorem in the non-degenerate case, cf. [15, 33], and we omit the proof here.

3. Reduction theorem

Throughout all of this section we assume that

- there are given a smooth manifold M and a 2–form F on M not necessary of constant rank neither closed
- there is given an action $\psi : G \times M \to M$ of a Lie group G $(\dim(G) = d)$ on M which preserves F
- there are given a momentum map $\mu : M \to \mathfrak{g}^*$ and $\phi_0 \in \mathfrak{g}^*$ which is G–invariant and such that $\mu^{-1}(\phi_0) \neq \emptyset$; G acts freely and properly on $\mu^{-1}(\phi_0)$. The action of G on M determines the orbits and we denote by $\mathcal{O}_x$ the orbit of the action of G passing through x.

The proper action of G on $\mu^{-1}(\phi_0)$ means that the map $\psi \times \mathrm{id} : G \times \mu^{-1}(\phi_0) \to \mu^{-1}(\phi_0) \times \mu^{-1}(\phi_0)$ is a proper map, cf. [8]. We also will use the following notation: $\pi : \mu^{-1}(\phi_0) \to \bar{M} := G\backslash\mu^{-1}(\phi_0)$ for the canonical projection and $u : \mu^{-1}(\phi_0) \hookrightarrow M$ for the canonical inclusion.

Theorem 3.1. *If there exists an open neighbourhood U of $\mu^{-1}(\phi_0)$ such that $U \ni x \mapsto \dim(T_x\mathcal{O} \cap N(F_x)) = const. = k$ then*

(A1) *$\mu^{-1}(\phi_0)$ is a regular submanifold of M of dimension $\dim(M) - d + k$*

(A2) *$\mu^{-1}(\phi_0)$ is G–invariant, $\bar{M}$ is a manifold of dimension $\dim(M) - 2d + k$ and the projection $\pi : \mu^{-1}(\phi_0) \to \bar{M}$ is a G-left principal fibre bundle*

(A3) *there exits a unique 2–form $\bar{F}$ on $\mu^{-1}(\phi_0)$ such that $\pi^*\bar{F} = u^*F$*

(A4) *if $x \in \mu^{-1}(\phi_0)$, $\pi(x) = \bar{x}$ then $\operatorname{rank}(F_x) = \operatorname{rank}(\bar{F}_{\bar{x}}) + 2(d-k)$ and $\dim(N(F_x)) = \dim(N(\bar{F}_{\bar{x}})) + k$; in particular, if $\operatorname{rank}(F) = constant$ on an open neighbourhood of $\mu^{-1}(\phi_0)$ so is $\operatorname{rank}(\bar{F})$ and $\dim(N(\bar{F}))$ throughout all of $\bar{M}$*

(A5) *u^*F is closed if and only if $\bar{F}$ is closed*

(A6) *if $\mu^{-1}(\phi_0)$ is connected then $\bar{F}$ is exact if and only if there exists a 1–form η on $\mu^{-1}(\phi_0)$ such that $\eta(T\mathcal{O}|_{\mu^{-1}(\phi_0)}) = 0$ and $d\eta = u^*F$.*

Proof. (A1) Since the action of G is free when restricted to $\mu^{-1}(\phi_0)$, we may assume that $d_x\psi : \mathfrak{g} \to T_x\mathcal{O}$ is an isomorphism for each $x \in U$. We fix $x \in U$ and consider a map $\alpha : T_x\mathcal{O} \to T_x^*M$ defined by $\alpha(X) := X \lrcorner F$. Then the point is that $\dim(\mathrm{Im}\alpha) = d - k$. To prove this we consider the dual map $\alpha^* : (T_x^*M)^* \to T_x^*\mathcal{O}$ which under the identification $(T_x^*M)^* \cong T_xM$ becomes $\alpha^*(X) = \alpha^*((X^*)^*) = -(X \lrcorner F)|_{T_x\mathcal{O}}$ for each $X \in T_xM$. Therefore, α^* has the rank equal to $d-k$. On the other hand, we have

$$\begin{aligned}
\mathrm{Im}(d_x\mu) &= \{d_x\mu(X) \in \mathfrak{g}^* | \; X \in T_xM\} \\
&= \{-(X \lrcorner F) \circ d_x\psi \in \mathfrak{g}^* | \; X \in T_xM\} \\
&\cong \{(X \lrcorner F)|_{T_x\mathcal{O}} \in T_x^*\mathcal{O} | \; X \in T_xM\} = \mathrm{Im}\alpha^*.
\end{aligned}$$

It follows that the map $U \ni x \mapsto \dim(\mathrm{Im} d_x\mu)$ is of constant rank equal to $d-k$. Hence from the local expression of the maps of constant rank, cf. page 41 of [35], it follows that $\mu^{-1}(\phi_0)$ is a regular closed submanifold of M of dimension $\dim(M) - d + k$.
(A2) Since for each $x \in \mu^{-1}(\phi_0)$ and each $a \in G$ we have $\mu(a \cdot x) = a \cdot \mu(x) = a \cdot \phi_0 = \phi_0$ then $\mu^{-1}(\phi_0)$ is G-invariant; this implies that $T_x\mathcal{O} \subset T_x\mu^{-1}(\phi_0)$. The second part of the assertion (A2) are well known theorems in differential topology, cf. [35, 9].
(A3) For each $a \in G$, $X \in T\mu^{-1}(\phi_0)$ and each $A \in \mathfrak{g}$ we have that

$$\begin{aligned}
L_a^*(u^*F) &= u^*(L_a^*F) = u^*F \\
(\widetilde{A} \lrcorner (u^*F))(X) &= (u^*(\widetilde{A} \lrcorner F))(X) = F(\widetilde{A}, X) = \langle d\mu(X), A \rangle = 0.
\end{aligned}$$

Therefore u^*F is a tensorial form on $\mu^{-1}(\phi_0)$, cf. [28], and then there exists a unique 2–form $\bar{F}$ on $\bar{M}$ such that $\pi^*\bar{F} = u^*F$.
(A4) If we fix $x \in \mu^{-1}(\phi_0)$ than we have that

$$T_x\mu^{-1}(\phi_0) = \ker d_x\mu = \bigcap_{A \in \mathfrak{g}} \ker(\widetilde{A} \lrcorner F).$$

Since $T_x\mathcal{O} \subset T_x\mu^{-1}(\phi_0)$, cf. proof of (A2), then for each $A, B \in \mathfrak{g}$ we have $F(\widetilde{A}, \widetilde{B}) = 0$. There exists a vector subspace V_x of $T_x\mu^{-1}(\phi_0)$ such that $N(F_x) = V_x \oplus (N(F_x) \cap T_x\mathcal{O})$; we put $k' := \dim(V_x)$. There exists a vector subspace W_x of $T_x\mathcal{O}$ such that $T_x\mathcal{O} = W_x \oplus (N(F_x) \cap T_x\mathcal{O})$. From the assumptions it follows that $\dim(W_x) = d-k$ and F_x restricted to $W_x \times W_x$ vanish since $W_x \subset T_x\mathcal{O}$. We denote by $\mathcal{B}_1$ a basis of the vector space V_x and by $\mathcal{B}_2$ a basis of $N(F_x) \cap (T_x\mathcal{O})$. Then we will construct convenient vectors which together with the elements of $\mathcal{B}_1 \cup \mathcal{B}_2$ give a basis of $T_x\mu^{-1}(\phi_0)$. Since F_x vanishes on $N(F_x)$ and since $N(F_x) \cap W_x = 0$ then there exist vectors

$$X_1, \ldots, X_{d-k}, Y_1, \ldots, Y_l, Z_1, \ldots, Z_l$$

in $T_x\mu^{-1}(\phi_0)$ which together with those of $\mathcal{B}_1 \cup \mathcal{B}_2$ give a basis of $T_x\mu^{-1}(\phi_0)$ and the following equalities hold

$$\begin{aligned} &\operatorname{span}\{X_1, \ldots, X_{d-k}\} = W_x, \quad F(Y_\alpha, Z_\beta) = \delta_{\alpha\beta}, \quad F(X_i, Y_\alpha) = 0, \\ &F(X_i, Z_\alpha) = F(X_i, X_j) = F(Y_\alpha, Y_\beta) = F(Z_\alpha, Z_\beta) = 0 \end{aligned} \tag{3.1}$$

for each $i, j = 1, \ldots, d-k$ and $\alpha, \beta = 1, \ldots, l$. We observe that the rank of F_x equals to $2l + 2(d-k)$. Then we put $\mathcal{B}_3 := \{X_1, \ldots, X_{d-k}\}$, $\mathcal{B}_4 := \{Y_1, \ldots, Y_l\}$, $\mathcal{B}_5 := \{Z_1, \ldots, Z_l\}$. The set $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5$ is a basis of $T_x\mu^{-1}(\phi_0)$. The point is that the matrix of F_x restricted to $T_x\mu^{-1}(\phi_0)$ and taken with respect to the constructed basis is the following:

$$F_x \sim \begin{pmatrix} \mathbb{O}^{k'}_{k'} & \mathbb{O}^{k'}_{k} & \mathbb{O}^{k'}_{d-k} & \mathbb{O}^{k'}_{l} & \mathbb{O}^{k'}_{l} \\ \mathbb{O}^{k}_{k'} & \mathbb{O}^{k}_{k} & \mathbb{O}^{k}_{d-k} & \mathbb{O}^{k}_{l} & \mathbb{O}^{k}_{l} \\ \mathbb{O}^{d-k}_{k'} & \mathbb{O}^{d-k}_{k} & \mathbb{O}^{d-k}_{d-k} & \mathbb{O}^{d-k}_{l} & \mathbb{O}^{d-k}_{l} \\ \mathbb{O}^{l}_{k'} & \mathbb{O}^{l}_{k} & \mathbb{O}^{l}_{d-k} & \mathbb{O}_{l} & \mathbb{I}_{l} \\ \mathbb{O}^{l}_{k'} & \mathbb{O}^{l}_{k} & \mathbb{O}^{l}_{d-k} & -\mathbb{I}_{l} & \mathbb{O}^{l}_{l} \end{pmatrix} \tag{3.2}$$

where $\mathbb{O}^p_q$ denotes the null $p \times q$ matrix and $\mathbb{I}_p$ denotes p–dimensional identity matrix. We observe that the projection $d\pi : T_x\mu^{-1}(\phi_0) \to T_{\bar{x}}\bar{M}$ sends bijectively the set of vectors $\mathcal{B}_1 \cup \mathcal{B}_4 \cup \mathcal{B}_5$ onto the basis $\bar{\mathcal{B}}_1 \cup \bar{\mathcal{B}}_4 \cup \bar{\mathcal{B}}_5$ of $T_{\bar{x}}\bar{M}$ where $\bar{\mathcal{B}}_i := d\pi(\mathcal{B}_i)$ for $i = 1, 4, 5$. Since $u^*F = \pi^*\bar{F}$ then the matrix of $\bar{F}$ with respect to the basis $\bar{\mathcal{B}}_1 \cup \bar{\mathcal{B}}_4 \cup \bar{\mathcal{B}}_5$ is the following

$$\bar{F}_{\bar{x}} \sim \begin{pmatrix} \mathbb{O}^{k'}_{k'} & \mathbb{O}^{k'}_{l} & \mathbb{O}^{k'}_{l} \\ \mathbb{O}^{l}_{k'} & \mathbb{O}^{l}_{l} & \mathbb{I}_{l} \\ \mathbb{O}^{l}_{k'} & -\mathbb{I}_{l} & \mathbb{O}^{l}_{l} \end{pmatrix}. \tag{3.3}$$

Therefore the rank of $\bar{F}_{\bar{x}}$ equals to $2l$ and hence

$$\operatorname{rank}(F_x) = 2l + 2(d-k) = \operatorname{rank}(\bar{F}_{\bar{x}}) + 2(d-k).$$

It also follows from (3.3) that $\dim(N(\bar{F}_{\bar{x}})) = k'$. Then from the decomposition $N(F_x) = V_x \oplus (T_x\mu^{-1}(\phi_0) \cap T_x\mathcal{O})$, cf. proof of (A4), we have that

$$\dim(N(F_x)) = \dim(V_x) + \dim(T_x\mu^{-1}(\phi_0) \cap T_x\mathcal{O}) = k' + k = \dim(N(\bar{F}_{\bar{x}})) + k.$$

This ends the proof of (A4).

(A5) This follows immediately from the facts that $\pi^*\bar{F} = u^*F$ and that π is a

submersion.
(A6) If there exists $\bar{\eta} \in \Gamma(T^*\bar{M})$ such that $d\bar{\eta} = \bar{F}$ then we put $\eta := \pi^*\bar{\eta}$ which satisfies the required conditions. Vice versa, suppose that there exists $\eta \in \Gamma(T^*\mu^{-1}(\phi_0))$ such that $\eta(T\mathcal{O}|_{\mu^{-1}(\phi_0)}) = 0$ and $d\eta = u^*F$. If $A \in \mathfrak{g}$ and $X \in T\mu^{-1}(\phi_0)$ then $(L_{\widetilde{A}}\eta)(X) = (\widetilde{A}\lrcorner(d\eta))(X) + (d(\widetilde{A}\lrcorner\eta))(X) = (\widetilde{A}\lrcorner(u^*F))(X) = (u^*(\widetilde{A}\lrcorner F))(X) = 2F(\widetilde{A}, X) = 0$. Since $\mu^{-1}(\phi_0)$ is connected then from the above equation it follows that η is G–invariant on $\mu^{-1}(\phi_0)$. Therefore there exists a 1–form $\bar{\eta}$ on $\bar{M}$ such that $\pi^*\bar{\eta} = \eta$. Then

$$\pi^*\bar{F} = u^*F = d\eta = d(\pi^*\bar{\eta}) = \pi^*(d\bar{\eta}) \tag{3.4}$$

and hence $d\bar{\eta} = \bar{F}$ since π is a submersion. This ends the proof of (A6) and the theorem. ■

4. Reduction theorems for $\mathcal{K}, \mathcal{C}, \mathcal{S}$–structures

Throughout all of this section we assume that

- there are given a manifold M with a Riemannian metric g and an f–structure φ which are compatible; hence there is also given the Sasaki 2–form, cf. (1.2)
- there is given an action $\psi : G \times M \to M$ of the d-dimensional Lie group G on M which preserves g and φ; hence ψ preserves also F
- there is given a momentum map $\mu : M \to \mathfrak{g}^*$ and $\phi_0 \in \mathfrak{g}^*$ which is G–invariant and such that $\mu^{-1}(\phi_0) \neq \emptyset$; G acts freely and properly on $\mu^{-1}(\phi_0)$.

To shorten the notation we put $\bar{M} := G\backslash\mu^{-1}(\phi_0)$ for the quotient space, $u : \mu^{-1}(\phi_0) \hookrightarrow M$ for the canonical immersion and $\pi : \mu^{-1}(\phi_0) \to \bar{M}$ for the canonical projection.

Theorem 4.1. *If there exists an open neighbourhood U of $\mu^{-1}(\phi_0)$ such that $U \ni x \mapsto \dim(N(F_x) \cap T_x\mathcal{O}) = const. = k$ then*

(B1) *$\mu^{-1}(\phi_0)$ is a regular submanifold of M of dimension $\dim(M) - d + k$*

(B2) *$\mu^{-1}(\phi_0)$ is G–invariant, $\bar{M}$ is a manifold of dimension $\dim(M) - 2d + k$ and $\pi : \mu^{-1}(\phi_0) \to \bar{M}$ is a G-left principal fibre bundle; moreover π is a Riemannian fibration where $\bar{M}$ carries the natural metric $\bar{g}$ obtained via the isometric action of G on $\mu^{-1}(\phi_0)$*

(B3) *there exits a unique 2–form $\bar{F}$ on $\mu^{-1}(\phi_0)$ such that $\pi^*\bar{F} = u^*F$*

(B4) *if $x \in \mu^{-1}(\phi_0)$, $\pi(x) = \bar{x}$ then $\mathrm{rank}(F_x) = \mathrm{rank}(\bar{F}_{\bar{x}}) + 2(d-k)$ and $\dim(N(F_x)) = \dim(N(\bar{F}_{\bar{x}})) + k$; in particular, if $\mathrm{rank}(F) = const.$ on an open neighbourhood of $\mu^{-1}(\phi_0)$ so is $\mathrm{rank}(\bar{F})$ and $\dim(N(\bar{F}))$ throughout all of $\bar{M}$*

(B5) *u^*F is closed if and only if $\bar{F}$ is closed*

(B6) *if $\mu^{-1}(\phi_0)$ is connected then $\bar{F}$ is exact if and only if there exists a 1–form η on $\mu^{-1}(\phi_0)$ such that $\eta(T\mathcal{O}|_{\mu^{-1}(\phi_0)}) = 0$ and $d\eta = u^*F$*

(B7) *$\bar{F}$, $\bar{g}$ are compatible; φ is projectable via π on the unique metric f-structure $\bar{\varphi}$ on $\bar{M}$; moreover $\bar{\varphi}$ is determined by $\bar{g}$ and $\bar{F}$.*

Proof. Points (B1), (B2), ..., (B6) follow immediately from Theorem 3.1. We remark only that in point (A2) we cite a well known fact that is: an isometric, free and proper action of G on $(\mu^{-1}(\phi_0), g)$ determines the Riemannian metric on the quotient space $\bar{M}$ and π is a G-left principal bundle which is also a Riemannian submersion.
(B7). We fix $x \in \mu^{-1}(\phi_0)$ and put $\bar{x} := \pi(x)$. To prove this point we need only to refine a little the proof of point (A4) of Theorem 3.1. In fact, in that proof we consider the decomposition

$$T_x\mu^{-1}(\phi_0) = V_x \oplus (N(F_x) \cap T_x\mathcal{O}) \oplus W_x \oplus \mathrm{span}\mathcal{B}_4 \oplus \mathrm{span}\mathcal{B}_5. \tag{4.1}$$

Since g and φ are compatible then we can find V_x, W_x, $\mathcal{B}_4$ and $\mathcal{B}_5$ in such a way that the decomposition (4.1) is orthogonal and such that the bases $\mathcal{B}_4$ and $\mathcal{B}_5$ are g–orthonormal. Then we choose $\mathcal{B}_1$ a basis of V_x, $\mathcal{B}_2$ a basis of $N(F_x) \cap T_x\mathcal{O}$ and $\mathcal{B}_3$ a basis of W_x consisting of orthonormal vectors. Hence equations (3.1) are satisfied and the matrix of F_x with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5$ of $T_x\mu^{-1}(\phi_0)$ is the same as the matrix in (3.2). The projections $\bar{\mathcal{B}_1} \cup \bar{\mathcal{B}_4} \cup \bar{\mathcal{B}_5}$ give an orthonormal basis of $T_{\bar{x}}\bar{M}$. Then the matrix of $\bar{F}_{\bar{x}}$ with respect to the basis $\bar{\mathcal{B}_1} \cup \bar{\mathcal{B}_4} \cup \bar{\mathcal{B}_5}$ is equal to the matrix defined in (3.3). From Observation 1.2 it follows that $\bar{g}$ and $\bar{F}$ are compatible. Since $\bar{F}$ and $\bar{g}$ are compatible then there exits a unique f-structure $\bar{\varphi}$ on $\bar{M}$ determined by them; it is easy to observe that $\bar{\varphi}$ is the projection via π of φ. ■

Theorem 4.2. *Suppose that*

- $\dim(M) = 2n + s$ *and there is given an $f.pk$–structure $(M, g, \varphi, \xi_i, \eta^j)$ $(i, j = 1, \dots, s)$ on M*
- *the action of G on M leaves invariant the vector fields $\xi_1, \dots, \xi_s$; hence $\eta^1, \dots, \eta^s$ are also G–invariant*
- $\mathrm{span}\{\xi_1, \dots, \xi_s\} \cap T\mathcal{O} = \mathrm{span}\{\xi_{s-k+1}, \dots, \xi_s\}$*; this condition means that the intersection of the tangent space to the orbit of G and $N(F_x)$ is always spanned by $\xi_{s-k+1}, \dots, \xi_s$*

then

(C1) *$\mu^{-1}(\phi_0)$ is a regular submanifold of M of dimension* $\dim(M) - d + k$

(C2) *$\mu^{-1}(\phi_0)$ is G–invariant and $\bar{M}$ is a manifold of dimension* $\dim(M) - 2d + k$ *and $\pi : \mu^{-1}(\phi_0) \to \bar{M}$ is a G-left principal fibre bundle; moreover π is a Riemannian fibration where $\bar{M}$ carries the natural metric $\bar{g}$ obtained via the isometric action of G on $\mu^{-1}(\phi_0)$*

(C3) *there exits a unique 2–form $\bar{F}$ on $\mu^{-1}(\phi_0)$ such that $\pi^*\bar{F} = u^*F$*

(C4) $\operatorname{rank}(F) = const. = \operatorname{rank}(\bar{F}) + 2(d-k)$ *and* $\dim(N(F)) = const. = \dim(N(\bar{F})) + k$

(C5) u^*F *is closed if and only if* $\bar{F}$ *is closed*

(C6) *if* $\mu^{-1}(\phi_0)$ *is connected then* $\bar{F}$ *is exact if and only if there exists a* 1*–form* η *on* $\mu^{-1}(\phi_0)$ *such that* $\eta(T\mathcal{O}|_{\mu^{-1}(\phi_0)}) = 0$ *and* $d\eta = u^*F$

(C7) $\bar{F}$ *and* $\bar{g}$ *are compatible and* φ *is projectable via* π *on the unique metric* f*-structure* $\bar{\varphi}$ *on* $\bar{M}$*; moreover* $\bar{\varphi}$ *is determined by* $\bar{g}$ *and* $\bar{F}$

(C8) *the vector fields* $\xi_1, \dots, \xi_{s-k}$ *project via* π *onto vector fields* $\bar{\xi}_1, \dots, \bar{\xi}_{s-k}$*,* $\eta^1, \dots, \eta^{s-k}$ *project onto* $1-$*forms* $\bar{\eta}_1, \dots, \bar{\eta}_{s-k}$*, and* $(\bar{M}, \bar{g}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}_\beta)$*,* $(\alpha, \beta = 1, \dots, s-k)$ *is a f.pk–structure; we denote this structure by* $\bar{\mathcal{Z}}$

(C9) *if* $\mathcal{Z}$ *is almost* $\mathcal{S}, \mathcal{C}, \mathcal{K}$*–structure then so is* $\bar{\mathcal{Z}}$*, respectively*

(C10) *if* $\mathcal{Z}$ *is* $\mathcal{S}, \mathcal{C}, \mathcal{K}$*–structure then so is* $\bar{\mathcal{Z}}$*, respectively.*

Proof. Since the assumptions of Theorem 4.1 are satisfied then the properties (C1), (C2), (C3), (C5), (C6) and (C7) follow immediately. Since $N(F) = \operatorname{span}\{\xi_1, \dots, \xi_s\}$ then

$$M \ni x \mapsto \dim(T_x\mathcal{O} \cap \operatorname{span}N(F)) = \dim(\operatorname{span}\{\xi_{s-k+1} \dots, \xi_s\}) = const. = k$$

Then (C4) is clear too since the rank of F is constant equal to $2n$.
(C8) We fix $x \in \mu^{-1}(\phi_0)$ and $\bar{x} = \pi(x)$. We use some part of the proof of the condition (B7) of Theorem 4.1, in particular we have the following orthogonal decomposition

$$T_x\mu^{-1}(\phi_0) = V_x \oplus (N(F_x) \cap T_x\mathcal{O}) \oplus W_x \oplus \operatorname{span}\mathcal{B}_4 \oplus \operatorname{span}\mathcal{B}_5. \tag{4.2}$$

for a convenient choice of V_x, W_x and sets of vectors $\mathcal{B}_4$, $\mathcal{B}_5$. From the assumptions of the present theorem we have that $N(F_x) \cap T_x\mathcal{O} = \operatorname{span}\{\xi_{s-k+1}, \dots, \xi_s\}$ and $V_x = \operatorname{span}\{\xi_1, \dots, \xi_{s-k}\}$. Hence $d\pi$ sends isometrically $\xi_1, \dots, \xi_{s-k}$ onto $\bar{\xi}_1, \dots, \bar{\xi}_{s-k}$; the projected vectors do not depend on the choice of $x \in \mu^{-1}(\phi_0)$ since the frame $\xi_1, \dots, \xi_s$ is invariant with respect to the action of G. Moreover, it is clear that $N(\bar{F}) = \operatorname{span}\{\bar{\xi}_1, \dots, \bar{\xi}_{s-k}\}$. Since $\eta^i(-) = g(\xi_i, -)$ for $i = 1, \dots, s$, g projects on $\bar{g}$ and ξ_i projects on $\bar{\xi}_i$ for $i = 1, \dots, s-k$ then η^i projects onto $\bar{\eta}^i$ and $\bar{\eta}^i = \bar{g}(\bar{\xi}_i, -)$ for $i = 1, \dots, s-k$. We need to verify conditions (1.6) and (1.7) which may be proved using similar methods so we prove here only the latter one. In fact, for each X, Y horizontal vectors in $T_x\mu^{-1}(\phi_0)$ which projects on $\bar{X}, \bar{Y}$ we have

$$\begin{aligned} \bar{g}(\bar{X}, \bar{Y}) &= g(X,Y) = g(\varphi(X), \varphi(Y)) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \\ &= \bar{g}(\bar{\varphi}(\bar{X}), \bar{\varphi}(\bar{Y})) + \sum_{\alpha=1}^{s-k} \eta^\alpha(X)\eta^\alpha(Y) + \overbrace{\sum_{i=s-k+1}^{s} \eta^i(X)\eta^i(Y)}^{=0} \\ &= \bar{g}(\bar{\varphi}(\bar{X}), \bar{\varphi}(\bar{Y})) + \sum_{\alpha=1}^{s-k} \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y}). \end{aligned}$$

Therefore, $(\bar{M}, \bar{g}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\xi}_\beta)$ $(\alpha, \beta = 1, \ldots, s-k)$ is a metric f–structure with parallelizable kernel.
(C9) Essentially this point follows from $u^*\eta^\alpha = \pi^*\bar{\eta}^\alpha$ and $u^*F = \pi^*\bar{F}$ for $\alpha = 1, \ldots, s-k$. In fact, if $\mathcal{Z}$ is an almost $\mathcal{C}$–manifold then $0 = u^*d\eta^\alpha = du^*\eta^\alpha = d\pi^*\bar{\eta}^\alpha = \pi^*d\bar{\eta}^\alpha$ and hence $\bar{\mathcal{Z}}$ is an almost $\mathcal{C}$–manifold. If $\mathcal{Z}$ is an almost $\mathcal{K}$–manifold then $0 = u^*dF = du^*F = d\pi^*\bar{F} = \pi^*d\bar{F}$ and hence $\bar{\mathcal{Z}}$ is an almost $\mathcal{K}$–manifold. Similarly, if $\mathcal{Z}$ is an almost $\mathcal{S}$–manifold then $\pi^*d\bar{\eta}^\alpha = u^*d\eta^\alpha = u^*F = \pi^*\bar{F}$ and hence $\mathcal{Z}$ is an almost $\mathcal{S}$–manifold.
(C10) Since $\pi : \mu^{-1}(\phi_0) \to \bar{M}$ is a Riemannian fibration then there exists the operator of the horizontal liftings of the vector fields from $T\bar{M}$ to $T\mu^{-1}(\phi_0)$: we denote this operator by H. Let $\bar{X}, \bar{Y}$ be vector fields on $\bar{M}$ and let $X = \bar{X}^H$, $Y = \bar{Y}^H$ be their horizontal liftings. With this notation we have the following formulas for the Nijenhuis torsion

$$\begin{aligned}([\varphi,\varphi])(X,Y) &= \big([\bar{\varphi},\bar{\varphi}](\bar{X},\bar{Y})\big)^H + \text{ terms belonging to} \\ &\qquad (N(F)\cap T\mathcal{O}) \oplus W \oplus (T\mu^{-1}(\phi_0))^\perp.\end{aligned}$$

On the other hand, since $\operatorname{span}\{\xi_{s-k+1}, \ldots, \xi_s\} \subset \ker(d\pi)$ then

$$\sum_{i=1}^{s}(d\eta^i \otimes \xi_i)(X,Y) = \sum_{\alpha=1}^{s-k}(d\bar{\eta}^\alpha \otimes \bar{\xi}_\alpha)(\bar{X},\bar{Y})^H;$$

it follows that if $\mathcal{N}_\varphi = 0$ then $\bar{\mathcal{N}}_{\bar{\varphi}} = 0$. This means that the normality of $\mathcal{Z}$ implies the normality of $\bar{\mathcal{Z}}$; together with (C9) this completes the proof of assertion (C10). ∎

Remark 4.1. The symplectic reduction for manifolds equipped with supplementary structures have been considered by various mathematicians and is still an area of intensive research, e.g. a Kähler reduction [26], a hyper-Kähler reduction [30, 25], a quaternion-Kähler reduction [19, 20] Sasakian and 3-Sasakian reduction [10, 11, 22]. There are also many other applications and generalizations of the reduction procedure which we do not mention here cf. [33].

5. Construction of examples

Example 5.1. Let (M_0, F_0) be a smooth manifold with a 2-form and $\psi_0 : G \times M_0 \to M_0$ be an action of the Lie group G preserving F_0. Let $\mu_0 : M_0 \to \mathfrak{g}^*$ be a momentum map for the action ψ_0. Let (M_1, F_1) be another manifold with a 2–form F_1. We put $M := M_0 \times M_1$ and $\pi_i : M \to M_i$ the projection on the i–th component for $i = 0, 1$; moreover we put $F := \pi_0^*F_0 + \pi_1^*F_1$ and define the action $\psi : G \times M \to M$ such that $\psi(a, (x_0, x_1)) := (\psi_1(a, x_0), x_1)$ for each $a \in G$ and $(x_0, x_1) \in M$. Then it is easy to observe that $\mu_0 \circ \pi_1$ is a momentum map for ψ.

The above example may be generalized in the following way.

Example 5.2. Let (M_i, F_i), $i = 0, 1$, be manifolds equipped with 2–forms and let $\psi_i : G \times M_i \to M_i$ be the actions of a Lie group G on M_i leaving invariant the forms F_i. Suppose that there is given a G-equivariant map $\pi : M_1 \to M_0$ such that $F_1 = \pi^* F_0$ and suppose that $\mu_0 : M_0 \to \mathfrak{g}^*$ is a momentum map for ψ_0 then it is easy to observe that $\mu_0 \circ \pi$ is a momentum map for the action ψ_1 on (M_1, F_1).

We have the following proposition which is a good instrument to construct examples of momentum maps.

Proposition 5.1. *Let (M_i, F_i), $i = 0, 1$, be manifolds equipped with 2–forms and let $\psi_i : G_i \times M_i \to M_i$ be the actions of the Lie groups G_i on M_i leaving invariant the forms F_i for $i = 0, 1$. Suppose that there is given a homomorphism $p : G_1 \to G_0$ which is a local diffeomorphism (hence p is a covering map) and that there is given a local diffemomorphism $\pi : M_1 \to M_0$ such that $F_1 = \pi^* F_0$ and such that π is p–equivariant. Let us also suppose that there are given maps $\mu_i : M_i \to \mathfrak{g}_i^*$ $(i = 0, 1)$ such that the diagram*

$$\begin{array}{ccccc} G_1 \times M_1 & \xrightarrow{\psi_1} & M_1 & \xrightarrow{\mu_1} & \mathfrak{g}_1^* \\ {\scriptstyle p\times\pi}\downarrow & & {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle (dp)^*} \\ G_0 \times M_0 & \xrightarrow{\psi_0} & M_0 & \xrightarrow{\mu_0} & \mathfrak{g}_0^* \end{array}$$

commutes. Then μ_1 is a momentum map for ψ_1 if and only if μ_0 is a momentum map for ψ_0.

Proof. Suppose that μ_0 is a momentum map for ψ_0 then for a given $A \in \mathfrak{g}_1$ we have that

$$\begin{aligned} \langle d\mu_1, A\rangle &= \langle d(\pi^*\mu_0) \circ dp, A\rangle = \langle \pi^*(d\mu_0), dp(A)\rangle = \pi^*\langle d\mu_0, dp(A)\rangle \\ &= \pi^*(\widetilde{dp(A)} \lrcorner F_0) = \pi^*(d\pi(\widetilde{A}) \lrcorner F_0) = \widetilde{A} \lrcorner (\pi^* F_0). \end{aligned} \tag{5.1}$$

On the other hand, for each $a_1 \in G_1$ and each $x_1 \in M_1$

$$\begin{aligned} \mu_1(a_1 \cdot x_1) &= \mu_0(p(a_1) \cdot \pi(x_1)) \circ dp = \mu_0(\pi(x_1)) \circ Ad_{p(a_1)^{-1}} \circ dp \\ &= \mu_0(\pi(x_1)) \circ dp \circ Ad_{a_1^{-1}} = a_1 \cdot \mu_1(x_1). \end{aligned} \tag{5.2}$$

Therefore from (5.1) and (5.2) it follows that μ_1 is a momentum map. In a similar way it may be proved the opposite implication of our proposition. ∎

From the previous proposition it follows, as a particular case, the next example.

Example 5.3. Let (M, F) be a connected manifold with a 2–form F and let $\psi : G \times M \to M$ be an action of the connected Lie group G preserving F. Then we consider the universal cover $\widetilde{M}$ and the canonical projection $\pi : \widetilde{M} \to M$. Then there is given the 2–form $\widetilde{F} := \pi^* F$ on $\widetilde{M}$. Moreover the action ψ lifts to the action $\widetilde{\psi} : \widetilde{G} \times \widetilde{M} \to \widetilde{M}$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{M} & \xrightarrow{\widetilde{\psi}} & \widetilde{M} \\ {\scriptstyle p\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times M & \xrightarrow{\psi} & M. \end{array}$$

Here $\widetilde{G}$ is the universal cover Lie group and $p : \widetilde{G} \to G$ is the canonical covering map. Clearly π is p–equivariant and $\tilde{F}$ is G-invariant. It follows from Propositon 5.1 that a momentum map for ψ lifts to a momentum map for $\widetilde{\psi}$; moreover, any momentum map for $\widetilde{\psi}$ which is $\pi_1(M)$-invariant may be pushed down to a momentum map for ψ.

In the next example we apply the suspension construction to get momentum maps, cf. [34].

Example 5.4. Let (M_i, F_i), $i = 0, 1$, be two manifolds with 2–forms F_i and let $\psi_i : G_i \times M_i \to M_i$ be actions of the Lie groups G_i preserving the forms; suppose also that there exist momentum maps $\mu_i : M_i \to \mathfrak{g}^*$. Let $\rho : \pi_1(M_0) \to \mathrm{Diff}(M_1, F_1, \mu_1)$ be a homomorphism of groups where $\mathrm{Diff}(M_1, F_1, \mu_1)$ denotes the group of diffeomorphisms of M_1 preserving the form F_1 and leaving invariant the map μ_1. Then $\widetilde{M}_0$ carries the induced 2–form $\tilde{F}_0 = \pi^* F_0$, the action of the group $\widetilde{G}_0$ on $\widetilde{M}_0$ and the momentum map $\widetilde{\mu}_0 : \widetilde{M}_0 \to \mathfrak{g}_0^*$, cf. Example 5.3. The group $\pi_1(M_0)$ acts freely and properly on $\widetilde{M}_0 \times M_1$ in the following way: for each $l \in \pi_1(M_0)$ and each $(\tilde{x}_0, x_1) \in \widetilde{M}_0 \times M_1$

$$l \cdot (\tilde{x}_0, x_1) := (l \cdot \tilde{x}_0, \rho(l) x_1)$$

and the quotient space is a smooth manifold; we put $\bar{M} := \pi_1(M_0)\backslash(\widetilde{M}_0 \times M_1)$ and we have also the canonical map $\pi : \widetilde{M}_0 \times M_1 \to \bar{M}$. The 2–form $\widetilde{F}_0 + F_1$ projects via π to a form $\bar{F}$ on $\bar{M}$. Since the map $\widetilde{\mu}_0 \times \mu_1$ is invariant by the action of $\pi_1(M_0)$ then it may be pushed down to a map $\Lambda : \bar{M} \to \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$. Suppose that the action $\tilde{\psi}_0$ commutes with the deck transformations determined by $\pi_1(M_0)$ and suppose also that the action of $\rho(\pi_1(M_0))$ and that one of G_1 on M_1 commute with each other. Then it is possible to define an action Ψ of the Lie group $\widetilde{G}_0 \times G_1$ on $\bar{M}$ such that for each $(\widetilde{a}, b) \in \widetilde{G}_0 \times G_1$ and $[\widetilde{x}_0, x_1] \in \bar{M}$

$$\Psi((\widetilde{a}, b), [\widetilde{x}_0, x_1]) := [\widetilde{\psi}_0(\widetilde{a}, \widetilde{x}_0), \psi_1(b, x_1)].$$

Hence the following diagram of maps

$$\begin{array}{ccccc}
(\widetilde{G}_0 \times G_1) \times (\widetilde{M}_0 \times M_1) & \xrightarrow{\widetilde{\psi}_0 \times \psi_1} & (\widetilde{M}_0 \times M_1) & \xrightarrow{\widetilde{\mu}_0 \times \mu_1} & \mathfrak{g}_0^* \oplus \mathfrak{g}_1^* \\
{\scriptstyle \mathrm{id} \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \| \\
(\widetilde{G}_0 \times G_1) \times \bar{M} & \xrightarrow{\Psi} & \bar{M} & \xrightarrow{\Lambda} & \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*
\end{array}$$

is commutative. It is easy to observe that Λ is a momentum map for the action of $\widetilde{G}_0 \times G_1$ on $\bar{M}$.

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