Journal of Lie Theory
Volume **17** (2007) 145–161

Infinite Fusion Products and $\widehat{\mathfrak{sl}_2}$ Cosets

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Communicated by G. Olshanski

Abstract. In this paper we study an approximation of tensor product of irreducible integrable $\widehat{\mathfrak{sl}_2}$ representations by infinite fusion products. This gives an approximation of the corresponding coset theories. As an application we represent characters of spaces of these theories as limits of certain restricted Kostka polynomials. This leads to the bosonic (which is known) and fermionic (which is new) formulas for the $\widehat{\mathfrak{sl}_2}$ branching functions.
Mathematics Subject Classification: 17B67, 81R10;
Key Words and Phrases: Cosets, branching functions, Kostka polynomials.

Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\widehat{\mathfrak{g}}$ be the corresponding affine algebra,

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is a central element and $[d, x_i] = -ix_i$. We set $\widehat{\mathfrak{g}}' = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$. Let L_{λ_1}, L_{λ_2} be two integrable irreducible $\widehat{\mathfrak{g}}$-modules. Then one has the decomposition of the tensor product

$$L_{\lambda_1} \otimes L_{\lambda_2} = \bigoplus_{\mu} C^{\mu}_{\lambda_1,\lambda_2} \otimes L_{\mu}$$

into the direct sum of integrable irreducible representations of $\widehat{\mathfrak{g}}'$ (see [13]). Here $C^{\mu}_{\lambda_1,\lambda_2}$ are spaces of highest weight vectors of the weight μ in the tensor product $L_{\lambda_1} \otimes L_{\lambda_2}$. Therefore $C^{\mu}_{\lambda_1,\lambda_2}$ are naturally graded by the operator d. This gives the character (branching function)

$$c^{\mu}_{\lambda_1,\lambda_2}(q) = \operatorname{Tr} q^d|_{C^{\mu}_{\lambda_1,\lambda_2}}. \tag{1}$$

Note that the GKO construction (see [11]) endows spaces $C^{\mu}_{\lambda_1,\lambda_2}$ with the structure of the representation of the Virasoro algebra Vir. This also gives a grading which differs from (1) by certain constant.

There exist different formulas for $c^{\mu}_{\lambda_1,\lambda_2}(q)$. For the case $\mathfrak{g} = \mathfrak{sl}_2$ the bosonic (alternating sign) formula was obtained in [1, 14, 15] using the representation

ISSN 0949–5932 / \$2.50

theory of Virasoro algebra (Feigin-Fuchs construction [8]). Another approach is based on the connection of the branching functions with configuration sums of RSOS model (see [3], [17], [18]). This also gives different types formulas, in particular the fermionic formula for $\mathfrak{g} = \mathfrak{sl}_2$. One of the important points in this approach is a construction of some finitization (approximation) of branching functions. The same method is used in [19], where for the type A affine Kac-Moody algebra the finitization of some branching functions is constructed by means of the combinatorics of crystal bases. This allows to obtain $c^{\mu}_{\lambda_1,\lambda_2}(q)$ as certain limits of restricted Kostka polynomials. In our paper we construct the representation theoretical approximation of the spaces $C^{\mu}_{\lambda_1,\lambda_2}$ for $\mathfrak{g} = \mathfrak{sl}_2$. We give some details below.

Let $L_{i,k}$, $0 \le i \le k$ be irreducible integrable level k representations with highest weight i with respect to $h \otimes 1 \in \widehat{\mathfrak{sl}_2}$ (h is a standard generator of the Cartan subalgebra of $\mathfrak{sl}_2$). Then one has the isomorphism of $\widehat{\mathfrak{sl}_2}'$ modules

$$L_{i_1,k_1} \otimes L_{i_2,k_2} = \bigoplus_{j=0}^{k_1+k_2} C^{j}_{i_1,i_2} \otimes L_{j,k_1+k_2}.$$

Our main tool is a construction of the filtration

$$L(0) \hookrightarrow L(1) \hookrightarrow L(2) \hookrightarrow \ldots = L_{i_1,k_1} \otimes L_{i_2,k_2} \tag{2}$$

of the tensor product, where $L(p)$ are certain integrable representations of $\widehat{\mathfrak{sl}_2}$. Namely, let $v_n \in L_{i_1,k_1}$, $w_n \in L_{i_2,k_2}$, $n,m \in \mathbb{Z}$ be sets of extremal vectors (the orbits of the highest weight vectors with respect to the action of the $\widehat{\mathfrak{sl}_2}$ Weyl group). Then obviously

$$L_{i_1,k_1} \otimes L_{i_2,k_2} = \bigcup_{n,m\in\mathbb{Z}} U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m),$$

where $U(\widehat{\mathfrak{sl}_2})$ is the universal enveloping algebra. In addition it is easy to find $n(p), m(p)$, $p \ge 0$ such that for $L(p) = U(\widehat{\mathfrak{sl}_2}) \cdot (v_{n(p)} \otimes w_{m(p)})$ the following holds:

$$L(0) \hookrightarrow L(1) \hookrightarrow L(2) \hookrightarrow \ldots = L_{i_1,k_1} \otimes L_{i_2,k_2}. \tag{3}$$

This procedure reduces the decomposition of the right hand side of (3) to the decomposition of $L(p)$ into the direct sum of irreducible representations. This can be done using the results from [5, 7].

We recall that in [5] the spaces $U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m)$ were identified with infinite fusion products (the inductive limits of finite-dimensional fusion products, see [10]). The infinite fusion products were decomposed in [7] and the corresponding q-multiplicities of irreducible representations were expressed in terms of the restricted Kostka polynomials. Therefore from (2) we obtain that branching functions are equal to the appropriate limits of restricted Kostka polynomials.

This Kostka polynomial approximation gives two different formulas for branching functions. From one hand we can use the alternating sum formula, which expresses the restricted Kostka polynomials in terms of the unrestricted Kostka polynomials (see [19, 9]). The latter are related to the characters of the representations of $\widehat{\mathfrak{sl}_2}$. Namely, certain limits of unrestricted Kostka polynomials can be expressed as a difference of two $\widehat{\mathfrak{sl}_2}$ string functions. This leads to the following formula

$$c^j_{i_1,i_2}(q) = q^{\gamma_1(i_1,i_2,j)}\times \\ \sum_{p\in\mathbb{Z}} q^{-(k_1+k_2+2)p^2-(j+1)p}(\mathrm{ch}_q L^{2(k_1+k_2+2)p+j-i_2}_{i_1,k_1} - \mathrm{ch}_q L^{2(k_1+k_2+2)p+j+i_2+2}_{i_1,k_1}),$$

where $L^a_{i,k}(q) = \{v\in L_{i,k} :\ (h\otimes 1)v = av\}$ and $\gamma_1(i_1,i_2,j)$ is some constant. We show that this bosonic formula can be rewritten in a form of [1, 14, 15].

Another possibility is to use the fermionic formula for the restricted Kostka polynomials (see [19, 9]). In the appropriate limit this approach gives the following type formula:

$$c^j_{i_1,i_2}(q) = q^{\gamma_2(i_1,i_2,j)}\times \\ \sum_{\substack{s_i\geq 0\\ i\in\{1,\dots,k_1+k_2\}\setminus\{k_1\}}} \frac{q^{\mathbf{s}B\mathbf{s}+\mathbf{u}\mathbf{s}}\begin{bmatrix}C\mathbf{s}+\mathbf{v}+\mathbf{s}\\ \mathbf{s}\end{bmatrix}_q}{(\min(j,k_2)-i_2+2\sum_{\beta\neq k_1}s_\beta(\beta-\min(k_1,\beta)))_q!}, \tag{4}$$

where B and C are some $(k_1+k_2-1)\times(k_1+k_2-1)$ matrices and $\mathbf{u},\mathbf{v}$ are some vectors. The following notations are used: for two vectors $\mathbf{n},\mathbf{m}\in\mathbb{Z}^N_{\geq 0}$ we set

$$\begin{bmatrix}\mathbf{n}\\ \mathbf{m}\end{bmatrix}_q = \prod_{i=1}^N \begin{bmatrix}n_i\\ m_i\end{bmatrix}_q = \prod_{i=1}^N \frac{(n_i)_q!}{(m_i)_q!(n_i-m_i)_q!},\quad (k)_q! = \prod_{i=1}^k (1-q^i).$$

Our paper is organized in the following way.

In Section 1 we recall the main definitions and properties of the representations of $\widehat{\mathfrak{sl}_2}$ and of the fusion products.

Section 2 is devoted to the description of the Kostka polynomial approach to the computation of the branching functions. This gives bosonic (Section 3) and fermionic (Section 4) formulas for $\widehat{\mathfrak{sl}_2}$ branching functions.

Acknowledgments. This work was partially supported by the RFBR grant 06-01-00037 and LSS 4401.2006.2.

1. Preliminaries

In this section we fix our notations and collect the main properties of fusion products (see [10, 4, 5, 6]).

Let $V_1,\dots,V_n$ be cyclic representations of the Lie algebra $\mathfrak{g}$ with cyclic vectors $v_1,\dots,v_n$. Fix $Z=(z_1,\dots,z_n)\in\mathbb{C}^n$ with $z_i\neq z_j$ for $i\neq j$. The fusion product $V_1(z_1)*\dots*V_n(z_n)$ is the adjoint graded $\mathfrak{g}\otimes\mathbb{C}[t]$ module with respect to the filtration F_m on the tensor product $V_1(z_1)\otimes\dots\otimes V_n(z_n)$:

$$F_m = \mathrm{span}\{g^{(1)}_{k_1}\cdots g^{(p)}_{k_p}(v_1\otimes\dots\otimes v_n) :\ k_1+\dots+k_p\leq m, g^{(i)}\in\mathfrak{g}\}. \tag{5}$$

Here $g_k = g\otimes t^k$ and $V_i(z_i)$ is the evaluation representation of $\mathfrak{g}\otimes\mathbb{C}[t]$, which is isomorphic to V_i as vector space and the action is defined via the map $\mathfrak{g}\otimes\mathbb{C}[t]\to\mathfrak{g}$, $g\otimes t^j\mapsto z_i^j g$, $g\in\mathfrak{g}$.

The most important property of fusion product is its independence on Z in some special cases. We will deal with the case $\mathfrak{g} = \mathfrak{sl}_2$.

Let $A = (a_1, \dots, a_n)$. Denote by

$$M_A = \pi_{a_1} * \dots * \pi_{a_n} \tag{6}$$

the fusion product of finite-dimensional irreducible representations of $\mathfrak{sl}_2$ ($\dim \pi_j = j+1$). Let v_A be the highest weight vector of (6) which is the image of the tensor product of highest weight vectors of π_{a_i}. We set

$$(\pi_{a_1} * \dots * \pi_{a_n})_m = \operatorname{span}\{(x^{(1)}_{i_1} \dots x^{(k)}_{i_k}) \cdot v_A, i_1 + \dots + i_k = m, x^{(i)} \in \mathfrak{sl}_2\}.$$

Let h be the standard generator of the Cartan subalgebra of $\mathfrak{sl}_2$. For $\mathfrak{sl}_2$-module M we denote $M^\alpha = \{w \in M : \ hw = \alpha w\}$ and set

$$\operatorname{ch}_q(\pi_{a_1} * \dots * \pi_{a_n})^\alpha = \sum_{m=0}^{\infty} q^m \dim\bigl((\pi_{a_1} * \dots * \pi_{a_n})^\alpha \cap (\pi_{a_1} * \dots * \pi_{a_n})_m\bigr).$$

We will need the following generalization of the standard embedding $\pi_{a+b} \hookrightarrow \pi_a \otimes \pi_b$. Let $A = (a_1 \le \dots \le a_n)$, $B = (b_1 \le \dots \le b_m)$, $m \le n$. Then we have an embedding

$$\begin{gathered} \pi_{a_1} * \dots * \pi_{a_{n-m}} * \pi_{a_{n-m+1}+b_1} * \dots * \pi_{a_n+b_m} \hookrightarrow M_A \otimes M_B, \\ v_{a_1,\dots,a_{n-m},a_{n-m+1}+b_1,\dots,a_n+b_m} \mapsto v_A \otimes v_B. \end{gathered} \tag{7}$$

Let us discuss one class of submodules of fusion products (see [6]). Let $A = (a_1 \le \dots \le a_n)$. Then for any $1 \le i < n$ there exists $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-module $S_i(A)$ such that the following sequence is exact:

$$\begin{multline} 0 \to S_i(A) \to \pi_{a_1} * \dots * \pi_{a_n} \to \\ \pi_{a_1} * \dots * \pi_{a_{i-1}} * \pi_{a_i - 1} * \pi_{a_{i+1}+1} * \pi_{a_{i+2}} * \dots * \pi_{a_n} \to 0 \end{multline} \tag{8}$$

For example, for $i = 1$

$$S_1(A) \simeq \pi_{a_2 - a_1} * \pi_{a_3} * \dots * \pi_{a_n}.$$

We will also need the case $i = n-1$. In this case

$$S_{n-1}(A) \simeq \pi_{a_1} * \dots * \pi_{a_{n-2}} \otimes \pi_{a_n - a_{n-1}}. \tag{9}$$

Therefore one has an exact sequence

$$\begin{multline} 0 \to \pi_{a_1} * \dots * \pi_{a_{n-2}} \otimes \pi_{a_n - a_{n-1}} \to \pi_{a_1} * \dots * \pi_{a_n} \to \\ \pi_{a_1} * \dots * \pi_{a_{n-2}} * \pi_{a_{n-1}-1} * \pi_{a_n+1} \to 0 \end{multline} \tag{10}$$

We now fix our notations about $\widehat{\mathfrak{sl}_2}$. Let

$$\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is a central element and $[d, x_i] = -ix_i$, where for $x \in \mathfrak{sl}_2$ we put $x_i = x \otimes t^i$. Let e, h, f be standard basis of $\mathfrak{sl}_2$. Consider nilpotent subalgebras

$$\mathfrak{n}_+ = \mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}f, \ \mathfrak{n}_- = \mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}e.$$

We denote by $L_{l,k}$, $0 \le l \le k$ integrable irreducible $\widehat{\mathfrak{sl}_2}$-module with highest weight vector $v_{l,k}$ such that

$$h_0 v_{l,k} = l v_{l,k}, \ K v_{l,k} = k v_{l,k}, \ d v_{l,k} = 0, \ \mathfrak{n}_- v_{l,k} = 0, \ U(\mathfrak{n}_+) \cdot v_{l,k} = L_{l,k},$$

where $U(\mathfrak{n}_+)$ is the universal enveloping algebra. Representations $L_{l,k}$ are bigraded by operators d and h_0. We set

$$L_{l,k} = \bigoplus_{\alpha, s \in \mathbb{Z}} (L_{l,k})^\alpha_s = \bigoplus_{\alpha, s \in \mathbb{Z}} \{v : \ dv = sv, h_0 v = \alpha v\}.$$

This determines the character $\mathrm{ch}_{q,z} L_{l,k} = \sum_{\alpha, s \in \mathbb{Z}} q^s z^\alpha \dim (L_{l,k})^\alpha_s$. For any graded subspace $V \hookrightarrow L_{l,k}$ we set

$$\mathrm{ch}_q V = \sum_{s \ge 0} q^s \dim \{v : \ dv = sv\}.$$

We now recall the Sugawara construction for the representation of the Virasoro algebra in the space of the level k $\widehat{\mathfrak{sl}_2}$-module. Namely, following operators form the Virasoro algebra:

$$L_n = \frac{1}{2(k+2)} \sum_{m \in \mathbb{Z}} : e_m f_{n-m} + f_m e_{n-m} + \frac{1}{2} h_m h_{n-m} :, \ n \in \mathbb{Z},$$

where $: \, :$ is the normal ordering sign,

$$: \ x_i y_j \ := \begin{cases} x_i y_j, \ i \ge j, \\ y_j x_i, \ j \ge i, \\ \frac{1}{2}(x_i y_j + y_j x_i). \end{cases}$$

The central charge is equal to $c(k) = \frac{3k}{k+2}$. We denote by $\triangle_{l,k}$ the conformal weight of the highest weight vector $v_{l,k} \in L_{l,k}$, i.e. $L_0 v_{l,k} = \triangle_{l,k} v_{l,k}$.

We consider the decomposition of the tensor product

$$L_{i_1,k_1} \otimes L_{i_2,k_2} = \bigoplus_{j=0}^{k_1+k_2} C^j_{i_1,i_2} \otimes L_{j, k_1+k_2},$$

where $C^j_{i_1,i_2}$ is the space of highest weight vectors of the weight j in the tensor product $L_{i_1,k_1} \otimes L_{i_2,k_2}$. Then by the GKO construction (see [11]) the space $C^j_{i_1,i_2}$ is a representation of the Virasoro algebra

$$L_n = L_n^{(1)} \otimes \mathrm{Id} + \mathrm{Id} \otimes L_n^{(2)} - L_n^{(diag)},$$

where $L_n^{(1)}, L_n^{(2)}, L_n^{(diag)}$ are the Sugawara operators acting on L_{i_1,k_1}, L_{i_2,k_2} and $L_{i_1,k_1} \otimes L_{i_2,k_2}$ respectively. We put

$$c^j_{i_1,i_2}(q) = \mathrm{ch}_q C^j_{i_1,i_2} = \mathrm{Tr}\, q^{L_0}|_{C^\mu_{\lambda_1,\lambda_2}}.$$

Remark 1.1. Note that the degree operator $d \in \widehat{\mathfrak{g}}$ acts on $C^{\mu}_{\lambda_1,\lambda_2}$ and $L_0 = d + \triangle_{i_1,k_1} + \triangle_{i_2,k_2} - \triangle_{j,k_1+k_2}$.

For the k-tuple $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k_{\geq 0}$ we set

$$V_{\mathbf{m}} = \pi_1^{*m_1} * \ldots * \pi_k^{*m_k} = \underbrace{\pi_1 * \ldots * \pi_1}_{m_1} * \ldots * \underbrace{\pi_k * \ldots * \pi_k}_{m_k}.$$

We also use the notation $\mathbf{m} = (1^{m_1} \ldots k^{m_k})$.

Consider the decomposition of $V_{\mathbf{m}}$ into the direct sum of irreducible representations of $\mathfrak{sl}_2 \otimes 1 \hookrightarrow \mathfrak{sl}_2 \otimes \mathbb{C}[t]$:

$$V_{\mathbf{m}} = \bigoplus_{l \geq 0} \pi_l \otimes \widetilde{K}_{l,\mathbf{m}},$$

where $\widetilde{K}_{l,\mathbf{m}} \hookrightarrow V_{\mathbf{m}}$ is a space of highest weight vectors of weight l. We note that each $\widetilde{K}_{l,\mathbf{m}}$ inherits a grading from $V_{\mathbf{m}}$. It is proved in [7] that $\mathrm{ch}_q \widetilde{K}_{l,\mathbf{m}} = \widetilde{K}_{l,\mathbf{m}}(q)$, where $\widetilde{K}_{l,\mathbf{m}}(q)$ is unrestricted Kostka polynomial. These polynomials are related to ones from [9] by

$$\widetilde{K}_{l,\mathbf{m}}(q) = q^{h(\mathbf{m})} K_{l,\mathbf{m}}(q^{-1}),$$

where

$$h(\mathbf{m}) = (\mathbf{m}A\mathbf{m} - p(\mathbf{m}))/4, \quad A = (A_{i,j}) = \min(i,j), \tag{11}$$
$$p(\mathbf{m}) = \#\{\alpha : \ m_\alpha + \ldots + m_k \text{ is odd }\}.$$

We note that $h(\mathbf{m})$ can be defined as follows:

$$h(\mathbf{m}) = \max\{s : \ (\pi_1^{*m_1} * \ldots * \pi_k^{*m_k})_s \neq 0\}.$$

Thus the "reversed" character of the fusion product is given by

$$\widetilde{\mathrm{ch}}_q V^{\alpha}_{\mathbf{m}} = q^{h(\mathbf{m})} \mathrm{ch}_{q^{-1}} V^{\alpha}_{\mathbf{m}}.$$

We proceed with a limit construction of fusion products. It is proved in [5] that there exists an embedding of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules:

$$V_{(m_1,\ldots,m_k)} \hookrightarrow V_{(m_1,\ldots,m_{k-1},m_k+2)}.$$

This allows to define an injective limit

$$L_{\mathbf{m},k} = \lim_{N \to \infty} V_{(m_1,\ldots,m_{k-1},m_k+2N)}. \tag{12}$$

It turns out that $L_{\mathbf{m},k}$ has the natural structure of level k $\widehat{\mathfrak{sl}_2}$ module. For example, $L_{i,k} = \lim_{N\to\infty} \pi_i * \pi_k^{*2N}$. In general representations $L_{\mathbf{m},k}$ are reducible. Consider the decomposition

$$L_{\mathbf{m},k} = \bigoplus_{l=0}^{k} C_{l,\mathbf{m}} \otimes L_{l,k},$$

where $C_{l,\mathbf{m}} \hookrightarrow L_{\mathbf{m},k}$ is the space of the highest weight vectors of the weight l. Note that $C_{l,\mathbf{m}}$ are naturally graded by the operator d. We set $\mathrm{ch}_q C_{l,\mathbf{m}} = \mathrm{Tr}\, q^d|_{C_{l,\mathbf{m}}}$. It is shown in [7] that

$$\mathrm{ch}_q C_{l,\mathbf{m}} = \widetilde{K}^{(k)}_{l,\mathbf{m}}(q), \tag{13}$$

where $\widetilde{K}^{(k)}_{l,\mathbf{m}}(q)$ is the restricted Kostka polynomials. These $\widetilde{K}^{(k)}_{l,\mathbf{m}}(q)$ are related to $K^{(k)}_{l,\mathbf{m}}(q)$ from [9] by

$$\widetilde{K}^{(k)}_{l,\mathbf{m}}(q) = q^{h(\mathbf{m})} K^{(k)}_{l,\mathbf{m}}(q^{-1}).$$

2. Kostka polynomials approximation

In this section we obtain the Kostka polynomials approximation of the characters of the $\widehat{\mathfrak{sl}_2}$ coset models, using the injective limits of fusion products. Our main idea is to construct an injective sequence of $\widehat{\mathfrak{sl}_2}$ subrepresentations W_i, $i \ge 1$ (i.e. $W_i \hookrightarrow W_{i+1}$) inside $L_{i_1,k_1} \otimes L_{i_2,k_2}$ such that each of W_i is isomorphic to some $L_{\mathbf{m},k_1+k_2}$ ($\mathbf{m}$ depends on i) and a limit (a union) of all W_i coincides with $L_{i_1,k_2} \otimes L_{i_2,k_2}$.

Let $v_n \in L_{i_1,k_1}$, $n \in \mathbb{Z}$ be the set of extremal vectors. This means that the weight of v_n is equal to the weight of the highest weight vector $v_{i,k}$ shifted by the n-th power of the translation operator T from the Weyl group W of $\widehat{\mathfrak{sl}_2}$. Recall that W contains a lattice $\mathbb{Z}$ and T is one of its generators. We fix T by the condition $h_0 v_n = i_1 - 2k_1 n$. We also denote the set of extremal vectors of L_{i_2,k_2} by w_n, $h_0 w_n = i_2 - 2k_2 n$.

Lemma 2.1. *Let $n \ge m$. Then*
a) $U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m) \hookrightarrow U(\widehat{\mathfrak{sl}_2}) \cdot (v_{n+1} \otimes w_m)$*;*
b) $\lim_{n\to\infty} U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m) = L_{i_1,k_1} \otimes L_{i_2,k_2}$.

Proof. We note that $(e_{2n-1})^{k_1-i_1}(e_{2n})^{i_1} v_n$ is proportional to v_{n-1} and also $e_i w_m = 0$ if $i > 2m$. Therefore $(e_{2n-1})^{k_1-i_1}(e_{2n})^{i_1}(v_n \otimes w_m)$ is proportional to $v_{n-1} \otimes w_m$. So $a)$ is proved.

To prove $b)$ we show that

$$U(\widehat{\mathfrak{sl}_2}) \cdot \mathrm{span}\{v_n \otimes w_m :\ n \ge m\} = L_{i_1,k_1} \otimes L_{i_2,k_2}. \tag{14}$$

In fact, $e_i w_m = 0$ for $i > 2m$ and so the space

$$(\mathbb{C}[e_{2m+1}, e_{2m+2}, \ldots] \cdot v_n) \otimes w_m$$

is a subspace of the left hand side of (14). Therefore the same is true for $L_{i_1,k_1} \otimes w_m$, because

$$L_{i_1,k_1} = \lim_{n\to\infty} \mathbb{C}[e_{2m+1}, e_{2m+2}, \ldots] \cdot v_n.$$

Now $b)$ follows. ∎

Lemma 2.1 provides us with an injective sequence of $\widehat{\mathfrak{sl}_2}$ subrepresentations which converges to the whole tensor product of two integrable modules. In the next Lemma we show that all these subrepresentations are of the type (12).

Lemma 2.2. *Let $n \ge m > 0$. Then*

$$U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m) \simeq \lim_{s\to\infty} \pi_{i_1} * \pi_{k_1}^{*(2(n-m)-1)} * \pi_{k_1+i_2} * \pi_{k_1+k_2}^{*2s}. \tag{15}$$

Proof. We note that

$$U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot v_n \simeq \pi_{i_1} * \pi_{k_1}^{*2n},\ U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot w_m \simeq \pi_{i_2} * \pi_{k_2}^{*2m}.$$

Therefore, because of (7),

$$U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot (v_n \otimes w_m) \simeq \pi_{i_1} * \pi_{k_1}^{*(2(n-m)-1)} * \pi_{k_1+i_2} * \pi_{k_1+k_2}^{*2m}.$$

We now obtain our lemma from

$$U(\widehat{\mathfrak{sl}_2}) \cdot (v_n \otimes w_m) \simeq \lim_{s\to\infty} U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot (v_{n+s} \otimes w_{m+s}).$$

∎

In the rest of this section we combine together Lemma 2.1, Lemma 2.2 and formula (13) to obtain the Kostka polynomials approximation of the coset characters. We introduce special notation for the right hand side of (15). Denote by $\mathbf{m}(N)$ a (k_1+k_2)-tuple such that

$$V_{\mathbf{m}(N)} = \pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2}. \tag{16}$$

For example, for $i_1 = i_2 = 0$ we have

$$\mathbf{m}(N) = (\underbrace{0,\dots,0}_{k_1-1}, 2N, 0, \dots, 0)$$

and for $i_1 = 0$ and $i_2 = k_2$

$$\mathbf{m}(N) = (\underbrace{0,\dots,0}_{k_1-1}, 2N-1, 0, \dots, 0, 1).$$

From the definition (12), formula (15) and Lemma 2.1 we obtain embeddings of $\widehat{\mathfrak{sl}_2}$ submodules of $L_{i_1,k_1} \otimes L_{i_2,k_2}$

$$L_{\mathbf{m}(1),k_1+k_2} \hookrightarrow L_{\mathbf{m}(2),k_1+k_2} \hookrightarrow \dots$$

and

$$L_{i_1,k_1} \otimes L_{i_2,k_2} = \lim_{N\to\infty} L_{\mathbf{m}(N),k_1+k_2}. \tag{17}$$

Now (13) and (17) gives

Corollary 2.3.

$$c^j_{i_1,i_2}(q) = q^{\triangle_{i_1,k_1}+\triangle_{i_2,k_2}-\triangle_{j,k_1+k_2}} \lim_{N\to\infty} \widetilde{K}^{(k_1+k_2)}_{j,\mathbf{m}(N)}.$$

3. Bosonic formula

We use the alternating sign formula for the restricted Kostka polynomials in terms of the unrestricted Kostka polynomials (see [19, 9])

$$K^{(k)}_{j,\mathbf{m}}(q) = \sum_{p\geq 0} q^{(k+2)p^2+(j+1)p} K_{2(k+2)p+j,\mathbf{m}}(q) - \sum_{p>0} q^{(k+2)p^2-(j+1)p} K_{2(k+2)p-j-2,\mathbf{m}}(q). \tag{18}$$

Recall the notations from [7]

$$\widetilde{K}^{(k)}_{l,\mathbf{m}}(q) = q^{h(\mathbf{m})} K^{(k)}_{l,\mathbf{m}}(q^{-1}),\ \ \widetilde{K}_{l,\mathbf{m}}(q) = q^{h(\mathbf{m})} K_{l,\mathbf{m}}(q^{-1}).$$

Lemma 3.1.

$$\lim_{N\to\infty} \widetilde{K}_{j,\mathbf{m}(N)}(q) = \mathrm{ch}_q L^{j-i_2}_{i_1,k_1} - \mathrm{ch}_q L^{j+i_2+2}_{i_1,k_1}. \tag{19}$$

Proof. It is shown in [7] that $\widetilde{K}_{j,\mathbf{m}}(q)$ is a multiplicity of π_j in $V_{\mathbf{m}}$. Consider embeddings

$$\pi_{i_1} * \pi_{k_1}^{*2(N-1)} \otimes \pi_{i_2} \hookrightarrow \pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2} \hookrightarrow \pi_{i_1} * \pi_{k_1}^{*2N} \otimes \pi_{i_2}, \tag{20}$$

where the first embedding comes from (7) and the second from (9). We note that (20) for N and $N+1$ can be combined into the commutative diagram

$$\begin{tikzcd}
\pi_{i_1} * \pi_{k_1}^{*2(N-1)} \otimes \pi_{i_2} \arrow[r] \arrow[d] & \pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2} \arrow[r] & \pi_{i_1} * \pi_{k_1}^{*2N} \otimes \pi_{i_2} \arrow[d] \\
\pi_{i_1} * \pi_{k_1}^{*2N} \otimes \pi_{i_2} \arrow[r] & \pi_{i_1} * \pi_{k_1}^{*(2N+1)} * \pi_{k_1+i_2} \arrow[r] & \pi_{i_1} * \pi_{k_1}^{*2(N+1)} \otimes \pi_{i_2}.
\end{tikzcd}$$

In view of $L_{i,k} = \lim_{N\to\infty} \pi_i * \pi_k^{*2N}$ we obtain

$$\widetilde{K}_{j,\mathbf{m}(N)}(q) = \widetilde{\mathrm{ch}}_q(V_{\mathbf{m}(N)} \otimes \pi_{i_2})^j - \widetilde{\mathrm{ch}}_q(V_{\mathbf{m}(N)} \otimes \pi_{i_2})^{j+2} = \\ \widetilde{\mathrm{ch}}_q V_{\mathbf{m}(N)}^{j-i_2} - \widetilde{\mathrm{ch}}_q V_{\mathbf{m}(N)}^{j+i_2+2} \to \mathrm{ch}_q L_{i_1,k_1}^{j-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{j+i_2+2}, \ N \to \infty. \tag{21}$$

Lemma is proved. ∎

Now we need to know how "fast" the left hand side of (19) converges to the right hand side.

Proposition 3.2. $\mathrm{ch}_q L_{i,k}^n - \widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2N})^n = \mathrm{O}(q^{N+1+\frac{n^2}{4k}-\frac{k}{4}})$.

Proof. We first consider the difference

$$\widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2(N+1)})^n - \widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2N})^n.$$

Recall (see (10)) that there exists an exact sequence

$$0 \to \pi_i * \pi_k^{*2N} \to \pi_i * \pi_k^{*2(N+1)} \to \pi_i * \pi_{k-1} * \pi_k^{*2N} * \pi_{k+1} \to 0.$$

We also note that the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-homomorphism

$$\pi_i * \pi_k^{*2(N+1)} \to \pi_i * \pi_{k-1} * \pi_k^{*2N} * \pi_{k+1}$$

is determined by the condition that the highest weight vector (with respect to h_0) maps to the highest weight vector. Therefore

$$\widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2(N+1)})^n - \widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2N})^n = \\ q^{h(i^1 k^{2(N+1)}) - h(i^1(k-1)k^{2N}(k+1))} \widetilde{\mathrm{ch}}_q(\pi_i * \pi_{k-1} * \pi_k^{*2N} * \pi_{k+1})^n. \tag{22}$$

Using the formula $h(\mathbf{m}) = (\mathbf{m}A\mathbf{m} - p(\mathbf{m}))/4$ (see (11)) we obtain

$$h(i^1 k^{2(N+1)}) - h(i^1(k-1)k^{2N}(k+1)) = \\ (N+1)(k(N+1)+i) - (N+1)(k(N+1)+i-1) = N+1. \tag{23}$$

To evaluate $\widetilde{\mathrm{ch}}_q(\pi_i * \pi_{k-1} * \pi_k^{*2N} * \pi_{k+1})^n$ we use the following formula for the character of the fusion product from [4]

$$\widetilde{\mathrm{ch}}_q(\pi_1^{*m_1} * \ldots * \pi_k^{*m_k})^n = q^{-\frac{p(\mathbf{m})}{4}} \sum_{\substack{j_1,\ldots,j_k \geq 0 \\ 2\sum_{l=1}^k (j_l-\alpha_l)=n}} q^{\sum_{l=1}^k (j_l-\alpha_l)^2} \begin{bmatrix} m_k \\ j_k \end{bmatrix}_q \prod_{l=1}^{k-1} \begin{bmatrix} m_{k-l}+j_{k-l+1} \\ j_{k-l} \end{bmatrix}_q, \quad (24)$$

where $2\alpha_l = m_l + \ldots + m_k$, and

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{(m)_q!}{(j)_q!(m-j)_q!}, \ (m)_q! = \prod_{i=1}^m (1-q^i).$$

In view of $\sum_{l=1}^k (j_l - \alpha_l) = n/2$ we obtain

$$-\frac{p(\mathbf{m})}{4} + \sum_{l=1}^k (j_l - \alpha_l)^2 \geq \frac{n^2}{4k} - \frac{k+1}{4}.$$

Therefore

$$\widetilde{\mathrm{ch}}_q(\pi_i * \pi_{k-1} * \pi_k^{*2N} * \pi_{k+1})^n = \mathrm{O}(q^{\frac{n^2}{4k}-\frac{k}{4}}). \quad (25)$$

From (23) and (25) we obtain

$$\widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2(N+1)})^n - \widetilde{\mathrm{ch}}_q(\pi_i * \pi_k^{*2N})^n = \mathrm{O}(q^{N+1+\frac{n^2}{4k}-\frac{k}{4}}). \quad (26)$$

Now using the limit construction

$$L_{i,k} = \pi_i \hookrightarrow \ldots \hookrightarrow \pi_i * \pi_k^{*2N} \hookrightarrow \pi_i * \pi_k^{*2(N+1)} \hookrightarrow \ldots$$

we obtain our proposition. ∎

Corollary 3.3. *Let $l \geq i_2$.*

$$\widetilde{K}_{l,\mathbf{m}(N)}(q) - (\mathrm{ch}_q L_{i_1,k_1}^{l-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{l+i_2+2}) = \mathrm{O}(q^{N+\frac{(l-i_2)^2}{4k_1}-\frac{k_1}{4}}).$$

Proof. We recall that

$$\widetilde{K}_{l,\mathbf{m}(N)}(q) = \widetilde{\mathrm{ch}}_q(\pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2})^l - \widetilde{\mathrm{ch}}_q(\pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2})^{l+2}. \quad (27)$$

Using (7) and (9) we obtain embeddings

$$\pi_{i_1} * \pi_{k_1}^{*(2N-2)} \otimes \pi_{i_2} \hookrightarrow \pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2} \hookrightarrow \pi_{i_1} * \pi_{k_1}^{*(2N)} \otimes \pi_{i_2}. \quad (28)$$

From Proposition (3.2) we have

$$\widetilde{\mathrm{ch}}_q(\pi_{i_1} * \pi_{k_1}^{*(2N)} \otimes \pi_{i_2})^l - \mathrm{ch}_q(L_{i_1,k_1} \otimes \pi_{i_2})^l = \sum_{s=0}^{i_2} \big(\widetilde{\mathrm{ch}}_q(\pi_{i_1} * \pi_{k_1}^{*(2N)} \otimes \pi_{i_2})^{l-i_2+2s} - \mathrm{ch}_q(L_{i_1,k_1} \otimes \pi_{i_2})^{l-i_2+2s}\big) = \mathrm{O}(q^{N+1+\frac{(l-i_2)^2}{4k_1}-\frac{k_1}{4}}), \quad (29)$$

because $l \geq i_2$. Therefore from (28) and (29) follows that

$$\widetilde{\mathrm{ch}}_q(\pi_{i_1} * \pi_{k_1}^{*(2N-1)} * \pi_{k_1+i_2})^l - \mathrm{ch}_q(L_{i_1,k_1} \otimes \pi_{i_2})^l = \mathrm{O}(q^{N+\frac{(l-i_2)^2}{4k_1}-\frac{k_1}{4}}), \tag{30}$$

where the approximation $L_{i_1,k_1} \otimes \pi_{i_2} = \lim_{N\to\infty} \pi_{i_1} * \pi_{k_1}^{*2N} \otimes \pi_{i_2}$ is used. Now our corollary follows from (27) and

$$\mathrm{ch}_q(L_{i_1,k_1} \otimes \pi_{i_2})^l - \mathrm{ch}_q(L_{i_1,k_1} \otimes \pi_{i_2})^{l+2} = \mathrm{ch}_q L_{i_1,k_1}^{l-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{l+i_2+2}.$$

■

Theorem 3.4.

$$\lim_{N\to\infty} \widetilde{K}_{j,\mathbf{m}(N)}^{(k_1+k_2)} = \sum_{p\in\mathbb{Z}} q^{-(k_1+k_2+2)p^2-(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j+i_2+2}\Big). \tag{31}$$

Proof. We use the alternating sign formula

$$\widetilde{K}_{j,\mathbf{m}(N)}^{(k_1+k_2)}(q) = \sum_{p\geq 0} q^{-(k_1+k_2+2)p^2-(j+1)p} \widetilde{K}_{2(k_1+k_2+2)p+j,\mathbf{m}(N)}(q) - \sum_{p>0} q^{-(k_1+k_2+2)p^2+(j+1)p} \widetilde{K}_{2(k_1+k_2+2)p-j-2,\mathbf{m}(N)}(q). \tag{32}$$

Using Corollary (3.3) we rewrite this expression as

$$\sum_{p\geq 0} q^{-(k_1+k_2+2)p^2-(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j+i_2+2} + \mathrm{O}\big(q^{N+\frac{(2(k_1+k_2+2)p+j-i_2)^2-k_1^2}{4k_1}}\big)\Big) - \sum_{p>0} q^{-(k_1+k_2+2)p^2+(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p-j-2-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p-j+i_2} + \mathrm{O}\big(q^{N+\frac{(2(k_1+k_2+2)p-j-2-i_2)^2-k_1^2}{4k_1}}\big)\Big). \tag{33}$$

We note that for p big enough we have

$$(k_1+k_2+2)p^2 + (j+1)p < \frac{(2(k_1+k_2+2)p-j-2-i_2)^2 - k_1^2}{4k_1},$$
$$(k_1+k_2+2)p^2 - (j+1)p < \frac{(2(k_1+k_2+2)p-j-2-i_2)^2 - k_1^2}{4k_1}.$$

Therefore we obtain

$$\lim_{N\to\infty} \widetilde{K}_{j,\mathbf{m}(N)}^{(k_1+k_2)}(q) = \sum_{p\geq 0} q^{-(k_1+k_2+2)p^2-(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j+i_2+2}\Big) - \sum_{p>0} q^{-(k_1+k_2+2)p^2+(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p-j-2-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p-j+i_2}\Big). \tag{34}$$

We now rewrite the second sum replacing p by $-p$. This gives

$$-\sum_{p<0} q^{-(k_1+k_2+2)p^2-(j+1)p}\Big(\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j-i_2} - \mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j+i_2+2}\Big) \tag{35}$$

(because $\mathrm{ch}_q L_{i,k}^a = \mathrm{ch}_q L_{i,k}^{-a}$). Our theorem is proved. ∎

Corollary 3.5. *The right hand side of* (31) *coincides with*

$$q^{-\triangle_{i_1,k_1}-\triangle_{i_2,k_2}+\triangle_{j,k_1+k_2}} c_{i_1,i_2}^j(q). \tag{36}$$

We finish this section with the identification of our bosonic formula with the known one (see [1],[14],[15]).

Let $\mathfrak{h}$ be Cartan subalgebra of $\widehat{\mathfrak{sl}_2}$ and define elements $(a,k,s)\in\mathfrak{h}^*$ by

$$(a,k,s)h_0 = i,\ (i,k,s)K = k,\ (a,k,s)d = s.$$

We consider a translation element t from the Weyl group W of $\widehat{\mathfrak{sl}_2}$ defined by

$$t(a,k,s) = (a+2k, k, s+k+a).$$

Therefore we have an isomorphism of vector spaces $L_{i,k}^a \simeq L_{i,k}^{a+2k}$ and for the corresponding characters we obtain

$$\mathrm{ch}_q L_{i,k}^{a+2k} = q^{k+a}\mathrm{ch}_q L_{i,k}^a. \tag{37}$$

This gives

$$\mathrm{ch}_q L_{i,k}^{a+2\lambda k} = q^{\lambda(\lambda k+a)}\mathrm{ch}_q L_{i,k}^a. \tag{38}$$

We now rewrite the right hand side of (31) using (38). Namely, let

$$2(k_1+k_2+2)p + j - i_2 = 2m + 2\lambda k_1, \tag{39}$$

where $0\le m\le k_1/2$ is integer for even i_1 and half-integer for odd i_1. Using (38) and (39) we obtain

$$q^{-(k_1+k_2+2)p^2-(j+1)p}\mathrm{ch}_q L_{i_1,k_1}^{2(k_1+k_2+2)p+j-i_2} = \\ q^{\frac{(j-i_2)^2}{4k_1^2}} q^{\frac{-m^2}{k_1}} q^{\frac{p}{k_1}(p(k_1+k_2+2)(k_2+2)+(k_2+2)(j+1)-(k_1+k_2+2)(i_2+1))}\mathrm{ch}_q L_{i_1,k_1}^m.$$

Combining this with the similar formula for the second term in the right hand side of (31) we obtain that up to a power of q the branching function $c_{i_1,i_2}^j(q)$ equals to

$$\sum_{0\le m\le k_1/2} q^{-\frac{m^2}{4}}\mathrm{ch}_q L_{i_1,k_1}^{2m}\times \\ \Bigg(\sum_{\substack{p\in\mathbb{Z}\\ m_{j-i_2}(p)\equiv\pm m \pmod{k_1}}} q^{\frac{p}{k_1}(p(k_1+k_2+2)(k_2+2)+(k_2+2)(j+1)-(k_1+k_2+2)(i_2+1))} - \\ \sum_{\substack{p\in\mathbb{Z}\\ m_{j+i_2}(p)\equiv\pm m \pmod{k_1}}} q^{\frac{1}{k_1}(p(k_1+k_2+2)+j+1)(p(k_2+2)+(i_2+1))}\Bigg),$$

where $m_a(p) = a/2 + (k_1+k_2+2)p$ and m runs over integers if i_1 is even and over half-integers if i_1 is odd. Identifying $q^{-\frac{m^2}{4}}\mathrm{ch}_q L_{i_1,k_1}^{2m}$ with the $\widehat{\mathfrak{sl}_2}$ string functions we obtain the known bosonic formula.

4. Fermionic formula

We now compute the limit $\lim_{N\to\infty}\widetilde{K}^{(k_1+k_2)}_{j,\mathbf{m}(N)}(q)$ using the fermionic formula from [7]:

$$q^{p(\mathbf{m})/4}\widetilde{K}^{(k)}_{j,\mathbf{m}} = \sum_{\substack{\mathbf{s}\in\mathbb{Z}^k_{\geq 0}\\ 2|\mathbf{s}|=|\mathbf{m}|-j}} q^{(\frac{\mathbf{m}}{2}-\mathbf{s})A(\frac{\mathbf{m}}{2}-\mathbf{s})}\begin{bmatrix} A(\mathbf{m}-2\mathbf{s})-\nu+\mathbf{s}\\ \mathbf{s}\end{bmatrix}_q, \tag{40}$$

where $A=(A_{i,j})^k_{i,j=1}=(\min(i,j))$, $\nu=(\nu_a)^k_{a=1}=(\max(0,a-k+j))$, $|\mathbf{m}|=\sum_{i=1}^k im_i$. For $\mathbf{v}\in\mathbb{Z}^k_{\geq 0}$ we put

$$\begin{bmatrix}\mathbf{v}\\ \mathbf{s}\end{bmatrix}_q=\prod_{i=1}^k\begin{bmatrix}v_i\\ s_i\end{bmatrix}_q=\prod_{i=1}^k\frac{(v_i)_q!}{(s_i)_q!(v_i-s_i)_q!}.$$

Now let $k=k_1+k_2$, $\mathbf{m}=\mathbf{m}(N)$ (see (16)). Then $|\mathbf{m}(N)|=i_1+i_2+2k_1N$ and for $\mathbf{s}$ from the right hand side of (40) we have $|\mathbf{s}|=k_1N+\frac{i_1+i_2-j}{2}$. This gives

$$s_{k_1}=N+\frac{i_1+i_2-j}{2k_1}-\frac{1}{k_1}\sum_{\substack{1\leq\alpha\leq k_1+k_2\\ \alpha\neq k_1}}\alpha s_\alpha. \tag{41}$$

We now rewrite the fermionic formula for $\widetilde{K}^{(k_1+k_2)}_{j,\mathbf{m}(N)}(q)$ using (41).

We start with the power $(\frac{\mathbf{m}(N)}{2}-\mathbf{s})A(\frac{\mathbf{m}(N)}{2}-\mathbf{s})$. Note that

$$\frac{\mathbf{m}(N)}{2}A\frac{\mathbf{m}(N)}{2}=N^2k_1+Ni_1+\frac{i_1+i_2}{4}.$$

Therefore

$$\begin{aligned}\left(\frac{\mathbf{m}(N)}{2}-\mathbf{s}\right)A\left(\frac{\mathbf{m}(N)}{2}-\mathbf{s}\right)&=N^2k_1+Ni_1+\frac{i_1+i_2}{4}+\\ &\sum_{\substack{1\leq\alpha,\beta\leq k_1+k_2\\ \alpha,\beta\neq k_1}}\min(\alpha,\beta)s_\alpha s_\beta-\\ &\sum_{\alpha\neq k_1}s_\alpha(\min(\alpha,i_1)+(2N-1)\min(\alpha,k_1)+\min(\alpha,k_1+i_2))+\\ &k_1s_{k_1}^2+2\sum_{\alpha\neq k_1}\min(\alpha,k_1)s_\alpha s_{k_1}-(i_1+2Nk_1)s_{k_1}.\end{aligned} \tag{42}$$

Using (41) we rewrite the last line as

$$\begin{aligned}&k_1(N+\frac{i_1+i_2-j}{2k_1}-\frac{1}{k_1}\sum_{\alpha\neq k_1}\alpha s_\alpha)^2+\\ &(-i_1-2Nk_1+2\sum_{\alpha\neq k_1}\min(\alpha,k_1)s_\alpha)(N+\frac{i_1+i_2-j}{2k_1}-\frac{1}{k_1}\sum_{\alpha\neq k_1}\alpha s_\alpha).\end{aligned} \tag{43}$$

Combining together (42) and (43) we obtain

$$\begin{aligned}
&\left(\frac{\mathbf{m}(N)}{2}-\mathbf{s}\right)A\left(\frac{\mathbf{m}(N)}{2}-\mathbf{s}\right)=\\
&\quad\sum_{\alpha,\beta\neq k_1}s_\alpha s_\beta\left[\min(\alpha,\beta)+\frac{1}{k_1}(\alpha\beta-\min(\alpha,k_1)\beta-\min(\beta,k_1)\alpha)\right]+\\
&\quad\sum_{\alpha\neq k_1}s_\alpha\Big[-\min(\alpha,i_1)-\min(\alpha,k_1+i_2)+\min(\alpha,k_1)\frac{i_1+i_2-j+k_1}{k_1}+\\
&\qquad\qquad\alpha\frac{j-i_2}{k_1}\Big]+\frac{i_1+i_2}{4}+\frac{(i_1+i_2-j)^2}{4k_1}-\frac{i_1(i_1+i_2-j)}{2k_1}. \qquad (44)
\end{aligned}$$

We now rewrite the binomial coefficient

$$\begin{bmatrix}(A(\mathbf{m}(N)-2\mathbf{s}))_\alpha-\nu_\alpha+s_\alpha\\ s_\alpha\end{bmatrix}_q \qquad (45)$$

using (41). Let $\alpha\neq k_1$. Then

$$\begin{aligned}
&(A(\mathbf{m}(N)-2\mathbf{s}))_\alpha-\nu_\alpha+s_\alpha=\\
&\quad\min(\alpha,i_1)+(2N-1)\min(\alpha,k_1)+\min(\alpha,k_1+i_2)-\\
&\quad-2\sum_{\beta\neq k_1}\min(\alpha,\beta)s_\beta-2\min(\alpha,k_1)s_{k_1}-\nu_\alpha+s_\alpha=\\
&\quad\min(\alpha,i_1)-\min(\alpha,k_1)(\frac{i_1+i_2-j+k_1}{k_1})+\min(\alpha,k_1+i_2)+\\
&\quad 2\sum_{\beta\neq k_1}s_\beta\left(-\min(\alpha,\beta)+\frac{\min(\alpha,k_1)\beta}{k_1}\right)-\nu_\alpha+s_\alpha. \qquad (46)
\end{aligned}$$

We note that the result is independent on N. Now let $\alpha=k_1$. Then

$$\begin{aligned}
&(A(\mathbf{m}(N)-2\mathbf{s}))_{k_1}-\nu_{k_1}=\\
&\quad i_1+(2N-1)k_1+k_1-2\sum_{\beta\neq k_1}\min(k_1,\beta)s_\beta-2k_1s_{k_1}-\nu_{k_1}=\\
&\quad j-i_2-\nu_{k_1}+2\sum_{\beta\neq k_1}s_\beta(\beta-\min(k_1,\beta)). \qquad (47)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\begin{bmatrix}(A(\mathbf{m}(N)-2\mathbf{s}))_{k_1}-\nu_{k_1}+s_{k_1}\\ s_{k_1}\end{bmatrix}_q=\\
&\quad\frac{1+O(q^{1+s_{k_1}})}{(j-i_2-\nu_{k_1}+2\sum_{\beta\neq k_1}s_\beta(\beta-\min(k_1,\beta)))_q!}. \qquad (48)
\end{aligned}$$

Note that

$$s_{k_1} + \left(\frac{\mathbf{m}(N)}{2} - \mathbf{s}\right) A \left(\frac{\mathbf{m}(N)}{2} - \mathbf{s}\right) =$$
$$s_{k_1} + \sum_{\alpha=1}^{k_1+k_2} \left[\sum_{\beta=k_1+k_2-\alpha+1}^{k_1+k_2} \left(\frac{\mathbf{m}(N)}{2} - \mathbf{s}\right)_\beta \right]^2 \geq$$
$$s_{k_1} + \left[\sum_{\beta=k_1+1}^{k_1+k_2} \left(\frac{\mathbf{m}(N)}{2} - \mathbf{s}\right)_\beta \right]^2 + \left[\sum_{\beta=k_1}^{k_1+k_2} \left(\frac{\mathbf{m}(N)}{2} - \mathbf{s}\right)_\beta \right]^2 =$$
$$s_{k_1} + (s_{k_1+1} + \ldots + s_{k_1+k_2} - 1/2)^2 + (s_{k_1} + \ldots + s_{k_1+k_2} - N)^2. \quad (49)$$

The expression in the last line is greater than or equal to $N/3$ (because if $s_{k_1} < N/3$ and $s_{k_1+1}+\ldots+s_{k_1+k_2}-1/2 < N/3$ then $(s_{k_1}+\ldots+s_{k_1+k_2}-N)^2 > N/3$). Therefore

$$q^{(\frac{\mathbf{m}}{2}-\mathbf{s})A(\frac{\mathbf{m}}{2}-\mathbf{s})} \begin{bmatrix} A(\mathbf{m}-2\mathbf{s})_{k_1} - \nu_{k_1} + s_{k_1} \\ s_{k_1} \end{bmatrix}_q =$$
$$\frac{1}{(j - i_2 - \nu_{k_1} + 2\sum_{\beta\neq k_1} s_\beta(\beta - \min(k_1,\beta)))_q!} + O(q^{N/3}). \quad (50)$$

Using $p(\mathbf{m}(N)) = i_1 + i_2$ we obtain the following theorem.

Theorem 4.1.

$$\lim_{N\to\infty} q^{-\frac{(i_1+i_2-j)(i_2-i_1-j)}{4k_1}} \widetilde{K}^{(k_1+k_2)}_{j,\mathbf{m}(N)} =$$
$$\sum_{\substack{s_i\geq 0 \\ i\in\{1,\ldots,k_1+k_2\}\setminus\{k_1\}}} \frac{q^{\mathbf{s}B\mathbf{s}+\mathbf{u}\mathbf{s}} \begin{bmatrix} C\mathbf{s}+\mathbf{v}+\mathbf{s} \\ \mathbf{s} \end{bmatrix}_q}{(\min(j,k_2) - i_2 + 2\sum_{\beta\neq k_1} s_\beta(\beta - \min(k_1,\beta)))_q!}, \quad (51)$$

where

$$B_{\alpha,\beta} = \min(\alpha,\beta) + \frac{\alpha\beta - \min(\alpha,k_1)\beta - \min(\beta,k_1)\alpha}{k_1},$$
$$\mathbf{u}_\alpha = -\min(\alpha,i_1) - \min(\alpha,k_1+i_2) + \frac{\min(\alpha,k_1)(i_1+i_2-j+k_1) + \alpha(j-i_2)}{k_1},$$
$$C_{\alpha,\beta} = 2\frac{\min(\alpha,k_1)\beta}{k_1} - 2\min(\alpha,\beta),$$
$$\mathbf{v}_\alpha = \min(\alpha,i_1) - \min(\alpha,k_1)\frac{i_1+i_2-j+k_1}{k_1} + \min(\alpha,k_1+i_2) - \max(0,\alpha-k_1-k_2+j).$$

Corollary 4.2. *The branching function* $c^j_{i_1,i_2}(q)$ *equals to the product of the right hand side of* (51) *and*

$$q^{\triangle_{i_1,k_1}+\triangle_{i_2,k_2}-\triangle_{j,k_1+k_2}+\frac{(i_1+i_2-j)(i_2-i_1-j)}{4k_1}}.$$

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Received May 3, 2006
and in final form September 11, 2006