Nearly Integrable SO(3) Structures
on 5-Dimensional Lie Groups

Simon G. Chiossi and Anna Fino

Communicated by B. Ørsted

Abstract. Recent work Bobieński and Nurowski on 5-dimensional Riemannian manifolds with an $SO(3)$ structure prompts us to investigate which Lie groups admit such a geometry. The case in which the $SO(3)$ structure admits a compatible connection with torsion is considered. This leads to a classification with respect to special behaviour of the connection, which enables us to recover all known examples, plus others bearing torsion of pure type. Suggestive relations with special structures in other dimensions are highlighted, with attention to eight-dimensional $SU(3)$ geometry.
Mathematical Subject Classification 2000: 53A40 (primary), 53C10, 53B15, 53C35, 53C25 (secondary).
Keywords and Phrases: $SO(3)$ structure, connections with skewsymmetric torsion, symmetric space.

1. Introduction

Given an oriented Riemannian manifold of dimension five (M^5, g), an $SO(3)$ structure is the reduction of the structure group of the frame bundle to the Lie group $SO(3)$ sitting inside $SO(5)$. The inclusion we choose is the one based on the *irreducible* 5-dimensional representation of $SO(3)$, determined by the decomposition

$$\mathfrak{so}(5) = \mathfrak{so}(3) \oplus V,$$

where V is the unique irreducible 7-dimensional representation of $SO(3)$. It is known that the homogeneous space $SO(5)/SO(3)$ has an $SO(3)$-connection $\tilde{\nabla}$ whose torsion tensor T is skew-symmetric (and unique), by which one intends that

$$T(X, Y, Z) = g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], Z) \tag{1}$$

is a three-form for any X, Y, Z vector fields, for which refer to [2, 9]. Indeed this was already noticed in [15], to the effect that the skew-symmetry of T can be used to align $SO(3)$ structures on five-manifolds with other kinds of torsion geometries. In the general framework of geometric structures on Riemannian n-manifolds, G-reductions are distinguished by the irreducible components of the representation

$\mathbb{R}^n \otimes \mathfrak{so}(n)/\mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of G. The decomposition of this space depends on a tensorial object, most of the times a differential form. Well known is the archetypal description of almost Hermitian structures [11] in terms of the Kähler form. Other G-geometries have been discussed using the same approach, see [6] amongst others for $G = SU(n)$.

We review $SO(3)$ geometry in section 2. In some sense it is slightly different from more familiar G-structures, for it is defined not by means of a skew form but rather a symmetric tensor, denoted $\mathbb{T}$. Apart from $\mathbb{T}$, one can not expect to find other invariants for the representation of $SO(3)$ on $\mathbb{R}^5$, in particular no differential form. A rather diverse situation occurs in higher dimensions, such as eight for instance, where both totally symmetric and totally skew 3-forms play a relevant role [14], see section 9.

In the special case of concern, the so-called *nearly integrable* structures, this tensor behaves – strikingly – like the almost complex structure J of a nearly Kähler manifold, whence the similar name.

For a general $SO(3)$-reduction the tensor product $\mathbb{R}^5 \otimes \mathfrak{so}(3)^\perp$ decomposes into the analogues of the Gray–Hervella classes, minding that here the complement $\mathfrak{so}(3)^\perp$ is reducible (hence not V). As the torsion T of a nearly integrable $SO(3)$ structure is uniquely defined and skew-symmetric, and the spaces of three- and two-forms on M^5 are Hodge-isomorphic, one decomposes the latter under $SO(3)$ in the sum of two irreducible modules

$$\Lambda^2 M^5 = \Lambda^2_3 \oplus \Lambda^2_7$$

of dimensions three, seven respectively. When the components therein are trivial, the characteristic connection $\tilde{\nabla}$ is in fact the Riemannian one, and [5] proves that the simply connected manifolds M^5 admitting a torsion-free $SO(3)$ structure are $\mathbb{R}^5$, $SU(3)/SO(3)$ or $SL(3,\mathbb{R})/SO(3)$. It is not surprising that these models are symmetric spaces, as proven by Berger's holonomy theorem. The paper of Bobieński and Nurowski classifies 5-dimensional Lie groups with type Λ^2_3, and finds examples with closed torsion of the complementary type Λ^2_7.

The present note pertains to 5-dimensional real connected Lie groups L with invariant metric g and $SO(3)$ structure having anti-symmetric torsion T. More precisely, we characterise the Lie groups $(L, \mathbb{T}, g)$ admitting a splitting $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p}$ of the Lie algebra of L defined via an adapted frame, see section 5., in terms of the structure constants. This in turn yields a classification (theorem 4.4) assuming that the $SO(3)$ connection $\tilde{\nabla}$ satisfies

$$\begin{aligned}
\tilde{\nabla}_X \mathfrak{h} \subseteq \mathfrak{h}, \quad \tilde{\nabla}_X \mathfrak{p} \subseteq \mathfrak{p}, \quad \forall X \in \mathfrak{h}, \\
\tilde{\nabla}_Y \mathfrak{h} \subseteq \mathfrak{p}, \quad \tilde{\nabla}_Y \mathfrak{p} \subseteq \mathfrak{h}, \quad \forall Y \in \mathfrak{p}.
\end{aligned}$$

This special behaviour is displayed by the Levi–Civita connection ∇ in all instances of [5], and corresponds to demanding that the group L act transitively on a Riemannian symmetric surface. We then prove that either the Levi-Civita connection fulfills the same algebraic conditions as $\tilde{\nabla}$, or $\tilde{\nabla}$ is identically zero (theorem 7.1). In the latter case the Lie group is essentially $SO(3) \times \mathbb{R}^2$, and thus the unique instance up to isomorphisms. Our results are in line with the main reference [5] and in some sense attempt to complete that classification. All examples have $d*T = 0$, and it would be thus natural to ask whether this were

always the case. We prove that a large class of Lie groups, comprising those of [5], satisfies the equivalent requirement of symmetry of the $SO(3)$-connection's Ricci tensor. In every case the torsion type, whether Λ_3^2, Λ_7^2 or generic, is determined. In section 8. we prove the existence of structures of type Λ_7^2 with non-closed torsion, by finding an explicit example. This is once again realised by a Lie group that acts transitively on a 3-dimensional symmetric space by way of

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{p}.$$

Section 9. is devoted to understanding the geodesic equation $\nabla_X X = 0$, for any X in $\mathfrak{l}$, which arises from 'near integrability' and has remarkable consequences. Using this property it is possible to construct nearly integrable geometry of higher dimension, namely $SU(3)$ structures on the Riemannian products $L \times \mathbb{R}^3$ and $L \times SO(3)$.

The authors would like to dedicate this paper to the memory of Giulio Minervini (University of Bari).

2.   Irreducible SO(3) geometry

The vector space $\mathbb{R}^5$ is isomorphic to the set of real 3×3 symmetric matrices with no trace $\mathcal{S}_0^2 \mathbb{R}^3$; we fix the isomorphism as follows

$$X = (x_1, \ldots, x_5) \longleftrightarrow \begin{pmatrix} \frac{x_1}{\sqrt{3}} - x_4 & x_2 & x_3 \\ x_2 & \frac{x_1}{\sqrt{3}} + x_4 & x_5 \\ x_3 & x_5 & -\frac{2}{\sqrt{3}}x_1 \end{pmatrix}, \qquad (2)$$

the square root only being a convenient factor. The irreducible representation on $\mathbb{R}^5$ is given by

$$\rho(h)X = hXh^{-1}, \quad h \in SO(3),$$

where the vector X is thought of as matrix of the above form. An $SO(3)$ structure on (M, g) can be identified [5] with an element of $\mathbb{T} \in \bigotimes^3 \mathbb{R}^5$ such that

   i) $\mathbb{T} = \sum_{i,j,k=1}^5 \mathbb{T}_{ijk} \, dx_i \otimes dx_j \otimes dx_k$ is symmetric in all arguments,

   ii)   the endomorphisms $X \mapsto \mathbb{T}_X = X \lrcorner \mathbb{T}$ are trace-free and

   iii) $(\mathbb{T}_X)^2 X = g(X, X)X$, for all $X \in \mathbb{R}^5$.

The contraction $X \lrcorner \mathbb{T}$ prescribes to fill the first argument of $\mathbb{T}$, so $\mathbb{T}_X(\cdot, \cdot) = \mathbb{T}(X, \cdot, \cdot)$. Notice by the way that ii) is expected, given that $\sum_j \mathbb{T}_{ijj}$ defines the components of an $SO(3)$-invariant vector in $\mathbb{R}^5$, so it must vanish for each $i = 1 \ldots 5$ because the representation is irreducible.

These properties together with the identification (2) determine an adapted frame, i.e. an orthomormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of tangent vectors so that $X = \sum_{j=1}^{5} x_j e_j$ and the metric automatically assumes the canonical form $g = \sum_{i=1}^{5}(e^i)^2$. Adopting the choice made in [5], the $SO(3)$ structure is completely described by the tensor

$$\mathbb{T}(X, X, X) = x_1(3x_2^2 + 3x_4^2 - x_1^2 - \tfrac{3}{2}x_3^2 - \tfrac{3}{2}x_5^2) + \tfrac{3\sqrt{3}}{2}x_4(x_5^2 - x_3^2) + 3\sqrt{3}x_2x_3x_5, \quad (3)$$

given by the determinant of X.

The action of $SO(3)$ on $\mathbb{C}^2$, like the one of $SU(2)$, endows the symmetric tensor product $\mathcal{S}^4\mathbb{C}^2 \cong \mathbb{C}^5$ with a real structure, and $\mathcal{S}^3(\mathcal{S}^4\mathbb{C}^2)$ contains only one copy of $\mathbb{C}$, generated in fact by $\mathbb{T}$. The recipe to tackle a general G-structure prescribes first to determine the form(s) with isotropy $G = SO(3)$, the lack of which makes this whole matter quite complicated.

Definition 2.1. [5] An $SO(3)$ structure is said *nearly integrable* (NI) when

$$(\nabla_X\mathbb{T})(X, X, X) = 0 \quad (4)$$

for all vector fields $X \in \mathbb{R}^5$.

This relation is – at least formally – similar to that defining a nearly Kähler structure on an even dimensional manifold: $(\nabla_X J)X = 0$. Just as the almost complex structure J is a Killing form [17] there, (4) is saying that $\mathbb{T}$ is a symmetric Killing 3-tensor.

The theoretical interest of such a structure lies in the fact that it admits a uniquely defined *characteristic* connection

$$\tilde{\nabla} = \nabla - \tfrac{1}{2}T \quad (5)$$

with torsion T a three-form, in the sense of (1). We finally ought to remind that the existence of a nearly integrable $SO(3)$ structure is actually the same [5] as having *skew-symmetric torsion T*. Among the simplest examples of G-invariant metric connections with anti-symmetric torsion one counts those of naturally reductive spaces. These are homogeneous spaces with a reductive decomposition $\mathfrak{h} \oplus \mathfrak{p}$ possessing a connection $\tilde{\nabla} = \nabla + [\cdot, \cdot]_\mathfrak{p}$ and characteristic torsion tensor $-g([\cdot, \cdot]_\mathfrak{p}, \cdot)$, skew-symmetric by very definition. Because the latter and the induced curvature are parallel, naturally reductive spaces generalise Riemannian symmetric spaces. For a G-structure though, there is no Lie group acting transitively in general. Besides, there are sound reasons to believe that Riemannian manifolds M^n endowed with metric connections $\tilde{\nabla}$ with skew torsion are useful in string and supergravity theories, see [10, 1] and references.

The space $\Lambda^3\mathbb{R}^5$ is isomorphic via the Hodge operator to that of two-forms, and the latter decomposes under $SO(3)$ into the direct sum of the irreducible modules

$$\Lambda_3^2 = span\{E_1, E_2, E_3\} \cong \mathfrak{so}(3), \qquad \Lambda_7^2 = \left(\Lambda_3^2\right)^\perp. \quad (6)$$

The orthogonal complement is taken with respect to the pairing

$$\langle\alpha, \beta\rangle = *(\hat{\mathbb{T}}(\alpha) \wedge *\beta), \quad (7)$$

via the endomorphism $\hat{\mathbb{T}}(e^i \wedge e^k) = 4\sum_{j,l,m} \mathbb{T}_{ijm}\mathbb{T}_{klm}e^j \wedge e^l$ of $\Lambda^2(\mathbb{R}^5)^*$ from [5]. Wedge products of basis vectors will be henceforth expressed by juxtaposed indexes.

The forms E_i are given in matrix form by

$$
E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\sqrt{3} & 0 & 0 & -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},
$$

or

$$E_1 = \sqrt{3}e^{15} + e^{23} + e^{45}, \quad E_2 = \sqrt{3}e^{13} + e^{25} + e^{34}, \quad E_3 = 2e^{24} + e^{35} \qquad (8)$$

when seen as elements of $\Lambda^2(\mathbb{R}^5)^*$.

We shall say that *a NI structure* $\mathbb{T}$, *the corresponding torsion* T *or the manifold carrying* $\mathbb{T}$ *has/is of type* $\mathcal{W}$ if $*T$ belongs to the subspace $\mathcal{W} \subseteq \Lambda^2$. The type can thus be $\Lambda^2_3 \oplus \Lambda^2_7$ generically, Λ^2_3, Λ^2_7 which we shall both refer to as 'pure' type, or $\{0\}$ i.e. the torsion-free case.

3.   The characteristic connection of a 5-dimensional Lie group

We begin by taking a generic Lie group of dimension five and write its structure equations

$$
\begin{cases}
de^1 & = & b_1 e^{12} + \ldots + b_{10}e^{45} \\
de^2 & = & b_{11} e^{12} + \ldots + b_{20}e^{45} \\
de^3 & = & \ldots \\
de^4 & = & \ldots \\
de^5 & = & b_{41} e^{12} + \ldots + b_{50}e^{45}.
\end{cases}
\qquad (9)
$$

Taking $(e_1, \ldots e_5)$ to be the adapted frame means fixing the $SO(3)$ structure and varying the Lie algebra equations in terms of real numbers b_α, $\alpha = 1, \ldots, 50$.

Polarising the expression (3) gives the non-zero components of $\mathbb{T}$

$$\mathbb{T}_{111} = -1, \quad \mathbb{T}_{122} = 1, \quad \mathbb{T}_{144} = 1, \quad \mathbb{T}_{133} = -\tfrac{1}{2}, \quad \mathbb{T}_{155} = -\tfrac{1}{2},$$
$$\mathbb{T}_{433} = -\tfrac{\sqrt{3}}{2}, \quad \mathbb{T}_{455} = \tfrac{\sqrt{3}}{2}, \quad \mathbb{T}_{235} = \tfrac{\sqrt{3}}{2}.$$

Lemma 3.1.   *The $SO(3)$-tensor $\mathbb{T}$ is NI if and only if the structure equations satisfy the set of linear relations (10) below.*

Proof.   Allow X to be a generic linear combination of the adapted basis (e_i) of the Lie algebra $\mathfrak{l}$ of L, i.e. $X = \sum_1^5 \lambda_j e_j$. Imposing (4) means solving the equation $(\nabla_X \mathbb{T})(X, X, X) = 0$ for all values of the λ_j's.

Consider, heuristically, $X = \lambda e_3 + \mu e_4$ and let $Y = \nabla_X X$. Then (4) becomes

$$
\begin{aligned}
0 = & \tfrac{1}{3}(\nabla_X \mathbb{T})(X, X, X) = \mathbb{T}(Y, X, X) = \lambda^2 \mathbb{T}(Y, e_3, e_3) + 2\lambda\mu\, \mathbb{T}(Y, e_3, e_4) \\
+ & \mu^2 \mathbb{T}(Y, e_4, e_4) = \lambda^2(-\tfrac{1}{2}g(Y, e_1) - \tfrac{\sqrt{3}}{2}g(Y, e_4)) + 2\lambda\mu\, \tfrac{\sqrt{3}}{2}g(Y, e_3) + \mu^2 g(Y, e_1) \\
= & \lambda^2\left(\tfrac{1}{2}g([X, e_1], X) + \tfrac{\sqrt{3}}{2}g([X, e_4], X)\right) - \sqrt{3}\lambda\mu\, g([X, e_3], X) - \mu^2 g([X, e_1], X),
\end{aligned}
$$

eventually resulting in a fourth order polynomial $\sum_{i+j=4} \lambda^i \mu^j P_{ij}$. Given that λ, μ are arbitrary, the conditions translate into $P_{ij} = 0$ for all $i, j = 0, \ldots, 4$. Explicitly

$$P_{40} = \tfrac{1}{2}(b_{22} - \sqrt{3}b_{28}) = 0, \quad P_{04} = -b_{33} = 0, \quad P_{22} = \tfrac{1}{2}b_{33} - b_{22} + \sqrt{3}b_{28} = 0$$
$$P_{31} = \tfrac{1}{2}(b_{32} + b_{23} - \sqrt{3}b_{38}) = 0, \quad P_{13} = -(b_{23} + b_{32}) + \sqrt{3}b_{38} = 0.$$

The story is similar when all λ_i's are present. Passing from linear combinations X of p elements to $p+1$ is not that more involved, since most of the new information is trivial by previous relations, justifying the fact that the system is linear. The set of conditions imposed by (4) hence reads:

$$\left.\begin{aligned}
& b_1 = b_{11} = b_3 = b_{33} = 0, \ b_{20} = -b_{37}, \ b_{13} + b_{31} = 0, \\
& b_2 = \sqrt{3}(b_{23} + b_8), \ b_4 = \sqrt{3}(-b_{43} + b_{10}), \ b_{22} = \sqrt{3}b_{28}, \ b_{44} = \sqrt{3}b_{50}, \\
& b_{21}+b_{12}=\sqrt{3}b_{17}, b_{14}+b_{41}=\sqrt{3}b_{15}, b_4 = \sqrt{3}(b_5 - b_{21}), b_2=\sqrt{3}(b_7 - b_{41}), \\
& 2b_{22} + \sqrt{3}b_{16} = 2\sqrt{3}(b_{19} + b_{27}), \ 2b_{44} - \sqrt{3}b_{16} = 2\sqrt{3}(b_{45} - b_{19}), \\
& 2b_{29} + b_{17} = b_{26} + b_{18}, \ 2b_{29} + b_{40} = b_{26} + b_{35}, \ 2b_{49} - b_{15} - b_{37} = b_{46}, \\
& b_{28} + b_{50} = b_{45} + b_{27}, \ b_{24} + b_{42}=\sqrt{3}(b_{25} + b_{47}), \ b_{48} - b_{30} = b_{47} - b_{25}, \\
& 2(b_{24} + b_9) = 2\sqrt{3}b_{25} + b_{13} + b_6, \ 2(b_{42} - b_9) = 2\sqrt{3}b_{47} - (b_{13} + b_6), \\
& 2(b_{39} + b_{30} - b_{25})=b_{36}, \ b_{35} + b_{17} = b_{40} + b_{18}, \ 2(b_{48} + b_{39} - b_{47})=b_{36}, \\
& b_{38} = -b_{15}, \ \sqrt{3}(b_{40} + b_{18} - b_{35}) = b_{21} + b_{12}, \ b_{32} = -b_{23} - \sqrt{3}b_{15},
\end{aligned}\right\} \quad (10)$$

which is easily handled by computer programs.   ■

Further constraints on the coefficients in (9) derive from $d^2 = 0$, but due to computational complexity we reserve the Jacobi identity to when strictly necessary, typically at the very end of the classifying process.

Since $\tilde{\nabla}$ preserves the metric and the tensor $\mathbb{T}$, it is clear that

$$g(\tilde{\nabla}_X Y, Z) + g(\tilde{\nabla}_X Z, Y) = 0,$$
$$\mathbb{T}(\tilde{\nabla}_X Y, Z, W) + \mathbb{T}(\tilde{\nabla}_X Z, Y, W) + \mathbb{T}(\tilde{\nabla}_X W, Y, Z) = 0.$$

The characteristic torsion (5) of the structure is then given by

$$\begin{aligned}
T = \ & (b_{43} - b_{10} + b_{12})e^{123} - b_6 e^{124} + (\sqrt{3}b_{15} - b_7)e^{125} + \\
& (\sqrt{3}b_{15} - b_8)e^{134} + (b_{24} - \sqrt{3}b_{47} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}b_6)e^{135} + \\
& (\sqrt{3}b_{40} - b_{10})e^{145} + (2b_{29} - b_{17} - b_{35})e^{234} + (b_{28} - b_{50} - b_{19})e^{235} + \\
& (b_{37} - b_{15} - 2b_{49})e^{245} + (\tfrac{\sqrt{3}}{6}b_{13} + \tfrac{\sqrt{3}}{6}b_6 - \tfrac{\sqrt{3}}{3}b_9 - \tfrac{\sqrt{3}}{3}b_{24} + b_{39} - b_{47})e^{345}.
\end{aligned}$$

The $SO(3)$-connection $\tilde{\nabla}$ can be thus reconstructed, and is written here as an $\mathfrak{so}(3)$-valued form

$$\Gamma = \gamma^1 E_1 + \gamma^2 E_2 + \gamma^3 E_3$$

where

$$\gamma^1 = \Gamma_3^2, \qquad \gamma^2 = \Gamma_5^2, \qquad \gamma^3 = \Gamma_5^3 \qquad\qquad (11)$$

and

$$\Gamma_5^2 = \tfrac{1}{\sqrt{3}}\Gamma_3^1 = \Gamma_4^3 = -(b_{23}+b_8)e^1 - b_{17}e^2 - b_{28}e^3 + b_{15}e^4 - b_{47}e^5,$$

$$\Gamma_3^2 = \tfrac{1}{\sqrt{3}}\Gamma_5^1 = \Gamma_5^4 = (b_{43}-b_{10})e^1 - b_{15}e^2 + \tfrac{1}{\sqrt{3}}(\tfrac{1}{2}b_{13}-b_9+\tfrac{1}{2}b_6-b_{24})e^3 - b_{40}e^4 - b_{50}e^5,$$

$$\Gamma_5^3 = \tfrac{1}{2}\Gamma_4^2 = -\tfrac{1}{2}(b_6+b_{13})e^1 + (b_{45}-b_{50}-b_{19})e^2 - b_{29}e^3 + (b_{47}-b_{39}-b_{48})e^4 - b_{49}e^5$$

are the connection 1-forms. All Γ_j^i's vanish precisely when

$$b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{40} = b_{49} = 0,$$

$$b_{28} = b_{47} = b_{50} = 0, b_{45} = b_{19}, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9. \tag{12}$$

Now the curvature $K = r^1 E_1 + r^2 E_2 + r^3 E_3 \in \Lambda^2 \mathfrak{l}^* \otimes \mathfrak{so}(3)$ is determined by

$$r^1 = d\gamma^1 + \gamma^2 \wedge \gamma^3, \quad r^2 = d\gamma^2 + \gamma^3 \wedge \gamma^1, \quad r^3 = d\gamma^3 + \gamma^1 \wedge \gamma^2,$$

and $r^1 = K_3^2, r^2 = K_5^2, r^3 = K_5^3$ are then non-zero components of K. If $T \equiv 0$ then K is determined by a constant $F \in \mathbb{R}$, since [5]

$$r^j = F E_j \qquad j = 1, 2, 3$$

and the $SO(3)$ structure is locally isometric to that of a symmetric space $Q/SO(3)$, with

$$Q = \begin{cases} SO(3) \times_\rho \mathbb{R}^5 & \text{if } F = 0 \\ SL(3, \mathbb{R}) & \text{if } F < 0 \\ SU(3) & \text{if } F > 0. \end{cases} \tag{13}$$

A purely formal but still suggestive relation between near integrability and lower dimensional geometry is the fact that the number of parameters b_α upon which the Riemannian connection of a NI structure depends, is precisely half of the total, exactly as for self-dual connections on a four-manifold [14]. Whether this has to do with the universal covering $SU(2) \to SO(3)$ remains to be seen – though the fact that the Berger sphere $SO(5)/SO(3)$ is diffeomorphic to $Sp(2)/SU(2)$ is a clear indication – and will be dealt with elsewhere by the authors. But here is an

Example. Consider the Lie algebra $\mathfrak{l}$ with structure equations

$$\begin{cases} de^1 = b_7 e^{25} - b_{37}e^{35} - b_{45}e^{45}, \\ de^2 = -b_7 e^{15} + b_{45}e^{35} - b_{37}e^{45}, \\ de^3 = b_{37}e^{15} - b_{45}e^{25} + \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{45}, \\ de^4 = b_{45}e^{15} + b_{37}e^{25} - \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{35}, \\ de^5 = b_7 e^{12} - b_{37}e^{24} - b_{45}e^{14} - b_{37}e^{13} + b_{45}e^{23} + \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{34}, \end{cases} \tag{14}$$

and the NI structure whose three-form T is given by

$$T = -b_7 e^{125} + b_{37}e^{135} + b_{45}e^{145} - b_{45}e^{235} + b_{37}e^{245} - \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{345}.$$

First of all this torsion is harmonic. But more crucially, the associated Lie group L is thus endowed with an $SU(2)$ structure defined by

$$\alpha = e^5, \quad \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23}.$$

To be precise, this is actually hypo [8] as

$$d\omega_1 = 0, \quad d(\omega_2 \wedge \alpha) = d(\omega_3 \wedge \alpha) = 0.$$

Examples of this sort are noteworthy because they induce local integrable $SU(3)$ structures on $L \times \mathbb{R}$. Moreover, they yield half-flat geometries, which also evolve in one dimension higher, but to holonomy G_2. In fact, (14) is part of a more general family, as shown by

Proposition 3.2. *Let L be a Lie group whose Maurer-Cartan structure (9) satisfies (12) plus $b_6 = b_8 = b_{12} = b_{35} = 0, b_{10} = -b_{45}, b_9 = -b_{37}, b_{39} = (b_{37}^2 + b_{45}^2)/b_7$. Then for some closed 1-form e^6, the invariant $SU(3)$ structure on $L \times \mathbb{R}$*

$$\omega = \omega_3 + \alpha \wedge e^6, \qquad \psi_+ = \omega_2 \wedge \alpha - \omega_1 \wedge e^6$$

is half-flat.

Consequently $\omega \wedge e^7 + \psi_+$ is an integrable G_2 structure on a local product $L \times \mathbb{R} \times \mathbb{R}$ for a closed 1-form e^7.

Proof. Almost immediate, once one recalls that half-flat is a fancy name for the closure of ψ_+ and $\omega \wedge \omega$. ∎

4. Classification

The action of the endomorphisms $\mathbb{T}_X = \mathbb{T}(X, \cdot, \cdot)$ indicates that whilst $\mathbb{T}_{e_1}, \mathbb{T}_{e_2}$ and $\mathbb{T}_{e_4}$ preserve the decomposition

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p} = span\{e_1, e_2, e_4\} \oplus \ span\{e_3, e_5\}, \tag{15}$$

$\mathbb{T}_{e_3}, \mathbb{T}_{e_5}$ on the contrary do not. This reflects the irreducibility of the rotational action.

We shall capture the Lie algebras $\mathfrak{l}$ for which the $SO(3)$-connection satisfies the conditions:

$$\begin{aligned} \tilde{\nabla}_X \mathfrak{h} \subseteq \mathfrak{h}, \quad \tilde{\nabla}_X \mathfrak{p} \subseteq \mathfrak{p}, \quad \forall X \in \mathfrak{h} \\ \tilde{\nabla}_Y \mathfrak{h} \subseteq \mathfrak{p}, \quad \tilde{\nabla}_Y \mathfrak{p} \subseteq \mathfrak{h}, \quad \forall Y \in \mathfrak{p}. \end{aligned} \tag{16}$$

Otherwise said, $\mathfrak{h}$-derivatives preserve the splitting (15), but differentiation in the $\mathfrak{p}$-direction swaps the subspaces. The geometric motivation of this condition are readily explained.

In order to classify 5-dimensional Lie groups L with an invariant $SO(3)$ structure $(g, \mathbb{T})$ whose characteristic connection $\tilde{\nabla}$ satisfies (16), it is quite useful to distinguish whether $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair or not. A Lie algebra $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{q}$, or better $(\mathfrak{l}, \mathfrak{m})$ is called a *symmetric pair* when

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{q}] \subseteq \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{m}.$$

If H is the connected Lie subgroup of L with Lie algebra $\mathfrak{h}$, we have a Riemannian fibration of Lie groups

$$H \longrightarrow L \longrightarrow L/H \tag{17}$$

whose fibres H are totally geodesic submanifolds of L and whose base is a 2-dimensional symmetric space. Therefore L/H is a finite quotient of $\mathbb{R}^2$, the Riemann sphere or the Poincaré disc according to the curvature. Since $\mathfrak{p}$ is indeed the tangent 2-plane spanned by e_3, e_5, the curvature of L/H is given by

$$k(\mathfrak{p}) = -g(e_5, [[e_3, e_5]_{\mathfrak{h}}, e_3]_{\mathfrak{p}}) = b_9(\tfrac{1}{2}b_6 + \tfrac{1}{2}b_{13} - b_9 - \sqrt{3}b_{47}) - b_{19}b_{45} + b_{39}b_{48}, \quad (18)$$

and its sign decides which of the space forms one is looking at.

Lemma 4.1. *If $\mathfrak{p}$ is $ad(\mathfrak{h})$-invariant then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{l}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$.*

Proof. If $\mathfrak{p}$ is $ad(\mathfrak{h})$-invariant, the structure coefficients

$$b_2, b_4, b_5, b_7, b_8, b_{10}, \ b_{12}, b_{14}, b_{15}, b_{17}, b_{18}, \ b_{32}, b_{34}, b_{35}, b_{37}, b_{38}, b_{40}$$

all vanish. As a consequence of (10) also $b_{23}, b_{43}, b_{21}, b_{41}$ are zero. So the Maurer–Cartan system (9) simplifies to

$$\begin{cases}
de^1 = & b_6 e^{24} + b_9 e^{35} \\
de^2 = & b_{13} e^{14} + 2(b_{50} + b_{19} - b_{45})e^{24} + b_{19} e^{35} \\
de^3 = & \sqrt{3}b_{28} e^{13} + b_{24} e^{15} + \tfrac{\sqrt{3}}{3}(b_9 + b_{24} - \tfrac{1}{2}(b_{13} + b_6))e^{23} + 2b_{29} e^{24} \\
& + (b_{50} + b_{28} - b_{45})e^{25} + b_{28} e^{34} + b_{29} e^{35} \\
& + \tfrac{\sqrt{3}}{3}(b_9 + b_{24} - \tfrac{1}{2}(b_{13} + b_6) + \sqrt{3}b_{48} - \sqrt{3}b_{47})e^{45} \\
de^4 = & -b_{13} e^{12} + 2(b_{39} + b_{48} - b_{47})e^{24} + b_{39} e^{35} \\
de^5 = & (b_9 + \sqrt{3}b_{47} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}b_6)e^{13} + \sqrt{3}b_{50} e^{15} + b_{45} e^{23} + 2b_{49} e^{24} + b_{47} e^{25} \\
& + b_{48} e^{34} + b_{49} e^{35} + b_{50} e^{45}.
\end{cases}$$

Using the vanishing of the coefficients of e^{234}, e^{245} in $d(de^i), i = 1, 2, 4$, and of $e^{124}, e^{135}, e^{235}, e^{345}$ for $i = 3, 5$, then either

- $b_{29} = b_{49} = 0$ (implying $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$), or

- $b_{29}^2 + b_{49}^2 \neq 0$ and $b_9 = b_{19} = b_{39} = 0, b_{13} = -b_6, b_{45} = -b_{28}, b_{48} = b_{47}$, $b_{50} = -b_{28}$ (which force $\mathfrak{p}$ to become a Lie subalgebra of $\mathfrak{l}$).

But in the latter case, the Jacobi equation ends up annihilating both b_{49} and b_{29}, contradicting the assumption. ∎

Corollary 4.2. *Demanding $\mathfrak{p}$ to be $ad(\mathfrak{h})$-invariant is equivalent to the pair $(\mathfrak{l}, \mathfrak{h})$ being symmetric.*

Proof. In terms of the structure equations $\tilde{\nabla}$ satisfies (16) if and only if

$$b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{37} = b_{40} = b_{49} = 0.$$

If, in addition, $b_7 = b_8 = b_{10} = b_{12} = b_{35} = 0$ then $[\mathfrak{p}, \mathfrak{h}] \subseteq \mathfrak{p}$ and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{l}$. ∎

The algebraic structures appearing later will often be *solvable* Lie algebras $\mathfrak{g}$, whose derived series

$$\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \ldots \supseteq \mathfrak{g}^q = 0$$

collapses at some stage $q \in \mathbb{N}$ called the step length. Each ideal is defined by bracketing the preceding one in the sequence with itself, $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$, beginning from $\mathfrak{g}^0 = \mathfrak{g}$. The first subspace $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is the commutator and such a $\mathfrak{g}$ is referred to as q-step (solvable).

Example 4.3. There is only one solvable (non-nilpotent) Lie algebra of dimension two, modulo isomorphisms:
$$\mathfrak{s}_2 = (0, e^{12}),$$
and similarly, only one 3-dimensional 2-step one, namely
$$\mathfrak{s}_3 = (0, 0, e^{13}).$$
This notation for Lie algebras expresses the bracket of $\mathfrak{g}$ via the exterior differential of $\mathfrak{g}^*$, so in the latter example one should understand the basis e_1, e_2, e_3 of $\mathfrak{s}_3$ to satisfy $de^1 = de^2 = 0, de^3 = e^{13}$, or equivalently $[e_1, e_2] = [e_2, e_3] = 0, [e_1, e_3] = -e_3$.

Solvable Lie algebras and groups are intimately linked to non-positive curvature, as predicted by Alekseevskiĭ [3]. This phenomenon manifests itself patently in the coming sections, goal of which is to prove

Theorem 4.4. *If the characteristic connection behaves as in (16), the torsion is always coclosed*
$$d*T = 0.$$

Moreover
a) if $(\mathfrak{l}, \mathfrak{h})$ is symmetric, $\mathfrak{l}$ is either solvable or isomorphic to

$$\mathfrak{l}_1 = \mathfrak{so}(3) \oplus \mathbb{R}^2$$
$$\mathfrak{l}_2 = (2e^{12}, e^{14}, e^{15} - e^{23}, 2e^{24}, e^{25} + e^{34}),$$
$$\mathfrak{l}_3 = \mathfrak{so}(3) \oplus \mathfrak{s}_2$$

b) If $(\mathfrak{l}, \mathfrak{h})$ is not symmetric instead, then $\tilde{\nabla} \equiv 0$ and $\mathfrak{l}$ is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$.

The appearance of algebras isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ in distinct contexts depends upon the underlying $SO(3)$ structure, which changes in *a)* and *b)*. The proof of this statement is scattered over sections 5. and 7.

The type of $SO(3)$ geometry is determined by the presence of intrinsic torsion in the modules (6). Because the two-forms E_i's are eigenvectors of the endomorphism $\hat{\mathbb{T}}$ of (7), one has that $*T \in \Lambda^2_7$ if and only if

$$T \wedge E_i = 0, \quad i = 1, 2, 3. \tag{19}$$

The Lie algebras of type Λ^2_7 are therefore those for which this linear system in the b_α's holds. They will be highlighted in the rest of the paper.

To conclude the section we recall two basic equations [13] for general metric connections with torsion $\tilde{\nabla}$. One of these relates the covariant and exterior derivatives of T

$$dT(x,y,z,w) = \mathfrak{S}_{x,y,z}(\tilde{\nabla}_x T)(y,z,w) - (\tilde{\nabla}_w T)(x,y,z) + 2\sigma_T(x,y,z,w), \quad (20)$$

where $\mathfrak{S}$ is the cyclic sum. The four-form

$$2\sigma_T = \sum_1^5 (e_i \lrcorner T)^2 \quad (21)$$

also shows up in the Bianchi identity for the curvature $\tilde{R}$ of $\tilde{\nabla}$

$$\mathfrak{S}_{x,y,z}\tilde{R}(x,y,z,w) = dT(x,y,z,w) - \sigma_T(x,y,z,w) + (\tilde{\nabla}_w T)(x,y,z).$$

Corollary 4.5. [13] *When the torsion is $\tilde{\nabla}$-parallel then $dT = 2\sigma_T$.*

If additionally the manifold happens to be flat $\tilde{R} \equiv 0$, then it has closed torsion. ∎

For other manifestations and the significance of σ_T see [2].

5. $(\mathfrak{l}, \mathfrak{h})$ symmetric pair

The case when $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair corresponds to the Levi-Civita connection satisfying a bunch of relations similar to (16):

Proposition 5.1. *The Levi-Civita connection ∇ satisfies*

$$\nabla_X \mathfrak{h} \subseteq \mathfrak{h}, \quad \nabla_X \mathfrak{p} \subseteq \mathfrak{p}, \quad \forall X \in \mathfrak{h}$$
$$\nabla_Y \mathfrak{h} \subseteq \mathfrak{p}, \quad \nabla_X \mathfrak{p} \subseteq \mathfrak{h}, \quad \forall Y \in \mathfrak{p}$$

if and only if $\mathfrak{p}$ is $\mathrm{ad}(\mathfrak{h})$-invariant and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{l}$.

Proof. The conditions are equivalent to the vanishing of the coefficients b_7, b_8, b_{10}, b_{12}, b_{15}, b_{17}, b_{23}, b_{29}, b_{35}, b_{37}, b_{40}, b_{43}, b_{49} in (9). ∎

The first relation is an old acquaintance, for $0 = (\nabla_{\mathfrak{h}}\mathfrak{h})_{\mathfrak{p}}$ is the second fundamental form of $H \subset L$.

Imposing (16) reduces the characteristic connection to the treatable

$$
\left.
\begin{aligned}
\tilde{\nabla}e_1 &= -\sqrt{3}(b_{28}e_3 + Ae_5) \otimes e^3 + \sqrt{3}(b_{47}e_3 + b_{50}e_5) \otimes e^5 \\
\tilde{\nabla}e_2 &= 2Be_4 \otimes e^1 + 2Ce_4 \otimes e^2 + (Ae_3 + b_{28}e_5) \otimes e^3 + 2De_4 \otimes e^4 + \\
&\quad (b_{50}e_3 + b_{47}e_5) \otimes e^5, \\
\tilde{\nabla}e_3 &= Be_5 \otimes e^1 + Ce_5 \otimes e^2 + \left(b_{28}(e_4 - \sqrt{3}e_1) - Ae_2\right) \otimes e^3 + De_5 \otimes e^4 + \\
&\quad \left(b_{47}(e_4 - \sqrt{3}e_1) - b_{50}e_2\right) \otimes e^5, \\
\tilde{\nabla}e_4 &= -2Be_2 \otimes e^1 - 2Ce_2 \otimes e^2 + (-b_{28}e_3 + Ae_5) \otimes e^3 - 2De_2 \otimes e^4 + \\
&\quad (b_{50}e_5 - b_{47}e_3) \otimes e^5, \\
\tilde{\nabla}e_5 &= -Be_3 \otimes e^1 - Ce_3 \otimes e^2 - \left(b_{28}e_2 + A(e_4 - \sqrt{3}e_1)\right) \otimes e^3 - \\
&\quad De_3 \otimes e^4 - \left(b_{47}e_2 + b_{50}(e_4 + \sqrt{3}e_1)\right) \otimes e^5,
\end{aligned}
\right\} \quad (22)
$$

where capitals are merely used to abbreviate the constants

$$B = \tfrac{1}{2}(b_6 + b_{13}), \qquad A = \tfrac{\sqrt{3}}{3}(b_9 + b_{24} - B),$$
$$C = b_{50} - b_{45} + b_{19}, \quad D = b_{39} + b_{48} - b_{47}.$$

Corollary 5.2.   *The splitting $\mathfrak{h} \oplus \mathfrak{p}$ is curvature-invariant:*

$$R_{XY}(\mathfrak{h}) \subseteq \mathfrak{h}, \ R_{XY}(\mathfrak{p}) \subseteq \mathfrak{p} \qquad \forall X, Y \in \mathfrak{l}.$$

Proof.   From (16) one induces easily that

$$R_{\mathfrak{hh}}\mathfrak{h}, \ R_{\mathfrak{hp}}\mathfrak{h}, \ R_{\mathfrak{pp}}\mathfrak{h} \in h,$$
$$R_{\mathfrak{hp}}\mathfrak{p}, \ R_{\mathfrak{hh}}\mathfrak{p}, \ R_{\mathfrak{pp}}\mathfrak{p} \in \mathfrak{p}.$$

Using the symmetries of R, this means $R_{\mathfrak{hhhp}} = R_{\mathfrak{hppp}} = 0$, i.e. whenever any three indexes in the components R_{ijkl} denote vectors in $\mathfrak{p}$ (or $\mathfrak{h}$), the curvature is zero. In other words the curvature 2-form preserves (15). ∎

As a consequence, the $SO(3)$-curvature $\tilde{R}$ fulfills similar conditions $\tilde{R}_{\mathfrak{hhhp}} = \tilde{R}_{\mathfrak{hphh}} = \tilde{R}_{\mathfrak{hppp}} = \tilde{R}_{\mathfrak{pphp}} = 0$, despite it no longer being a symmetric endomorphism of $\Lambda^2 \mathfrak{l}$.

The vector space decomposition (15) also transfers to the algebraic level as follows

Corollary 5.3.   *When $\mathfrak{p}$ is Abelian $\mathfrak{l}$ is a semidirect sum*

$$\mathfrak{l} = \mathfrak{h} \oplus_\alpha \mathfrak{p},$$

where $\alpha : \mathfrak{h} \to \mathfrak{Der}(\mathfrak{p})$ is a homomorphism of Lie algebras.

Proof.   This descends from the $ad(\mathfrak{h})$-invariance of $\mathfrak{p}$. ∎

For better handling the discussion now divides into the mutually exclusive cases

$$T = 0, \qquad dT = 0 \ (\text{with } T \neq 0, \text{ called 'strong'}), \quad \text{and} \quad dT \neq 0,$$

the simplest being the torsion-free one.

Proposition 5.4.   *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair with $T \equiv 0$. Then*

  (i) *[5] the $SO(3)$ structure of L is locally isometric to that of a symmetric space of non-positive curvature, hence either $\mathbb{R}^5$ or $SL(3, \mathbb{R})/SO(3)$;*

 (ii) *$\mathfrak{l}$ is solvable;*

(iii) *the symmetric space L/H is flat.*

Proof.   The structure equations are

$$\begin{cases} de^1 = b_9 e^{35} \\ de^2 = b_{13}e^{14} + 2(b_{50} + b_{19} - b_{45})e^{24} + (b_{28} - b_{50})e^{35} \\ de^3 = \sqrt{3}b_{28}e^{13} + (\frac{1}{2}b_{13} + \sqrt{3}b_{47})e^{15} + (-b_{47} + \frac{\sqrt{3}}{3}b_9)e^{23} + (b_{50} + b_{28} - b_{45})e^{25} + \\ \qquad b_{28}e^{34} + (\frac{\sqrt{3}}{3}b_9 + b_{48})e^{45} \\ de^4 = -b_{13}e^{12} + 2(b_{39} + b_{48} - b_{47})e^{24} + b_{39}e^{35} \\ de^5 = (b_9 + 2\sqrt{3}b_{47} - b_{24})e^{13} + \sqrt{3}b_{50}e^{15} + b_{45}e^{23} + b_{47}e^{25} + b_{48}e^{34} + b_{50}e^{45}. \end{cases}$$

The inspection of the Jacobi identity tells that there are four solutions for which $\nabla = \tilde{\nabla}$. Only the non-zero coefficients are indicated and serve to distinguish the Lie algebras:

1. $b_{28} = b_{50} = a \neq 0$

   $(0, 2ae^{24}, a(\sqrt{3}e^{13} + e^{34} + 2e^{25}), 0, a(\sqrt{3}e^{15} + e^{45}))$

2. $b_{28} = b_{50} = \frac{1}{2}(b_{45}^2 + b_{48}^2)/b_{45}, b_{45} = a \neq 0, b_{48} = b$

   $(0, \frac{-a^2+b^2}{a}e^{24}, be^{45} + \frac{a^2+b^2}{2a}(e^{34} + \sqrt{3}e^{13}) + \frac{b^2}{a}e^{25}, 2be^{24}, \frac{a^2+b^2}{2a}(e^{45} + \sqrt{3}e^{15}) + be^{34} + ae^{23})$.

3. $b_{50} = -\frac{1}{2}b_{19}, b_{28} = \frac{1}{2}b_{19}, b_{39} = 2b_{47}, b_{24} = \sqrt{3}b_{47}, b_{19} = a, b_{47} = b, a^2 + b^2 \neq 0$

   $(0, a(e^{24} + e^{35}), \frac{1}{2}a(\sqrt{3}e^{13} + e^{34}) + b(\sqrt{3}e^{15} + e^{23}), b(2e^{24} + 2e^{35}),$
   $b(\sqrt{3}e^{13} + e^{25}) - \frac{1}{2}a(\sqrt{3}e^{15} + e^{45}))$.

4. $b_{13} = 2b_{24} = a \neq 0$

   $(0, ae^{14}, \frac{1}{2}ae^{15}, -ae^{12}, -\frac{1}{2}ae^{13})$.

The first three instances are 3-step solvable Lie algebras with 3-dimensional commutator $\mathfrak{l}^1$ and $\mathfrak{h} \cong \mathbb{R} \oplus \mathfrak{s}_2$. The Lie algebra (1) is isomorphic to (2) with $b = 0$. The Lie algebra (4) is 2-step solvable with 4-dimensional commutator $\mathfrak{l}^1$ and $\mathfrak{h}$ Abelian. Instead $\mathfrak{p}$ is an Abelian subalgebra only for the Lie algebras (1), (2) and (4), so the curvature formula (18) implies sectional flatness. In the remaining case, where $\mathfrak{p}$ is not Abelian, we still have $k(\mathfrak{p}) = 0$. The constant F of (13) is negative for (1)-(3) and vanishes for (4), so [5] implies that the first three are isometric to $SL(3, \mathbb{C})/SO(3)$, the last to $\mathbb{R}^5$. ∎

The Lie algebras listed (in roman numerals) in the sequel have simpler structure equations than those generated in the proofs. They are attained by standard changes of bases – omitted not to bore the reader senseless – and may not be optimal: for example $\sqrt{3}e^1 + e^4$ could reasonably be a basic 1-form, as variously hinted by, e.g., (22). This depends essentially on the choice of representation (2).

Nevertheless, it should be all but clear that doing this will alter the $SO(3)$ structure rather dramatically, so retaining the original expressions, prior to any algebra isomorphism, allows to read off the geometry of interest. In order to truly distinguish algebras up to $SO(3)$ equivalence, Cartan-Kähler theory seems the only reasonable way. This was pursued in [5].

Lemma 5.5. *If* $(\mathfrak{l}, \mathfrak{h})$ *is a symmetric pair with strong torsion, then* $\mathfrak{l}$ *must be isomorphic to one of*

I. $(0, -2e^{24}, (e^{13} + e^{34}) + m(e^{15} + e^{23}), -2me^{24}, m(e^{13} + e^{25}) - (e^{15} + e^{45}))$

II. $(e^{24}, e^{14}, 0, e^{12}, 0)$

III. $(e^{35}, 0, e^{45} + e^{23}, e^{24}, -e^{25})$

IV. $(-ce^{35}, ae^{35}, ce^{15} - ae^{25} + be^{45}, -be^{35}, -ce^{13} + ae^{23} + be^{34})$,

where $m, a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0$. *The first two algebras give rise to a flat quotient* L/H, *whereas* $k(\mathfrak{p})$ *is negative for III and IV. The torsion is of pure type* Λ_7^2 *in the family of Lie algebras IV when* $c = 0$, *and always in I.*

Proof. Suppose that T be closed, and not identically zero. After imposing the Jacobi identity one reduces to four instances:

1. $b_{50} = -b_{28} = -a \neq 0, b_{24} = \sqrt{3}b_{47} = \sqrt{3}b$
   $(0, -2ae^{24}, a(\sqrt{3}e^{13} + e^{34}) + b(\sqrt{3}e^{15} + e^{23}), -2be^{24}, b(\sqrt{3}e^{13} + e^{25}) - a(\sqrt{3}e^{15} + e^{45}))$ with $T = 2(ae^2 + be^4)e^{35}$ and $k(\mathfrak{p}) = 0$. Example 6.3.3 in [5] is contained in this family of algebras by requiring $b \neq 0$.

2. $b_6 = -b_{13} = a \neq 0$
   $(-ae^{24}, ae^{14}, 0, -ae^{12}, 0)$ with $T = ae^{124}$ and $k(\mathfrak{p}) = 0$.

3. $b_{47} = -\frac{1}{3}\sqrt{3}b_9 = -a \neq 0$
   $(\sqrt{3}ae^{35}, 0, 2ae^{45} + ae^{23}, 2ae^{24}, -ae^{25})$ with $T = \sqrt{3}ae^{135}$ and $k(\mathfrak{p}) = -3a^2$.

4. $b_{45} = b_{19} = a, b_{39} = -b_{48} = -b, b_9 = -b_{24} = -c, a^2 + b^2 + c^2 \neq 0$
   $(-ce^{35}, ae^{35}, ce^{15} - ae^{25} + be^{45}, -be^{35}, -ce^{13} + ae^{23} + be^{34})$ with $T = (ce^1 - ae^2 + be^4)e^{35}$ and $k(\mathfrak{p}) = -a^2 - b^2 - c^2$.

The following observations guarantee that three cases are algebraically distinct. The first Lie algebra is 2-step solvable with $\dim \mathfrak{l}^1 = 3$, has an Abelian $\mathfrak{p}$ and $\mathfrak{h} \cong \mathbb{R} \oplus \mathfrak{s}_2$. The third one is 3-step solvable with 4-dimensional commutator. Numbers (2), (4) are both essentially $\mathfrak{so}(3) \oplus \mathbb{R}^2$, and account for $\mathfrak{l}_1$ in theorem 4.4. ∎

Lemma 5.6. *If $dT \neq 0$ and $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair then $\mathfrak{l}$ is one of*

I. $(e^{24}, 0, -e^{23}, e^{24}, e^{25} + e^{34})$

II. $(6e^{24}, 2e^{14}, e^{15} - \sqrt{3}e^{23} - \sqrt{3}e^{45}, -2e^{12}, -e^{13} + \sqrt{3}e^{25} + \sqrt{3}e^{34})$

III. $\big((2c + \sqrt{3}(b^2 + 1))e^{24} + ce^{35}, -2be^{24}, -\frac{1}{2}(b^2 + 1)e^{23} - be^{25} - b^2e^{45},$
      $(1 - b^2)e^{24}, be^{23} + \frac{1}{2}(b^2 + 1)e^{25} + e^{34}\big)$

IV. $\big(-2(b^2 + c^2)e^{24} - (b^2 + c^2 + 1)e^{35}, -2be^{24}, e^{15} - be^{25} + ce^{45}, 2ce^{24},$
      $-e^{13} + be^{23} + ce^{34}\big)$

up to isomorphisms, where $b, c \in \mathbb{R}$. The surface L/H has zero curvature in cases I, II and negative in the others.

Proof. If $dT \neq 0$ the following possibilities come out of $d^2 = 0$:

1. $b_6 = 2\sqrt{3}b_{47} = 2\sqrt{3}a \neq 0$;
   $(2\sqrt{3}ae^{24}, 0, -ae^{23} - 2ae^{45}, -2ae^{24}, ae^{25})$ and

$$T = -2\sqrt{3}a(e^{124} + e^{135}), \quad dT = 12a^2e^{2345}.$$

2. $b_{48} = 2b_{47}, b_6 = 2\sqrt{3}b_{47}, b_{47} = a \neq 0$
$(2\sqrt{3}ae^{24}, 0, -ae^{23}, 2ae^{24}, ae^{25} + 2ae^{34})$ and

$$T = -2\sqrt{3}a(e^{124} + e^{135}), \quad dT = 12a^2 e^{2345}.$$

3. $b_{24} = \frac{1}{2}b_{13}, b_{48} = \frac{\sqrt{3}}{2}b_{13}, b_6 = 3b_{13}, b_{47} = \frac{\sqrt{3}}{2}b_{13}, b_{13} = a \neq 0$
$(3ae^{24}, ae^{14}, \frac{1}{2}ae^{15} - \frac{\sqrt{3}}{2}ae^{23} - \frac{\sqrt{3}}{2}ae^{45}, -ae^{12}, -\frac{1}{2}ae^{13} + \frac{\sqrt{3}}{2}ae^{25} + \frac{\sqrt{3}}{2}ae^{34})$ and

$$T = -3a(e^{124} + e^{135}), \quad dT = 9a^2 e^{2345}.$$

4. $b_9 = a, b_{45} = b, b_{48} = c \neq 0, b_6 = \frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}, b_{47} = \frac{b^2+c^2}{2c}$
$\left(\frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{24}+ae^{35}, -2be^{24}, -\frac{b^2+c^2}{2c}e^{23}-be^{25}-\frac{b^2}{c}e^{45}, \frac{c^2-b^2}{c}e^{24}, be^{23}+\frac{b^2+c^2}{2c}e^{25} + ce^{34}\right)$ with

$$T = -\frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{124} - \frac{ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{135}, \quad dT = \frac{(2ac+\sqrt{3}b^2+\sqrt{3}c^2)^2}{c^2}e^{2345}$$

and $k(\mathfrak{p}) = -a^2$. The algebra $\mathfrak{l}$ is 3-step solvable. For $b \neq 0$ the dimension of $\mathfrak{l}^1$ is four. Further asking $a \neq 0$ recovers [5, Example 6.3.2, $\delta = 1$]. Taking $b = 0$ gives a 3-dimensional commutator instead.

5. $b_{24} = a \neq 0, b_{45} = b, b_{48} = c, b^2 + c^2 \neq 0, b_6 = -2\frac{b^2+c^2}{a}, b_9 = -\frac{b^2+c^2+a^2}{a}$,
$\left(-2\frac{b^2+c^2}{a}e^{24} - \frac{b^2+c^2+a^2}{a}e^{35}, -2be^{24}, ae^{15}-be^{25}+ce^{45}, 2ce^{24}, -ae^{13}+be^{23}+ce^{34}\right)$

$$T = 2\frac{b^2+c^2}{a}e^{124} + \frac{b^2+c^2+a^2}{a}e^{135}, \quad dT = 4\frac{(b^2+c^2)(a^2+b^2+c^2)}{a^2}e^{2345}.$$

This has 4-dimensional commutator such that $\mathfrak{l}^2 \cong \mathfrak{so}(3)$, and curvature $k(\mathfrak{p}) = -\left(\frac{b^2+c^2}{a} + 1\right)$. It corresponds to [5, Example 6.3.2] with $\delta = 0$.

The first two Lie algebras are 3-step solvable with 3-dimensional commutator and isomorphic. The third one is perfect, i.e. $\mathfrak{l}^1 = \mathfrak{l}$. For the Lie algebras (1)–(3), and for the family (4) with $a = 0$, $\mathfrak{p}$ is Abelian. The algebra $\mathfrak{h}$ is always $\mathfrak{s}_3$, except for (3) where $\mathfrak{h} \cong \mathfrak{so}(3)$. The first three instances fibre onto a Euclidean surface L/H, for the remaining ones $k(\mathfrak{p})$ is negative. Suitable normalisation of (3) and (5) give precisely $\mathfrak{l}_2, \mathfrak{l}_3$ of Theorem 4.4. ∎

The torsion has type Λ_3^2 for (4) with $-\sqrt{3}ac = b^2 + c^2$, and (5) with $a^2 = 3(b^2 + c^2)$. Up to $SO(3)$ equivalence these appear, though in disguise, in [5, Theorem 6.7].

Let $\widetilde{Ric}, Ric$ indicate the Ricci tensors of the characteristic and Levi-Civita connections. Although in general there is no reason for the former to be symmetric, this is what happens at present

Proposition 5.7. *If $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair, the torsion is always coclosed.*

Proof. One checks without effort that all Lie algebras in this section have a symmetric Ricci curvature $\widetilde{Ric}$. The general formula for a metric connection with skew torsion

$$\widetilde{Ric}(X,Y) = Ric(X,Y) + \tfrac{1}{2}*d*T(X,Y) + \tfrac{1}{4}\sum_{i,j}^{1,5} g(T(X,e_i),T(Y,e_j))$$

allows to conclude.   ■

The same reasoning holds in the non-symmetric case. Since the characteristic connection will be identically zero there, every $SO(3)$-curvature tensor vanishes (theorem 7.1), turning $*T$ into a closed form.

Lemmas 5.6, 5.5 can be modified and proved differently, for it is actually possible to spot the closure of dT a priori. A lengthy computation involving (20) shows that $dT = \sum(e_i \lrcorner T)^2$ everywhere. Now T can be either decomposable as $h \wedge e^{ij}$, for some $h \in \mathfrak{h}$, or proportional to $e^1 \wedge (e^{24} + e^{35})$. In the first case the contraction with any basis element is itself decomposable and simple, so $\sigma_T = 0$ eventually and the torsion must be strong by corollary 4.5. Alternatively, one obtains a non-zero differential e^{2345}.

6. Parallel forms

6.1. Parallel torsion.

Whenever in presence of a characteristic connection, it is relevant to consider whether it annihilates the torsion 3-form. Evidence of this very restrictive condition can be found in [7, 4]. In the former Cleyton and Swann prove that parallel torsion implies $d*T = 0$, which seems to pervade our classification.

Proposition 6.1. *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair. Then $\mathfrak{l}$ admits (non-zero) parallel characteristic torsion if and only if it is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$. For these $\mathfrak{l}$'s the characteristic connection is actually zero.*

Proof. As for lemma 5.5, when T is closed (and coclosed) the Lie algebras (2) and (4) fulfill $\tilde{\nabla}T = 0$. An obvious coordinate change proves them both Abelian extensions of $\mathfrak{so}(3)$. If $dT \neq 0$ instead (see lemma 5.6), the torsion is never parallel.   ■

6.2. Reduced holonomy.

The holonomy of the characteristic connection will reduce to a subgroup only in presence of a parallel vector, and because of lack of space inside $SO(3)$ there is one non-trivial case, that of the circle

$$Hol(\tilde{\nabla}) = U(1) \iff \exists \xi \in \mathfrak{l} : \tilde{\nabla}\xi = 0.$$

The next result concerns groups L other than the 5-torus:

Proposition 6.2. *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair whose Lie group L has characteristic holonomy $U(1) \subset SO(3)$ by way of a $\tilde{\nabla}$-parallel vector field ξ. Then $\mathfrak{l}$ is isomorphic to*

     1) $\mathfrak{so}(3) \oplus \mathbb{R}^2$

2) $(0, e^{14}, e^{15}, e^{12}, e^{13})$,

3) $(e^{24} + e^{13} + e^{35}, 0, e^{15}, e^{24}, e^{31})$

4) $(0, 0, 0, 0, e^{15} + e^{23})$

5) $\mathfrak{s}_3 \oplus \mathbb{R}^2$

6) $(0, 0, e^{15}, e^{53}, e^{34})$.

All others have full holonomy.

Proof.   First of all a parallel vector-to-be ξ must belong to $\mathfrak{h}$: the coefficients' relations imply that components in e_3, e_5 cannot appear. Disregarding situations where the connection itself is zero, imposing $\tilde{\nabla}\xi = 0$ yields either that

- $\mathfrak{p}$-derivatives are zero (precisely when $\xi = e_1$), or
- $\tilde{\nabla}_{\mathfrak{h}} \equiv 0$ (i.e. ξ has e_2-, e_4-components as well).

a) e_1 is parallel for these types of algebras:

$$(0, 0, b_{24}e^{15} + b_{48}e^{45}, -b_{48}e^{35}, -b_{24}e^{13} + b_{48}e^{34})$$
$$\big(b_9 e^{35}, (b_{19} - b_{45})e^{24} + b_{19}e^{35}, b_{24}e^{15} - b_{45}e^{25} + b_{48}e^{45}, -b_{48}e^{35},$$
$$-b_{24}e^{13} + b_{45}e^{23} + b_{48}e^{34}\big)\big(b_6 e^{24}, -b_6 e^{14}, 0, b_6 e^{12}, 0\big)$$
$$(0, b_{13}e^{14}, b_{24}e^{15}, -b_{13}e^{12}, -b_{24}e^{13}),$$
$$(b_6 e^{24} + b_9 e^{35}, -2b_{45}e^{24}, b_{24}e^{15} - b_{45}e^{25} + b_{48}e^{45}, 2b_{48}e^{24}, b_{24}e^{13} + b_{45}e^{23} + b_{48}e^{34})$$

with $b_{24} = -2(b_{45}^2 + b_{48}^2)/b_6$. The first three are isomorphic.

b) When the line ξ intersects the plane $span\{e_2, e_4\}$, the candidates for covariantly constant fields have the form

$$xe_1 + ye_2 + te_4,$$

subjected to $t^2 + y^2 = 3x^2 \neq 0$ (we omit other relations imposed on (22)). Apart from the Abelian algebra $\mathfrak{a}_5$, here are the solutions:

$$\big(b_9 e^{35}, 0, b_9 e^{51}, \tfrac{\sqrt{3}}{3}b_9 e^{35}, b_9 e^{13} - \tfrac{\sqrt{3}}{3}b_9 e^{34}\big), \quad \big(b_6 e^{24}, 0, b_6 e^{41}, 0, b_6 e^{12}\big),$$
$$\big(0, 0, b_{24}(e^{15} + \sqrt{3}e^{45}), 0, 0\big), \quad \big(0, 0, 0, 0, b_{45}(\sqrt{3}e^{15} + e^{23} + e^{45})\big),$$
$$(0, b_{19}e^{35}, b_{19}e^{52} + \sqrt{3}b_{39}e^{45}, b_{39}e^{35}, b_{19}e^{23} + b_{39}e^{34}).$$

Recognizing the isomorphisms is easy now.   ∎

7. $(\mathfrak{l}, \mathfrak{h})$ non-symmetric pair

All examples in this section will have $\tilde{\nabla} = 0$. In particular they have flat characteristic connection, so their structure coincides with [5, eqns (6.10)]. We will also determine which ones have torsion of type Λ_7^2.

Remember that the $SO(3)$-connection $\tilde{\nabla}$ satisfies (16) if and only if

$$b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{40} = b_{49} = 0.$$

One can assume at least one of $b_7, b_8, b_{10}, b_{12}, b_{35}, b_{37}$ to be non-zero to prevent $\mathfrak{p}$ from being $ad(\mathfrak{h})$-invariant and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{l}$. The structure equations then are

$$
\begin{cases}
de^1 &= -b_{12}e^{23} + b_6e^{24} + b_7e^{25} + b_8e^{34} + b_9e^{35} + b_{10}e^{45}, \\
de^2 &= b_{12}e^{13} + b_{13}e^{14} - b_7e^{15} + 2(b_{50} - b_{45} + b_{19})e^{24} + b_{35}e^{34} + b_{19}e^{35} - b_{37}e^{45}, \\
de^3 &= -b_{12}e^{12} + \sqrt{3}b_{28}e^{13} - b_8e^{14} + b_{24}e^{15} + \frac{\sqrt{3}}{3}(b_9 + b_{24} - \frac{1}{2}b_{13} - \frac{1}{2}b_6)e^{23} \\
     &\quad -b_{35}e^{24} + (b_{50} + b_{28} - b_{45})e^{25} + b_{28}e^{34} \\
     &\quad +(\frac{\sqrt{3}}{3}b_9 + \frac{\sqrt{3}}{3}b_{24} - \frac{\sqrt{3}}{6}b_{13} - \frac{\sqrt{3}}{6}b_6 + b_{48} - b_{47})e^{45}, \\
de^4 &= -b_{13}e^{12} + b_8e^{13} - b_{10}e^{15} + b_{35}e^{23} + 2(b_{39} + b_{48} - b_{47})e^{24} + b_{37}e^{25} + b_{39}e^{35}, \\
de^5 &= b_7e^{12} + (b_9 + \sqrt{3}b_{47} - \frac{1}{2}b_{13} - \frac{1}{2}b_6)e^{13} + b_{10}e^{14} + \sqrt{3}b_{50}e^{15} + b_{45}e^{23} \\
     &\quad -b_{37}e^{24} + b_{47}e^{25} + b_{48}e^{34} + b_{50}e^{45},
\end{cases}
$$

and the torsion reads

$$
\begin{aligned}
T &= b_{12}e^{123} - b_6e^{124} - b_7e^{125} - b_8e^{134} + (b_{24} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}b_6 - \sqrt{3}b_{47})e^{135} - \\
&\quad b_{10}e^{145} - b_{35}e^{234} + (b_{28} - b_{50} - b_{19}e^{235} + b_{37}e^{245} + \\
&\quad \tfrac{\sqrt{3}}{3}(\tfrac{1}{2}b_{13} + \tfrac{1}{2}b_6 - b_9 - b_{24} + b_{39} - b_{47})e^{345}.
\end{aligned}
$$

(23)

Theorem 7.1. *If* $(\mathfrak{l}, \mathfrak{h})$ *is non-symmetric, then* $\tilde{\nabla} = 0$ *and* T *is harmonic*

$$ dT = 0, \qquad d{*}T = 0. $$

In addition, $\mathfrak{l}$ *has 3-dimensional commutator* $\mathfrak{l}^1 = \mathfrak{l}^2$.

Proof. By imposing $d^2 = 0$ one gets the extra conditions:

$$ b_{50} = b_{47} = b_{28} = 0, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9, b_{45} = b_{19}, $$

and then $\tilde{\nabla} = 0$ by (11). Thus the above equations reduce to (6.10) of [5] with

$$ t_1 = -b_{12}, t_2 = b_6, t_3 = b_7, t_4 = b_8, t_5 = b_9, t_6 = b_{10}, t_7 = b_{35}, t_8 = b_{19}, t_9 = 0, t_{10} = -b_{39} $$

subject to the constraints

$$
\begin{cases}
-b_{10}b_{19} + b_7b_{39} - b_9b_{37} = 0, \\
b_7b_{35} - b_6b_{19} + b_{12}b_{37} = 0, \\
-b_{12}b_{39} + b_8b_{19} - b_9b_{35} = 0, \\
-b_6b_{39} + b_{10}b_{35} + b_8b_{37} = 0, \\
-b_7b_8 + b_{12}b_{10} + b_6b_9 = 0.
\end{cases}
$$

This system implies $dT = d({*}T) = 0$. This can be alternatively seen by computing the form σ_T of (21), as done previously.

A standard computation yields

1. $b_6 = -\frac{(-b_7b_8 + b_{12}b_{10})}{b_9}, b_{37} = \frac{(b_7b_{39} - b_{19}b_{10})}{b_9}, b_{35} = -\frac{(-b_{19}b_8 + b_{12}b_{39})}{b_9}, b_9 \neq 0,$

2. $b_9 = 0, b_{35} = -\frac{(-b_6 b_{19} + b_{12} b_{37})}{b_7}, b_8 = \frac{b_{12} b_{10}}{b_7}, b_{39} = \frac{b_{19} b_{10}}{b_7}, b_7 \neq 0$,

3. $b_7 = 0, b_9 = 0, b_{12} = 0, b_{19} = 0, b_{35} = \frac{(-b_8 b_{37} + b_6 b_{39})}{b_{10}}, b_{10} \neq 0$,

4. $b_7 = 0, b_{39} = \frac{b_{12} b_{37}}{b_6}, b_9 = b_{10} = 0, b_{39} = \frac{b_8 b_{37}}{b_6}, b_6 \neq 0$,

5. $b_{19} = \frac{b_{12} b_{39}}{b_8}, b_6 = b_7 = b_9 = b_{10} = b_{37} = 0, b_8 \neq 0$,

6. $b_6 = b_7 = b_8 = b_9 = b_{10} = b_{12} = 0$,

7. $b_6 = b_7 = b_8 = b_9 = b_{10} = b_{39} = 0$,

whose corresponding Lie algebras have all 3-dimensional commutator $\mathfrak{l}^1 = [\mathfrak{l}^1, \mathfrak{l}^1]$. Tedious details have been left out. ∎

Since $\tilde{\nabla}$ is flat, [5, proposition 6.6] has $\mathfrak{l}$ isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ provided that $b_{39} \neq 0$ in (23). We can generalise this result regardless of coefficients

Theorem 7.2. $\tilde{\nabla} = 0$ *on* $\mathfrak{l}$ *if and only if* $\mathfrak{l}$ *is isomorphic to* $\mathfrak{so}(3) \oplus \mathbb{R}^2$.

Proof. The connection is zero if and only if $g([X, Y], Z)$ is a totally skew. In other words if

$$g(\mathrm{ad}_X Y, Z) = -g(Y, \mathrm{ad}_X Z),$$

for any $X, Y, Z \in \mathfrak{l}$. Then $\mathfrak{l}$ is compact and Weyl's theorem says that $\mathfrak{l} = \mathfrak{l}^1 \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{l}$. By theorem 7.1 the commutator $\mathfrak{l}^1$ is a 3-dimensional centerless compact Lie algebra, whence it is semisimple. Thus it is in fact isomorphic to $\mathfrak{so}(3)$. ∎

Remark 7.3. The characteristic connection being zero is reminiscent of compact semisimple Lie groups of even dimension equipped with the complex structure of Samelson [16] and metric given by the negified Killing form. They have in fact zero Bismut connection and the corresponding torsion is always harmonic, exactly as in 7.1, see [18].

Remark 7.4. Note incidentally that if $\tilde{\nabla} \equiv 0$ then $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p}$ defines a naturally reductive space, for both the characteristic curvature and the torsion are obviously parallel [19].

As equations (23) translate now in to

$$\sqrt{3} b_{35} = b_{10} - b_{12}, \quad b_8 + b_7 = \sqrt{3} b_{37}, \quad b_6 = -2b_9,$$

Corollary 7.5. *If* $(\mathfrak{l}, \mathfrak{h})$ *is non-symmetric, the torsion* T *is of pure type* Λ_7^2 *when:*

1. $b_6 = -2b_9, b_{12} = -\sqrt{3} b_{35} + b_{10}, b_{39} = \frac{\sqrt{3}}{6b_9}(2b_9^2 - b_{10}^2 - b_8^2)$,
   $b_{37} = -\frac{\sqrt{3}}{3b_8}(-b_8^2 + 2b_9^2 - 3b_{10}^2 + \sqrt{3} b_{10} b_{35})$,
   $b_{19} = \frac{8\sqrt{3}}{3}(-b_{10} b_8^2 + 2b_{10} b_9^2 - b_{10}^3 + \sqrt{3} b_{10}^2 b_{35} + \sqrt{3} b_8^2 b_{35})$,
   $b_7 = -\frac{1}{b_8}(2b_9^2 - b_{10}^2 + \sqrt{3} b_{10} b_{35})$

2. $b_6 = -2b_9, b_8 = 0, b_{19} = -\frac{b_{10}b_{37}}{2b_9}, b_{39} = \frac{\sqrt{3}}{6b_9}(2b_9^2 - b_{10}^2),$
   $b_{35} = -\frac{\sqrt{3}}{3b_{10}}(2b_9^2 - b_{10}^2), b_{12} = \frac{2b_9^2}{b_{10}}, b_7 = \sqrt{3}b_{37}$

3. $b_8 = b_{35} = b_{39} = 0, b_{10}^2 = 2b_9^2, b_6 = -2b_9, b_{12} = b_{10}, b_7 = \sqrt{3}b_{37}, b_{37}^2 = 2b_{19}^2$

4. $b_6 = b_8 = b_9 = b_{10} = b_{39} = 0, b_{12} = -\sqrt{3}b_{35}, b_7 = \sqrt{3}b_{37}.$ ∎

8. Example of pure type Λ_7^2 with non-closed torsion

The majority of Lie algebras found are strongly NI, in the sense that T is closed. It is tempting to think that strongness would hold in general, but this is not the case. The disproving example is constructed using a fibration similar to (17) and will be required to have type Λ_7^2.

We consider Lie algebras $\mathfrak{l}$ whose Levi-Civita connection satisfies

$$\begin{aligned} \nabla_X \mathfrak{h} \subseteq \mathfrak{p}, \quad \nabla_X \mathfrak{p} \subseteq \mathfrak{h}, \quad \forall X \in \mathfrak{h} \\ \nabla_Y \mathfrak{h} \subseteq \mathfrak{h}, \quad \nabla_Y \mathfrak{p} \subseteq \mathfrak{p}, \quad \forall Y \in \mathfrak{p}. \end{aligned} \tag{24}$$

These are in some sense 'dual' to those of proposition 5.1. The constraints (24) are equivalent to $(\mathfrak{l}, \mathfrak{p})$ being a symmetric pair this time, i. e.

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{p}.$$

The connected Lie subgroup $P \subset L$ with Lie algebra $\mathfrak{p}$ is the typical fibre of the Riemannian submersion

$$L \longrightarrow L/P$$

that renders L/P a 3-dimensional locally symmetric space. With the aid of O'Neill formulas the Ricci tensor can be expressed via

$$Ric(X, X) = -([X, [X, e_1]_\mathfrak{p}]_\mathfrak{h}, e_1) - ([X, [X, e_2]_\mathfrak{p}]_\mathfrak{h}, e_2) - ([X, [X, e_4]_\mathfrak{p}]_\mathfrak{h}, e_4),$$

for any $X \in \mathfrak{h}$, and yields the scalar curvature in terms of the b_α's.

The Lie algebra

$$\mathfrak{l} = \left(-\tfrac{3}{4}e^{15} + \tfrac{3\sqrt{3}}{4}e^{23} - \tfrac{\sqrt{3}}{4}e^{45}, e^{25}, \sqrt{3}e^{12} - e^{24} - e^{35}, -\tfrac{5}{4}\sqrt{3}e^{15} - \tfrac{9}{4}e^{23} - \tfrac{5}{4}e^{45}, 0 \right)$$

is a particular solution of the system (19) coupled with the Jacobi identity. It has highest step length (four) and commutator $\mathfrak{l}^1 = span\{e_1, \dots, e_4\}$. Its torsion

$$T = \tfrac{\sqrt{3}}{4}e^{123} - \sqrt{3}e^{145} - \tfrac{3}{4}e^{234}$$

is not strong

$$dT = -\tfrac{3}{2}(e^{2345} + \sqrt{3}e^{1235}), \quad d*T = 0.$$

The Ricci curvature

$$Ric(L) = \begin{pmatrix} -\frac{141}{32} & 0 & 0 & -\frac{99}{32}\sqrt{3} & 0 \\ 0 & -\frac{27}{8} & 0 & 0 & 0 \\ 0 & 0 & -\frac{27}{8} & 0 & 0 \\ -\frac{99}{32}\sqrt{3} & 0 & 0 & -\frac{99}{32}\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{15}{2} \end{pmatrix}$$

is non-positive, and L/P is scalar-flat.

9.   Higher dimensional geometry

The added value of $SO(3)$ geometry comes from links with other G-reductions. Interesting seven-dimensional manifolds can be built starting from an $SO(3)$ five-manifold L: non-integrable CR-structures when L has pure type Λ_7^2, and G_2-metrics on sphere bundles over L [5]. We concentrate on 8-manifolds now, motivated by the observation [14] that a natural generalisation of an irreducible $SO(3)$ structure on a five-manifold is to consider the action of $SU(3)$ in real dimension eight. We therefore take the product $L \times K$, with $K = \mathbb{R}^3$ or $SO(3)$, equipped with the left-invariant metric

$$\tilde{g} = g + (e^6)^2 + (e^7)^2 + (e^8)^2,$$

where g is the adapted metric on L and e_6, e_7, e_8 span the Lie algebra $\mathfrak{k}$ of K. When $K = SO(3)$ this basis is the standard one $[e_6, e_7] = e_8, [e_7, e_8] = e_6, [e_8, e_6] = e_7$. The vector space $\mathbb{R}^8$ is identified with the set of traceless skew-Hermitian 3×3 matrices

$$\tilde{X} = (x_1, \ldots, x_8) \leftrightarrow \begin{pmatrix} x_1 - \sqrt{3}x_4 & \sqrt{3}(x_2 + ix_8) & \sqrt{3}(x_3 + ix_7) \\ \sqrt{3}(x_2 - ix_8) & x_1 + \sqrt{3}x_4 & \sqrt{3}(x_5 + ix_6) \\ \sqrt{3}(x_3 - ix_7) & \sqrt{3}(x_5 - ix_6) & -2x_1 \end{pmatrix}.$$

The irreducible representation of $SU(3)$ on $\mathbb{R}^8$ is given – as for $SO(3)$ – by $\tilde{\rho}(h)\tilde{X} = h\tilde{X}h^{-1}, h \in SU(3)$. An $SU(3)$ structure on $(L \times K, \tilde{g})$ is defined [14] by an element $\tilde{\mathbb{T}} \in \otimes^3 \mathbb{R}^8$ satisfying similar relations to those of $\mathbb{T}$

$$\begin{aligned} \tilde{\mathbb{T}}(\tilde{X}, \tilde{X}, \tilde{X}) \ &= \mathbb{T}(X, X, X) + \tfrac{3}{2}x_1(x_6^2 + x_7^2 - 2x_8^2) - \tfrac{3\sqrt{3}}{2}x_4(-x_7^2 + x_6^2) \\ &\quad + 3\sqrt{3}(x_3x_6x_8 - x_5x_6x_7 + x_2x_7x_6), \\ &= \tfrac{1}{2}\det \tilde{X} \end{aligned}$$

with $X = (x_1, \ldots, x_5)$. Setting x_6, x_7, x_8 to zero induces the same structure on $\mathbb{R}^5 \subset \mathbb{R}^8$ as (2), because the position of $\sqrt{3}$ is more or less only cosmetic. Yet in stark contrast to dimension five [5], the reduction is equivalently determined by the differential 3-form

$$\psi = E_1 \wedge e^6 + E_2 \wedge e^7 + E_3 \wedge e^8 + e^{678},$$

where the E_j's come from (8). The isotropy of ψ is $SU(3)/\mathbb{Z}_3$ embedded in $GL(8, \mathbb{R})$. We refer the reader to [12] where the action of $SU(3)$ on $\mathbb{R}^8$ was first examined. In this respect we can show that under (24) the 3-form ψ is never closed: in particular the Lie algebra of the previous section does not have an integrable $PSU(3)$ structure à la [20].

As in the lower dimension, a nearly integrable $SU(3)$ structure is defined in terms of a symmetric Killing tensor $\tilde{\mathbb{T}}$. This only implies the existence of a connection with totally skew torsion, but is no longer equivalent to it. We shall see under which circumstances one can get hold of a nearly integrable $SU(3)$ structure on $L \times K$, and that the induced characteristic connection is the zero connection, making ψ obviously parallel.

The explicit components of $\tilde{\mathbb{T}} = \sum_{i,j,k=1}^{8} \tilde{\mathbb{T}}_{ijk}\, dx_i \otimes dx_j \otimes dx_k$ are:

$$\tilde{\mathbb{T}}_{ijk} = \mathbb{T}_{ijk}, \quad i,j,k \leqslant 5;$$

$$\tilde{\mathbb{T}}_{166} = -\tfrac{1}{2} = \tilde{\mathbb{T}}_{177}, \quad \tilde{\mathbb{T}}_{188} = 1, \quad \tilde{\mathbb{T}}_{466} = -\tilde{\mathbb{T}}_{477} = \tilde{\mathbb{T}}_{267} = -\tilde{\mathbb{T}}_{368} = \tilde{\mathbb{T}}_{578} = \tfrac{\sqrt{3}}{2}.$$
$$(25)$$

Let ∇ indicate the Levi-Civita connection on $\mathfrak{l} + \mathfrak{k}$ and also its restriction to $\mathfrak{l}$ and $\mathfrak{k}$, so

$$\nabla_{e_6} e_7 = \tfrac{1}{2} e_8 = -\nabla_{e_7} e_6, \quad \nabla_{e_6} e_8 = -\tfrac{1}{2} e_7 = -\nabla_{e_8} e_6, \quad \nabla_{e_7} e_8 = \tfrac{1}{2} e_6 = -\nabla_{e_8} e_7$$

for $K = SO(3)$.

Lemma 9.1. *Let $(L^5, g, \mathbb{T})$ be NI. Then the $SU(3)$ structure $(L \times K, \tilde{g}, \tilde{\mathbb{T}})$, with $K = \mathbb{R}^3$ or $SO(3)$ is NI if and only if*

$$\nabla_X X = 0 \quad \textit{for all } X \in \mathfrak{l}.$$

Proof. Set $\tilde{X} = X + Y \in \mathfrak{l} + \mathfrak{k}$. By construction $\nabla_X Y = 0$, and both when $\mathfrak{k} = \mathbb{R}^3$ and $\mathfrak{k} = \mathfrak{so}(3)$, $\nabla_Y Y$ vanishes as well. Thus

$$0 = \tilde{\mathbb{T}}(\nabla_{\tilde{X}} \tilde{X}, \tilde{X}, \tilde{X}) = \tilde{\mathbb{T}}(\nabla_X X, \tilde{X}, \tilde{X})$$
$$= \mathbb{T}(\nabla_X X, X, X) + 2\tilde{\mathbb{T}}(\nabla_X X, X, Y) + \tilde{\mathbb{T}}(\nabla_X X, Y, Y).$$

The first term (all X's) vanishes by near integrability of L, and the second too because the components (25) yield $\tilde{\mathbb{T}}(\mathfrak{l}, \mathfrak{l}, \mathfrak{k}) = 0$. By linearity $\tilde{\mathbb{T}}(\nabla_X X, e_i, e_j) = 0$, $i, j = 6, 7, 8$. Using (25) once more gives $g(\nabla_X X, \mathfrak{l}) = 0$. The argument also works backwards. $\blacksquare$

The relation of the lemma says that $\nabla_{e_i} e_j + \nabla_{e_j} e_i = 0$ for all $i, j = 1, \ldots, 5$, which implies (12). The first string of the latter equations characterises (16), so

Corollary 9.2. *The geodesic equation $\nabla_X X = 0$ induces the split behaviour of $\tilde{\nabla}$ on $\mathfrak{l}$ described by (16) and forces $\tilde{\nabla}$ to be zero.* $\blacksquare$

In this case, the skew-symmetry of ∇ can be simply detected by considering the relation $2g(\nabla_{e_i} e_j + \nabla_{e_j} e_i, e_k) = T(e_i, e_j, e_k) + T(e_j, e_i, e_k)$, $\forall i, j, k$.
That said, generating nearly integrable $SU(3)$ structures on the 8-manifold $(L \times K, \tilde{g}, \tilde{\mathbb{T}})$ is possible only starting with one candidate

Proposition 9.3. *Up to isomorphisms, the unique symmetric pair giving rise to NI products $L \times \mathbb{R}^3, L \times SO(3)$ is $\left(\mathfrak{l} = \mathfrak{so}(3) \oplus \mathbb{R}^2, \mathfrak{h} = \mathfrak{so}(3)\right)$.*

Proof. In the symmetric case $\nabla_X X = 0$ reduces the structure equations of L to

$$\begin{cases} de^1 = b_6 e^{24} + b_9 e^{35} \\ de^2 = -b_6 e^{14} + b_{19} e^{35} \\ de^3 = e^5 \wedge (b_9 e^1 + b_{19} e^2 + b_{39} e^4) \\ de^4 = b_6 e^{12} + b_{39} e^{35} \\ de^5 = (b_9 e^1 + b_{19} e^2 + b_{39} e^4) \wedge e^3. \end{cases}$$

From the Jacobi identity either $b_6 = 0$ or $b_9 = b_{19} = b_{39}$. Both structures are isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ and crop up as II, IV in lemma 5.5. $\blacksquare$

Proposition 9.4.   *Non-symmetric pairs with $b_{37} = 0$ always induce a nearly integrable $SU(3)$ structure on the product $L \times K$.*

Proof.   All algebras of section 7. satisfy, by way of $d^2 = 0$, the extra requirements $b_{50} = b_{47} = b_{28} = 0, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9, b_{45} = b_{19}$.   ∎

References

[1]   Agricola, I., *The Srní lectures on non-integrable geometries with torsion*, Rend. Circ. Mat. Palermo eprint `arXive:math.DG/0606705`, to appear.

[2]   Agricola, I., and Th. Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. **328** (2004), 711–748.

[3]   Alekseevskiĭ, D. V., *Classification of quaternionic spaces with a transitive solvable group of motions*, Izv. Akad. Nauk. SSSR Ser. Mat. **9** (2) (1975), 315–362. English translation: Math. USSR-Izv. **9** (1976), 297–339.

[4]   Alexandrov, B., Th. Friedrich. and N. Schoemann, *Almost Hermitian 6-manifolds revisited*, J. Geom. Phys. **53** (2005), 1–30.

[5]   Bobieński, M., and P. Nurowski, *Irreducible SO(3) geometry in dimension five*, J. Reine Angew. Math., eprint `arXive:math.DG/0507152`, to appear.

[6]   Martín Cabrera, F., *Special almost Hermitian geometry*, J. Geom. Phys. **55** (2005), 450–470.

[7]   Cleyton, R., and A. F. Swann, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), 513–528.

[8]   Conti, D., and S. Salamon, *Generalized Killing spinors in dimension 5*, Trans. Amer. Math. Soc., eprint `arXive:math.DG/0508375`, to appear.

[9]   Friedrich, Th., *On types of non-integrable geometries*, Proceedings of the 22nd Winter School "Geometry and Physics", Srní 2002, Rend. Circ. Mat. Palermo (2) Suppl. **71** (2003), 99–113.

[10]   Friedrich, Th., and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.

[11]   Gray, A., and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4), **123** (1980), 35–58.

[12]   Hitchin, N. J., *Stable forms and special metrics*, in: "Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)", Contemp. Math. **288**, Amer. Math. Soc., Providence, RI, 2001, 70–89.

[13]   Ivanov, S., and G. Papadopoulos, *Vanishing theorems and string backgrounds*, Class. Quantum Gravity **18** (2001), 1089–1110.

[14]   Nurowski, P., *Distinguished dimensions for special Riemannian geometries*, eprint `arXive:math.DG/0601020`.

[15]   Salamon, S. M., "Riemannian geometry and holonomy groups, " Pitman Research Notes in Mathematics **201**, Longman, Harlow, 1989.

[16]   Samelson, H., *A class of complex-analytic manifolds*, Portugaliae Math. **12**, 1953, 129–132.

[17] Semmelmann, U., *Conformal Killing forms on Riemannian manifolds*, Math. Z. **245** (2003), 503–527.

[18] Spindel, Ph., A. Sevrin, W. Troost, and A. Van Proeyen, *Extended supersymmetric σ-models on group manifolds, I: The complex structures*, Nuclear Phys. B **308** (1988), 662–698.

[19] Tricerri, F., and L. Vanhecke, *Naturally reductive homogeneous spaces and generalized Heisenberg groups*, Comp. Math. **52** (1984), 389–408.

[20] Witt, F., *Special metrics and Triality*, eprint `arXive:math.DG/0602414`.

Simon G. Chiossi
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
sgc@math.hu-berlin.de

Anna Fino
Dipartimento di Matematica
Università di Torino
via Carlo Alberto 10
10123 Torino, Italy
annamaria.fino@unito.it