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Weyl Groups with Coxeter Presentation and Presentation by Conjugation

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Abstract. We investigate which Weyl groups have a Coxeter presentation and which of them at least have the presentation by conjugation with respect to their root system. For most concepts of root systems the Weyl group has both. In the context of extended affine root systems (EARS) there is a small subclass allowing a Coxeter presentation of the Weyl group and a larger subclass allowing the presentation by conjugation. We give necessary and sufficient conditions for both classes. Our results entail that every extended affine Weyl group (EAWeG) has the presentation by conjugation with respect to a suitable EARS.
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1. Introduction

Although some of our results are applicable to several kinds of root systems, our main focus are the extended affine root systems (EARS). These root systems belong to the theory of extended affine Lie algebras (EALA). The class of EALAs is a generalization of the class of affine Kac-Moody algebras. In [1] EALAs are introduced axiomatically and the EARs are classified. Lie algebras in this class were studied by physicists under the name of irreducible quasi-simple Lie algebras in [8]. Several years before a certain subclass of EALAs was already studied in [15].

If $\mathcal{W}$ is a group acting on a set R then R is called a *root set* if the following conditions are satisfied:

(i) For every $\alpha \in R$ there is an involution $r_\alpha \in \mathcal{W}$, and the elements $(r_\alpha)_{\alpha\in R}$ generate $\mathcal{W}$.

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(ii) We have

$$r_\alpha r_\beta (r_\alpha)^{-1} = r_{r_\alpha(\beta)}$$

for all α and $\beta \in R$.

The elements of R are called *roots*. For any root α we call r_α the involution *associated* to α. The group $\mathcal{W}$ is referred to as the *Weyl group* of R and group elements of the form r_α are called *root involutions*.

We have introduced the new notion of root sets to encompass different concepts of root systems. For instance R is a root set for the Weyl group involved if R

a) is a finite (not necessarily crystallographic) root system.

b) consists of the roots associated to a *set of root data* in the sense of MOODY and PIANZOLA (see [14]), or if R is the root system of a Coxeter group in the sense of DEODHAR (see [6]) or a root system of the slightly more general kind studied in [9] based on VINBERG's idea of a *discrete linear group* $\mathcal{W}$ *generated by reflections* (see [16]).

c) consists of the nonzero roots of a locally finite root system in the sense of [12].

d) consists of the anisotropic roots of an extended affine root system (EARS).

We will say that $\mathcal{W}$ has a *Coxeter presentation with respect to the root set*, if there is a subset $\mathcal{S}$ of root involutions in $\mathcal{W}$ such that $(\mathcal{W}, \mathcal{S})$ is a Coxeter system.

In cases a) and b) the Weyl group has a Coxeter presentation with respect to the root set. (See [14] Proposition 5.7.(i), [6] Proposition 3.1.(i) and [16] §3 Theorem 2.1 for respective results for the case b).) In this paper we focus on the case d). We will prove the following theorem. Let $\mathcal{W}$ be the Weyl group of an EARS with nullity ν. (This number is the index of the $\mathcal{W}$-invariant quadratic form.)

Theorem 1.1. *The group $\mathcal{W}$ is a Coxeter group if and only if $\nu < 2$. In the case $\nu < 2$ a Coxeter presentation with respect to the root set of anisotropic roots exists.*

We will investigate whether the Weyl group $\mathcal{W}$ of a root set R has the presentation

$$\begin{aligned} \Big\langle (\widehat{r}_\alpha)_{\alpha \in R^\times} \;\Big|\; & \widehat{r}_\alpha = \widehat{r}_\beta \quad \text{if } \alpha \text{ and } \beta \text{ are linearly dependent,} \\ & \widehat{r}_\alpha^2 = 1, \\ & \widehat{r}_\alpha \widehat{r}_\beta \widehat{r}_\alpha^{-1} = \widehat{r}_{r_\alpha(\beta)}; \text{ for } \alpha, \beta \in R^\times \Big\rangle, \end{aligned} \tag{1}$$

which is called the *presentation by conjugation*. The answer is yes in the cases a) and b) since the existence of a Coxeter presentation with respect to the root set implies the existence of the presentation by conjugation. The answer is also yes in the case of c) (see [12] Theorem 5.12).

The answer is also known to be yes for various subclasses of EARSs and their EAWeGs (see [3], [4], [11]). In this article we prove the following, which we consider the main result:

Theorem 1.2. *Let R be an EARS and let $\mathcal{W}$ be its Weyl group. Then $\mathcal{W}$ has a presentation by conjugation with respect to R if and only if R is minimal, i.e. the set of root reflections is a minimal $\mathcal{W}$-invariant generating set.*

We also show that every EARS contains a subset which is a minimal EARS for the same Weyl group. As a consequence we obtain the following result: Every EAWeG has the presentation by conjugation with respect to a suitable EARS.

It has come to our attention that a similar result has been obtained independently in [2]. There it is proved for the sub-case of EARSs of type A_1 that the minimality of a certain generating set of $\mathcal{W}$ is equivalent to the existence of the presentation by conjugation.

2. Extended affine root systems

In this section we provide the definition of an extended affine root system (EARS) and some basic facts taken from [1].

Definition 2.1. Let $\mathcal{V}$ be a finite dimensional real vector space with a positive semi-definite symmetric bilinear form $(\cdot,\cdot)$ and let R be a subset of $\mathcal{V}$. Set

$$\mathcal{V}^0 := \{v \in \mathcal{V} : (v,v) = 0\}, \quad R^0 := R \cap \mathcal{V}^0, \quad \text{and} \quad R^\times := R \setminus R^0$$

The subset R is called an *extended affine root system* in $\mathcal{V}$ if it satisfies the following axioms:

(R1) $0 \in R$.

(R2) $-R = R$.

(R3) R spans $\mathcal{V}$.

(R4) $\alpha \in R^\times \implies 2\alpha \notin R$.

(R5) R is discrete in $\mathcal{V}$.

(R6) If $\alpha \in R^\times$ and $\beta \in R$, then there exist $d, u \in \mathbb{Z}_{\geq 0}$ satisfying

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, \dots, \beta + u\alpha\} \text{ and } d - u = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}.$$

(R7) $R^\times$ cannot be decomposed as a disjoint union $R_1 \dot\cup R_2$, where R_1 and R_2 are nonempty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$.

(R8) If $\sigma \in R^0$, then there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

Elements of $R^\times$ are called *anisotropic roots* and elements of R^0 are called *isotropic roots*. The *nullity* of R is the dimension of $\mathcal{V}^0$.

By $\bar{\cdot} : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0$ we denote the quotient map. Then $\overline{R^\times}$ is a finite crystallographic root system. It contains a fundamental system $\overline{\Pi} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_l\}$ with $l = \dim(\mathcal{V}/\mathcal{V}^0)$. For each $\overline{\alpha}_i$ we pick a preimage $\dot{\alpha}_i$ in $R^\times$. Then we set

$$\dot{\mathcal{V}} := \operatorname{span}_{\mathbb{R}}\{\dot{\alpha}_1, \dots, \dot{\alpha}_l\}. \tag{2}$$

Definition 2.2. A nonempty subset S of a real vector space V is called a *translated semilattice* if it spans V, is discrete and satisfies

$$S = -S \quad \text{and} \quad S + 2S \subseteq S.$$

A translated semilattice is called a *semilattice* if it contains zero. If S is a semilattice and $\pi : \langle S\rangle \to \langle S\rangle/\langle 2S\rangle$ is the quotient map, then the *index* of S is the number $|\pi(S)|-1$. (Here $\langle S\rangle$ stands for the additive subgroup of V generated by S.)

Since we will make use of it and because of its significance for the structure of extended affine root systems we repeat their classification obtained in [1].

Construction 2.3. *Suppose that $\dot{R}$ is an irreducible finite root system of type X in a finite dimensional real vector space $\dot{\mathcal{V}}$ with positive definite symmetric bilinear form $(\cdot,\cdot)$. We decompose the set $\dot{R}^\times$ of nonzero elements of $\dot{R}$ according to length as*

$$\dot{R}^\times = \dot{R}_{sh} \dot{\cup} \dot{R}_{lg} \dot{\cup} \dot{R}_{ex}. \tag{3}$$

Let $\mathcal{V}^0$ be a finite dimensional real vector space, set $\mathcal{V} := \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and extend $(\cdot,\cdot)$ to $\mathcal{V}$ in such a way that $(\mathcal{V},\mathcal{V}^0) = \{0\}$.

*(a) (*The simply laced construction*) Suppose that X is simply laced, i.e.*

$$X = A_\ell(\ell \geq 1),\ D_\ell(\ell \geq 4),\ E_6,\ E_7,\ \text{or } E_8.$$

Suppose that S is a semilattice in $\mathcal{V}^0$. If $X \neq A_1$ suppose further that S is a lattice in $\mathcal{V}^0$. Put

$$R = R(X,S) := (S+S) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}^\times} (\dot\alpha + S) \Big).$$

*(b) (*The reduced non-simply laced construction*) Suppose that X is reduced and non-simply laced, i.e.*

$$X = B_\ell(\ell \geq 2),\ C_\ell(\ell \geq 3),\ F_4,\ \text{or } G_2.$$

Suppose that S and L are semilattices in $\mathcal{V}^0$ such that

$$L + kS \subseteq L \quad \text{and} \quad S + L \subseteq S,$$

where k is defined by

$$k := \begin{cases} 2 & \text{if } R \text{ has type } B_\ell(\ell \geq 2),\ C_\ell(\ell \geq 3), \\ & \qquad F_4,\ \text{or } BC_\ell(\ell \geq 2), \\ 3 & \text{if } R \text{ has type } G_2. \end{cases} \tag{4}$$

Further, if $X = B_\ell(\ell \geq 3)$ suppose that L is a lattice, if $X = C_\ell(\ell \geq 3)$ suppose that S is a lattice, and if $X = F_4$ or G_2 suppose that both S and L are lattices. Put

$$R = R(X,S,L) := (S+S) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{sh}} (\dot\alpha + S) \Big) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{lg}} (\dot\alpha + L) \Big).$$

(c) (The BC_ℓ construction, $\ell \geq 2$) *Suppose that* $X = BC_\ell(\ell \geq 2)$. *Suppose that* S *and* L *are semilattices in* $\mathcal{V}^0$ *and* E *is a translated semilattice in* $\mathcal{V}^0$ *such that* $E \cap 2S = \varnothing$ *and*

$$L + 2S \subseteq L, \quad S + L \subseteq S,$$
$$E + 2L \subseteq E, \quad L + E \subseteq L.$$

If $\ell \geq 3$, *suppose further that* L *is a lattice. Put*

$$\begin{aligned} R = R(BC_\ell, S, L) \;\; &:= \;\; (S+S) \cup \Big(\bigcup_{\dot\alpha \in \dot R_{sh}} (\dot\alpha + S) \Big) \\ &\quad \cup \Big(\bigcup_{\dot\alpha \in \dot R_{lg}} (\dot\alpha + L) \Big) \cup \Big(\bigcup_{\dot\alpha \in \dot R_{ex}} (\dot\alpha + E) \Big). \end{aligned}$$

(d) (The BC_1 construction, $\ell \geq 2$) *Suppose that* $X = BC_1$. *Suppose that* S *is a semilattices in* $\mathcal{V}^0$ *and* E *is a translated semilattice in* $\mathcal{V}^0$ *such that* $E \cap 2S = \varnothing$ *and*

$$E + 4S \subseteq E \quad \textit{and} \quad S + E \subseteq S.$$

Put

$$\begin{aligned} R = R(BC_1, S, E) \;\; &:= \;\; (S+S) \cup \Big(\bigcup_{\dot\alpha \in \dot R_{sh}} (\dot\alpha + S) \Big) \\ &\quad \cup \Big(\bigcup_{\dot\alpha \in \dot R_{ex}} (\dot\alpha + E) \Big). \end{aligned}$$

Theorem 2.4. *Let* X *be one of the types for a finite root system. Starting from a finite root system* $\dot R$ *of type* X *and up to three semilattices or translated semilattices (as indicated in the construction), Construction 2.3 produces an extended affine root system of type* X. *Conversely, any extended affine root system of type* X *is isomorphic to a root system obtained from the part of Construction 2.3 corresponding to type* X.

We need to do some preparations for the definition of the extended affine Weyl group of an EARS R. Set

$$V := \mathcal{V}^0 \oplus \dot{\mathcal{V}} \oplus (\mathcal{V}^0)^* \tag{5}$$

and define a bilinear form $\langle \cdot, \cdot \rangle$ on V in the following way:

- $\langle \cdot, \cdot \rangle$ extends $(\cdot, \cdot)$ on $\mathcal{V} = \mathcal{V}^0 \oplus \dot{\mathcal{V}}$,
- $\big\langle \dot{\mathcal{V}}, (\mathcal{V}^0)^* \big\rangle = \{0\}$, $\big\langle (\mathcal{V}^0)^*, (\mathcal{V}^0)^* \big\rangle = \{0\}$,
- $\langle \cdot, \cdot \rangle$ is the natural pairing on $\mathcal{V}^0 \times (\mathcal{V}^0)^*$.

In other words we have

$$\big\langle (v_1, v_2, v_3), (u_1, u_2, u_3) \big\rangle = u_3(v_1) + (v_2, u_2) + v_3(u_1)$$

with respect to the decomposition (5). Now we have the vector space V with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a totally isotropic subspace $\mathcal{V}^0$.

For every element $\alpha \in R^\times$ we define an associated reflection via

$$r_\alpha : \; V \to V, r_\alpha(v) = v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = v - \langle v, \alpha^\vee \rangle \alpha \quad \text{with} \quad \alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle}\alpha.$$

A reflection associated to some root is called a *root reflection*.

Definition 2.5. The *extended affine Weyl group (EAWeG)* $\mathcal{W}$ is the subgroup of $\mathrm{Aut}(V)$ generated by the root reflections.

3. The Coxeter Presentation

In this section we introduce the definition of a root set. This new notion is designed to encompass different concepts of root systems as well as the concept of a Coxeter system. We prove that an EAWeG has a Coxeter presentation with respect to its EARS if and only if the nullity is not greater than 1.

Definition 3.1. Let $\mathcal{W}$ be a group acting on a set R. We call R a *root set* for $\mathcal{W}$ if the following conditions are satisfied

(i) For every $\alpha \in R$ there is an involution $r_\alpha \in \mathcal{W}$, and the elements $(r_\alpha)_{\alpha\in R}$ generate $\mathcal{W}$.

(ii) We have
$$r_\alpha r_\beta (r_\alpha)^{-1} = r_{r_\alpha(\beta)}$$
for all α and $\beta \in R$.

If R is a root set for $\mathcal{W}$, then we define
$$\alpha \sim \beta :\Longleftrightarrow r_\alpha = r_\beta.$$

The elements of R are called *roots*. For any root α we call r_α the reflection *associated* to α. The group $\mathcal{W}$ is referred to as the *Weyl group* of R and group elements of the form r_α are called *root involutions*.

Remark 3.2. *The relation $\sim$ is a congruence relation, i.e. an equivalence relation satisfying*
$$\alpha \sim \beta \iff w.\alpha \sim w.\beta,$$
for every $w \in \mathcal{W}$.

Example 3.3. We refer the reader to the list of examples a) - d) given in the introduction. We add one example to this list:

e) If $(\mathcal{W}, \mathcal{S})$ is a Coxeter system, then
$$T := \{wsw^{-1} : w \in W, s \in \mathcal{S}\} \tag{6}$$
is a root set for $\mathcal{W}$ if we set $r_t = t$. The equivalence relation in this setting is equality.

Let R be a root set with Weyl group $\mathcal{W}$. Let S be a subset of R such that the group root involutions associated to the elements in S generate W and such that
$$(\forall \alpha, \beta \in S)\ \alpha \sim \beta \implies \alpha = \beta.$$

Set

$$\widehat{\mathcal{W}}_S := \big\langle (\widehat{r}_\alpha)_{\alpha\in S} \ \mid\ \widehat{r}_\alpha^2 = 1 \quad \text{for } \alpha \in S, \qquad (\widehat{r}_\alpha \widehat{r}_\beta)^{\mathrm{ord}(r_\alpha r_\beta)} = 1 \quad \text{for } \alpha,\, \beta \in R \text{ and } \mathrm{ord}(r_\alpha r_\beta) < \infty \big\rangle \tag{7}$$

where $\mathrm{ord}(\cdot)$ stands for the order of a group element. Since the relations in this presentation are true for the generators $(r_\alpha)_{\alpha\in S}$ of $\mathcal{W}$, there is a group homomorphism $\varphi : \widehat{\mathcal{W}}_S \to \mathcal{W}$ that maps $\widehat{r}_\alpha$ to r_α for all $\alpha \in S$. Since φ has a generating set of $\mathcal{W}$ in its image, it is surjective.

Definition 3.4. We say that $\mathcal{W}$ has a *Coxeter presentation (with respect to R and S)* if φ is injective.

The requirement in this definition is equivalent to saying that there is a subset $\mathcal{S}$ of root involutions such that $(\mathcal{W}, \mathcal{S})$ is a Coxeter system.

Example 3.5. In the list of examples discussed in the Introduction and in Example 3.3 the following Weyl groups have a Coxeter presentation with respect to their root sets: a), b), e).

Theorem 3.6. *Let $\mathcal{W}$ be the Weyl group of an EARS with nullity ν. Then $\mathcal{W}$ is a Coxeter group if and only if $\nu < 2$. In the case $\nu < 2$ a Coxeter presentation with respect to the root set of anisotropic roots exists.*

Proof. The anisotropic roots of an EARS with nullity 0 form a finite root system. So the required Coxeter presentation exists. (The Coxeter group is spherical.) The anisotropic roots of an EARS with nullity 1 can be viewed as a root system in the sense of [14]. Again, a Coxeter presentation with respect to the underlying root set exists. (The Coxeter group is affine.)

So now we suppose the nullity ν is greater than 1. We consider the natural homomorphism $\mathcal{W} \to \mathcal{W}_0$ onto the finite reflection group $\mathcal{W}_0$ associated to $\mathcal{W}$. In the case $\nu > 1$ its kernel H is a 2-step nilpotent group and its center Z as well as the quotient $Q = H/Z$ are free abelian groups of finite (non-zero) rank. (These facts about the structure of $\mathcal{W}$ can be found, for instance, in Remark 4.4.23 and Lemma 4.4.30 of [10].)

We suppose that $\mathcal{W}$ is a Coxeter group. According to Theorem 1 of [13], the group $\mathcal{W}$ either contains an abelian subgroup of finite index, or contains a non-abelian free subgroup.

We suppose the first alternative. Then H must contain an abelian subgroup A of finite index. Since AZ is abelian and H is not, we may assume that A contains the center Z. Since H is 2-step nilpotent its commutator map $H \times H \to Z$ is bimultiplicative. The same is true for the factored map

$$\varphi : Q \times Q \to Z,\ (h_1 Z, h_2 Z) \mapsto h_1 h_2 h_1^{-1} h_2^{-1}.$$

Since A/Z has finite index in Q, it has the same rank as Q. Thus $\langle \varphi(A/Z, A/Z) \rangle$ has the same rank as $\langle \varphi(Q, Q) \rangle$, which is the commutator subgroup of H, a nontrivial free abelian group. This is a contradiction to the fact that A is abelian.

Now we suppose the second alternative. That means that $\mathcal{W}$ contains a non-abelian free subgroup. Since H has finite index in $\mathcal{W}$, this is also true for H. That is a contradiction to the fact that H is 2-step nilpotent. ∎

4. The Presentation by Conjugation

In this section we introduce the notion of the presentation by conjugation of a group with respect to a root set. If a group $\mathcal{W}$ has a Coxeter presentation with respect to a root set R then $\mathcal{W}$ has a presentation by conjugation with respect to this root set. The key result of this section is that if $\mathcal{W}$ has the presentation by conjugation with respect to R then R is minimal.

Let R be a root set with Weyl group $\mathcal{W}$. Set

$$\widehat{\mathcal{W}} := \Big\langle (\widehat{r}_\alpha)_{\alpha\in R} \;\Big|\; \begin{array}{l} \widehat{r}_\alpha = \widehat{r}_\beta \quad \text{for } \alpha \text{ and } \beta \in R \text{ with } \alpha \sim \beta, \\ \widehat{r}_\alpha^2 = 1 \quad \text{for } \alpha \in R, \\ \widehat{r}_\alpha \widehat{r}_\beta \widehat{r}_\alpha^{-1} = \widehat{r}_{r_\alpha(\beta)} \quad \text{for } \alpha \text{ and } \beta \in R \Big\rangle. \end{array} \tag{8}$$

Since the relations in this presentation are true for the generators $(r_\alpha)_{\alpha\in R}$ of $\mathcal{W}$, there is a group homomorphism $\varphi : \widehat{\mathcal{W}} \to \mathcal{W}$ that maps $\widehat{r}_\alpha$ to r_α for all $\alpha \in R$. Since φ has a generating set of $\mathcal{W}$ in its image, it is surjective.

Definition 4.1. We say that $\mathcal{W}$ has the *presentation by conjugation (with respect to R)* if φ is injective.

The following result is known. It is used in a slightly different version, for instance, in [7] for the proof of 1.43. Since no source for it is known to the author we will include a proof.

Proposition 4.2. *Let $(\mathcal{W}, S)$ be a Coxeter system. Consider the root set T for W described in Example 3.3. Then $\mathcal{W}$ has the presentation by conjugation with respect to T.*

Proof. Set $\widehat{S} = \{\widehat{r}_s : s \in S\}$. We want to show that the map $S \to \widehat{S},\ s \mapsto \widehat{r}_s$ can be extended to a map $\psi : \widehat{\mathcal{W}} \to \mathcal{W}$. For that purpose we need to show that the relations defining the Coxeter group $\mathcal{W}$ (see (7)) hold for the generators $\widehat{S}$ in $\widehat{\mathcal{W}}$. The first defining relation in (7) is clear so let's look at the second:

In $\mathcal{W}$ we have:

$$(s_1 s_2)^{m(s_1,s_2)} \;=\; 1.$$

Let's consider the case that $m(s_1, s_2)$ is odd, say $m(s_1, s_2) = 2k + 1$. Then this relation together with the fact that s_1 and s_2 are involutions entails

$$(s_1 s_2)^k s_1 (s_2^{-1} s_1^{-1})^k \;=\; s_2.$$

From the defining relations of $\widehat{\mathcal{W}}$ it follows that in $\widehat{\mathcal{W}}$ we have

$$(\widehat{r}_1 \widehat{r}_2)^k \widehat{r}_1 (\widehat{r}_2^{-1} \widehat{r}_1^{-1})^k \;=\; \widehat{r}_2$$

where we have written $\widehat{r}_1$ for $\widehat{r}_{s_1}$ and $\widehat{r}_2$ for $\widehat{r}_{s_2}$ to avoid a notational overload. In turn, this implies

$$(\widehat{r}_1 \widehat{r}_2)^{m(\widehat{r}_1,\widehat{r}_2)} \;=\; 1.$$

The case $m(s_1, s_2)$ even is similar and is left to the reader.

So we have the map $\psi : \mathcal{W} \to \widehat{\mathcal{W}}$ and need to prove that $\psi \circ \varphi$ is the identity on $\widehat{\mathcal{W}}$. It is clear that it is the identity on $\widehat{S}$, so we need to show that $\widehat{S}$ generates $\widehat{\mathcal{W}}$. We do this by proving that the subgroup $\langle \widehat{S} \rangle$ of $\widehat{\mathcal{W}}$ generated by $\widehat{S}$ contains the set $\widehat{T} := \{\widehat{r}_t : t \in T\}$.

So let $\widehat{r}_t \in \widehat{T}$. this means $t = wsw^{-1}$ for some $w \in \mathcal{W}$ and $s \in S$. The group element w can be written as $w = s_1 s_2 \cdots s_k$ for some $s_1,\ s_2, \ldots, s_k \in S$. So we have

$$t \;=\; s_1 s_2 \cdots s_k \cdot s \cdot s_k^{-1} s_{k-1}^{-1} \cdots s_1^{-1}.$$

Using arguments similar to those above we deduce

$$\widehat{r}_t \;=\; \widehat{r}_1 \widehat{r}_2 \cdots \widehat{r}_k \cdot \widehat{r}_s \cdot \widehat{r}_k^{-1} \widehat{r}_{k-1}^{-1} \cdots \widehat{r}_1^{-1}.$$

This shows $\widehat{r}_t \in \langle \widehat{S} \rangle$, which concludes the proof. ∎

Example 4.3. (i) In the list of examples discussed in the Introduction the root sets in a) and b) have a Coxeter presentation. By the previous proposition, they have a presentation by conjugation.

(ii) The underlying root set of a locally finite root system described in the introduction in c) has the presentation by conjugation (see [12] Theorem 5.12).

Definition 4.4. Let R be a root set with Weyl group $\mathcal{W}$. We call the root set R *minimal* if the set of root involutions is a minimal $\mathcal{W}$-invariant generating set.

Theorem 4.5. *If the group $\mathcal{W}$ has a presentation by conjugation with respect to R, then the root set R is minimal.*

Proof. Suppose that R is not minimal, i.e. there is a $\mathcal{W}$-invariant generating set of $\mathcal{W}$ which is a strict subset of r_R, the set of all root reflections. So there is a root γ such that $r_\gamma \in r_R \setminus S$. Then there must be roots $\delta_1, \ldots, \delta_k$ with

$$r_\gamma = r_{\delta_1} r_{\delta_2} \cdots r_{\delta_k} \quad \text{where } r_{\delta_1}, \ldots, r_{\delta_k} \in S. \tag{9}$$

Now let $F(R)$ be the free group generated by the elements $\widehat{r}_\alpha$ with $\alpha \in R$. Consider the normal subgroup N of $F(R)$ generated by

$$\begin{aligned} X := \quad & \{\widehat{r}_\alpha (\widehat{r}_\beta)^{-1} : \alpha, \beta \in R \text{ with } \alpha \sim \beta\} \\ \cup \quad & \{\widehat{r}_\alpha \widehat{r}_\alpha : \alpha \in R\} \\ \cup \quad & \{\widehat{r}_\alpha \widehat{r}_\beta \widehat{r}_\alpha (\widehat{r}_{r_\alpha(\beta)})^{-1} : \alpha, \beta \in R\}. \end{aligned}$$

The quotient $F(R)/N$ is the group $\widehat{\mathcal{W}}$.

Let's consider the map

$$\psi : R \to \ \mathbb{Z}_2,\ \alpha \mapsto \begin{cases} 0 & \text{if } r_\alpha \in S \\ 1 & \text{else.} \end{cases}$$

This map extends to a group homomorphism $\widetilde{\psi} : F(R) \to \mathbb{Z}_2$. It is easily verified that X is in $\ker \psi$ and thus $N \leq \ker \psi$. This means that the map $\widetilde{\psi}$ factors to a map $\widehat{\psi} : F(R)/N \to \mathbb{Z}_2$.

Note that

$$\widehat{\psi}(\widehat{r}_\gamma) = 1 \quad \text{whereas} \quad \widehat{\psi}(\widehat{r}_{\delta_1}\widehat{r}_{\delta_2}\cdots\widehat{r}_{\delta_k}) = 0.$$

So $\widehat{r}_\gamma$ and $\widehat{r}_{\delta_1}\widehat{r}_{\delta_2}\cdots\widehat{r}_{\delta_k}$ are two different group elements of $\widehat{\mathcal{W}}$. Due to (9), they are mapped to the same element in $\mathcal{W}$ by the group homomorphism $\varphi : \widehat{\mathcal{W}} \to \mathcal{W}$. So this homomorphism is not injective. ∎

The converse of the previous theorem is not true. In the following we present an example of a reflection group with a minimal root set which does not have a presentation by conjugation.

Example 4.6. Let R be the EARS of type A_1, nullity $\nu = 2$ such that the index of the involved semilattice S (see Theorem 2.4) is 2. Let $\mathcal{W}$ be its Weyl group. Then $\mathcal{V}$ is a $\mathcal{W}$-invariant subspace of V. Let $\overline{\mathcal{W}}$ be the restriction of $\mathcal{W}$ to $\mathcal{V}$. Since $R^\times \subseteq \mathcal{V}$, the set $R^\times$ is a root set for $\overline{\mathcal{W}}$. The restriction induces a group homomorphism

$$\mathcal{W} \to \overline{\mathcal{W}},$$

more precisely, the group $\mathcal{W}$ is a non-trivial central extension of $\overline{\mathcal{W}}$. The groups $\widehat{\mathcal{W}}$ and $\widehat{\overline{\mathcal{W}}}$ have the same presentation. Since the homomorphism $\widehat{\mathcal{W}} \to \mathcal{W}$ is surjective, the composition of homomorphisms $\widehat{\mathcal{W}} \to \mathcal{W} \to \overline{\mathcal{W}}$ is not injective. This means that $\widehat{\overline{\mathcal{W}}} \to \overline{\mathcal{W}}$ is not injective, so $\overline{\mathcal{W}}$ does not have the presentation by conjugation with respect to $R^\times$.

By considering the orbits of $\overline{\mathcal{W}}$ in $R^\times$ it is not hard to see that R is a minimal root set for $\overline{\mathcal{W}}$.

Example 4.7. For all concepts of root systems R and their Weyl groups $\mathcal{W}$ such that the underlying root set provides the presentation by conjugation for the Weyl group, the root set is minimal. This is true for the examples a) - c) in the Introduction as well as Example 3.3 e).

We will look at the case of a finite crystallographic root system more closely and derive a result about the orbits of its Weyl group which we need later on:

Lemma 4.8. *Let R be an irreducible finite crystallographic root system with Weyl group $\mathcal{W}$. Suppose $\widetilde{R}$ is a subset of R that is invariant under $\mathcal{W}$ and such that the reflections associated to the roots in $\widetilde{R}$ generate $\mathcal{W}$. Then $\widetilde{R}$ is an irreducible finite crystallographic root system and one of the following is true:*

(i) $R = \widetilde{R}$,

(ii) *R is of type BC_1 and $\widetilde{R}$ is of type A_1,*

(iii) *R is of type BC_2 and $\widetilde{R}$ is of type B_2,*

(iv) *R is of type BC_ℓ, $\ell \geq 3$ and $\widetilde{R}$ is of type B_ℓ or C_ℓ.*

Proof. Let $R_{\rm sh}$, $R_{\rm lg}$ and $R_{\rm ex}$ be the set of all short, long and extra long roots in R. Then R is the disjoint union of these three sets. Each of these sets is empty or acted on transitively by $\mathcal{W}$. (See [5] Ch. VI. n^0 1.4.10.) So $\widetilde{R}$ must be the union of some of the three sets.

Since R is a minimal root set for $\mathcal{W}$, the reflections $r_{\widetilde{R}}$ associated to $\widetilde{R}$ are precisely the reflections r_R associated to R. If $\widetilde{R}$ not equal to R, then only one of the cases (ii) - (iv) is possible, since $r_{R_{\rm sh}} \cap r_{R_{\rm lg}} = r_{R_{\rm ex}} \cap r_{R_{\rm lg}} = \varnothing$. In each of these cases $\widetilde{R}$ is an irreducible finite root system. ■

In the following we will present another result needed later on to prove that certain EAWeGs have the presentation by conjugation.

Let R be a minimal root system with Weyl group $\mathcal{W}$. Let $\widetilde{R}$ be a $\mathcal{W}$-invariant subset of R and denote by $\mathcal{W}_{\widetilde{R}}$ the subgroup of $\mathcal{W}$ generated by the reflections associated to the elements of $\widetilde{R}$. Suppose the following property is satisfied:

$$(\forall\alpha \in R \setminus \widetilde{R})\ (\forall w \in \mathcal{W})\ (\exists w' \in \mathcal{W}_{\widetilde{R}})\ w\alpha \sim w'\alpha$$

Proposition 4.9. *If the group $\mathcal{W}_{\widetilde{R}}$ has a presentation by conjugation with respect to $\widetilde{R}$, then the group $\mathcal{W}$ has a presentation by conjugation with respect to R.*

Proof. We will start with a relation

$$\widehat{r}_*\widehat{r}_*\cdots\widehat{r}_* = 1 \tag{10}$$

satisfied in the group $\mathcal{W}$ and show that the relation follows from the relations of the presentation by conjugation:

$$\widehat{r}_\alpha = \widehat{r}_\beta \quad \text{for } \alpha \text{ and } \beta \in R \text{ with } \alpha \sim \beta, \tag{11}$$
$$\widehat{r}_\alpha^2 = 1 \quad \text{for } \alpha \in R, \tag{12}$$
$$\widehat{r}_\alpha\widehat{r}_\beta\widehat{r}_\alpha^{-1} = \widehat{r}_{r_\alpha(\beta)} \quad \text{for } \alpha \text{ and } \beta \in R \tag{13}$$

First we suppose that relation (10) contains no elements $\widehat{r}_\alpha$ with $\alpha \in R\setminus\widetilde{R}$. But then we know by hypothesis that relation (10) follows from the relations in (11) to (13) with α and $\beta \in \widetilde{R}$. So in this case there is nothing to prove.

Now we suppose that relation (10) contains at least one $\widehat{r}_\alpha$ with $\alpha \in R\setminus\widetilde{R}$. Moreover, we assume it contains no $\widehat{r}_{\alpha'}$ such that $\alpha \sim w.\alpha'$ for some $w \in \mathcal{W}$. Then the relation can be written as

$$\widehat{r}_\alpha \quad = \quad \widehat{r}_*\widehat{r}_*\cdots\widehat{r}_*,$$

where all the generators on the right side are associated to roots in

$$R_\alpha = R \setminus \{\gamma : (\exists w \in \mathcal{W})\gamma \sim w.\alpha\}.$$

This means that the reflections associated to this set actually generate $\mathcal{W}$, which is a contradiction to the fact that R is minimal.

So, if we are supposing that relation (10) contains an $\widehat{r}_\alpha$ with $\alpha \in R \setminus \widetilde{R}$ then it must contain another element $\widehat{r}_{\alpha'}$ with $\alpha \sim w.\alpha'$ for some $w \in \mathcal{W}$. So it is of the form

$$\widehat{r}_* \cdots \widehat{r}_* \widehat{r}_\alpha \widehat{r}_* \cdots \widehat{r}_* \widehat{r}_{\alpha'} \widehat{r}_* \cdots \widehat{r}_* \quad = \quad 1. \tag{14}$$

By hypothesis, we may assume $w \in \mathcal{W}_{\widetilde{R}}$. So we have

$$w \quad = \quad r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_n}$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \widetilde{R}$. The conjugation relations (11) and (13) imply

$$\begin{aligned} \widehat{r}_\alpha \quad &= \quad \widehat{r}_{w.\alpha'} = \widehat{r}_{r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_n}.\alpha'} \\ &= \quad \widehat{r}_{\alpha_1} \widehat{r}_{\alpha_2} \cdots \widehat{r}_{\alpha_n} \; \widehat{r}_{\alpha'} \; \widehat{r}_{\alpha_n}^{-1} \widehat{r}_{\alpha_{n-1}}^{-1} \cdots \widehat{r}_{\alpha_1}^{-1}, \end{aligned}$$

which can be rewritten as

$$\widehat{r}_{\alpha'} \quad = \quad \widehat{r}_{\alpha_n} \widehat{r}_{\alpha_{n-1}} \cdots \widehat{r}_{\alpha_1} \; \widehat{r}_\alpha \; \widehat{r}_{\alpha_1} \widehat{r}_{\alpha_2} \cdots \widehat{r}_{\alpha_n}$$

using relation (12). So our relation (14) can be written as

$$\widehat{r}_* \cdots \widehat{r}_* \widehat{r}_\alpha \widehat{r}_* \cdots \widehat{r}_* \widehat{r}_\alpha \widehat{r}_* \cdots \widehat{r}_* \quad = \quad 1. \tag{15}$$

Note that number of generators in this relation may have increased but the number of generators elements $\widehat{r}_\beta$ with $\beta \in R \setminus \widetilde{R}$ has not changed.

For any $\beta \in R$ the conjugation relation $\widehat{r}_\alpha \widehat{r}_\beta \widehat{r}_\alpha^{-1} = \widehat{r}_{r_\alpha(\beta)}$ can be rewritten as

$$\widehat{r}_\beta \widehat{r}_\alpha \quad = \quad \widehat{r}_\alpha \widehat{r}_{r_\alpha(\beta)}$$

using relation (12). This means that the relation (15) can be rewritten as

$$\widehat{r}_* \cdots \widehat{r}_* \widehat{r}_\alpha \widehat{r}_\alpha \widehat{r}_* \cdots \widehat{r}_* \quad = \quad 1.$$

The two generators $\widehat{r}_\alpha$ in this relation cancel out reducing it to a relation which has fewer generators $\widehat{r}_\beta$ with $\beta \in R \setminus \widetilde{R}$. This process can be repeated until the number of generators $\widehat{r}_\beta$ with $\beta \in R \setminus \widetilde{R}$ is zero. Then we are done since we are in a case that has been discussed earlier. ■

5. Presentation by conjugation for extended affine Weyl groups

The main result of this work is derived in this section: Given an EARS R, the corresponding Weyl group $\mathcal{W}$ has the presentation by conjugation with respect to $R^\times$ if and only if R is minimal. Moreover, every EARS R contains a minimal one. In particular, every EAWeG $\mathcal{W}$ has a minimal $\mathcal{W}$-invariant generating set of root reflections. Taken together, this means that every EAWeG has a presentation by conjugation with respect to some EARS.

First we need to study the orbits under an EAWeG $\mathcal{W}$ of certain points in the span of the root system.

Proposition 5.1. *Let $\alpha \in \mathcal{V}$ and $\dot\alpha \in \dot R$ with $\alpha - \dot\alpha \in \mathcal{V}^0$. We set*

$$T := \big(\dot\alpha, \dot R^\vee_{\mathrm{sh}}\big)\langle S\rangle + \big(\dot\alpha, \dot R^\vee_{\mathrm{lg}}\big)\langle L\rangle + \big(\dot\alpha, \dot R^\vee_{\mathrm{ex}}\big)\langle E\rangle.$$

Then

$$\mathcal{W}.\alpha = \alpha - \dot\alpha + \dot{\mathcal{W}}\dot\alpha + T \tag{16}$$

Proof. To prove that a set is an orbit of a group, two things need to be proved: The set is invariant under the group and the group acts transitively on the set.

For invariance we suppose that $\alpha - \dot\alpha + \dot w.\dot\alpha + \tau$ is an element of the set on the right side of (16), i.e. $\dot w \in \dot{\mathcal{W}}$ and $\tau \in T$. Let $\beta \in R$. Then it can be written as $\beta = \dot\beta + \sigma$ with $\dot\beta \in \dot R$ and $\sigma \in \mathcal{V}^0$.

$$\begin{aligned} r_{\dot\beta+\sigma}(\overbrace{\alpha - \dot\alpha}^{\in\mathcal{V}^0} + \dot w.\dot\alpha + \tau) &= \alpha - \dot\alpha + \dot w.\dot\alpha + \tau - (\dot w.\dot\alpha, \dot\beta^\vee)(\dot\beta + \sigma) \\ &= \alpha - \dot\alpha + \underbrace{r_{\dot\beta}\dot w.\dot\alpha}_{\in\dot{\mathcal{W}}.\dot\alpha} + \underbrace{\tau}_{\in T} - \underbrace{(\dot\alpha, \dot w^{-1}.\dot\beta^\vee)\sigma}_{\in T} \end{aligned}$$

Thus, since $\mathcal{W}$ is generated by the reflections associated to the roots, the right hand side of (16) is invariant under $\mathcal{W}$.

Now we prove that $\mathcal{W}$ acts transitively on this set. It suffices to show that for any $\tau \in T$, any $\dot\beta \in \dot R^\times$ and any $\sigma \in S_{\dot\beta} = \{\rho \in \mathcal{V}^0 : \dot\beta + \rho \in R\}$ there is a $w \in \mathcal{W}$ such that

$$w.(\alpha + \tau) = \alpha + \tau + (\alpha, \beta^\vee)\sigma.$$

We have

$$\begin{aligned} r_{\dot\beta} r_{\dot\beta-\sigma}.(\alpha + \tau) &= r_{\dot\beta}\Big(\alpha + \tau - (\dot\alpha, \dot\beta^\vee)(\dot\beta - \sigma)\Big) \\ &= r_{\dot\beta}\Big(r_{\dot\beta}.\alpha + \tau + (\dot\alpha, \dot\beta^\vee)\sigma\Big) \\ &= \alpha + \tau + (\dot\alpha, \dot\beta^\vee)\sigma \end{aligned}$$

This concludes the proof. ∎

Definition 5.2. Let $R^\times$ be a subset of $\mathcal{V}$. Then the set

$$\mathrm{IRC}(R^\times) := \big((R^\times - R^\times) \cap \mathcal{V}^0\big) \cup R^\times$$

is called the *isotropic root closure*.

Lemma 5.3. *If R is an EARS and $R^\times$ is the set of its anisotropic roots then* $\mathrm{IRC}(R^\times) = R$.

Proof. The inclusion $\supseteq$ follows from axiom (R6). For the converse inclusion let $\sigma \in \big((R^\times - R^\times) \cap \mathcal{V}\big)$. So there are α and $\beta \in R^\times$ with $\sigma = \beta - \alpha \in \mathcal{V}^0$. This implies $(\alpha, \beta) = (\alpha, \alpha) > 0$. So by (R6) the set $\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R$ contains $\sigma = \beta - \alpha$, since $d = 2\frac{(\alpha,\beta)}{(\alpha,\beta)} + u > u \geq 0$. This implies $\sigma \in R$, and we are done. ∎

Definition 5.4. An EARS R is called *minimal* if the root set $R^\times$ for the Weyl group $\mathcal{W}$ is minimal.

Example 5.5. Let R be the EARS of Type A_1, nullity $\nu = 3$ such that the index of the involved semilattice S (see Theorem 2.4) is 7. This EARS is not minimal. For any anisotropic root γ, the reflections associated to the set

$$R_\gamma^\times := R^\times \setminus \mathcal{W}.\gamma$$

generate the Weyl group $\mathcal{W}$. This is proven in [10] by realizing that r_γ is a so-called ghost reflection (i.e. a reflection that is not associated to a root) for the EARS $\mathrm{IRC}(R_\gamma^\times)$ (see Example 4.4.83). In [2] it is proven that apart from this example there are many more EARS of type A_1 that are not minimal: for every nullity $\nu \geq 3$, there is at least one (See Corollary 5.19 in view of Theorem 5.16).

In the proof of the next theorem we will need to handle EARS of type BC_ℓ with special care. They are best dealt with by trimming them into an EARS of reduced type. This motivates the following definition.

Definition 5.6. Let R be an EARS of non-reduced type, i.e. of type BC_ℓ, $\ell \geq 1$. We set

$$\mathrm{trim}(R) \quad = \quad \mathrm{IRC}(R_{\mathrm{sh}} \cup R_{\mathrm{lg}} \cup \frac{1}{2} R_{\mathrm{ex}})$$

and call $\mathrm{trim}(R)$ the *trimmed root system* of R.

The following result justifies this terminology.

Lemma 5.7. (i) *The set* $\mathrm{trim}(R)$ *is an EARS of type* B_ℓ.

(ii) *The root system* $\mathrm{trim}(R)$ *has the same Weyl group as* R.

(iii) *The root system* $\mathrm{trim}(R)$ *has the same group* $\widehat{\mathcal{W}}$ *(see (8)) as* R. *In particular* $\mathcal{W}$ *has a presentation by conjugation with respect to* R *if and only if it has a presentation by conjugation with respect to* $\mathrm{trim}(R)$.

(iv) *The root system* $\mathrm{trim}(R)$ *is minimal if and only if* R *is minimal.*

Proof. In order to show that $\mathrm{trim}(R)$ is an EARS it suffices to see that $R_{\mathrm{sh}} \cup R_{\mathrm{lg}} \cup \frac{1}{2} R_{\mathrm{ex}}$ is the set of anisotropic roots of an EARS in view of Lemma 5.3. If we set $S' := S \cup \frac{1}{2} E$, we have

$$R_{\mathrm{sh}} \cup R_{\mathrm{lg}} \cup \frac{1}{2} R_{\mathrm{ex}} = \left(S' + S'\right) \cup \left(\bigcup_{\dot\alpha \in \dot R_{\mathrm{sh}}} (\dot\alpha + S') \right) \cup \left(\bigcup_{\dot\alpha \in \dot R_{\mathrm{lg}}} (\dot\alpha + L) \right). \tag{17}$$

A look at Theorem 2.4 reveals that it suffices to show that S' is a semilattice satisfying

$$L + 2S' \subseteq L \quad \text{and} \quad S' + L \subseteq S'.$$

We have

$$\begin{aligned}
S' + 2S' &= (S \cup \frac{1}{2}E) + 2(S \cup \frac{1}{2}E) \\
&= \underbrace{(S+2S)}_{\subseteq S} \cup \underbrace{(S+E)}_{\subseteq S} \cup \underbrace{(\frac{1}{2}E + 2S)}_{\subseteq \frac{1}{2}E} \cup \underbrace{(\frac{1}{2}E + E)}_{\subseteq \frac{1}{2}E} \\
&\subseteq S \cup \frac{1}{2}E = S', \\
L + 2S' &= L + 2(S \cup \frac{1}{2}E) \\
&= \underbrace{(L+2S)}_{\subseteq L} \cup \underbrace{(L+E)}_{\subseteq L} \\
&\subseteq L \quad \text{and} \\
S' + L &= (S \cup \frac{1}{2}E) + L \\
&= \underbrace{(S+L)}_{\subseteq S} \cup \underbrace{(L + \frac{1}{2}E)}_{\subseteq \frac{1}{2}E} \\
&\subseteq S \cup \frac{1}{2}E = S'
\end{aligned}$$

From (17) it is clear that $\operatorname{trim}(R)$ is of the type that you obtain by omitting the extra long roots in a BC_ℓ type finite root system. This is type B_ℓ.

Item (ii) follows from the fact that the two Weyl groups have the same generating set and (iii) and (iv) follow from the one-to-one correspondence of orbits of the Weyl group in R and $\operatorname{trim}(R)$. ■

Theorem 5.8. *Let R be an EARS and let $\mathcal{W}$ be its Weyl group. Then $\mathcal{W}$ has a presentation by conjugation with respect to R if and only if R is minimal.*

Proof. Let R be an EARS and let $\mathcal{W}$ be its Weyl group. In view of Lemma 5.7, we may assume that R is of reduced type.

In [3] a subset R_Π of R is introduced which is also an EARS (See Proposition 4.12 of [3]). Furthermore it is proven that the Weyl group $\mathcal{W}_\Pi$ of that root system R_Π has a presentation by conjugation (See Theorem 5.15 of [3]). If we prove that

$$(\forall w \in \mathcal{W})\ (\forall \alpha \in R^\times)\ (\exists w' \in \mathcal{W}_{R_\Pi})\ w.\alpha = w'.\alpha \tag{18}$$

then we are done by Proposition 4.9.

Due to Proposition 4.41 and Equation (4.16) in [3] we only need to consider EARSs R of type $X = A_1$, $X = B_\ell$ ($\ell \geq 2$) and $X = C_\ell$ ($\ell \geq 3$). According to [3] (4.11), if we have

$$\begin{aligned}
R &= \big(S+S\big) \cup (\dot{R}_{\mathrm{sh}} + S) \cup (\dot{R}_{\mathrm{lg}} + L) \quad \text{then} \\
R_\Pi &= \big(S_\Pi + S_\Pi\big) \cup (\dot{R}_{\mathrm{sh}} + S_\Pi) \cup (\dot{R}_{\mathrm{lg}} + L_\Pi) \quad \text{with} \\
&\quad \langle S_\Pi \rangle = \langle S \rangle \quad \text{and} \quad \langle L_\Pi \rangle = \langle L \rangle
\end{aligned}$$

In view of Proposition 5.1 this means that the two Weyl groups $\mathcal{W}$ and $\mathcal{W}_\Pi$ have the same orbits in $R^\times$. This entails our requirement in (18). ■

The following characterization of EARSs will be useful when we need to identify certain subsets of EARSs as EARSs.

Proposition 5.9. *Let $R^\times$ be a set of anisotropic vectors in $\mathcal{V}$ such that*

(i) $\mathcal{W}_{R^\times}.R^\times \subseteq R^\times$,

(ii) $\overline{R^\times}$ *is an irreducible finite root system,*

(iii) $\langle R^\times \rangle$ *is a lattice in* $\mathcal{V}$,

(iv) $\alpha \in R^\times$ *implies* $2\alpha \notin R^\times$.

Then $R := \mathrm{IRC}(R^\times)$ *is the EARS having* $R^\times$ *as its set of anisotropic roots.*

Proof. We need to verify the axioms (R1) - (R8). Axiom (R1) follows from the definition of $\mathrm{IRC}(R^\times)$. Due to the fact that $r_\gamma.\gamma = -\gamma$ for every $\gamma \in R^\times$ we have $-R^\times = R^\times$. By the definition of $\mathrm{IRC}(R^\times)$, this implies (R2). Item (iii) implies (R3) and (iv) implies (R4). Since R is contained in $\langle R^\times \rangle$, it is discrete, so (R5) is satisfied. Since $\overline{\dot{R}^\times}$ is irreducible (R7) holds. Axiom (R8) follows from the definition of $\mathrm{IRC}(R^\times)$.

The only axiom remaining to be proved is (R6). So let $\alpha \in R^\times$ and $\beta \in R$. We will look at three different cases.

Case $\overline{\alpha}$ and $\overline{\beta}$ linearly independent: Then let $\mathcal{W}_{\alpha,\beta}$ be the subgroup of $\mathcal{W}$ generated by α and β. This subgroup stabilizes the subspace $\mathcal{V}_{\alpha,\beta} = \mathrm{span}(\alpha, \beta)$ of $\mathcal{V}$. Set $R_{\alpha,\beta} = \mathcal{W}_{\alpha,\beta}.\{\alpha, \beta\}$. Since

$$2\frac{(\gamma,\delta)}{(\delta,\delta)} = 2\frac{(\overline{\gamma},\overline{\delta})}{(\overline{\delta},\overline{\delta})} \in \mathbb{Z}$$

for all δ, $\gamma \in R_{\alpha,\beta}$. So the set $R_{\alpha,\beta}$ is a finite root system in $\mathcal{V}_{\alpha,\beta}$. In this situation (R6) follows from [5] Ch. VI n^0 1.3 Proposition 9.

Case $\beta \in R^0$: Then

$$2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 0 \quad \text{and} \quad r_\alpha.(\beta + n\alpha) = \beta + n\alpha - 2\frac{(n\alpha,\alpha)}{\alpha,\alpha)}\alpha = \beta - n\alpha$$

for any $n \in \mathbb{Z}$. As a result of this symmetry and due to the fact that

$$\overline{\{\beta + n\alpha : n \in \mathbb{Z} \cap R^\times\}} = \{n\overline{\alpha} : n \in \mathbb{Z}\} \cap \overline{R}^\times$$

is a subset of the finite root system $\overline{R}^\times$ there can only be two cases:

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \begin{cases} \{\beta - \alpha, \beta, \beta + \alpha\} & \text{or} \\ \{\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\}. \end{cases}$$

In both cases (R6) is satisfied.

Case $\beta \in R^\times$ and $\overline{\alpha}$ and $\overline{\beta}$ linearly dependent: Since

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta + n(-\alpha) : n \in \mathbb{Z}\} \cap R \quad \text{and} \quad 2\frac{(-\alpha,\beta)}{(-\alpha,-\alpha)} = -2\frac{(\alpha,\beta)}{(\alpha,\alpha)},$$

We only need to consider the following cases:

(i) $\overline{\alpha} = \overline{\beta}$,

(ii) $2\overline{\alpha} = \overline{\beta}$ and

(iii) $\overline{\alpha} = 2\overline{\beta}$.

For each of these cases, it suffices to show that there is an $n \in \mathbb{Z}$ such that $\beta' := \beta + n\alpha \in R^0$, since

$$\begin{aligned}
\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R &= \{\beta' + n\alpha : n \in \mathbb{Z}\} \cap R \\
\{\beta - d\alpha, \dots, \beta + u\alpha\}, &= \{\beta' - (d+n)\alpha, \dots, \beta' + (u-n)\alpha\} \text{ and} \\
(d+n) - (u-n) = d - u + 2n &= 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} + 2\frac{(\alpha,n\alpha)}{(\alpha,\alpha)} = 2\frac{(\alpha,\beta')}{(\alpha,\alpha)}.
\end{aligned}$$

So the proof of this case is reduced to case discussed previously.

Case (i): Then $\beta - \alpha \in R^0$ by the definition of $\mathrm{IRC}(R^\times)$.

Case (ii): Then

$$r_\beta(-\alpha) = -\alpha - 2\frac{(-\alpha,\beta)}{(\beta,\beta)}\beta = -\alpha - 2\frac{(-\frac{1}{2}\overline{\beta},\overline{\beta})}{(\overline{\beta},\overline{\beta})}\beta = \beta - \alpha \in R.$$

Then by the definition of $\mathrm{IRC}(R^\times)$, we have $\beta - 2\alpha \in R^0$.

Case (iii): We have

$$r_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \alpha.$$

So

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R \supseteq \{\beta, \beta - \alpha\}.$$

We see that we must have equality, since any element in the left hand side and not in the right hand side would have at least three times the length of α. ■

Lemma 5.10. *Let R be an EARS with EAWeG $\mathcal{W}$. Suppose there is a root $\beta \in R^\times$ such that the reflections associated to*

$$\widetilde{R^\times} := R^\times \setminus \mathcal{W}.\beta$$

generate $\mathcal{W}$. Then there is an EARS $\widetilde{R}$, whose anisotropic roots are $\widetilde{R^\times}$. When passing from R to $\widetilde{R}$, only the following changes in types are possible:

$$BC_1 \to A_1, \quad BC_2 \to B_2, \quad BC_\ell \to B_\ell, \quad BC_\ell \to C_\ell, \quad \textit{where } \ell \geq 3.$$

Proof. We use Proposition 5.9 for $\widetilde{R^\times}$. So we need to verify conditions (i) - (iv) of that proposition. Conditions (i), and (iv) are evident so first we focus on (iii). Discreteness of $\langle\widetilde{R^\times}\rangle$ follows from the fact that $\langle R^\times\rangle$ is discrete. (This can be seen, for instance from Theorem 2.4.) Now we suppose that $\mathrm{span}(\widetilde{R^\times})$ is a real subset U of $\mathcal{V}$. This means that U is invariant under $\mathcal{W}_{\widetilde{R^\times}}$ and thus also under $\mathcal{W}_{R^\times}$. That is a contradiction to axiom (R7) for the EARS R.

Now we verify (ii). The group $\overline{\mathcal{W}_{R^\times}}$ obtained by reducing the elements of the group $\mathcal{W}_{R^\times}$ to automorphisms of $\mathrm{Aut}(\mathcal{V}/\mathcal{V}^0)$ is the same as the subgroup $\mathcal{W}_{\overline{R^\times}}$

of $\mathrm{Aut}(\mathcal{V}/\mathcal{V}^0)$ generated by the reflections associated to the roots in $\overline{R^\times}$. So by hypothesis we have

$$\mathcal{W}_{\overline{R^\times}} = \mathcal{W}_{\overline{\widetilde{R^\times}}}$$

By Lemma 4.8, the set $\overline{\widetilde{R^\times}}$ is an irreducible finite crystallographic root system. The restrictions about the type of $\widetilde{R}$ follow from this lemma, as well. ■

Theorem 5.11. *Every EARS R contains a minimal one R' with the same Weyl group. When passing from R to R' only the following changes in type are possible:*

$$BC_1 \to A_1, \quad BC_2 \to B_2, \quad BC_\ell \to B_\ell, \quad BC_\ell \to C_\ell, \quad \textit{where } \ell \geq 3.$$

Proof. Due to Proposition 5.1 the Weyl group $\mathcal{W}$ of R has only a finite number of orbits in R. So Lemma 5.10 can be applied repeatedly until a minimal EARS $R' \subseteq R$ is obtained. The only possible changes in type are those indicated in Lemma 5.10. ■

It is tempting to think that an EARS contained in a minimal EARS is minimal. But this is not necessarily the case, as the following example reveals:

Example 5.12. Let R be a minimal EARS of type A_1 and nullity 3. (Such an EARS exists by Theorem 2.4 and Theorem 5.11. If R is given by

$$R = (S + S) \cup (\dot{R}_{\mathrm{sh}} + S)$$

then the set

$$R = (S' + S') \cup (\dot{R}_{\mathrm{sh}} + S') \quad \text{with} \quad S' = 2\langle S \rangle$$

is an EARS contained in R by Theorem 2.4. Since this EARS is isomorphic to the EARS R discussed in Example 5.5, it is not minimal.

The statements of Theorem 5.8 and Theorem 5.11 taken together imply:

Theorem 5.13. *Every EAWeG $\mathcal{W}$ has the presentation by conjugation with respect to some EARS.*

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