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Lie Group Invariants of Inhomogeneous Polynomial Vector Spaces

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Abstract. We present a method which efficiently generates Lie group invariants in the classical invariant theory of polynomials and its extensions to vector spaces of inhomogeneous polynomials under the actions of the general affine group and pseudo-Euclidean subgroups. Our derivation of the invariants uses the classical Cartan method of moving frames and requires no assumption on the degree of the polynomial or the number of variables. Consequently, we are able to express the invariants in a compact indicial notation. We employ our results to solve the equivalence and canonical forms problems for the vector space of inhomogeneous cubic polynomials in two real variables under the action of the Euclidean group. We show that the space partitions into twelve distinct classes of canonical forms, each admitting a system of invariants which globally separates its associated orbits.
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1. Introduction

A motivational quote from Gian-Carlo Rota's book *Indiscrete Thoughts* beautifully encapsulates the essence of this paper:

> *"It is a task for the present generation to recreate the lost life of invariant theory. Whether this task will be accomplished by rereading and reinterpreting the classics or whether it will be reinvented, as some physicists are now doing, will be one of the dramas of the coming years which which will be watched with interest..."* [18, pg 222]

The quintessential problem of invariant theory is the determination of a complete set of *invariant functions* for a given group action. Precisely speaking, if G is a transformation group acting on a space X, one requires functions $\mathcal{I} : X \to F$, where F is a field, satisfying

$$\mathcal{I}(g \cdot x) = \mathcal{I}(x)$$

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for all $g \in G$. This central problem is intimately linked to the problems of *equivalence* and *canonical forms*. For the former, one asks if a group element $g \in G$ exists such that $g \cdot x = y$, given $x, y \in X$. Evidently, this notion defines an equivalence relation on X. Thus, the canonical form problem seeks a "suitable" or "simple" representative in each equivalence class. In the setting of *classical invariant theory* (CIT) (see Olver [16] for more details), the space X is typically a vector space of homogeneous polynomials of some fixed degree m in n variables, usually over $\mathbb{C}$ or $\mathbb{R}$. The group G is commonly the general linear group $\mathrm{GL}(n, F)$ or a subgroup thereof. In this paper, we shall present an algorithm which streamlines the generation of invariants in CIT and its extensions to vector spaces of *inhomogeneous polynomials* under the action of the general affine group and some of its subgroups.

The mathematical foundations of invariant theory can be cast into the frameworks of Klein and Cartan geometry. Klein's synthesis of geometry, as introduced in his famous *Erlangen Program* [12, 13], stipulates how the essential properties of a given geometry (e.g. Euclidean, affine, projective, etc.) can be represented by the transformation group which preserves these properties. In the Cartan approach to differential geometry, the philosophies due to Klein together with the concepts of a Riemannian manifold and a metric tensor are fused into a single theory [19]. Invariant theory can thus be rightfully placed into the theory initiated by Cartan. We shall elaborate on Cartan geometries and its applicability to CIT in Section 2.

The primary goal in the study of a particular group action over a space is a proof of a Hilbert finiteness theorem or "first fundamental theorem" (FFT). Such an FFT establishes that the space admits a finite Hilbert basis, i.e. a finite basis of invariants with the property that any other invariant is a polynomial function of the basis elements. In the setting of CIT, FFT proofs are well known for spaces of homogeneous polynomials under the action of the general linear, orthogonal and symplectic groups (see [7, ch 4] for a modern treatment). Methods for generating invariants and Hilbert bases are also widely known, some of which include the use of "transvection" and the so-called Ω-process, symbolic methods and infinitesimal methods [7, 16, 22]. These approaches are all highly sensitive to the underlying dimension of the vector space of polynomials. The latter method in particular involves solving linear partial differential equations arising from the associated Lie algebra action. Forming a suitable polynomial ansatz for the invariants, one obtains a (sparse) system of linear equations for the undetermined coefficients [22, theorem 4.5.2]. Although this method is straightforward to implement in any computer algebra system, it inherently suffers from the "curse of dimensionality".

In contrast to the dimensionally dependent methods for generating invariants, the method we present in this paper reveals that the invariants can be expressed in a tantalizingly simple form. Our method can be realized as a generalization of the transvection process (see for example, [16, ch 5]). We shall see that the invariants can be expressed using a compact tensorial or indicial notation, irrespective of the degree of the polynomial or the number of variables. Our results will equally apply to vector spaces of inhomogeneous polynomials under the action of the affine group and the pseudo-Euclidean subgroups. To the knowledge of the author, the inhomogeneous case has yet to be fully treated in the literature. In fact, our result is so simple and compact that the task of computing invariants is now likely to be perceived as a triviality. Quoting Rota again,

> *"Not only is every mathematical problem solved, but eventually every mathematical problem is proved trivial. The quest for ultimate triviality is characteristic of the mathematical enterprise."* [18, pg 93]

The main results concerning the generation of invariants are stated in Theorems 4.2 and 4.3 and Corollary 4.4, the proofs of which are a direct consequence of the representation of the group on the underlying space of polynomials derived in Section 3. For simplicity, we have restricted our attention to the real number field throughout the paper; our results easily generalize to the complex case. The proof of these theorems uses a "hybrid" version of the classical Cartan method of moving frames [4, 5] in conjunction with the recursive construction of the moving frame recently introduced by Kogan [14].

Once a set of fundamental (functionally independent) group invariants is found, one can employ these invariants to classify the orbits of the associated space. This procedure is often challenging to fully implement because invariants are largely generated by *local* methods. Indeed, the moving frame method requires a choice of cross-section, however the resulting moving frame map is generally not globally defined. Thus, the resulting invariants need not separate the orbits globally. This lack of discriminating power in the invariants is also apparent as orbits generally do not all have the same dimension. One way to fix these deficiencies in the group action is to extend the space on which it acts, for example, by prolonging the group action to copies of the original space, thus leading to the concept of *joint invariants* [2]. Another possible type of prolongation (and one that is emphasized in this paper) leads to *covariants*. In the context of CIT, covariants are similar to invariants but may also have explicit dependence on the coordinates of the polynomial. In this paper, we shall see that covariants and their transvectants are extremely useful in providing a *global* classification of the orbits of the original unprolonged space.

In Section 5, we employ the theory of this paper to solve the equivalence and canonical forms problems for the space of inhomogeneous cubic polynomials in two variables, under the action of the Euclidean group of rigid motions on the plane. Moreover, we shall demonstrate that there exist sets of invariants which separate the orbits of the vector space globally. The analogous problem for homogeneous cubics in three variables under the action of $\mathrm{GL}(3,\mathbb{C})$ is treated in [15], however we are unaware of any classification schemes for inhomogeneous spaces of cubic polynomials and thus believe our results to be new.

The results of this paper extend to the *invariant theory of Killing tensors* (ITKT) (see [9] and the relevant references therein), which have wide applicability to separation of variable theory and integrability of finite-dimensional Hamiltonian systems. An article to this effect is currently in preparation [8]. The origin of the idea to write invariants in a compact indicial notation actually stems from ITKT. The first time such a notation was employed in ITKT was in the derivation of fundamental isometry group invariants for the vector space of valence three Killing tensors defined on the Euclidean plane [10] (see also [9]). The example of Section 5, in particular, will provide further insight into the problem of classifying the orbits of the aforementioned vector space of Killing tensors and shall prove useful for analyzing the associated cubic first integrals of motion in the momenta.

2. Mathematical foundations

The classical invariant theory of polynomials can be realized as an extension of Cartan's geometry. Recall that the traditional Cartan approach to differential geometry [6, 11] is based on the following key elements. Let G be a Lie group acting transitively on a homogeneous space $\mathcal{M}$, identified as the left coset space G/H, where H is a closed subgroup of G. In the Cartan philosophy, one is interested in the geometric properties of submanifolds $\mathcal{N}$ of $\mathcal{M}$ which are invariant under G. The study of these intrinsic properties of $\mathcal{N}$ is established in the study of the principal fibre H-bundle

$$\pi : G \to G/H \simeq \mathcal{M},$$

together with a map $F : \mathcal{N} \to \mathcal{M}$, describing the position of the submanifold in the homogeneous space. Finally, a map $f : \mathcal{N} \to G$, called a *lift*, is defined so that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\pi"] & G/H \simeq \mathcal{M} \\ & \mathcal{N} \arrow[ul, "f"] \arrow[u, "F"'] \end{tikzcd} \tag{1}$$

Cartan realized that in many cases the group G may be identified as a set of frames (or orthonormal bases) of $\mathcal{M}$. Consequently, associated to the submanifold $\mathcal{N}$ is a "natural" set of frames or, equivalently, a "cross-section" of the fibration π over $\mathcal{N}$. In such cases, the Mauer-Cartan forms of G, when restricted to this natural frame, become a complete set of invariants for $\mathcal{N}$ in $\mathcal{M}$. The map f is thus the "moving frame map" which, given any point $p \in \mathcal{N}$, yields the group action which maps the frame at the point p to to its natural or "canonical" frame. The invariants are thus the coordinates of these "canonical forms". The following lemma due to Cartan (and restated and proved in [6]) gives necessary and sufficient conditions on the existence of a moving frame map.

Lemma 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let ω denote the $\mathfrak{g}$-valued left-invariant Maurer-Cartan forms on G. Suppose φ is a $\mathfrak{g}$-valued one-form on a connected (or simply connected) manifold $\mathcal{N}$. Then there exists a C^∞-map $f : \mathcal{N} \to G$ with $f^*\omega = \varphi$ iff $\mathrm{d}\varphi = \varphi \wedge \varphi$. Moreover, the resulting map is unique up to left translation.*

Thus, the Mauer-Cartan form ω in Lemma 2.1 is what carries the information "upstairs" about G. When a map f exists, we have at the same time "downstairs" on $\mathcal{N}$ a form φ (e.g. a frame of vectors) which is connected by the same structure constants as those of the Lie algebra of G. The lift or moving frame map f pulls back ω to φ.

The use of the modern language of fibre bundles, invariant sections and homogeneous spaces to describe the framework of classical invariant theory has been developed by Fels and Olver [3]. In what follows, we shall use this language of Cartan geometry to extend the diagram (1) to spaces of polynomials defined on a real n-dimensional manifold $\mathcal{M}$. Our framework is analogous to that used in the extension of Cartan geometry to the invariant theory of Killing tensors, first introduced by Adlam, McLenaghan and Smirnov in [1]. In order to proceed, it is necessary first to precisely define a polynomial function on a manifold $\mathcal{M}$, independent of the choice of coordinates. To achieve such a definition, we shall henceforth restrict our attention to the case when $\mathcal{M}$ is the n-fold product space $\mathbb{R} \times \cdots \times \mathbb{R}$ defined by the set of n-tuples $(x^1, \ldots, x^n)$.

Definition 2.2. A function $f : \mathcal{M} \to \mathbb{R}$ defined on an n-dimensional manifold $\mathcal{M} = \mathbb{R} \times \cdots \times \mathbb{R}$ is a *homogeneous polynomial of degree* m iff for a given set of coordinates x^i, there exist constants $c_{i_1 \cdots i_m}$ such that

$$f(p) = \sum_{i_1,\ldots,i_m=1}^{n} c_{i_1 \cdots i_m} x^{i_1} \cdots x^{i_m}, \tag{2}$$

for all $p \in \mathcal{M}$.

Here and in what follows, we will use the summation convention: any repeated upper and lower index implies summation over that index from 1 to n. Thus, (2) can be written compactly as

$$f(p) = c_{i_1 \cdots i_m} x^{i_1} \cdots x^{i_m}. \tag{3}$$

The use of the notation x^i to denote the i-th coordinate name (rather than x_i) is standard in tensor analysis because the x^i transform like the components of a contravariant tensor under a change of coordinates. We also note that one may assume without loss of generality that the m-index object $c_{i_1 \cdots i_m}$ in (3) is completely symmetric in all its indices, i.e. $c_{i_1 \cdots i_m} = c_{(i_1 \cdots i_m)}$.

It is obvious that Definition 2.2 is independent of the choice of coordinates. Indeed, if $\tilde{x}^i$ is another set of coordinates, then $x^i = A^i{}_j \tilde{x}^j$ for some $A^i{}_j \in \mathrm{GL}(n)$. This coordinate transformation preserves (3).

Let $\mathcal{P}^m(\mathcal{M})|_{\boldsymbol{x}}$ denote the set of homogeneous polynomials of degree m defined on $\mathcal{M}$ with respect to the coordinates $\boldsymbol{x} = (x^1, \ldots, x^n)$. It is straightforward to extend Definition 2.2 to $\mathcal{P}_m(\mathcal{M})|_{\boldsymbol{x}}$, the set of inhomogeneous polynomials of degree at most m. Both $\mathcal{P}^m(\mathcal{M})|_{\boldsymbol{x}}$ and $\mathcal{P}_m(\mathcal{M})|_{\boldsymbol{x}}$ define *real vector spaces* with the following dimensions:

$$\dim \mathcal{P}^m(\mathcal{M})|_{\boldsymbol{x}} = \binom{n+m-1}{m}, \quad \dim \mathcal{P}_m(\mathcal{M})|_{\boldsymbol{x}} = \frac{m+1}{n}\binom{n+m}{m+1}. \tag{4}$$

The philosophy of Cartan provides a natural framework to study spaces of inhomogeneous polynomials. The formulation begins by considering a Lie group G which acts transitively on $\mathcal{M}$ and a closed subgroup H of G. Thus, as in the standard model of Cartan geometry, we have the principal fibre H-bundle

$$\pi_1 : G \to G/H \simeq \mathcal{M}.$$

In analogy to the definitions of the tangent space $T(\mathcal{M})|_{\boldsymbol{x}}$ at $\boldsymbol{x} \in \mathcal{M}$ and the tangent bundle $T(\mathcal{M})$, we can define $\mathcal{P}_m(\mathcal{M})$ to be the *bundle of inhomogeneous polynomials of degree* m *on* $\mathcal{M}$ viz

$$\mathcal{P}_m(\mathcal{M}) = \big\{(\boldsymbol{x}, Q) \mid \boldsymbol{x} \in \mathcal{M},\ Q \in \mathcal{P}_m(\mathcal{M})|_{\boldsymbol{x}}\big\}.$$

The structure of $\mathcal{P}_m(\mathcal{M})$ induces two additional fibrations. Firstly, we have the natural structure of a vector bundle

$$\pi_2 : \mathcal{P}_m(\mathcal{M}) \to \mathcal{M}$$

in which the fibres are isomorphic to the vector space $\mathbb{R}^d$, $d = \dim \mathcal{P}_m(\mathcal{M})|_{\boldsymbol{x}}$. Secondly,

$$\pi_3 : \mathcal{P}_m(\mathcal{M}) \to \mathcal{P}_m(\mathcal{M})/G$$

is the principal fibre G-bundle corresponding to the orbit space $\mathcal{P}_m(\mathcal{M})/G$. Finally, we can define a map $f : \mathcal{P}_m(\mathcal{M})/G \to G$ so that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\pi_1"] & G/H \simeq \mathcal{M} \\ \mathcal{P}_m(\mathcal{M})/G \arrow[u, "f"] & \mathcal{P}_m(\mathcal{M}) \arrow[l, "\pi_3"] \arrow[u, "\pi_2"'] \end{tikzcd} \tag{5}$$

The basic equivalence problem can be formulated as follows. Two polynomials $Q_1, Q_2 \in \mathcal{P}_m(\mathcal{M})$ are said to be *equivalent* iff there exists a group element $g \in G$ such that $g \cdot Q_1 = Q_2$. Note that the transitive action $G \circlearrowright \mathcal{M}$ induces a corresponding non-transitive action $G \circlearrowright \mathcal{P}_m(\mathcal{M})$. Thus, solving the equivalence problem amounts to studying the orbit space $\mathcal{P}_m(\mathcal{M})/G$. The choice of a function f lifting the non-transitive action $G \circlearrowright \mathcal{P}_m(\mathcal{M})$ to G is equivalent to choosing a cross-section through the orbits or fixing the frame. Explicitly, the composition $\gamma = f \circ \pi_3 : \mathcal{P}_m(\mathcal{M}) \to G$ is the *moving frame map* corresponding to the the cross-section prescribed by a chosen f. The (local) invariants of the group action $G \circlearrowright \mathcal{P}_m(\mathcal{M})$ are the coordinates of the canonical forms obtained as the intersection of the orbits with the cross-section.

Example 2.3. The affine group $\mathrm{Aff}(n)$, a semi-direct product of $\mathrm{GL}(n)$ and the group of translations on $\mathbb{R}^n$, acts transitively on the manifold $\mathbb{R}^n$ and non-transitively on $\mathcal{P}_m(\mathbb{R}^n)$. Thus, in diagram (5), $G = \mathrm{Aff}(n)$, $H = \mathrm{GL}(n)$ and $\mathcal{M} = \mathbb{R}^n$.

Example 2.4. The pseudo-Euclidean group $\mathrm{E}(n-s, s)$ is the isometry group of the n-dimensional flat manifold $\mathbb{E}^{n-s,s}$ admitting a metric tensor with s minus signs in its signature. Note that $\mathrm{E}(n-s,s) = \mathrm{O}(n-s,s) \ltimes \mathbb{R}^n$, where $\mathrm{O}(n-s,s)$ is the associated group of pseudo-orthogonal rotations. In particular, for $s = 0$, $\mathrm{E}(n) \equiv \mathrm{E}(n, 0)$ is the group of rigid motions (rotations and translations) acting in Euclidean space $\mathbb{E}^n \equiv \mathbb{E}^{n,0}$. If $s = 1$, $\mathbb{E}^{n-1,1}$ is an n-dimensional Minkowski space and $\mathrm{O}(n-1, 1)$ is the familiar Lorentz group. In the context of diagram (5), $G = \mathrm{E}(n-s, s)$, $H = \mathrm{O}(n-s, s)$ and $\mathcal{M} = \mathbb{E}^{n-s,s}$.

Remark 2.5. The diagram (5) naturally extends to the study of *covariants* and *joint invariants* of $\mathcal{P}_m(\mathcal{M})$. Indeed, such studies can be realized in the context of Cartan geometry by replacing the space $\mathcal{P}_m(\mathcal{M})$ in (5) with one of the *extended spaces* $\mathcal{P}_m(\mathcal{M}) \times \mathcal{M}$ or $\mathcal{P}_m(\mathcal{M}) \times \cdots \times \mathcal{P}_m(\mathcal{M})$. In what follows, we shall use the terms covariants of $\mathcal{P}_m(\mathcal{M})$ and invariants of the extended space $\mathcal{P}_m(\mathcal{M}) \times \mathcal{M}$ interchangeably.

The action of the general linear group on the bundle $\mathcal{P}^m(\mathbb{R}^n)$ of homogeneous polynomials of degree m in n variables, the main object of study in classical invariant theory, can also be understood from the viewpoint of Cartan's geometry. As is well known, the action $\mathrm{GL}(n+1) \circlearrowright \mathcal{P}^m(\mathbb{R}^{n+1})$ is equivalent to the action of the group of projective (Möbius) transformations, $\mathrm{PGL}(n)$, on $\mathcal{P}_m(\mathbb{RP}^n)$. In this framework, the diagram (5) becomes the following:

$$\begin{tikzcd} \mathrm{PGL}(n) \arrow[r, "\pi_1"] & \mathrm{PGL}(n)/Z(\mathrm{GL}(n)) \simeq \mathbb{RP}^n \\ \mathcal{P}_m(\mathbb{RP}^n)/\mathrm{PGL}(n) \arrow[u, "f"] & \mathcal{P}_m(\mathbb{RP}^n) \simeq \mathcal{P}^m(\mathbb{R}^{n+1}) \arrow[l, "\pi_3"] \arrow[u, "\pi_2"'] \end{tikzcd} \tag{6}$$

In what follows, we shall study diagram (5) in the context of Examples 2.3 and 2.4 and diagram (6) in the traditional setting of classical invariant theory. Our primary goal will be to generate sets of fundamental invariants and covariants of these spaces by proposing a suitable cross-section in the orbit space, irrespective of degree, dimension and signature. Before we can begin the ensuing calculations, we will require an explicit form of the representation of the group G on the underlying space of polynomials. The derivation of these representations is the topic of the next section.

3. Group representations

Let V be a vector space of polynomials (or one of the extended spaces discussed in Remark 2.5) defined on the n-fold product manifold $\mathcal{M} = \mathbb{R} \times \cdots \times \mathbb{R}$. The dimension d of V can be computed using (4) and hence the general element of V can be represented by d arbitrary parameters $c_1, \ldots, c_d$, with respect to some system of coordinates. Each element g of the associated Lie group G acting on V induces, by the push forward map, a non-singular linear transformation $\rho(g)$ of V, where the map

$$\rho : G \to \mathrm{GL}(V)$$

defines a representation of G on V. Once the form of the general polynomial Q of V is available with respect to some system of coordinates on $\mathcal{M}$, the explicit form of the transformation $\rho(g)Q$ (written more conveniently as $g \cdot Q$) may be written in terms of the parameters $c_1, \ldots, c_d$. Equipped with these transformation rules, one can begin the search for group invariants. The precise definition of a group invariant of $\mathcal{P}_m(\mathbb{R}^n)$, for example, is as follows.

Definition 3.1. A smooth function $\mathcal{I} : \mathcal{P}_m(\mathbb{R}^n) \to \mathbb{R}$ is said to be an $\mathrm{Aff}(n)$-*invariant of* $\mathcal{P}_m(\mathbb{R}^n)$ iff it satisfies the condition

$$\mathcal{I}(g \cdot Q) = \mathcal{I}(Q), \tag{7}$$

for all $Q \in \mathcal{P}_m(\mathbb{R}^n)$ and for all $g \in \mathrm{Aff}(n)$.

We now derive the explicit form of the representation of $\mathrm{Aff}(n)$ on $\mathcal{P}_m(\mathbb{R}^n)$. The general polynomial $Q \in \mathcal{P}_m(\mathbb{R}^n)$ is of the form

$$Q = \sum_{k=0}^{m} \binom{m}{k} c^{\underline{k}}_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k}, \tag{8}$$

with respect to a given system of coordinates x^i; the k-index objects $c^{\underline{k}}_{i_1 \cdots i_k}$, for $k = 0, \ldots, m$, are completely symmetric. The use of the underlined index $\underline{k}$ is to remind the reader that it is simply a label and not an index to be summed over (the repeated indices $i_1, \ldots, i_k$ in (8) are subject to the usual summation convention and are summed from 1 to n). The transformation from the given set of coordinates x^i to another set $\tilde{x}^i$ is given by

$$x^i = A^i{}_j \tilde{x}^j + \delta^i, \tag{9}$$

where $A^i{}_j \in \mathrm{GL}(n)$ and $\delta^i \in \mathbb{R}^n$. The action $\mathrm{Aff}(n) \circlearrowright \mathbb{R}^n$ given by (9) induces a corresponding action $\mathrm{Aff}(n) \circlearrowright \mathcal{P}_m(\mathbb{R}^n)$. It follows from (8) and (9) together with the symmetry of the coefficients $\overset{k}{c}_{i_1 \cdots i_k}$ and elementary combinatorial considerations that

$$\tilde{\overset{k}{c}}_{i_1 \cdots i_k} = \sum_{j=k}^{m} \binom{m-k}{j-k} \overset{j}{c}_{\ell_1 \cdots \ell_j} A^{\ell_1}{}_{i_1} \cdots A^{\ell_k}{}_{i_k} \delta^{\ell_{k+1}} \cdots \delta^{\ell_j}, \tag{10}$$

for $k = 0, \ldots, m$. Therefore, by (7), an $\mathrm{Aff}(n)$-invariant of $\mathcal{P}_m(\mathbb{R}^n)$ is any function $\mathcal{I}$ satisfying

$$\mathcal{I}(\overset{0}{c}, \overset{1}{c}_{i_1}, \ldots, \overset{m}{c}_{i_1 \cdots i_m}) = \mathcal{I}(\tilde{\overset{0}{c}}, \tilde{\overset{1}{c}}_{i_1}, \ldots, \tilde{\overset{m}{c}}_{i_1 \cdots i_m}), \tag{11}$$

for all $A^i{}_j \in \mathrm{GL}(n)$ and $\delta^i \in \mathbb{R}^n$, while an $\mathrm{Aff}(n)$-covariant is any function $\mathcal{C}$ satisfying

$$\mathcal{C}(\overset{0}{c}, \overset{1}{c}_{i_1}, \ldots, \overset{m}{c}_{i_1 \cdots i_m}, x^i) = \mathcal{C}(\tilde{\overset{0}{c}}, \tilde{\overset{1}{c}}_{i_1}, \ldots, \tilde{\overset{m}{c}}_{i_1 \cdots i_m}, \tilde{x}^i), \tag{12}$$

for all $A^i{}_j \in \mathrm{GL}(n)$ and $\delta^i, x^i \in \mathbb{R}^n$.

Remark 3.2. Equation (10) specializes to the action $\mathrm{E}(n-s, s) \circlearrowright \mathcal{P}_m(\mathbb{E}^{n-s,s})$ whenever $A^i{}_j \in \mathrm{O}(n-s, s)$; the general polynomial $Q \in \mathcal{P}_m(\mathbb{E}^{n-s,s})$ is given by (8) where the x^i are pseudo-Cartesian coordinates of $\mathbb{E}^{n-s,s}$. For spaces of homogeneous polynomials, the analysis is even simpler. The general polynomial $Q \in \mathcal{P}^m(\mathbb{R}^n)$ is of the form

$$Q = c_{i_1 \cdots i_m} x^{i_1} \cdots x^{i_m}, \tag{13}$$

where $c_{i_1 \cdots i_m} = c_{(i_1 \cdots i_m)}$. The action $\mathrm{GL}(n) \circlearrowright \mathcal{P}^m(\mathbb{R}^n) \times \mathbb{R}^n$ is

$$x^i = A^i{}_j \tilde{x}^j, \quad \tilde{c}_{i_1 \cdots i_m} = c_{j_1 \cdots j_m} A^{j_1}{}_{i_1} \cdots A^{j_m}{}_{i_m}. \tag{14}$$

Notice that the coefficients $c_{i_1 \cdots i_m}$ in (14) transform like the components of a covariant tensor of valence m. This simple yet crucial observation will be of key importance in the theory of Section 4.

Example 3.3. In anticipation of the calculations in Section 5, let us specialize the results of this section to $\mathcal{P}_3(\mathbb{E}^2)$ using a more transparent notation. The general cubic $Q \in \mathcal{P}_3(\mathbb{E}^2)$ is of the form

$$Q = a_{ijk} x^i x^j x^k + 3 b_{ij} x^i x^j + 3 c_i x^i + e, \tag{15}$$

where $a_{ijk} = a_{(ijk)}$ and $b_{ij} = b_{(ij)}$. By (10), the action $\mathrm{E}(2) \circlearrowright \mathcal{P}_3(\mathbb{E}^2)$ is

$$\begin{aligned}
\tilde{a}_{ijk} &= a_{\ell mn} A^\ell{}_i A^m{}_j A^n{}_k, \\
\tilde{b}_{ij} &= b_{k\ell} A^k{}_i A^\ell{}_j + a_{k\ell m} A^k{}_i A^\ell{}_j \delta^m, \\
\tilde{c}_i &= c_j A^j{}_i + 2 b_{jk} A^j{}_i \delta^k + a_{jk\ell} A^j{}_i \delta^k \delta^\ell, \\
\tilde{e} &= e + 3 c_i \delta^i + 3 b_{ij} \delta^i \delta^j + a_{ijk} \delta^i \delta^j \delta^k,
\end{aligned} \tag{16}$$

where $A^i{}_j \in \mathrm{O}(2)$ and $\delta^i \in \mathbb{R}^2$.

4. Computation of invariants

In this section, we show that covariants of vector spaces of polynomials, both homogeneous and inhomogeneous alike, are formed by computing various contractions of the partial derivatives of the polynomial Q. Following the proofs of these results, we examine the orbit structure and argue that the invariants of the action $\mathrm{E}(n) \circlearrowright \mathcal{P}_m(\mathbb{E}^n) \times \mathbb{E}^n$ globally separate the orbits of the respective extended space, thereby demonstrating that the constructed covariants have the desired "discriminating power". Finally, we propose an alternate cross-section through the orbits which generate pure invariants of the action $\mathrm{E}(n-s, s) \circlearrowright \mathcal{P}_m(\mathbb{E}^{n-s,s})$.

Before stating and proving the main results, we need to introduce some further notation and one additional definition. It is convenient to adopt a compact notation for the partial derivatives of a polynomial (or any function) Q. To this effect, let

$$Q_{i_1 \cdots i_k} = \frac{\partial^k Q}{\partial x^{i_1} \cdots \partial x^{i_k}}. \tag{17}$$

In the construction of polynomial $\mathrm{Aff}(n)$- or $\mathrm{GL}(n)$-invariants, the functions are generally only invariant up to a determinantal factor. This leads to a modification of equations (11) and (12) and the definition of a weighted invariant.

Definition 4.1. An $\mathrm{Aff}(n)$*-invariant of weight* k *of* $\mathcal{P}_m(\mathbb{R}^n)$ is any function $\mathcal{I} : \mathcal{P}_m(\mathbb{R}^n) \to \mathbb{R}$ satisfying

$$\mathcal{I}(c^{\underline{0}}, c^{\underline{1}}_{i_1}, \ldots, c^{\underline{m}}_{i_1 \cdots i_m}) = |A|^k \, \mathcal{I}(\tilde{c}^{\underline{0}}, \tilde{c}^{\underline{1}}_{i_1}, \ldots, \tilde{c}^{\underline{m}}_{i_1 \cdots i_m}), \tag{18}$$

where $|A| = \det(A^i{}_j)$. Similarly, a $\mathrm{GL}(n)$*-invariant of weight* k *of* $\mathcal{P}^m(\mathbb{R}^n)$ is any function $\mathcal{I} : \mathcal{P}^m(\mathbb{R}^n) \to \mathbb{R}$ satisfying

$$\mathcal{I}(c_{i_1 \cdots i_m}) = |A|^k \, \mathcal{I}(\tilde{c}_{i_1 \cdots i_m}). \tag{19}$$

Weighted covariants are defined analogously.

Theorem 4.2. *Let* $Q \in \mathcal{P}^m(\mathbb{R}^n)$*. Any scalar formed from contractions of* Q*, its partial derivatives* Q_{i_1}*, ...,* $Q_{i_1 \cdots i_m}$ *and* k *completely antisymmetric permutation symbols* $\epsilon^{i_1 \cdots i_n}$ *is a* $\mathrm{GL}(n)$*-covariant of weight* k *of* $\mathcal{P}^m(\mathbb{R}^n)$*.*

Proof. The group action $\mathrm{GL}(n) \circlearrowright \mathcal{P}^m(\mathbb{R}^n) \times \mathbb{R}^n$ is given by Equation (14); the coefficients $c_{i_1 \cdots i_m}$ of the polynomial Q transform like the components of a covariant tensor of valence m. Notice that the same can be said for the partial derivatives $Q_{i_1 \cdots i_m}$, since the two are related by a factor of $m!$. By repeated differentiation of (13), it follows that the partial derivatives $Q_{i_1 \cdots i_k}$ transform like the components of covariant tensors of valence k, for $k = 1, \ldots, m$. This claim is also easily understood by realizing that the partial derivative operator $\partial_i = \partial / \partial x^i$ transforms like a tensor: $\tilde{\partial}_i = A^j{}_i \partial_j$, by the chain rule. By definition (see for example [23]), the permutation symbol $\epsilon^{i_1 \cdots i_n}$ is a completely antisymmetric contravariant tensor density of unit weight defined by

$$\epsilon^{i_1 \cdots i_n} = \epsilon^{[i_1 \cdots i_n]},$$

such that in any coordinate system

$$\tilde{\epsilon}^{12\cdots n} = 1.$$

This definition is consistent with the tensor transformation rule and the definition of the determinant of an $n \times n$ matrix, for

$$\tilde{\epsilon}^{12\cdots n} = |A|(A^{-1})^1{}_{j_1} \cdots (A^{-1})^n{}_{j_n} \epsilon^{j_1 \cdots j_n} = |A| \cdot |A^{-1}| = 1.$$

Therefore, the tensor product of any number of Q, Q_{i_1}, $\ldots$, $Q_{i_1 \cdots i_m}$ and k permutation symbols $\epsilon^{i_1 \cdots i_n}$ is a tensor density of weight k. Thus, any contraction yielding a scalar from this tensor product is necessarily a weighted invariant of the extended space, i.e. a $\mathrm{GL}(n)$-covariant of weight k of $\mathcal{P}^m(\mathbb{R}^n)$. ■

Theorem 4.3. *Let $Q \in \mathcal{P}_m(\mathbb{R}^n)$. Any scalar formed from contractions of Q, its partial derivatives Q_{i_1}, $\ldots$, $Q_{i_1 \cdots i_m}$ and k completely antisymmetric permutation symbols $\epsilon^{i_1 \cdots i_n}$ is an $\mathrm{Aff}(n)$-covariant of weight k of $\mathcal{P}_m(\mathbb{R}^n)$.*

Proof. We shall employ a "hybrid" version of the moving frame method [4, 5]. When applying the classical method to the full group $\mathrm{Aff}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$, one often encounters very complicated algebraic expressions in which it is not always clear how to compute a 'simple' moving frame map. To alleviate these computational obstructions, Kogan [14] formulates a recursive version of the moving frame method by taking advantage of any special topology the group might exhibit. We remark that this method was first applied to the study of Killing tensors in [20] and provides insight into how to treat the analogous problem in CIT. In our case, Kogan's method involves two steps. Firstly, we consider the subgroup of translations. The transformation rules become trivial and a convenient choice of cross-section through the orbits can be made leading to a set of covariants for the subgroup of translations. Secondly, we use the covariants obtained in the first step as new coordinates, applying the action of the second subgroup (i.e. $\mathrm{GL}(n)$) to obtain their respective transformation rules. Finally, one can compute covariants from the action of the second subgroup; these covariants are covariants of the full group. We now provide explicit details of this construction.

Specializing the full group action $\mathrm{Aff}(n) \circlearrowright \mathcal{P}_m(\mathbb{R}^n) \times \mathbb{R}^n$ given by Equations (9) and (10) to the subgroup of translations (i.e. $A = 1_n$) yields

$$\tilde{x}^i = x^i - \delta^i, \quad \tilde{c}^{\underline{k}}_{i_1 \cdots i_k} = \sum_{j=0}^{m-k} \binom{m-k}{j} c^{\underline{j+k}}_{i_1 \cdots i_k \ell_1 \cdots \ell_j} \delta^{\ell_1} \cdots \delta^{\ell_j}, \tag{20}$$

for $k = 0, \ldots, m$. The cross-section we choose through the orbits is extremely simple: $\tilde{x}^i = 0$, for $i = 1, \ldots, n$. The resulting globally defined moving frame map is just $\delta^i = x^i$. Substituting this map back into (20) yields the translational covariants

$$\tilde{c}^{\underline{k}}_{i_1 \cdots i_k} = \sum_{j=0}^{m-k} \binom{m-k}{j} c^{\underline{j+k}}_{i_1 \cdots i_k \ell_1 \cdots \ell_j} x^{\ell_1} \cdots x^{\ell_j},$$

for $k = 0, \ldots, m$. Up to a constant, these covariants are nothing more than the partial derivatives of $Q \in \mathcal{P}_m(\mathbb{R}^n)$. Indeed,

$$Q_{i_1 \cdots i_k} = \frac{m!}{(m-k)!} \tilde{c}^{\underline{k}}_{i_1 \cdots i_k}.$$

Therefore, $\{Q, Q_{i_1}, \ldots, Q_{i_1 \cdots i_m}\}$ constitute a set of fundamental translational covariants of $\mathcal{P}_m(\mathbb{R}^n)$. Thus, any partial derivative of any order of Q is a translational covariant and any partial derivative of any order of a translational covariant is also a translational covariant. We now apply the second step of the recursive moving frame method using the partial derivatives of Q as coordinates. However, no additional effort is required to implement this step. As in the proof of Theorem 4.2, we simply observe that Q and its partial derivatives all transform like tensors under the action of the subgroup $\mathrm{GL}(n)$ and thus the conclusion of this theorem holds. ∎

Corollary 4.4. *Let $Q \in \mathcal{P}_m(\mathbb{E}^{n-s,s})$. Any scalar formed from contractions of Q, its partial derivatives Q_{i_1}, ..., $Q_{i_1 \cdots i_m}$, the inverse metric tensor g^{ij} and the tensor product $\epsilon^{i_1 \cdots i_n} \epsilon^{j_1 \cdots j_n}$ is an $\mathrm{E}(n-s,s)$-covariant of $\mathcal{P}_m(\mathbb{E}^{n-s,s})$.*

Proof. By the tensor transformation rules, the components of the contravariant metric tensor transform according to

$$\tilde{g}^{ij} = g^{k\ell} (A^{-1})^i{}_k (A^{-1})^j{}_\ell = g^{ij},$$

since $A^i{}_j \in \mathrm{O}(n-s,s)$. Note that in the case $s=0$, the invariance of the metric tensor is a consequence of the identity $AA^t = 1_n$, for any orthogonal matrix A. Further, since $|A| = \pm 1$ for any pseudo-orthogonal matrix, it follows that

$$\tilde{\epsilon}^{i_1 \cdots i_n} = |A| (A^{-1})^{i_1}{}_{j_1} \cdots (A^{-1})^{i_n}{}_{j_n} \epsilon^{j_1 \cdots j_n} = \pm \epsilon^{i_1 \cdots i_n},$$

and hence the tensor product $\epsilon^{i_1 \cdots i_n} \epsilon^{j_1 \cdots j_n}$ is strictly invariant. The result now follows from the proof of Theorem 4.3 upon interchanging $\mathrm{Aff}(n) \leftrightarrow \mathrm{E}(n-s,s)$ and $\mathrm{GL}(n) \leftrightarrow \mathrm{O}(n-s,s)$. ∎

Using Theorems 4.2 and 4.3 and Corollary 4.4, one can immediately construct sets of functionally independent group covariants in which the number of covariants in the set is equal to the dimension of the space minus the dimension of the orbits [16, theorem 8.17]. We demonstrate this construction in Section 5, by computing a set of fundamental E(2)-covariants of $\mathcal{P}_3(\mathbb{E}^2)$. One can also use the results of this section to construct sets of polynomially independent covariants, also known as Hilbert bases. In this paper, we shall not pursue the problem of whether such bases are finite. The proof of such a first fundamental theorem or finiteness theorem is well known for the homogeneous case (see for example [7, 16]) and can probably be extended to the inhomogeneous vector spaces of polynomials treated in this paper.

In many applications, such as the $\mathcal{P}_3(\mathbb{E}^2)$ example treated in the next section, a complete Hilbert basis is often unnecessary and instead a functionally independent set of invariants or covariants is sufficient. The question of whether a set of functionally independent group invariants distinguishes between the orbits of the vector space is a fundamental problem. At best, invariants only distinguish *locally* between the orbits because they are essentially derived using local methods (such as the moving frame method) and orbits need not have the same dimension. However, for the special case of $\mathrm{E}(n)$-invariants of the extended space $\mathcal{P}_m(\mathbb{E}^n) \times \mathbb{E}^n$, one can compute a set of invariants using Corollary 4.4 which *globally* separate the orbits. Referring to the proof of Theorem 4.3, we first note that all orbits of the extended space $\mathcal{P}_m(\mathbb{E}^n) \times \mathbb{E}^n$ under the subgroup of translations have maximal dimension n; this is straightforward to verify by analyzing the infinitesimal generators of the subgroup. We emphasize that if we only consider the unprolonged space $\mathcal{P}_m(\mathbb{E}^n)$, then an orbit under the subgroup of translations need not have dimension n. Thus prolongation of the space (i.e. computing covariants of $\mathcal{P}_m(\mathbb{E}^n)$ rather than pure invariants) is the key! Further, the cross-section $\tilde{x}^i = 0$ intersects the orbits transversally. Therefore, the subgroup of translations acts *regularly* on the entire extended space with n-dimensional orbits. The standard theory of moving frames (as treated in [16, ch 8], for example) is thus applicable. Our cross-section defines a *global* moving frame map and yields $\frac{m+1}{n}\binom{n+m}{m+1}$ functionally independent translational invariants, which are simply the polynomial Q and its partial derivatives $Q_{i_1}, \dots, Q_{i_1 \cdots i_m}$. We now apply the action of the second subgroup $\mathrm{O}(n)$, using these translational invariants as new coordinates. Clearly, they all transform like tensors, thus we can easily construct a set of fundamental $\mathrm{O}(n)$-invariants (and hence $\mathrm{E}(n)$-invariants of the full extended space) upon forming appropriate contractions of the partial derivatives, as stated in Corollary 4.4. A result of Hilbert, restated in Onishchik and Vinberg [17], asserts the following:

> *"The orbits of a compact linear group acting in a real vector space are separated by the fundamental (polynomial) invariants."*

Therefore, due to the compactness of $\mathrm{O}(n)$, this result guarantees that our set of fundamental invariants constructed in the manner described separates or discriminates between the orbits globally.

Remark 4.5. The existence of a global cross-section in the construction of the translational covariants is a direct consequence of the topology of the extended space $\mathcal{P}_m(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s}$. We recall from fibre bundle theory that a principal bundle is trivial if and only if it admits a global cross-section [21]. Indeed, in our case, we have the fibration

$$\pi : \mathcal{P}_m(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s} \to \mathcal{P}_m(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s}/G,$$

where G is the subgroup of translations. Clearly, G is identifiable with $\mathbb{E}^{n-s,s}$. Therefore, each fibre of this principal bundle is the group itself, isomorphic to $\mathbb{E}^{n-s,s}$. The simplicity of the cross-section and the global moving frame map is thus an artifact of the triviality of this principal bundle.

To close this section, we address the problem of how group invariants of the prolonged space $\mathcal{P}_m(\mathcal{M}) \times \mathcal{M}$ can be used to distinguish between the orbits of the unprolonged space $\mathcal{P}_m(\mathcal{M})$. In some cases, certain polynomial combinations of covariants produce pure invariants, i.e. covariants independent of the coordinates x^i. These combinations are often easy to spot if one is attempting to distinguish between two different orbits in which their representatives take on a particularly simple form. Another technique, motivated from the "transvection" process, is to apply certain invariant differential operators to a covariant $\mathcal{C}$. For example, suppose $\mathcal{C}$ is an $\mathrm{E}(n-s,s)$-covariant of $\mathcal{P}_m(\mathbb{E}^{n-s,s})$. If $\mathcal{C}$ is linear in x^i, then $g^{ij}\mathcal{C}_i\mathcal{C}_j$, the "length" squared of its gradient, is a pure invariant. If $\mathcal{C}$ is quadratic in x^i, then its Laplacian (or more generally its d'Alembertian in the pseudo-Euclidean case $s \neq 0$), given by $g^{ij}\mathcal{C}_{ij}$, is a pure invariant. Finally, one can abandon the use of covariants altogether and instead compute invariants of the unprolonged space from the group action (10) by choosing an alternate cross-section through the orbits. Working in $\mathrm{E}(n-s,s)$ and proceeding as in the proof of Theorem 4.2, we use the recursive moving frame method and consider the restriction of the group action to the subgroup of translations given by (20). At the top level, we have

$$\tilde{c}^{m}_{i_1\cdots i_m} = c^{m}_{i_1\cdots i_m}, \quad \tilde{c}^{m-1}_{i_1\cdots i_{m-1}} = c^{m-1}_{i_1\cdots i_{m-1}} + c^{m}_{i_1\cdots i_{m-1}j}\delta^j.$$

One choice of cross-section is

$$g^{i_1 j_1}\cdots g^{i_{m-1}j_{m-1}}\tilde{c}^{m-1}_{i_1\cdots i_{m-1}}\tilde{c}^{m}_{j_1\cdots j_{m-1}k} = 0, \tag{21}$$

for $k = 1,\ldots,n$. The resulting normalization equations for δ^i are of the form $\alpha_{k\ell}\delta^\ell = \beta_k$, where

$$\begin{aligned}\alpha_{k\ell} &= g^{i_1 j_1}\cdots g^{i_{m-1}j_{m-1}} c^{m}_{i_1\cdots i_{m-1}k} c^{m}_{j_1\cdots j_{m-1}\ell},\\ \beta_k &= -g^{i_1 j_1}\cdots g^{i_{m-1}j_{m-1}} c^{m-1}_{i_1\cdots i_{m-1}} c^{m}_{j_1\cdots j_{m-1}k}.\end{aligned}$$

The $n\times n$ matrix whose components are $\alpha_{k\ell}$ is generally invertible, thus a (local) moving frame map exists. A set of translational invariants can be constructed from the remaining transformation rules in (20) and full $\mathrm{E}(n-s,s)$-invariants can be obtained be taking appropriate contractions. We stress that this construction is only local; the invariants constructed using the cross-section (21) only distinguish between those orbits satisfying $\det(\alpha_{k\ell}) \neq 0$.

5. Classification of inhomogeneous cubic polynomials in two variables

We now use the theory developed in this paper to give an invariant characterization of the orbits of $\mathcal{P}_3(\mathbb{E}^2)$, the bundle of inhomogeneous cubic polynomials in two variables, under the action of the Euclidean group $\mathrm{E}(2)$. Our choice to work in the subgroup $\mathrm{E}(2)$, rather than the full affine group as is standard in CIT, is motivated from the invariant theory of Killing tensors, as one treats analogous problems exclusively in the isometry group. Further, our choice of group leads to a wider variety of equivalence classes of polynomials and hence a more challenging classification problem and more thorough test of our method.

The general cubic $Q \in \mathcal{P}_3(\mathbb{E}^2)$ is given by equation (15), expressed in terms of ten independent parameters a_{ijk}, b_{ij}, c_i and e, where $a_{ijk} = a_{(ijk)}$ and $b_{ij} = b_{(ij)}$. The dimension of the extended space $\mathcal{P}_3(\mathbb{E}^2)\times\mathbb{E}^2$ is twelve and generic orbits have dimension three, thus we expect $12-3=9$ fundamental $\mathrm{E}(2)$-covariants. By Corollary 4.4, nine such covariants are

$$\begin{aligned}&\mathcal{C}_1 = Q^{ijk}Q_{ijk}, \quad \mathcal{C}_2 = \epsilon^{ik}\epsilon^{j\ell}Q^{m}{}_{ij}Q_{mk\ell},\\ &\mathcal{C}_3 = \epsilon^{i\ell}\epsilon^{jm}\epsilon^{kp}\epsilon^{nu}\epsilon^{qv}\epsilon^{rw}Q_{ijk}Q_{\ell mn}Q_{pqr}Q_{uvw},\\ &\mathcal{C}_4 = \epsilon^{ik}\epsilon^{j\ell}Q_{ij}Q_{k\ell}, \quad \mathcal{C}_5 = \epsilon^{ik}\epsilon^{j\ell}Q_iQ_jQ^{mn}{}_kQ_{mn\ell}, \quad \mathcal{C}_6 = Q^{ij}Q_{ij},\\ &\mathcal{C}_7 = \epsilon^{ik}\epsilon^{j\ell}Q_iQ_jQ^{m}{}_kQ_{m\ell}, \quad \mathcal{C}_8 = Q^iQ_i, \quad \mathcal{C}_9 = Q.\end{aligned} \tag{22}$$

In (22) and in what follows, we raise and lower all indices with the metric, e.g. $Q^{ij} = g^{ik}g^{j\ell}Q_{k\ell}$. We remind the reader that we are using (17) and employing the

Table 1: Canonical forms for $\mathcal{P}_3(\mathbb{E}^2)$

A1	$a_1x^3 + 3a_2xy(x+y) + a_3y^3 + b(x^2+y^2) + c_1x + c_2y + e,$ $a_1 \geqslant 0,\ a_3 \geqslant 0,\ a_2(a_1 - 2a_2 + a_3) \neq 0$
A2	$a_1x^3 + 3a_2xy(x+y) + a_3y^3 + bxy + c_1x + c_2y + e,$ $a_1 \geqslant 0,\ a_3 \geqslant 0,\ a_2(a_1 - 2a_2 + a_3) = 0$
A3	$ax^3 + (b_1x + b_2y)y + cx + e,\ a > 0,\ b_1 \geqslant 0,\ b_2 \neq 0$
A4	$ax^3 + bxy + cy + e,\ a > 0,\ b > 0$
A5	$ax^3 + c_1x + c_2y,\ a > 0,\ c_2 > 0$
A6	$ax^3 + cx + e,\ a > 0$
B1	$b_1x^2 + b_2y^2 + e$
B2	$bx^2 + cy,\ c > 0$
B3	$bx^2 + e$
C	$cx,\ c > 0$
E	e
Z	0

summation convention in the construction of these covariants. For example,

$$\mathcal{C}_6 = Q^{ij}Q_{ij} = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2 Q}{\partial x^i\,\partial x^j} \cdot \frac{\partial^2 Q}{\partial x^i\,\partial x^j}.$$

We remark that Equations (22) are also covariants for $\mathcal{P}_3(\mathbb{E}^{1,1})$ under the action of $\mathrm{E}(1,1)$; the only difference is that indices are raised and lowered with the Minkowski metric, $g_{ij} = \mathrm{diag}(-1,1)$. In both the Euclidean and Minkowski cases, it is straightforward to verify that the nine covariants (22) are functionally independent.

The first step in classifying the orbit space $\mathcal{P}_3(\mathbb{E}^2)/\mathrm{E}(2)$ is to find a suitable representative, or canonical form, on each orbit. In the language of Cartan geometry, we are implicitly choosing a cross-section through the orbits which leads to a moving frame map sending any element on the orbit to its canonical form. The construction of the moving frame map is rather elementary. We use the group freedom, specified by a rotation (and possibly a reflection) and translation on the Euclidean plane, to eliminate as many parameters of the cubic as possible. Once the group freedom has been exhausted, the resulting cubic defines a canonical form.

Proposition 5.1. *Any cubic in $\mathcal{P}_3(\mathbb{E}^2)$ is equivalent to one of the twelve classes of canonical forms listed in Table 1.*

Proof. The derivation of the canonical forms uses the action $\mathrm{E}(2) \circlearrowright \mathcal{P}_3(\mathbb{E}^2)$ specified by Equations (16) in terms of the parameters a_{ijk}, b_{ij}, c_i and e of $\mathcal{P}_3(\mathbb{E}^2)$, the rotation $A^i{}_j$ and the translation δ^i. As the ensuing calculations require only elementary algebra, we shall only give a brief outline of the details.

Suppose $Q \in \mathcal{P}_3(\mathbb{E}^2)$ and $\mathcal{C}_1 \neq 0$ (i.e. $a_{ijk} \neq 0$). Parametrizing the rotation ${A^i}_j$ by

$$ {A^i}_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{ij}, $$

it follows that one can fix θ so that the coefficients of the x^2y and xy^2 terms of Q are equal. Indeed, if

$$ Q = a_{111}x^3 + 3a_{112}x^2y + 3a_{122}xy^2 + a_{222}y^3 + \cdots , $$

then the condition $\tilde{a}_{112} = \tilde{a}_{122}$ yields a cubic equation for $\tan\theta$ with a leading coefficient of $a_{112} + a_{122}$. Thus, a real solution for $\tan\theta$ exists (if $a_{112} = -a_{122}$, we can take $\theta = \frac{\pi}{2}$). Without loss of generality,

$$ Q = a_1x^3 + 3a_2xy(x+y) + a_3y^3 + 3b_{ij}x^ix^j + 3c_ix^i + e. $$

By a reflection, we may assume that $a_1, a_3 \geqslant 0$. We now apply a translation. If $\mathcal{C}_2 = a_2(a_1 - 2a_2 + a_3) \neq 0$, we can set $b_{12} = 0$ and $b_{11} = b_{22}$ yielding case (A1). If $\mathcal{C}_2 = 0$, then either (i) $a_2 = 0$ or (ii) $a_3 = 2a_2 - a_1$. For case (i), if $a_1a_3 \neq 0$, we can set $b_{11} = b_{22} = 0$ giving case (A2), otherwise $a_3 = 0$, without loss of generality, and

$$ Q = ax^3 + 3b_{ij}x^ix^j + 3c_ix^i + e. $$

In this case, if $b_{22} \neq 0$, we can eliminate b_{11} and c_2 leading to case (A3). If $b_{22} = 0$ and $b_{12} \neq 0$, we can eliminate b_{11} and c_1 yielding case (A4). If both b_{12} and b_{22} vanish and $c_2 \neq 0$, then a translation exists which sends $b_{11} = 0$ and $e = 0$ and hence case (A5). Finally, if b_{12}, b_{22} and c_2 are all zero (i.e. Q depends only on x), then we can eliminate the quadratic term yielding case (A6). For case (ii), if $a_1 \neq a_2$, then a translation exists giving case (A2), as in case (i). If $a_1 = a_2$, then $Q = a_1(x+y)^3 + \cdots$ and thus reduces to one of (A3), (A4), (A5) or (A6) by a rotation with $\theta = \frac{\pi}{4}$.

Suppose that $a_{ijk} = 0$ and $b_{ij} \neq 0$. Then Q is an inhomogeneous quadratic in two variables and, as is well known, a rotation can always be found which transforms away the mixed xy term. By a further translation, one can always reduce the quadratic to one of the cases (B1), (B2) or (B3). Finally, if both a_{ijk} and b_{ij} identically vanish, then Q is at most linear and can always be reduced to either case (C), (E) or (Z). ∎

Having derived a set of canonical forms for $\mathcal{P}_3(\mathbb{E}^2)$, the problem of invariantly distinguishing between the orbits involves two main steps. Firstly, a characterization of each of the twelve classes of canonical forms in Table 1 is required. The solution to this problem is addressed in Proposition 5.2 and summarized in Table 2. Secondly, since each class of canonical forms itself depends generally on several parameters and thus contains infinitely many inequivalent orbits, classification schemes for each canonical form type are required. This step necessitates finding a set of invariants serving as global coordinates for each of the twelve classes. By definition, such invariant coordinates must enjoy the following property: two cubics belonging to the same class are equivalent if and only if they have the same invariants (coordinates). Computationally, this is a challenging step to implement. Firstly, if one finds a set of potential invariants $\mathcal{I}_1, \ldots, \mathcal{I}_d$ for the coordinates of a given canonical form class having d parameters $c_1, \ldots, c_d$, one must show that the only solutions to the system of algebraic equations $\mathcal{I}_i = \tilde{\mathcal{I}}_i$, $i = 1, \ldots, d$, define the same cubic up to equivalence. Provided the $\mathcal{I}_i$ are functionally independent, they will always serve as *local* coordinates of the canonical form. However, if one wishes to prove that the coordinates are *global*, then one must solve the generally non-linear system of equations $\mathcal{I}_i = \tilde{\mathcal{I}}_i$ for the parameters $\tilde{c}_i$ in terms of the c_i. Resource intensive Gröbner basis type calculations are usually required. Secondly, as discussed towards the end of Section 4, one must actually determine suitable "pure" invariants from the fundamental covariants. This procedure can be implemented in a number of different ways. For each class of canonical forms, one can search for (polynomial) combinations of the covariants (22) which have no explicit dependence on the Cartesian coordinates x and y. Alternatively, one can compute additional covariants using Corollary 4.4 having the aforementioned property. Such covariants would of course depend only on the (functionally independent) fundamental covariants $\mathcal{C}_1, \ldots, \mathcal{C}_9$, but not necessarily through a polynomial functions of these covariants. That being said, it is convenient to define a series of covariants $\mathcal{C}_{10}, \ldots, \mathcal{C}_{21}$ as follows:

$$
\begin{aligned}
&\mathcal{C}_{10} = Q^i{}_i, \quad \mathcal{C}_{11} = Q^i{}_{ij}Q^j, \quad \mathcal{C}_{12} = Q_i Q^{ij} Q_j, \quad \mathcal{C}_{13} = Q^i{}_j Q^j{}_k Q^k{}_i,\\
&\mathcal{C}_{14} = Q^i{}_{ij} Q^{jk} Q_k, \quad \mathcal{C}_{15} = Q^i{}_{ij} Q^j{}_k Q^k{}_\ell Q^\ell, \quad \mathcal{C}_{16} = Q^{ijk} Q_{ij} Q_{k\ell} Q^\ell,\\
&\mathcal{C}_{17} = \epsilon^{ik}\epsilon^{j\ell} Q_{ijm} Q^{mn} Q_n Q_{k\ell}, \quad \mathcal{C}_{18} = \epsilon^{ik}\epsilon^{j\ell} Q_{imn} Q^{mn} Q_j Q_{k\ell},\\
&\mathcal{C}_{19} = \epsilon^{ik}\epsilon^{j\ell} Q^m{}_{mi} Q_{jn} Q^n{}_k Q_\ell, \quad \mathcal{C}_{20} = Q^i{}_{ij} Q^{jk\ell} Q_{k\ell}, \quad \mathcal{C}_{21} = \epsilon^{ik}\epsilon^{j\ell} Q^m{}_{ij} Q^n{}_{nm} Q_{k\ell}.
\end{aligned}
$$

Moreover, we define nine auxiliary covariants written in terms of polynomials of the $\mathcal{C}_i$ according to

$$
\begin{aligned}
\mathcal{A}_1 &= \mathcal{C}_8 - 2\,\mathcal{C}_9\,\mathcal{C}_{10}, \quad \mathcal{A}_2 = \mathcal{A}_1\,\mathcal{C}_4 + 2\,\mathcal{C}_7, \quad \mathcal{A}_3 = 4\,\mathcal{C}_6 - 3\,\mathcal{C}_{10}{}^2 - 2\,\mathcal{C}_{11},\\
\mathcal{A}_4 &= \mathcal{C}_6\,\mathcal{C}_{10} - 6\,\mathcal{C}_1\,\mathcal{C}_9 - 3\,\mathcal{A}_3\,\mathcal{C}_{10}, \quad \mathcal{A}_5 = \mathcal{C}_{10}{}^3 - 3\,\mathcal{C}_{10}\,\mathcal{C}_{11} - \mathcal{C}_{13} + 3\,\mathcal{C}_{14},\\
\mathcal{A}_6 &= 6\,\mathcal{C}_1\,\mathcal{C}_4\,\mathcal{C}_9 + 12\,\mathcal{C}_6\,\mathcal{C}_{10}\,\mathcal{C}_{11} + 12\,\mathcal{C}_1\,\mathcal{C}_{12} - \mathcal{C}_4\,\mathcal{C}_{13} - 3\,\mathcal{C}_4\,\mathcal{C}_{14}\\
&\quad - 6\,\mathcal{C}_6\,\mathcal{C}_{14} - 6\,\mathcal{C}_{11}\,\mathcal{C}_{13} - 12\,\mathcal{C}_{11}\,\mathcal{C}_{14},\\
\mathcal{A}_7 &= \mathcal{C}_1\,\mathcal{C}_4 + 2\,\mathcal{C}_1\,\mathcal{C}_6 - 2\,\mathcal{C}_{10}\,\mathcal{C}_{20},\\
\mathcal{A}_8 &= \mathcal{C}_4{}^2 + 8\,(\mathcal{C}_{15} - \mathcal{C}_{16} - \mathcal{C}_{17} + \mathcal{C}_{18} + \mathcal{C}_{19}),\\
\mathcal{A}_9 &= 6\,\mathcal{C}_1\,\mathcal{C}_9\,\mathcal{C}_{21} + 3\,\mathcal{C}_6\,\mathcal{C}_{10}\,\mathcal{C}_{20} - 3\,\mathcal{C}_1\,\mathcal{C}_5 + 6\,\mathcal{C}_1\,\mathcal{C}_{15} + 6\,\mathcal{C}_1\,\mathcal{C}_{19}\\
&\quad - 3\,\mathcal{C}_{13}\,\mathcal{C}_{20} - \mathcal{C}_{13}\,\mathcal{C}_{21} - 6\,\mathcal{C}_{14}\,\mathcal{C}_{20} - 6\,\mathcal{C}_{14}\,\mathcal{C}_{21}.
\end{aligned}
$$

The "right combinations" of covariants which lead to global coordinates become increasingly complicated as the number of parameters in the canonical form grows. For cases (A1) and (A2), the two most general classes of canonical forms, we have found it easier to construct these coordinates using pure invariants derived using the alternate cross-section proposed at the end of Section 4. The cross-section (21) in our notation is $\tilde{a}_{ijk}\tilde{b}^{jk} = 0$ which leads to the normalization equations $\alpha_{ij}\delta^j = -a_{ijk}b^{jk}$ where

$$\alpha_{ij} = a^{k\ell}{}_i a_{jk\ell}. \tag{23}$$

Inverting the normalization equations using the definition of the matrix adjoint yields the moving frame map

$$\hat{\delta}^i = -\epsilon^{ik}\epsilon^{j\ell} a^{mn}{}_k a_{\ell mn} a_{jpq} b^{pq}, \tag{24}$$

Table 2: Invariant classification of $\mathcal{P}_3(\mathbb{E}^2)$ and global coordinates of its canonical forms

Class	Classification	Coordinates
A1	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 \neq 0$	$\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, $\mathcal{I}_7$, $\mathcal{I}_8$
A2	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 = 0$, $\mathcal{C}_3 \neq 0$	$\mathcal{I}_5$, $\mathcal{I}_6$, $\mathcal{I}_7$, $\mathcal{I}_8$, $\mathcal{I}_9$, $\mathcal{I}_{10}$
A3	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 = 0$, $\mathcal{C}_3 = 0$, $\lvert\nabla\mathcal{C}_4\rvert^2 \neq 0$	$\mathcal{C}_1$, $\mathcal{C}_{21}$, $\mathcal{A}_7$, $\mathcal{A}_8$, $\mathcal{A}_9$
A4	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 = 0$, $\mathcal{C}_3 = 0$, $\lvert\nabla\mathcal{C}_4\rvert^2 = 0$, $\mathcal{C}_4 \neq 0$	$\mathcal{C}_1$, $\mathcal{C}_4$, $\mathcal{A}_5$, $\mathcal{A}_6$
A5	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 = 0$, $\mathcal{C}_3 = 0$, $\mathcal{C}_4 = 0$, $\mathcal{C}_5 \neq 0$	$\mathcal{C}_1$, $\mathcal{C}_5$, $\mathcal{A}_3$
A6	$\mathcal{C}_1 \neq 0$, $\mathcal{C}_2 = 0$, $\mathcal{C}_3 = 0$, $\mathcal{C}_4 = 0$, $\mathcal{C}_5 = 0$	$\mathcal{C}_1$, $\mathcal{A}_3$, $\mathcal{A}_4$
B1	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 \neq 0$, $\nabla^2\mathcal{C}_7 \neq 0$	$\mathcal{C}_4$, $\mathcal{C}_{10}$, $\mathcal{A}_2$
B2	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 \neq 0$, $\nabla^2\mathcal{C}_7 = 0$, $\mathcal{C}_7 \neq 0$	$\mathcal{C}_7$, $\mathcal{C}_{10}$
B3	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 \neq 0$, $\mathcal{C}_7 = 0$	$\mathcal{C}_{10}$, $\mathcal{A}_1$
C	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 = 0$, $\mathcal{C}_8 \neq 0$	$\mathcal{C}_8$
E	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 = 0$, $\mathcal{C}_8 = 0$, $\mathcal{C}_9 \neq 0$	$\mathcal{C}_9$
Z	$\mathcal{C}_1 = 0$, $\mathcal{C}_6 = 0$, $\mathcal{C}_8 = 0$, $\mathcal{C}_9 = 0$	———

where $\hat{\delta}^i \equiv |\alpha|\, \delta^i$ and $|\alpha| \equiv \det(\alpha_{ij})$. Substituting (24) back into the group action (16), we obtain the following translational invariants in addition to a_{ijk}:

$$\begin{aligned} \beta_{ij} &\equiv |\alpha|\, \tilde{b}_{ij} = |\alpha|\, b_{ij} + a_{ijk}\hat{\delta}^k, \\ \gamma_i &\equiv |\alpha|^2\, \tilde{c}_i = |\alpha|^2\, c_i + 2\,|\alpha|\, b_{ij}\hat{\delta}^j + a_{ijk}\hat{\delta}^j\hat{\delta}^k, \\ \eta &\equiv |\alpha|^3\, \tilde{e} = |\alpha|^3\, e + 3\,|\alpha|^2 c_i\hat{\delta}^i + 3\,|\alpha|\, b_{ij}\hat{\delta}^i\hat{\delta}^j + a_{ijk}\hat{\delta}^i\hat{\delta}^j\hat{\delta}^k. \end{aligned} \tag{25}$$

Therefore, any scalar formed from contractions of a_{ijk}, β_{ij}, γ_i and η, the inverse metric g^{ij} and the tensor $\epsilon^{ij}\epsilon^{k\ell}$ is an E(2)-invariant of $\mathcal{P}_3(\mathbb{E}^2)$. The following invariants are used in the construction of global coordinates for cases (A1) and (A2):

$$\begin{aligned} &\mathcal{I}_1 = a^{ijk}a_{ijk}, \quad \mathcal{I}_2 = \epsilon^{ik}\epsilon^{j\ell}a^m{}_{ij}a_{mk\ell}, \\ &\mathcal{I}_3 = \epsilon^{i\ell}\epsilon^{jm}\epsilon^{kp}\epsilon^{nu}\epsilon^{qv}\epsilon^{rw}a_{ijk}a_{\ell mn}a_{pqr}a_{uvw}, \\ &\mathcal{I}_4 = \beta^i{}_i, \quad \mathcal{I}_5 = a^i{}_{ij}\gamma^j, \quad \mathcal{I}_6 = \beta^{ij}\beta_{ij}, \quad \mathcal{I}_7 = \eta, \\ &\mathcal{I}_8 = a^i{}_{ij}\beta^{jk}\gamma_k, \quad \mathcal{I}_9 = \gamma^i\gamma_i, \quad \mathcal{I}_{10} = \beta^{ij}\gamma_i\gamma_j. \end{aligned} \tag{26}$$

Proposition 5.2. *An invariant characterization of each of the twelve classes of canonical forms for $\mathcal{P}_3(\mathbb{E}^2)$ is listed in the second column of Table 2.*

Proof. It is straightforward to evaluate the covariants listed in the second column of Table 2 showing that they characterize their respective canonical form. Clearly $\mathcal{C}_1$ vanishes if and only if all cubic terms vanish. Thus, if $\mathcal{C}_1 \neq 0$, we necessarily have one of (A1), …, (A6). For (A1), $\mathcal{C}_2 = a_2(a_1 - 2a_2 + a_3) \neq 0$, while $\mathcal{C}_2 = 0$ for (A2), …, (A6). For (A2), $\mathcal{C}_3 = -2592{a_1}^2{a_3}^2 \neq 0$ if $a_2 = 0$, otherwise $\mathcal{C}_3 = -2592(a_1 - a_2)^2((a_1 - a_2)^2 + 4{a_2}^2) \neq 0$ if $a_3 = 2a_2 - a_1$. It follows that $\mathcal{C}_3 = 0$ for (A3), …, (A4). For (A3), $|\nabla\mathcal{C}_4|^2 = g^{ij}(\partial_i\mathcal{C}_4)(\partial_j\mathcal{C}_4) = 576a^2{b_2}^2 \neq 0$

($b_2 \neq 0$ for (A3) otherwise it reduces to a special case of (A4)), while $|\nabla \mathcal{C}_4|^2 = 0$ for (A4), (A5) and (A6). For (A4), $\mathcal{C}_4 = -2b^2 \neq 0$ and $\mathcal{C}_4 = 0$ for (A5) and (A6). Finally, $\mathcal{C}_5$ vanishes identically for (A6), while for (A5), $\mathcal{C}_5 = 36a^2{c_2}^2 \neq 0$.

Suppose now that $\mathcal{C}_1 = 0$ and $\mathcal{C}_6 \neq 0$; we necessarily have one of (B1), (B2) or (B3). For (B1), $\nabla^2 \mathcal{C}_7 = g^{ij}\partial_i\partial_j\mathcal{C}_7 = 64{b_1}^2{b_2}^2 \neq 0$, while $\nabla^2\mathcal{C}_7 = 0$ for (B2) and (B3). It follows that $\mathcal{C}_7 = 0$ for (B3) and $\mathcal{C}_7 = 4b^2c^2 \neq 0$ for (B2).

It remains to distinguish between classes (C), (E) and (Z). For these classes, $\mathcal{C}_1 = \mathcal{C}_6 = 0$. For (C), $\mathcal{C}_8 = c^2 \neq 0$, while $\mathcal{C}_8 = 0$ for (E) and (Z). Finally, $\mathcal{C}_9 = e$ for (E) and (Z), thereby separating these two classes. ∎

Proposition 5.3. *The quantities listed in the third column of Table 2 are pure invariants and globally separate the orbits in their respective canonical form class.*

Proof. The coordinates listed for classes (A1) and (A2) are given by Equations (26) and are, by definition, pure E(2)-invariants of $\mathcal{P}_3(\mathbb{E}^2)$. For the ten other classes of canonical forms, we evaluate the invariant coordinates listed in Table 2 and show that they contain no explicit dependence on the Cartesian coordinates x and y. Working from the bottom up, it follows that $\mathcal{C}_9 = e$ for case (E) and $\mathcal{C}_8 = c^2$ for case (C). For cases (B3), (B2) and (B1), $(\mathcal{C}_{10}, \mathcal{A}_1) = (2b, -4be)$, $(\mathcal{C}_7, \mathcal{C}_{10}) = (4b^2c^2, 2b)$ and $(\mathcal{C}_4, \mathcal{C}_{10}, \mathcal{A}_2) = (8b_1b_2, 2(b_1+b_2), -32eb_1b_2(b_1+b_2))$, respectively. For cases (A6), (A5), (A4) and (A3),

$$
\begin{aligned}
(\mathcal{C}_1, \mathcal{A}_3, \mathcal{A}_4) &= (36a^2, -12ac, -216a^2e),\\
(\mathcal{C}_1, \mathcal{C}_5, \mathcal{A}_3) &= (36a^2, 36a^2{c_2}^2, -12ac_1),\\
(\mathcal{C}_1, \mathcal{C}_4, \mathcal{A}_5, \mathcal{A}_6) &= \big(36a^2, -2b^2, 18abc, -36ab^2(bc+12ae)\big),\\
(\mathcal{C}_1, \mathcal{C}_{21}, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9) &= \big(36a^2, 72a^2b_2, 72a^2({b_1}^2+4{b_2}^2), 4{b_1}^4 - 192a{b_2}^2c,\\
&\qquad -144a^2b_2[36a(b_2c-3ae)+b_2(3{b_1}^2+4{b_2}^2)]\big),
\end{aligned}
$$

respectively. Finally, as discussed in the paragraph following the proof of Proposition 5.1, we can verify that the invariant coordinates in each of the twelve classes define a system of global coordinates. In most cases, these calculations can be readily done by hand. For example, for case (A4), we solve the system of equations

$$
\begin{gathered}
36a^2 = 36\tilde{a}^2, \quad -2b^2 = -2\tilde{b}^2, \quad 18abc = 18\tilde{a}\tilde{b}\tilde{c},\\
-36ab^2(bc+12ae) = -36\tilde{a}\tilde{b}^2(\tilde{b}\tilde{c}+12\tilde{a}\tilde{e}),
\end{gathered}
$$

for $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{e}$ in terms of a, b, c, e. The system admits four solutions given by

$$
\tilde{a} = \pm a,\ \tilde{b} = b,\ \tilde{c} = \pm c,\ \tilde{e} = e \quad \text{or} \quad \tilde{a} = \pm a,\ \tilde{b} = -b,\ \tilde{c} = \mp c,\ \tilde{e} = e.
$$

Clearly, all solutions define the same cubic up to a reflection. Therefore, the invariants $(\mathcal{C}_1, \mathcal{C}_4, \mathcal{A}_5, \mathcal{A}_6)$ define global coordinates for the (A4) class of canonical forms and hence globally separate the orbits. The other canonical form classes are treated similarly. ∎

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