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An Asymptotic Result on the A-Component in the Iwasawa Decomposition

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Abstract. Let G be a connected noncompact semisimple Lie group. For each $v', v, g \in G$, we prove that

$$\lim_{t\to\infty}[a(v'g^tv)]^{1/t} = s^{-1}\cdot b(g),$$

where $a(g)$ denotes the a-component in the Iwasawa decomposition of $g = kan$ and $b(g) \in A_+$ denotes the unique element that conjugate to the hyperbolic component h in the complete multiplicative Jordan decomposition of $g = ehu$. The element s in the Weyl group of (G, A) is determined by $yv \in G$ (not unique in general) in such a way that $yv \in N^- m_s MAN$, where $yhy^{-1} = b(g)$ and $G = \cup_{s\in W} N^- m_s MAN$ is the Bruhat decomposition of G.
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1. Introduction

Given $X \in \mathrm{GL}_n(\mathbb{C})$, the well-known QR decomposition asserts that $X = QR$, where Q is unitary and R is upper triangular with positive diagonal entries. The decomposition is unique. Let $a(X) := \operatorname{diag} R$. Recently it was shown in [2] that given $A, B \in \mathrm{GL}_n(\mathbb{C})$, $\lim_{t\to\infty}[a(AX^tB)]^{1/t}$ exists and the limit is related to the eigenvalue moduli of X. More precisely,

Theorem 1.1. [2] *Let $A, B, X \in \mathrm{GL}_n(\mathbb{C})$. Let $X = Y^{-1}JY$ be the Jordan decomposition of X, where J is the Jordan form of X,* $\operatorname{diag} J = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ *satisfying $|\lambda_1| \ge \cdots \ge |\lambda_n|$. Then*

$$\lim_{t\to\infty}[a(AX^tB)]^{1/t} = \operatorname{diag}(|\lambda_{\omega(1)}|, \dots, |\lambda_{\omega(n)}|),$$

where the permutation ω is uniquely determined by the $L\omega U$ decomposition of $YB = L\omega U$, such that L is lower triangular and U is unit upper triangular.

The $L\omega U$ decomposition is known as Gelfand-Naimark decomposition [1, p.434].

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The above asymptotic result relates three decompositions, namely, QR decomposition of X, Jordan decomposition of X, and Gelfand-Naimark decomposition of YB. Indeed the matrix Y (not unique in general) can be viewed from the standpoint of complete multiplicative Jordan decomposition (CMJD) of X [1]. Write $J = D + B$ where $D := \operatorname{diag} J$ is diagonal and B is the nilpotent part in the Jordan form J. Then

$$X = Y^{-1}JY = Y^{-1}[D(1 + D^{-1}B)]Y,$$

where $1 + D^{-1}B$ is unipotent. Decompose the diagonal

$$D = \operatorname{diag}(e^{i\theta_1}|\lambda_1|, \ldots, e^{i\theta_n}|\lambda_n|) = EH,$$

where

$$E := \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}), \qquad H := \operatorname{diag}(|\lambda_1|, \ldots, |\lambda_n|).$$

Now we have the CMJD

$$X = ehu,$$

where

$$e = Y^{-1}EY, \quad h = Y^{-1}HY, \quad u = Y^{-1}(1 + D^{-1}B)Y.$$

Notice that the diagonalizable e has eigenvalue moduli 1, and the diagonalizable h has positive eigenvalues and u is unipotent. They commute with each other and such decomposition is unique. Now Y is an element which via conjugation turns h into a positive diagonal matrix with nonincreasing diagonal entries.

Our goal is to extend Theorem 1.1 in the context of connected noncompact semisimple Lie group G. The three decompositions have their counterparts, namely Iwasawa decomposition, complete multiplicative Jordan decomposition (CMJD) and Bruhat decomposition. Motivated by Theorem 1.1, for any given $v', v, g \in G$, we study the sequence $\{[a(v'g^tv)]^{1/t}\}_{t\in\mathbb{N}}$ in which the a-component of a nonsingular matrix would be played by the a-component $a(g)$ of g, where $g = kan$ with respect to the Iwasawa decomposition $G = KAN$. The eigenvalue moduli $|\lambda|$ in nonincreasing order is replaced by the element $b(g) \in A_+$ that is conjugate to the hyperbolic element h in the CMJD of g. Here $A_+ := \exp \mathfrak{a}_+$ in which $\mathfrak{a}_+$ is a (closed) fundamental chamber. Finally the permutation ω would be provided by the Weyl group element s in the Bruhat decomposition of $yv \in N^- m_s MAN$ such that $yhy^{-1} = b(g)$.

2. CMJD, Iwasawa decomposition, Bruhat decomposition

Let G be a connected noncompact semisimple Lie group having $\mathfrak{g}$ as its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition. Let $K \subset G$ be the connected subgroup with Lie algebra $\mathfrak{k}$. Then K is closed and $\operatorname{Ad}_G(K)$ is compact [1, p.252-253]. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ so that the positive roots and thus the simple roots are fixed. Set $A_+ := \exp \mathfrak{a}_+$. Let Σ be the set of restricted roots.

Following the terminology in [4, p.419], an element $h \in G$ is called *hyperbolic* if $h = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\operatorname{ad} X \in \operatorname{End}(\mathfrak{g})$ is diagonalizable over $\mathbb{R}$. An element $u \in G$ is called *unipotent* if $u = \exp(N)$ where $N \in \mathfrak{g}$ is nilpotent, that is, $\operatorname{ad} N \in \operatorname{End}(\mathfrak{g})$ is nilpotent. An element $e \in G$ is *elliptic* if $\mathrm{Ad}\,(e) \in \mathrm{Aut}\,(\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition (CMJD) [4, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

A hyperbolic $h \in G$ is conjugate to a unique element $b(h) \in A_+$ [4, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

The group $A := \exp \mathfrak{a}$ is simply connected [3, p.317] and abelian so that the map $\mathfrak{a} \to A$ defined by exp is a diffeomorphism [3, p.63]. Thus $\log a \in \mathfrak{a}$ is well defined for any $a \in A$. Let $M = Z_K(A) = Z_K(\mathfrak{a})$ and $M' = N_K(A) = N_K(\mathfrak{a})$. The group $W = M'/M$ is the Weyl group . It acts on A by conjugation, and on $\mathfrak{a}$ via the adjoint action. In particular exp is a W-map. Let $\mathfrak{n} := \sum_{\alpha>0} \mathfrak{g}_\alpha$ be the sum of all positive root spaces. Set $N := \exp \mathfrak{n}$. Similarly let $\mathfrak{n}_- := \sum_{\alpha<0} \mathfrak{g}_\alpha$ and set $N^- := \exp \mathfrak{n}_-$. Let $G = KAN$ be the corresponding Iwasawa decomposition of G [3, p.317]. If $g \in G$, we write

$$g = kan,$$

where $k \in K$, $a \in A$, $n \in N$ are uniquely defined. For $G = \mathrm{SL}_n(\mathbb{C})$, the Iwasawa decomposition is just the QR decomposition if we choose AN as the group of upper triangular matrices with positive diagonal elements.

For $s \in W$, denote by $m_s \in M'$ a representative such that $s = m_s M$. The Bruhat decomposition of G asserts that

$$G = \cup_{s \in W} N^- m_s MAN$$

is a disjoint union. So for each $g \in G$, there is a unique $s \in W$ such that $g \in N^- m_s MAN$.

3. Asymptotic behavior of the Iwasawa component

Let G be a connected noncompact semisimple Lie group with Iwasawa decomposition $G = KAN$. Let $a(g)$ be the a-component of $g \in G$ with respect to the Iwasawa decomposition

$$g = k(g)a(g)n(g).$$

Given $v', v, g \in G$, we now prove the following main theorem concerning the asymptotic behavior of the sequence $\{[a(v'g^tv)]^{1/t}\}_{t \in \mathbb{N}}$. It turns out the limit $\lim_{t\to\infty}\{[a(v'g^tv)]^{1/t}\}$ exists and is independent of v'. We will make some remarks in the next section.

Theorem 3.1. *Let $v', v, g \in G$. Let $g = ehu$ be the complete multiplicative Jordan decomposition of g. Let $h = y^{-1}b(g)y$ for some $y \in G$, and $yv \in N^- m_s MAN$ in the Bruhat decomposition. Then*

$$\lim_{t\to\infty} [a(v'g^tv)]^{1/t} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s, \tag{1}$$

where the limit is independent of v' and the choice of y. If $b(g)$ is regular, that is, $b(g)$ is in the interior of A_+, then s is uniquely determined by g and v.

Proof. We may assume that G has trivial center since everything is independent of the center. We will make use of Theorem 1.1 by considering $\mathrm{Ad}\, G$ which can be viewed as a matrix group by choosing an appropriate orthonormal basis of $\mathfrak{g}$ with respect to the inner product $B_\theta(X,Y) = -B(X,\theta Y)$, where $B(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$ and $\theta \in \mathrm{Aut}\,(\mathfrak{g})$ is the Cartan involution $\theta(X+Y) = X - Y$, $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

It is known that [1, p.261] there is an orthonormal basis of $\mathfrak{g}$,

$$\mathcal{X} = \{X_i : i = 1, \ldots, d\}, \qquad d := \dim \mathfrak{g}, \tag{2}$$

compatible with the (restricted) root space decomposition of $\mathfrak{g}$ [3, p.313]

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

such that $X_i \in \mathfrak{g}_\alpha$ and $X_j \in \mathfrak{g}_\beta$ with $i < j$ implies $\alpha \geq \beta$ (by the lexicographic order $\mathcal{L}$ over the coordinates induced by pre-ordering the simple roots). Moreover, since $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ is an orthogonal sum [3, p.313], we can select $\mathcal{X}$ in a way that $\mathfrak{a}$ is spanned by some $\{X_i, X_{i+1}, \cdots, X_{i+\dim \mathfrak{a}-1}\} \subseteq \mathcal{X}$. With respect to $\mathcal{X}$, we view the elements in $\mathrm{GL}(\mathfrak{g})$ as matrices. The matrices $\mathrm{Ad}\,(K)$, $\mathrm{Ad}\,(A)$, $\mathrm{Ad}\,(N)$, and $\mathrm{Ad}\,(N^-)$ are orthogonal, positive diagonal, real unit upper triangular, and real unit lower triangular, respectively [3, p.317]. Because $\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$ is a representation of G, we may view the elements $\mathrm{Ad}\, g \in \mathrm{SL}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ as nonsingular matrices. Thus we have the following Iwasawa decomposition for $\mathrm{Ad}\, g$, that is, QR decomposition: $\mathrm{Ad}\, g = \mathrm{Ad}\, k\, \mathrm{Ad}\, a\, \mathrm{Ad}\, n$. Therefore,

$$a(\mathrm{Ad}\,(g)) = \mathrm{Ad}\,(a(g)), \tag{3}$$

for all $g \in G$, where $a(\mathrm{Ad}\,(g))$ is the diagonal part of the matrix R in the QR decomposition $\mathrm{Ad}\, g = QR$. By (3)

$$\mathrm{Ad}\,[a(v' g^t v)]^{1/t} = [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^t(\mathrm{Ad}\, v))]^{1/t}, \quad t \in \mathbb{N}. \tag{4}$$

Since the center of G is trivial, to prove (1) it suffices to show

$$\lim_{t\to\infty} [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^t(\mathrm{Ad}\, v))]^{1/t} = \mathrm{Ad}\,(s^{-1} \cdot b(g)), \tag{5}$$

where $s \in W$ is uniquely determined by the Bruhat decomposition of

$$yv = n^- m_s man \in N^- m_s MAN.$$

Since $\mathrm{Ad}\,(s^{-1} \cdot b(g)) = \mathrm{Ad}\,(m_s^{-1} b(g) m_s)$, (5) is equivalent to the following

$$\lim_{t\to\infty} [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^t(\mathrm{Ad}\, v))]^{1/t} = (\mathrm{Ad}\, m_s)^{-1} \mathrm{Ad}\,(b(g))(\mathrm{Ad}\, m_s). \tag{6}$$

In order to establish (6), we need several lemmas. For each $H \in \mathfrak{a}_+$, $\mathrm{ad}\, H = \mathrm{diag}\,(h_1, \cdots, h_d)$ is a diagonal matrix. The diagonal entries may *not* be in nonincreasing order so it is not readily to apply Theorem 1.1. The next two lemmas are obvious.

Lemma 3.2. *Let $H \in \mathfrak{a}_+$ and write* $\operatorname{ad} H = \operatorname{diag}(h_1, \cdots, h_d)$. *If $h_i > h_j$ for certain $i > j$, then the (i, j) entry of each element of* $\operatorname{ad} \mathfrak{n}^-$ *is always zero, where* $\mathfrak{n}^- = \sum_{\alpha<0} \mathfrak{g}_\alpha$.

Index the elements in $\Sigma \cup \{0\}$ in nonincreasing order as $\alpha_1 > \cdots > \alpha_{\frac{k+1}{2}}$ $(= 0) > \cdots > \alpha_k$ according to the lexicographic order $\mathcal{L}$, and let $n_i = \dim \mathfrak{g}_{\alpha_i}$, then we get a partition η of d

$$\eta := (n_1, n_2, \cdots, n_k). \tag{7}$$

The partition η is symmetric ($n_t = n_{k+1-t}$ for $1 \leq t \leq k$) since $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ [3, p.313]. This implies the following lemma.

Lemma 3.3. *For $H \in \mathfrak{a}$ and $a \in A$,* $\operatorname{ad} H$ *and* $\operatorname{Ad} a$ *are block diagonal matrices. More precisely,*

$$\begin{aligned} \operatorname{ad} H &= \operatorname{diag}(\alpha_1(H) I_{n_1}, \ldots, \alpha_k(H) I_{n_k}) && (8)\\ \operatorname{Ad} a &= \operatorname{diag}(e^{\alpha_1(\log a)} I_{n_1}, \cdots, e^{\alpha_k(\log a)} I_{n_k}). && (9) \end{aligned}$$

Let $e_i \in \mathbb{R}^d$ be the standard vector taking 1 at the i-th position and 0 elsewhere. For a permutation ω of d letters we associated the permutation matrix $P_\omega := [e_{\omega(1)}\, e_{\omega(2)} \cdots e_{\omega(d)}]$. Then $P_\omega^{-1}(x_{ij}) P_\omega = (x_{\omega(i)\omega(j)})$, and in particular

$$P_\omega^{-1} \operatorname{diag}(h_1, \ldots, h_d) P_\omega = \operatorname{diag}(h_{\omega(1)}, \ldots, h_{\omega(d)}). \tag{10}$$

We also view P_ω as an element of $\mathrm{GL}(\mathfrak{g})$ with respect to the basis $\mathcal{X}$. From now on, let $H := \log b(g) \in \mathfrak{a}_+$ and write

$$\operatorname{ad} H := \operatorname{diag}(h_1, \cdots, h_d) = \operatorname{diag}(\alpha_1(H) I_{n_1}, \cdots, \alpha_k(H) I_{n_k}).$$

Lemma 3.4. *Let $\omega \in S_d$ be the unique permutation that has the smallest number of transpositions in its factorization into products of $(1, 2)$, $(2, 3), \ldots,$ $(d-1, d)$, such that the diagonal entries of*

$$P_\omega^{-1}(\operatorname{ad} H) P_\omega = \operatorname{diag}(h_{\omega(1)}, \cdots, h_{\omega(d)}) \tag{11}$$

are in nonincreasing order. Then ω satisfies the following properties:

(1) If $h_{\omega(i)} = h_{\omega(j)}$ for $\omega(i) > \omega(j)$, then $i > j$. Here $h_{\omega(j)}$ and $h_{\omega(i)}$ are the $\omega(j)$-th and the $\omega(i)$-th diagonal entry of $\operatorname{ad} H$ *respectively; which will be mapped to the j-th and the i-th diagonal entry of* $P_\omega^{-1}(\operatorname{ad} H) P_\omega$ *respectively.*

(2) P_ω acts as an identity on $\mathfrak{g}_0 \supseteq \mathfrak{a} \supseteq \mathfrak{a}_+$.

(3) There is a permutation $\gamma \in S_k$ such that for $\eta = (n_1, n_2, \cdots, n_k)$

$$P_\omega^{-1} \operatorname{diag}(x_1 I_{n_1}, \cdots, x_k I_{n_k}) P_\omega = \operatorname{diag}(x_{\gamma(1)} I_{n_{\gamma(1)}}, \cdots, x_{\gamma(k)} I_{n_{\gamma(k)}})$$

for the free variables $x_1, \cdots, x_k$. If we partition the rows of P_ω by η, and partition the columns of P_ω by $\gamma(\eta) := (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$, then the $(i, \gamma^{-1}(i))$ block of P_ω is I_{n_i} for $i = 1, \cdots, k$, and the other blocks of P_ω are zero blocks.

Proof. Let $\omega \in S_d$ denote the unique permutation acting on the sequence $\{(-h_1, 1), (-h_2, 2), \cdots, (-h_d, d)\}$ in the way that the resulting sequence is increasing in lexicographic order:

$$\left\{(-h_{\omega(1)}, \omega(1)) < (-h_{\omega(2)}, \omega(2)) < \cdots < (-h_{\omega(d)}, \omega(d))\right\}.$$

Then ω is the permutation that has the smallest number of transpositions in its factorization, such that the diagonal entries of $P_\omega^{-1}(\operatorname{ad} H)P_\omega$ are in nonincreasing order. Moreover, statement (1) is true by the construction of ω.

If $h_j = h_{j+1}$, then by the construction of ω and statement (1), it is impossible to have $t \in \{1, \cdots, k\}$ such that $\omega(t)$ is a number between $\omega(j)$ and $\omega(j+1)$. So $\omega(j+1) = \omega(j) + 1$. This implies that P_ω is a block permutation matrix and statement (3) follows.

The matrix $\operatorname{ad} H$ is anti-symmetric about the anti-diagonal. So P_ω is symmetric about the anti-diagonal by (11). Then P_ω acts as an identity on $\mathfrak{g}_0$ by statement (3). This proves statement (2). ■

For each $X \in \mathfrak{n}^-$, write $\operatorname{ad} X = \big(x_{ij}\big)$ which is strictly lower triangular.

Lemma 3.5. *Let ω be determined by $b(g)$ as in Lemma 3.4. Then for all $X \in \mathfrak{n}^-$,*

$$P_\omega^{-1}(\operatorname{ad} X)P_\omega = \big(x_{\omega(i)\omega(j)}\big)$$

remains strictly lower triangular.

Proof. Clearly the diagonal entries of $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ are 0. The (i,j) entry of $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ is $x_{\omega(i)\omega(j)}$. Suppose on the contrary, $x_{\omega(i)\omega(j)} \neq 0$ for some $i < j$. Then $\omega(i) > \omega(j)$ since $\operatorname{ad} X$ is strictly lower triangular. Also $h_{\omega(i)} \geq h_{\omega(j)}$ by Lemma 3.4 (1). But $h_{\omega(i)} = h_{\omega(j)}$ contradicts Lemma 3.4 (2) since $\omega(i) > \omega(j)$ and $i < j$. On the other hand, if $h_{\omega(i)} > h_{\omega(j)}$, then it contradicts Lemma 3.2 since $\omega(i) > \omega(j)$ but the $(\omega(i), \omega(j))$ entry of $\operatorname{ad} X$ is $x_{\omega(i)\omega(j)} \neq 0$. This proves that $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ is strictly lower triangular for every $X \in \mathfrak{n}^-$. ■

Let us now prove (6). Taking exponential it follows that

$$P_\omega^{-1}(\operatorname{Ad} b(g))P_\omega \tag{12}$$

is a positive diagonal matrix with nonincreasing diagonal entries by Lemma 3.4, and if $n^- \in N^-$, then $P_\omega^{-1}(\operatorname{Ad} n^-)P_\omega$ is unit lower triangular by Lemma 3.5. Now

$$\begin{aligned}
\operatorname{Ad} g &= (\operatorname{Ad} e)(\operatorname{Ad} h)(\operatorname{Ad} u), && (13)\\
\operatorname{Ad} h &= (\operatorname{Ad} y)^{-1}(\operatorname{Ad} b(g))(\operatorname{Ad} y) &&\\
&= (\operatorname{Ad} y)^{-1}P_\omega[P_\omega^{-1}(\operatorname{Ad} b(g))P_\omega]P_\omega^{-1}(\operatorname{Ad} y), && (14)
\end{aligned}$$

where $y \in G$ such that $yhy^{-1} = b(g)$. Then

$$P_\omega^{-1}(\operatorname{Ad} y)(\operatorname{Ad} h)(\operatorname{Ad} y)^{-1}P_\omega = P_\omega^{-1}(\operatorname{Ad} b(g))P_\omega$$

whose diagonal entries are the eigenvalue moduli of $\operatorname{Ad}(g)$, in nonincreasing order. In order to apply Theorem 1.1 to $\operatorname{Ad}(g)$ we need to examine the Gelfand-Naimark decomposition of the matrix $P_\omega^{-1}(\mathrm{Ad}\, y)(\mathrm{Ad}\, v)$. Indeed it is sufficient to consider the Gelfand-Naimark decomposition of the matrix $P_\omega^{-1}\mathrm{Ad}\,(m_s m)$ since

$$\begin{aligned} P_\omega^{-1}\mathrm{Ad}\,(yv) &= P_\omega^{-1}(\mathrm{Ad}\, n^-)(\mathrm{Ad}\,(m_s m))(\mathrm{Ad}\,(an)) \\ &= L(P_\omega^{-1}\mathrm{Ad}\,(m_s m))(\mathrm{Ad}\,(an)) \end{aligned} \tag{15}$$

where $L := P_\omega^{-1}(\mathrm{Ad}\, n^-)P_\omega$ is unit lower triangular and $yv = n^- m_s man$.

Let us examine the matrix $P_\omega^{-1}\mathrm{Ad}\,(m_s m)$. On one hand, for each $m' \in M'$, $\mathrm{Ad}\, m'$ permutes the root spaces of the same dimensions [1, p.406] and $\mathrm{Ad}\, m' \in \mathrm{Ad}\, K$ is an orthogonal matrix. Since $m_s m \in M'$,

$$\mathrm{Ad}\,(m_s m) = P_\sigma D, \tag{16}$$

for some block permutation matrix P_σ and some block diagonal matrix D in which each diagonal block is an orthogonal matrix, all are in accordance with the partition η. On the other hand, $P_\omega^{-1} = P_\omega^T$ is a block permutation matrix with respect to the row partition $\gamma(\eta) = (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$ and the column partition $\eta = (n_1, \cdots, n_k)$. Let

$$D = L_D \Omega_D U_D$$

be the Gelfand-Naimark decomposition of D, where L_D, Ω_D, and U_D are block diagonal; L_D is unit lower triangular, Ω_D is a permutation matrix, and U_D is upper triangular; all are in accordance with the partition η. Then

$$P_\omega^{-1}\mathrm{Ad}\,(m_s m) = P_\omega^{-1} P_\sigma L_D \Omega_D U_D = L' P_\omega^{-1} P_\sigma \Omega_D U_D$$

where $L' := P_\omega^{-1} P_\sigma L_D P_\sigma^{-1} P_\omega$ is unit lower triangular and is block diagonal according to the row and column partition $\gamma(\eta) = (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$ (since $P_\sigma^{-1} P_\omega$ is a block permutation according to the row partition η and the column partition $\gamma(\eta)$). By (15), $P_\omega^{-1}(\mathrm{Ad}\, y)(\mathrm{Ad}\, v)$ has the Gelfand-Naimark decomposition

$$P_\omega^{-1}(\mathrm{Ad}\, y)(\mathrm{Ad}\, v) = (LL')(P_\omega^{-1} P_\sigma \Omega_D)(U_D \mathrm{Ad}\,(an)), \tag{17}$$

where $P_\omega^{-1} P_\sigma \Omega_D$ is the desired permutation matrix when Theorem 1.1 is applied. By Lemma 3.3 $P_\sigma^{-1}(\mathrm{Ad}\, b(g))P_\sigma$ is block diagonal in which each diagonal block is a scalar multiple of an identity matrix, all are in accordance with the partition η. So $P_\sigma^{-1}(\mathrm{Ad}\, b(g))P_\sigma$ commutes with L_D, Ω_D, U_D and D. Taking (12), (13), (14) and (17) into account,

$$\begin{aligned} &\lim_{t\to\infty} [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^t(\mathrm{Ad}\, v))]^{1/t} \\ = \;& (P_\omega^{-1} P_\sigma \Omega_D)^{-1}[P_\omega^{-1}(\mathrm{Ad}\, b(g))P_\omega](P_\omega^{-1} P_\sigma \Omega_D) \\ = \;& \Omega_D^{-1} P_\sigma^{-1}(\mathrm{Ad}\, b(g))P_\sigma \Omega_D \\ = \;& D^{-1} P_\sigma^{-1}(\mathrm{Ad}\, b(g))P_\sigma D \\ = \;& (\mathrm{Ad}\,(m_s m))^{-1}(\mathrm{Ad}\, b(g))(\mathrm{Ad}\,(m_s m)) \quad \text{by (17)} \\ = \;& (\mathrm{Ad}\, m_s)^{-1}\mathrm{Ad}\,(b(g))(\mathrm{Ad}\, m_s). \end{aligned} \tag{18}$$

So we prove (6) and thus (1).

The Weyl group element s in (1) is uniquely determined by g and v provided that $b(g) \in A_+$ is regular, that is, $b(g)$ is in the interior of A_+. If $yh(g)y^{-1} = b(g) = y'h(g)y'^{-1}$, then $z := y'y^{-1} \in Z_G(b)$ where $b := b(g)$. Write

$b = e^X$ where $X \in \mathfrak{a}_+^0$ the interior of $\mathfrak{a}_+$. So $e^{\operatorname{Ad}(z)X} = zbz^{-1} = e^X$. Since $\operatorname{Ad} z(X)$ is real semisimple, $\operatorname{Ad} z(X) = X$. By [3, Lemma 7.22] $\operatorname{Ad} p$ and $\operatorname{Ad} k$ fix X, where $z = kp$ is the Cartan decomposition of z. Hence $p, k \in Z_G(b)$. Thus, if $p = e^Y$, $Y \in \mathfrak{p}$, then $\operatorname{Ad}(b)Y = Y$ so that $\operatorname{ad} X(Y) = 0$ since X is regular. By [3, Lemma 6.50], $Y \in \mathfrak{a}$ so that $p \in A$. We also have $k \in M$ since $\exp : \mathfrak{k} \to K$ is surjective (K is compact connected as we assumed that G has trivial center). Now $z \in MA$ and $y' = zy$. If $yv = n^- m_s man \in N^- m_s MAN$ is the Bruhat decomposition of yv, then $y'v = zn^- m_s man = (zn^- z^{-1}) m_s (m_s^{-1} z m_s) man$. But m_s normalizes MA and MA normalizes N^- so that $zn^- z^{-1} \in N^-$, $m_s^{-1} z m_s \in MA$. So $s \in W$ is uniquely determined. ∎

Corollary 3.6. *Let $g = ehu$ be the complete multiplicative Jordan decomposition of $g \in G$. Let $h = y^{-1} b(g) y$ for some $y \in G$, and $y \in N^- m_s MAN$ in the Bruhat decomposition. Then*

$$\lim_{t \to \infty} [a(g^t)]^{1/t} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s,$$

where the limit is independent of the choice of y. If $b(g)$ is regular, that is, $b(g)$ is in the interior of A_+, then s is uniquely determined.

Proof. It follows immediately from Theorem 3.1 by setting v, v' to be the identity element. ∎

Corollary 3.7. *Given $b \in A_+$ and $v \in G$, if $y_1^{-1} b y_1 = y_2^{-1} b y_2$, $y_1 v \in N^- m_{s_1} MAN$ and $y_2 v \in N^- m_{s_2} MAN$, then $s_1^{-1} \cdot b = s_2^{-1} \cdot b$.*

Proof. Apply Corollary 3.6 on $g = y_1^{-1} b y_1 = y_2^{-1} b y_2$. ∎

By Corollary 3.6 the map $L : G \to A$ where $L(g) := \lim_{t\to\infty} [a(g^t)]^{1/t}$ is well defined. It is easy to see that L is not continuous. For example, $g_k = y_k d y_k^{-1}$, where $d := \operatorname{diag}(d_1, \ldots, d_n)$ with $d_1 > d_2 \ge d_3 \ge \cdots \ge d_n > 0$, $y_k := I_n/k + \omega \in \mathrm{GL}_n(\mathbb{R})$ where ω is the transposition $(1, 2)$. So $L(g_k) = d$ for all $k \in \mathbb{N}$ but $L(\lim_{k\to\infty} g_k) = \operatorname{diag}(d_2, d_1, d_3, \ldots, d_n)$. By appropriate scaling, L is not continuous for $\mathrm{SL}_n(\mathbb{R})$.

The next result asserts that the images under L of the orbits $O_G(g) := \{vgv^{-1} : v \in G\}$ and $O_K(g) := \{vgv^{-1} : v \in K\}$ of $g \in G$ are equal to $W\, b(g)$.

Corollary 3.8. *Let $g = ehu$ be the complete multiplicative Jordan decomposition of $g \in G$. Let $h = y^{-1} b(g) y$ for some $y \in G$, and $y \in N^- m_s MAN$ in the Bruhat decomposition. Then*

$$L(O_K(g)) = L(O_G(g)) = W\, b(g),$$

where $W\, b(g)$ is the orbit of $b(g)$ under the action of the Weyl group W.

Proof. Clearly $L(O_K(g)) \subset L(O_G(g)) \subset W\, b(g)$ because $b(vgv^{-1}) = b(g)$ for all $v \in G$. It suffices to show that $W\, b(g) \subset L(O_K(g))$.

Given $\ell \in G$ we denote by $s(\ell) := s \in W$ such that $\ell \in N^- m_s MAN$. Notice that $(vgv^{-1})^m = vg^m v^{-1}$ for $v \in G$ so that from Theorem 3.1

$$L(vgv^{-1}) = \lim_{t\to\infty} [a(vg^t v^{-1})]^{1/t} = [s(yv^{-1})]^{-1} \cdot b(g).$$

Since $G = G^{-1} = NAK$, one has $G = \theta(G) = N^- AK$, where $\theta : G \to G$ is the Cartan involution of G. So $y = n^- ak$ where $n^- \in N^-$, $a \in A$ and $k \in K$. Thus

$$s(yK) = s(n^- aK) = s(aK) \supset s(aM').$$

Since $M' \subset K$ normalizes A, for each m_s, there exists $a' \in A$ such that $m_s a' = am_s \in aM'$. So $s(am_s) = s$ for all $s \in W$. Hence $s(yK) = W$ and thus $W\, b(g) \subset L(O_K(g))$. ∎

4. Some remarks

Remark 4.1. We now take a closer look of (1):

1. Rewrite $[a(v'g^t v)]^{1/t} = [a(v'v(v^{-1}gv)^t)]^{1/t}$. The CMJD of $v^{-1}gv$ is $v^{-1}gv = (v^{-1}ev)(v^{-1}hv)(v^{-1}uv)$ so that $h(v^{-1}gv) = v^{-1}h(g)v$. Thus $b(v^{-1}gv) = b(g)$. Moreover $h(g) = y^{-1}b(g)y$ amounts to $h(v^{-1}gv) = v^{-1}y^{-1}b(g)yv$. So the validity of (1) is reduced to the special case that v is the identity:

$$\lim_{t\to\infty} [a(v'g^t)]^{1/t} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s, \tag{19}$$

where $y \in G$ such that $h(g) = y^{-1}b(g)y$ and $y \in N^- m_s MAN$.

2. We now explain why the element v' plays no role in (1), or equivalently, in (19). As before we may assume the center of G is trivial and so K is compact. Let $g^t = k_t a_t n_t$ be the Iwasawa decomposition of g^t, where $k_t \in K$, $a_t \in A$, $n_t \in N$, and let $v' = k_1 a' k_2$ be the Cartan decomposition of v', where $k_1, k_2 \in K$, $a' \in A_+$. Then

$$a(v'g^t) = a(k_1 a' k_2 k_t a_t n_t) = a(a' k_2 k_t a_t).$$

Let $a' k_2 k_t = k' a'_t n'$, $k' \in K$ ($a'_t \in A$, $n' \in N$) be the Iwasawa decomposition of $a' k_2 k_t \in a'K$. So $a(v'g^t) = a(k' a'_t n' a_t) = a(a'_t n' a_t)$. Now $a(a'_t n' a_t) = a(a'_t a_t \hat{n}) = a'_t a_t$ for some $\hat{n} \in N$ since A normalizes N. Thus

$$[a(v'g^t)]^{1/t} = (a'_t)^{1/t} a_t^{1/t} = (a'_t)^{1/t} [a(g^t)]^{1/t}.$$

Since the set $a(a'K)$ is compact (also see Kostant's theorem [4, Theorem 4.1]), $\lim_{t\to\infty} (a'_t)^{1/t}$ is the identity. So $\lim_{t\to\infty} [a(v'g^t)]^{1/t} = \lim_{t\to\infty} [a(g^t)]^{1/t}$. In other words, Corollary 3.6 and Theorem 3.1 are the same in essence.

3. When $G = \mathrm{SL}_n(\mathbb{C})$, the uniqueness of $s^{-1} \cdot b(g)$ in Theorem 3.1 is explained in [2] apart from Corollary 3.6. Recall that $yv \in N^- m_s MAN$. Notice that in (17) $P_\omega^{-1} P_\sigma \Omega_D$ is the desired permutation matrix in the Gelfand-Naimark decomposition of $P_\omega^{-1} \mathrm{Ad}\,(yv)$ and from (18)

$$(P_\omega^{-1} P_\sigma \Omega_D)^{-1} [P_\omega^{-1} (\mathrm{Ad}\, b(g)) P_\omega] (P_\omega^{-1} P_\sigma \Omega_D) = (\mathrm{Ad}\, m_s)^{-1} \mathrm{Ad}\,(b(g)) (\mathrm{Ad}\, m_s)$$

which is independent of the choice of $\mathrm{Ad}\, y$ by [2, Remark 2.3]. As before we may assume that G has trivial center. So $s^{-1} \cdot b(g) = m_s^{-1} b(g)\, m_s$ is independent of the choice of y.

Remark 4.2. Iwasawa decomposition may be expressed in the form $G = NAK$ in which we write $g = n'a'k' = kan$, $g \in G$, $n, n' \in N$, $a, a' \in A$ and $k, k' \in K$. Since $g^{-1} = n^{-1}a^{-1}k^{-1}$, by the uniqueness of Iwasawa decomposition $a'(g) = [a(g^{-1})]^{-1}$ so that

$$\begin{aligned} \lim_{t\to\infty} [a'(v' g^t v)]^{1/t} &= \lim_{t\to\infty} [a(v^{-1} (g^{-1})^t v'^{-1})]^{-1/t} \\ &= (\lim_{t\to\infty} [a(v^{-1} (g^{-1})^t v'^{-1})]^{1/t})^{-1}. \end{aligned}$$

Now the CMJD $g = ehu$ and $yhy^{-1} = b(g)$ imply $g^{-1} = e^{-1}h^{-1}u^{-1}$ and $yh^{-1}y^{-1} = (b(g))^{-1} \in A_+^{-1}$ respectively. So $b(g^{-1}) = m_\ell (b(g))^{-1} m_\ell^{-1} = \ell \cdot (b(g))^{-1} \in A_+$, where $\ell \in W$ is the longest element [1, p.406] which maps A_+ to A_+^{-1}. By Theorem 1

$$\lim_{t\to\infty} [a'(v' g^t v)]^{1/t} = [s^{-1} \cdot \ell \cdot (b(g))^{-1}]^{-1} = (s^{-1}\ell) \cdot b(g),$$

where $m_\ell y v'^{-1} \in N^- m_s MAN$.

The situation for the two variants $G = KNA$ and $G = ANK$ is identical to KAN and NAK, respectively, since A normalizes N (in general $G = AKN$ and $G = NKA$ are not true).

Remark 4.3. See [6] for an asymptotic result relating CMJD and the Cartan decomposition of $g \in G$:

$$\lim_{m\to\infty} a_+(g^m)^{1/m} = b(g),$$

where $a_+(g)$ denotes the unique element in $a_+ \in A_+$ such that $g = k_1 a_+ k_2$, $k_1, k_2 \in K$.

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