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Decomposition of a Tensor Product of a Higher Symplectic Spinor Module and the Defining Representation of $\mathfrak{sp}(2n,\mathbb{C})$

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Abstract. Let $L(\lambda)$ be the irreducible highest weight $\mathfrak{sp}(2n,\mathbb{C})$-module with a highest weight λ, such that $L(\lambda)$ is an infinite dimensional module with bounded multiplicities, and let $F(\varpi_1)$ be the defining representation of $\mathfrak{sp}(2n,\mathbb{C})$. In this article, the tensor product $L(\lambda)\otimes F(\varpi_1)$ is explicitly decomposed into irreducible summands. This decomposition may be used in order to define some invariant first order differential operators for metaplectic structures.
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1. Introduction

Let $L(\lambda)$ denote the irreducible highest weight module with a highest weight λ and let us write $F(\lambda)$ instead of $L(\lambda)$, if λ is integral dominant with respect to a choice of a Cartan subalgebra and of a set of positive roots. In this article, we shall study a decomposition of the tensor product $L(\lambda)\otimes F(\varpi_1)$ as a module over complex symplectic Lie algebras, where λ is some nonintegral weight from a suitable set, which will be denoted by $\mathbb{A}$, and ϖ_1 is the highest weight of the defining representation of the complex symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$.

This study was motivated by author's interest in certain first order invariant differential operators, which are symplectic analogues of orthogonal Dirac-type operators. In general, invariant differential operators are acting between sections of vector bundles associated to some principal fiber bundles via representations of the principal group. The operators, we were interested in, are acting between sections of vector bundles associated to projective contact or symplectic geometries via the so called higher symplectic spinor modules over complex symplectic Lie algebras. *Higher symplectic spinor modules* represent symplectic analogues of spinor

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representations of orthogonal complex Lie algebras $\mathfrak{so}(m, \mathbb{C})$, see Kostant [12]. Projective contact geometries belong to Cartan geometries defined by a contact grading of the tangent bundle and a projective class of partial affine connections, see Krýsl [13]. In physics, these geometries play a role of a phase-space of time dependent Hamiltonian mechanics, while the symplectic geometries are models of the time independent one. To classify invariant differential operators (on the infinitesimal level at least), one needs to decompose the mentioned tensor product $L(\lambda) \otimes F(\varpi_1)$, if the sections take their values in $L(\lambda)$. (See, e.g., Slovák, Souček [15].) One of the invariant differential operators serving as a motivation for our paper appeared already in Kostant [12] and is known as the Kostant Dirac operator. Analytical and geometrical aspects of the Kostant Dirac operator were studied by many authors, see, e.g., Habermann [4], Klein [10] and Kadlčáková [8]. The last author is studying also the so called symplectic twistor and symplectic Rarita-Schwinger operators, which are related to our decomposition as well. Let us mention that for the basic symplectic spinor modules, a kind of globalization is known. These globalizations are called Segal-Shale-Weil representations, see Kashiwara, Vergne [9], where these globalized modules are introduced as representations over the metaplectic group $Mp(2n, \mathbb{R})$. Let us also mention that the study of the corresponding first order differential operators has its application in theoretical physics, namely in the 10 dimensional super string theory, see Green, Hull [3], and in the theory of Dirac-Kähler fields, see, e.g., Reuter [14], where the author of this article found his motivation for this study.

In [1], Britten, Hooper and Lemire and in [2], Britten and Hooper described the decomposition of $L(\lambda_i) \otimes F(\nu)$ for $i = 0, 1$, where ν is a dominant integral weight, $\lambda_0 = -\frac{1}{2}\varpi_n$ and $\lambda_1 = \varpi_{n-1} - \frac{3}{2}\varpi_n$, i.e., λ_i are the highest weights of the so called *basic symplectic spinor modules* $L(\lambda_i)$ (for notation, see bellow). Britten, Hooper, Lemire in [1] and Britten, Hooper in [2] are giving a characterization of all infinite dimensional modules with bounded multiplicities over complex symplectic Lie algebras. The authors of these articles proved that the class of infinite dimensional highest weight modules with bounded multiplicities equals the set of higher symplectic spinor modules, i.e., the set $\{L(\lambda); \lambda \in \mathbb{A}\}$. In this article, we study a problem, which is in a sense complementary to that of Britten, Hooper and Lemire. Namely, we describe the decomposition of the tensor product $L(\lambda) \otimes F(\varpi_1)$ of an arbitrary infinite dimensional module with bounded multiplicities $L(\lambda)$, $\lambda \in \mathbb{A}$, and the defining representation $F(\varpi_1)$ of the complex symplectic Lie algebra. Techniques used to decompose the mentioned tensor product are based on a result on formal characters of tensor products of an irreducible highest weight module and an irreducible finite dimensional module over simple complex Lie algebras, described by Humphreys in [5]. The assumption under which his formula is valid is the same as that one used by Kostant, see [11], for a more general situation. The second ingredient we have used is the famous Kac-Wakimoto formula in Kac, Wakimoto [7], which was published for complex simple Lie algebras in Jantzen [6] earlier, but which is valid for slightly different set of weights.

In the second section of this article, some known results on formal characters of irreducible highest weight modules (Theorem 2.1), decomposition of tensor products (Theorems 2.2, 2.3) and formal character of a tensor product (Theorem 2.4) are presented. The second part contains also Lemma 2.7, in which Theorem 2.1 is adapted to the situation of our interest. The third part of this paper is devoted to the formulation of the decomposition of $L(\lambda) \otimes F(\varpi_1)$ for $\lambda \in \mathbb{A}$ and to its proof (Theorem 3.1).

2. Tensor products and higher symplectic modules

2.1. Tensor products decompositions.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and let $(\,,)$ denote the Killing form of $\mathfrak{g}$. Suppose a Cartan subalgebra $\mathfrak{h}$ together with a subset Φ^+ of positive roots of the set Φ of all roots are given. The set of roots determines its $\mathbb{R}$-linear span, denoted by $\mathfrak{h}_0^*$. With help of the Killing form on $\mathfrak{g}$, we can introduce a mapping $\langle\,,\rangle : \mathfrak{h}_0^* \times (\mathfrak{h}_0^* - \{0\}) \to \mathbb{R}$ by the following equation

$$\langle v, w\rangle := 2\frac{(v,w)}{(w,w)},$$

for $v \in \mathfrak{h}_0^*$ and $w \in \mathfrak{h}_0^* - \{0\}$. The half-sum of all positive roots will be denoted by δ, i.e., $\delta := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Further, let us denote the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ by $\mathcal{W}$. The determinant of an element $\sigma \in \mathcal{W}$ is denoted by $\epsilon(\sigma)$. If $\lambda \in \mathfrak{h}^*$ then the symbol $\mathcal{W}^\lambda$ is used for a subgroup of the Weyl group $\mathcal{W}$ generated by reflections in planes perpendicular to such simple roots γ, for which $\langle\lambda, \gamma\rangle \in \mathbb{Z}$. Further, let us denote the affine action of a Weyl group element by a dot, thus $\sigma.\lambda := \sigma(\lambda+\delta) - \delta$ is an affine action of an element $\sigma \in \mathcal{W}$ on $\lambda \in \mathfrak{h}^*$. For $\lambda, \mu \in \mathfrak{h}^*$, let us write $\lambda \sim \mu$, if there is an element $\sigma \in \mathcal{W}$ such that $\sigma.\lambda = \mu$. We will call such weights linked to each other. Let us denote the set of positive coroots by R_+ and the set $\{X \in R_+; \lambda(X) \in \mathbb{Z}\}$ for some $\lambda \in \mathfrak{h}^*$ by R_+^λ. Further, denote the basis of $R^\lambda := R_+^\lambda \cup -R_+^\lambda$ by $B^\lambda (\subseteq R_+^\lambda)$.

For a complex simple Lie algebra $\mathfrak{g}$, let $L(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ with a highest weight λ and $M(\lambda)$ be the Verma module with a highest weight λ. To stress that λ is integral and dominant for a choice of $(\mathfrak{h}, \Phi^+)$, i.e., the corresponding module $L(\lambda)$ is finite dimensional, we will denote $L(\lambda)$ by $F(\lambda)$ or simply by F, if the highest weight is not important or clear from the context. Let $\Pi(\lambda)$ be the set of all weights of the module $L(\lambda)$ and $n(\nu)$ be the multiplicity of weight $\nu \in \Pi(\lambda)$. For a weight $\lambda \in \mathfrak{h}^*$, symbol L_λ denotes the weight space of weight λ of a highest weight module L. Further, let us denote the formal character of a highest weight module L by $ch\ L$. The central character corresponding to a weight λ is denoted by χ_λ, i.e., we have $z.v = \chi_\lambda(z)v$ for each element v of a highest weight module with a highest weight λ and an element $z \in \mathfrak{Z} := Z(\mathfrak{U}(\mathfrak{g}))$ of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

Let L be a highest weight module over a complex semisimple algebra $\mathfrak{g}$. We call L module with bounded multiplicities, if there is $k \in \mathbb{N}$ such that $\dim L_\lambda \leq k$ for all weights λ of the module L. Such minimal k is called degree of L. We call a module with bounded multiplicities completely pointed provided its degree is 1. Let us mention that the basic symplectic spinor modules $L(\lambda_i)$, $i = 0, 1$ (see the Introduction for their definition via fundamental weights) are completely pointed and these are the only ones among infinite dimensional irreducible highest weight modules over the complex symplectic Lie algebra, see Britten, Hooper, Lemire [1].

There is a result on a formal character of an irreducible highest weight module over a complex semisimple algebra. In this theorem, the formal characters of Verma modules $M(\sigma.\lambda)$ for some Weyl group elements σ are related to the formal character of the irreducible module $L(\lambda)$.

Theorem 2.1. *Let $\lambda \in \mathfrak{h}_0^*$ be such that $(\lambda+\delta)\alpha > 0$ for all $\alpha \in B^\lambda$. Then we have*

$$ch\ L(\lambda) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.\lambda).$$

Proof. See Kac, Wakimoto [7], Theorem 1, pp. 4957. ∎

A version of the previous theorem appeared already in Jantzen [6], Theorem 2.23, pp. 70 but for a slightly different set of weights. We will refer to the formula in the preceding theorem as the Kac-Wakimoto formal character formula.

In the next theorem a decomposition of a tensor product of an irreducible highest weight module (possibly of infinite dimension) and a finite dimensional irreducible module into invariant summands is described, for further comments see Humphreys [5], pp. 1 - 64.

Theorem 2.2. *Let F be a finite dimensional module over a complex semisimple Lie algebra $\mathfrak{g}$ and $L(\lambda)$ be an irreducible highest weight module with a highest weight λ over $\mathfrak{g}$, then one has a canonical decomposition $F \otimes L(\lambda) = M^{(1)} \oplus \ldots \oplus M^{(k)}$, where $M^{(i)}$ is the generalized eigenspace corresponding to $\chi_{\lambda+\mu_i}$ and μ_i runs over a subset of the weights of F, so that the indicated central characters are distinct.*

Proof. See Humphreys [5], sect 4.4. and pp. 39. ∎

Let us recall the famous Harish-Chandra theorem, which says that $\chi_\lambda = \chi_\mu$, if and only if $\lambda \sim \mu$. In the next theorem, the generalized eigenspaces are specified more precisely.

Theorem 2.3. *Keep the above notation. Suppose $\mu := \mu_i$ is a weight of F such that for all weights $\nu \neq \mu$ of F, $\lambda+\nu$ and $\lambda+\mu$ are not linked to each other. Then $M := M^{(i)}$ is a direct sum of n copies of $L(\lambda+\mu)$, where $n = dim\ M_{\lambda+\mu}$.*

Proof. See Humphreys [5] sect. 6.3., pp. 40. ∎

In the next theorem, the formal character of the generalized eigenspace is related to formal characters of some Verma modules and to multiplicities of corresponding weights of the finite dimensional module F.

Theorem 2.4. *Keep the above notation and denote by $n(\mu)$ the multiplicity of the weight μ in the irreducible finite dimensional module F. Suppose that for all weights $\nu \neq \mu$ of F, $\lambda+\nu$ and $\lambda+\mu$ are not linked to each other. Further suppose that $(\lambda+\mu+\delta)\alpha >$ for each $\alpha \in B^{\lambda+\mu}$ and each weight μ of F. Then*

$$n(\mu) \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda+\mu)) = n\, ch\, L(\lambda+\mu).$$

Proof. See Humphreys [5] sect. 6.4., pp. 42 and use the Kac-Wakimoto formal character formula in the substitution for $a(w,\lambda)$ from Humphreys. ∎

2.2. The case of $\mathfrak{sp}(2n,\mathbb{C})$ and higher symplectic spinor modules.

In this subsection, we focus our attention to the complex symplectic Lie algebra, i.e., $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})(= C_n)$, and to a distinguished class of infinite dimensional irreducible highest weight modules. For a choice of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and of a system of positive roots Φ^+ of $\mathfrak{g}$, there is a set of fundamental weights, which will be denoted by $\{\varpi_i\}_{i=1}^n$. Having chosen the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we can define a subset $\{\epsilon_i\}_{i=1}^n$ of $\mathfrak{h}_0^*$, such that $\varpi_i = \sum_{j=1}^i \epsilon_j$, $i = 1,\dots,n$ which is an orthonormal basis of $\mathfrak{h}_0^*$ with respect to the restriction of the Killing form $(\,,)$ to the subspace $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$.

Now, let us describe modules we shall be dealing with.

Definition 2.5. Let us denote the set of weights

$$\{\lambda = \sum_{i=1}^n \lambda_i \varpi_i; \lambda_i \geq 0, i = 1,\dots,n-1, \lambda_n \in \mathbb{Z} + \frac{1}{2}, \lambda_{n-1} + 2\lambda_n + 3 > 0\} \subseteq \mathfrak{h}_0^*$$

by $\mathbb{A}$. We will call the modules $L(\lambda)$ for $\lambda \in \mathbb{A}$ *higher symplectic spinor modules.*

Theorem 2.6. *The following are equivalent:*

1.) $L(\lambda)$ is a higher symplectic spinor module, i.e., $\lambda \in \mathbb{A}$,

2.) $L(\lambda)$ has bounded multiplicities,

3.) $L(\lambda)$ is equivalent to a direct summand of the tensor product $L(-\frac{1}{2}\varpi_n) \otimes F(\nu)$ for some choice of dominant integral weight ν.

Proof. See Britten, Lemire [2], Theorem 2.1 pp. 3417 and Theorem 1.2 pp. 3415. ∎

In the next lemma, Theorem 2.1 is adapted to the situation we are studying.

Lemma 2.7. *Let $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda+\nu \in \mathbb{A}$, then*

$$ch\ L(\lambda+\nu) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda+\nu)).$$

Proof. We must check whether the assumption of Theorem 2.1 is satisfied. At first, we determine the set $R_+^{\lambda+\nu}$ for $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda+\nu \in \mathbb{A}$. Looking at the definition of the set $R_+^{\lambda+\nu}$, we easily obtain that

$$R_+^{\lambda+\nu} = \{e_i + e_j, 1 \leq i \leq j < n\} \cup \{e_i - e_j, 1 \leq i < j < n\} \cup \{e_k, 1 \leq k < n\},$$

where $\{e_i\}_{i=1}^n$ is the dual basis of $\mathfrak{h}_0$ to the basis $\{\epsilon_i\}_{i=1}^n$. The basis $B^{\lambda+\nu}$ of $R_+^{\lambda+\nu}$ is

$$B^{\lambda+\nu} = \{e_i - e_{i+1}, 1 \leq i \leq n-2\} \cup \{e_{n-1}\}.$$

Secondly, we need to compute $(\lambda+\nu+\delta)\alpha$ for $\alpha \in B^{\lambda+\nu}$. Suppose that $\nu = t\epsilon_p$ for some $p = 1,\dots,n$ and $t \in \{-1,1\}$.

1.) $A := (\lambda+\nu+\delta)(e_i - e_{i+1}) = [\sum_{r=1}^n (\sum_{s=r}^n \lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_i - e_{i+1}) = \lambda_i + 1 + t(\delta_{ip} - \delta_{i+1,p})$, $i = 1, \ldots, n-2$. We know that $\lambda + \nu \in \mathbb{A}$, from which it follows that $\lambda_i + t(\delta_{ip} - \delta_{i,p-1}) \geq 0$ for $i = 1, \ldots, n-1$, because $\epsilon_p = \varpi_p - \varpi_{p-1}, p = 1, \ldots, n$, where $\varpi_0 = 0$ and $\delta_{i,-1} := 0$ for $i = 1, \ldots, n$ are to be understood. Thus the condition $A > 0$, we have had to check, is satisfied.

2.) $B := (\lambda + \nu + \delta)(e_{n-1}) = [\sum_{r=1}^n (\sum_{s=r}^n \lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_{n-1}) = \lambda_{n-1} + \lambda_n + 2 + t\delta_{n-1,p}$. If $\lambda_n > 0$, then the inequality $B > 0$ is evidently satisfied. Now, suppose that $\lambda_n \leq -\frac{1}{2}$. If $p = n-1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 + t \geq 1$ $(\lambda + \nu \in \mathbb{A})$ and $\lambda_n \leq -\frac{1}{2}$, one obtains, that $\lambda_{n-1} + \lambda_n + \frac{3}{2} + t \geq 0$, from which $B > 0$ easily follows. If $p \neq n-1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 \geq 1$ $(\lambda \in \mathbb{A})$ and $\lambda_n \leq -\frac{1}{2}$, one obtains that $\lambda_{n-1} + \lambda_n + \frac{3}{2} \geq 0$, from which $B > 0$ follows.

Thus, we have proved that the assumption of Theorem 2.1 is satisfied and therefore the conclusion of this lemma follows. ■

3. Decomposition of $L(\lambda) \otimes F(\varpi_1)$ for $\lambda \in \mathbb{A}$

Theorem 3.1. *Let $L(\lambda)$ be a higher symplectic spinor module, i.e., $\lambda \in \mathbb{A}$. Then*

$$L(\lambda) \otimes F(\varpi_1) = \bigoplus_{\mu \in \mathbb{A}_\lambda} L(\mu),$$

where $\mathbb{A}_\lambda = \{\lambda + \nu; \nu \in \Pi(\varpi_1)\} \cap \mathbb{A}$. [2]

Proof. **Part I.** We would like to use Theorem 2.3. In this part, we shall verify its assumption. Thus we shall prove that $\lambda+\mu$ and $\lambda+\nu$ are not conjugated by the affine action of an element of the Weyl group $\mathcal{W}$ of the algebra C_n, if $\nu \neq \mu$ are arbitrary weights of $F(\varpi_1)$ and $\lambda \in \mathbb{A}$. Two elements $\phi, \psi \in \mathfrak{h}^*$ are conjugated by the affine action of an element of the Weyl group if and only if $\phi+\delta$ and $\psi+\delta$ are conjugated by an element of the Weyl group, i.e., if and only if $\sigma(\phi + \delta) = \psi + \delta$, for some $\sigma \in \mathcal{W}$.

Let us first prove that $\{\lambda + \nu + \delta, \lambda + \mu + \delta\} \subseteq \overline{W_1} \cup \overline{W_2}$, where

$$W_1 := \{\sum_{i=1}^n \beta_i \epsilon_i; \beta_1 > \ldots > \beta_n > 0\},$$

$$W_2 := \{\sum_{i=1}^n \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{n-1} > -\beta_n > 0\}$$

are two open neighbor Weyl chambers of C_n, and where $\overline{X}$ denotes the closure of $X \subseteq \mathfrak{h}_0^*$ wr. to the restriction of the Killing form $(\,,)$ to $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$. An arbitrary weight μ of $F(\varpi_1)$ is of the form $\mu = s\epsilon_p$ for $s \in \{-1, 1\}$ and some $p = 1, \ldots, n$. In the case of C_n, we have $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \ldots + \epsilon_n$. Using the relation

[2] One can easily compute that the (saturated) set $\Pi(\varpi_1)$ of weights of $F(\varpi_1)$ equals $\{\pm\epsilon_i; i = 1, \ldots, n\}$.

$\varpi_j = \sum_{i=1}^{j} \epsilon_i$ $(j = 1, \ldots, n)$, one easily computes that for $\lambda = \sum_{i=1}^{n} \lambda_i \varpi_i$, we have

$$\lambda + \mu + \delta =: \sum_{i=1}^{n} \beta_i \epsilon_i = \sum_{i=1}^{n} [(\sum_{j=i}^{n} \lambda_j) + n - i + 1 + s\delta_{ip}]\epsilon_i.$$

Thus the requirement $\lambda + \mu + \delta \in \overline{W_1}$ reduces to $\lambda_i + 1 \geq s(\delta_{i+1,p} - \delta_{ip})$ which is evidently satisfied for all $i = 1, \ldots, n-1$, see Definition 2.5. For $i = n$, the condition we need to check is $\beta_n \geq 0$ or $\beta_{n-1} \geq -\beta_n \geq 0$. If $\beta_n \geq 0$, we are done. Suppose $\beta_n < 0$, then the remaining condition we need to check is $\beta_{n-1} \geq -\beta_{n-1}$, because $-\beta_n \geq 0$ follows from our assumptions. The inequality $\beta_{n-1} \geq -\beta_{n-1}$ translates into

$$\lambda_{n-1} + 2\lambda_n + 3 + s(\delta_{n-1,p} + \delta_{np}) \geq 0. \tag{1}$$

Condition (1) is satisfied due to the last inequality in Definition 2.5 of higher symplectic spinor modules.

Suppose that there are some weights $\mu \neq \nu$ with $\mu, \nu \in \Pi(\varpi_1)$ for which $\lambda + \mu + \delta$ and $\lambda + \nu + \delta$ are conjugated by an element σ of the Weyl group of C_n, i.e., $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$.

(1) Suppose that $\lambda + \mu + \delta \in W_1$ and $\lambda + \nu + \delta \in W_2$ (or $\lambda + \mu + \delta \in W_2$ and $\lambda + \nu + \delta \in W_1$, which is analogous). The condition $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$ implies $\sigma W_1 = W_2$. It is evident that $\sigma_{\epsilon_n} W_1 = W_2$. The Weyl group acts simply transitively on the set of open (or closed) Weyl chambers. Hence $\sigma = \sigma_{\epsilon_n}$. The weight ϵ_n does not belong to the system of simple roots, but it is evident that we could have written $\sigma_{2\epsilon_n}$ instead of σ_{ϵ_n}. Now, $\sigma_{\epsilon_n}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\epsilon_n, \lambda + \mu + \delta)\epsilon_n = \lambda + \mu + \delta - 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$. This element equals to $\lambda + \nu + \delta$ if and only if $\mu - \nu = 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$ which is impossible due to the structure of the set $\Pi(\varpi_1)$ and the condition $\lambda_n \in \mathbb{Z} + \frac{1}{2}$.

(2) The case $\lambda + \mu + \delta, \lambda + \nu + \delta \in W_i$ and $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$ for $i = 1, 2$ leads to the condition $\sigma = id$, i.e., $\nu = \mu$ - a contradiction.

(3) The remaining case is $\lambda + \mu + \delta, \lambda + \nu + \delta \in \overline{W_1} \cup \overline{W_2} - (W_1 \cup W_2)$, i.e., the considered elements lie on the walls of the two Weyl chambers. (The other cases are impossible: if there is an element lying on a wall of a closed Weyl chamber and the other one is lying in the open Weyl chamber, then they cannot be conjugated.) The inspection of the fact $\lambda+\mu+\delta, \lambda+\nu+\delta \in \overline{W_1} \cup \overline{W_2}$ showed that if these elements lie on the walls of $\overline{W_1}$ and $\overline{W_2}$, then they lie in their interior (i.e., they do not lie on the walls of codimension 2): inequalities in the definition of $\overline{W_1}$ $(\beta_1 \geq \ldots \beta_n \geq 0)$ become equations only once and the same is true for $\overline{W_2}$. Let us define two families of open Weyl chambers

$$Y_r := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{r-1} > -\beta_r > \beta_{r+1} > \ldots > \beta_n > 0\},$$

$r = 1, \ldots, n-1$ and

$$Y'_t := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{t-1} > -\beta_t > \beta_{t+1} > \ldots > -\beta_n > 0\},$$

$t = 1, \ldots, n-1$.

(3.1) Suppose that $\lambda + \mu + \delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda + \nu + \delta \in \overline{W_2} \cap \overline{Y_t'}$ for some $r, t = 1, \ldots, n-1$. If we suppose that $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$, then the fact that these elements lie in the interior of the walls implies that $\sigma W_1 = W_2$ or $\sigma W_1 = Y_t'$. The first case leads to a contradiction as we have shown. Using the fact that the Weyl group acts simply transitively, we easily find that $\sigma = \sigma_{\epsilon_t}\sigma_{\epsilon_n}$ in the second case. Let us compute $\sigma_{\epsilon_t}\sigma_{\epsilon_n}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\epsilon_t, \lambda + \mu + \delta)\epsilon_t - 2(\epsilon_n, \lambda + \mu + \delta)\epsilon_n = \lambda + \mu + \delta - 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t - 2(\lambda_n + s\delta_{pn} + 1)\epsilon_n$. This element equals $\lambda + \nu + \delta$ if and only if $\mu - \nu = 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t + 2(\lambda_n + s\delta_{pn} + 1)\epsilon_n$. Because of the structure of $\Pi(\varpi_1)$, we obtain: $\mu - \nu \in \{\pm 2\epsilon_t, \pm 2\epsilon_n, \pm\epsilon_t \pm \epsilon_n, \pm\epsilon_t \mp \epsilon_n\}$. The first possibility leads to $0 = \lambda_n + s\delta_{np} + 1$, which is impossible because λ_n is half-integral. The second possibility implies $0 = \lambda_t + s\delta_{tp} + n - t + 1 \geq \lambda_t + n - t > 0$ - a contradiction. The third and fourth possibilities force $\pm 1 = 2(\lambda_t + s\delta_{tp} + n - t + 1)$ - an odd number equals an even one, which is a contradiction.

(3.2) Suppose that $\lambda + \mu + \delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda + \nu + \delta \in \overline{W_1} \cap \overline{Y_t}$. In this case, $\sigma W_1 = W_1$ or $\sigma W_1 = Y_t$. The first case leads to a contradiction as we already know. In the second case, one easily finds that $\sigma_{\epsilon_t} W_1 = Y_t$, i.e., using the simplicity of the Weyl group action, this implies $\sigma = \sigma_{\epsilon_t}$. Let us compute $\sigma_{\epsilon_t}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t$. This element equals $\lambda + \nu + \delta$ if and only if $\{\mu, \nu\} = \{\epsilon_t, -\epsilon_t\}$, i.e., $\mu - \nu = \pm 2\epsilon_t$. That means that $1 = \lambda_t + 1 + n - t + 1$ or $-1 = \lambda_t - 1 + n - t + 1$ which are impossible because $\lambda_t \geq 0$ and $t < n$ for $t = 1, \ldots, n-1$.

(3.3) The remaining cases are analogous to the previous ones and actually have been done.

Part II. Summarizing part I of the proof, we have proved that the assumption of Theorem 2.3 is satisfied, and therefore for each $\nu_i \in \Pi(\varpi_1)$ we have that the generalized eigenspace $M^{(i)}$ occurring in the canonical decomposition $L(\lambda) \otimes F(\varpi_1) = M^{(1)} \oplus \ldots \oplus M^{(k)}$ can be written as $M^{(i)} = n_i L(\lambda + \nu_i)$ for some nonnegative integer n_i. We should determine the numbers n_i for $i = 1, \ldots, k$. To do it, we should use Theorem 2.4. Let us suppose that $\nu_i \in \Pi(\varpi_1)$ is such that $\lambda + \nu_i \in \mathbb{A}$. It follows from the proof of Lemma 2.7 that for such weights, we have $(\lambda + \nu_i + \delta)\alpha > 0$ for each $\alpha \in B^{\lambda + \nu_i}$, i.e., the condition of Theorem 2.4 is satisfied. We may therefore write $n(\nu_i)\sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda + \nu_i)) = n_i ch L(\lambda + \nu_i)$. Because we know, that $n(\nu_i) = 1$ for all weights $\nu_i \in \Pi(\varpi_1)$, we get

$$\sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\, M(\sigma.(\lambda + \nu_i)) = n_i ch\, L(\lambda + \nu_i).$$

Using the formal character formula of Kac and Wakimoto from Lemma 2.7, we get $ch\, L(\lambda + \nu_i) = n_i ch\, L(\lambda + \nu_i)$, which implies $n_i = 1$ for such $\nu_i \in \Pi(\varpi_1)$ for which $\lambda + \nu_i \in \mathbb{A}$. From Theorem 2.3, we know that if $L(\mu)$ appears in the decomposition of $L(\lambda) \otimes F(\nu)$, then $\mu = \lambda + \eta$, where $\eta \in \Pi(\varpi_1)$. Still, we have shown that $\mu \in (\lambda + \Pi(\varpi_1)) \cap \mathbb{A} =: \mathbb{A}_\lambda$ occurs in the decomposition, we are interested in, with multiplicity 1. The remaining question is, whether a weight from $(\lambda + \Pi(\varpi_1)) \setminus \mathbb{A}$ may occur in the decomposition. But this is not possible, because the highest weight μ of an irreducible summand $L(\mu)$ of the decomposition lies in the set $\mathbb{A}$. To see it, consider an integral dominant weight $\nu \in \mathfrak{h}_0^*$ such that $L(\lambda) \subseteq L(\lambda_0) \otimes F(\nu)$. Such weight ν exists due to Theorem 2.6 (1. $\Rightarrow$ 3.). Using the associativity of a tensor product, we have $L(\mu) \subseteq L(\lambda_0) \otimes (F(\nu) \otimes F(\varpi_1))$. The tensor product $F(\nu) \otimes F(\varpi_1)$ decomposes into a finite direct sum of finite dimensional irreducible $\mathfrak{sp}(2n, \mathbb{C})$-modules, and therefore $L(\mu)$ is a direct summand in a tensor product $L(\lambda_0) \otimes F(\nu')$ for some integral dominant weight ν'. Using Theorem 2.6 (3. $\Rightarrow$ 1.) we get $\mu \in \mathbb{A}$. ■

Further research could be devoted to an investigation of real higher symplectic spinor representations of real symplectic Lie algebras and to their globalizations.

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