The Baker-Campbell-Hausdorff Formula
in the Free Metabelian Lie Algebra

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Abstract.   The classical Baker-Campbell-Hausdorff formula gives
a recursive way to compute the Hausdorff series $H = \ln(e^X e^Y)$ for
non-commuting X, Y. Formally H lives in the graded completion
of the free Lie algebra L generated by X, Y. We present a closed
explicit formula for $H = \ln(e^X e^Y)$ in a linear basis of the graded
completion of the free metabelian Lie algebra $L/[[L, L], [L, L]]$.
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1.   Introduction

1.1. Brief summary. The Baker-Campbell-Hausdorff (BCH) formula
arises naturally in the context of Lie groups and Lie algebras. Originally
the series $H = \ln(e^X e^Y)$ was used to define a multiplication law in a Lie
group associated to a given Lie algebra. If the variables X, Y commute
then $\ln(e^X e^Y) = X + Y$.

Let L be the free Lie algebra generated by X, Y. Then $H = \ln(e^X e^Y)$ belongs to the graded completion of L, i.e. H contains com-
mutators of all degrees (lengths). A linear basis of L contains exponen-
tially many elements of a fixed degree. That is why the classical BCH
formula is awkward for solving exponential equations in Lie algebras.

The author's interest in the BCH formula came from knot theory
and number theory. The Kontsevich integral is a powerful knot invariant
and can be computed combinatorially from a knot projection via a Drin-
feld associator [5]. A Drinfeld associator is a non-commutative 2-variable
series living in the graded completion of a Lie algebra and satisfying the

pentagon and hexagon equations involving 5 and 6 exponential factors, respectively. An outstanding problem in quantum algebra is to compute explicitly a Drinfeld associator with rational coefficients.

The set of Drinfeld associators modulo commutators of commutators contains a specific transcedental solution expressed via the classical zeta values $\zeta(n) = \sum_{k=1}^{\infty} \dfrac{1}{k^n}$. A closed metabelian version of the BCH formula was used to solve completely the pentagon and hexagon equations modulo commutators of commutators [10, Theorem 1.5c]. It turned out that these equations do not contain polynomial relations between odd zeta values.

Here we present a self-contained proof of the explicit BCH formula for $H = \ln(e^X e^Y)$ in the free metabelian Lie algebra $L/\big[[L,L],[L,L]\big]$.

1.2. Definitions and results. Classical notions of the Lie theory can be found in [12]. Lie algebras are considered over a field of characteristic 0. The free Lie algebra L generated by X,Y is graded by the *degree* : $\deg X = \deg Y = 1$ and $\deg[A,B] = \deg A + \deg B$ for all $A,B \in L$. The *graded* completion $\hat{L}$ of the Lie algebra L is the algebra of infinite series of elements of L.

The *Hausdorff* series is $H = \ln(e^X e^Y)$, where the logarithm and exponential are considered as formal power series, i.e. $e^X = \sum_{n=0}^{\infty} \dfrac{X^n}{n!}$ and $\ln(1+Y) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} Y^n$. The classical BCH formula states that $H \in \hat{L}$ [3, 4, 8], i.e. H can be expressed as an infinite sum of commutators, see Theorem 2.4. For $A_1, \ldots, A_m \in L$, the *long commutator* $[A_1 A_2 \ldots A_{m-1} A_m]$ is 'the right bracketing' $[A_1, [A_2, [\ldots [A_{m-1}, A_m] \ldots]]]]$, e.g. $[X^2 Y] = [X, [X,Y]]$ and $[YXY] = [Y, [X,Y]]$. Then

$$ H = X + Y + \frac{[XY]}{2} + \frac{[X^2 Y] - [YXY]}{12} - \frac{[XYXY]}{24} + \cdots $$

E. Dynkin found a closed formula for H [6], but not in a linear basis of the graded completion $\hat{L}$, see Theorem 2.5. Also the series H can be expressed via associative monomials W in the variables X,Y as follows: $H = X + Y + \sum c_W W$. The generating function for the coefficients c_W was computed by K. Goldberg [7], see Theorem 2.6.

If a Lie algebra L satisfies $\big[[L,L],[L,L]\big] = 0$, then L is said to be *metabelian*. For a free Lie algebra L, the quotient $L/\big[[L,L],[L,L]\big]$ is sometimes called the *free metabelian* Lie algebra.

Let L be the free Lie algebra generated by X,Y. Let $\hat{L}$ be the graded completion of L. Introduce the adjoint operators $x = \operatorname{ad} X$, $y = \operatorname{ad} Y$, i.e. $xA = [X,A]$, $yA = [Y,A]$ for $A \in \hat{L}$. Denote by $\bar{L}$ the metabelian quotient $\hat{L}/\big[[\hat{L},\hat{L}],[\hat{L},\hat{L}]\big]$.

Theorem 1.1. *Under $\hat{L} \to \bar{L}$ the Hausdorff series $H = \ln(e^X e^Y)$ maps onto*

$$\bar{H} = X + Y + \frac{1}{y}\left(1 - \frac{e^x - 1}{x} \cdot \frac{x + y}{e^{x+y} - 1}\right)[XY],$$

where the operator acting on $[XY]$ is considered as a commutative series in the adjoint operators x, y commuting on the Lie subalgebra $[\hat{L}, \hat{L}]$.

The series in x, y is a genuine power series with non-negative powers. The key advantage of the above *metabelian* BCH formula is that H is written in a linear basis of the free metabelian Lie algebra $\bar{L}$, see Lemma 3.1. The metabelian BCH formula can be effectively applied for solving exponential equations in Lie algebras, see Propositions 4.2, 4.4.

In section 2 we recall classical versions of the BCH formula. The original proof of Theorem 1.1 used the classical BCH formula and large combinatorial formulae involving extended Bernoulli numbers, see [10, Propositions 2.8, 2.12]. Section 3 contains a self-contained proof of Theorem 1.1. In section 4 we give applications to solving exponential equations in metabelian Lie algebras.

2. The classical Baker-Campbell-Hausdorff formula

2.1. The recursive Baker-Campbell-Hausdorff formula. Here we recall the classical version of the BCH formula (Theorem 2.4) proved by H. Baker [3], J. Campbell [4], F. Hausdorff [8].

Definition 2.1. The *Bernoulli* numbers B_n are defined by the generating function: $\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}$, e.g. $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$.

One can verify that $\frac{t}{e^t - 1} + \frac{t}{2}$ is an even function, hence $B_n = 0$ for all odd $n \geq 3$. The Bernoulli numbers can be easily computed from the recursive relation $\sum_{n=1}^{m} \binom{m+1}{n} B_n = -1$, $m \geq 1$, see [10, Lemma 2.2a].

Definition 2.2. A *derivation* of the graded completion $\bar{L}$ of a Lie algebra L is a linear function $D : \hat{L} \to \hat{L}$ satisfying the Leibnitz rule $D([A, B]) = [D(A), B] + [A, D(B)]$ for all $A, B \in \hat{L}$.

The *adjoint* operator $\operatorname{ad} A : \hat{L} \to \hat{L}$, $\operatorname{ad} A(B) = [A, B]$, is a derivation. The Leibnitz rule for the derivation $\operatorname{ad} A$ coincides with the Jacobi identity: $\operatorname{ad} A([B, C]) = [\operatorname{ad} A(B), C] + [B, \operatorname{ad} A(C)]$. Any continuous derivation of $\hat{L}$ is uniquely determined by its values on the topological generators.

Definition 2.3. Let L be the free Lie algebra generated by X and Y. Denote by D_Y the derivation of $\hat{L}$ *with respect to* Y such that $D_Y(X) = 0$ and $D_Y(Y) = H_1 = X + \sum_{n=1}^{\infty} \frac{B_n}{n!}[Y^n X] \in \hat{L}$.

Due to the property $B_n = 0$ for odd $n \geq 3$, the series H_1 can be rewritten as follows:

$$D_Y(Y) = H_1 = X + \frac{1}{2}[XY] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}[Y^{2n}X].$$

Theorem 2.4 is quoted from [12, Corollaries 3.24–3.25, p. 77–79].

Theorem 2.4. *[3, 4, 8] The Hausdorff series* $H = \ln(e^X e^Y)$ *is*

$$H = \sum_{m=0}^{\infty} H_m, \text{ where } H_0 = Y \text{ and } H_m = \frac{1}{m}D_Y(H_{m-1}) \text{ for } m \geq 1.$$

2.1. The BCH formulae in the forms of Dynkin, Goldberg.

Theorem 2.5. *[6] The Hausdorff series* $H = \ln(e^X e^Y)$ *equals*

$$H = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_i + q_i > 0 \\ p_i, q_i \geq 0}} \frac{\left[X^{p_1}Y^{q_1}X^{p_2}Y^{q_2}\ldots X^{p_m}Y^{q_m}\right]}{p_1!q_1!\ldots p_m!q_m!\sum_{i=1}^{m}(p_i+q_i)}.$$

The terms of the above Dynkin series are linearly dependent, e.g. $[XY] = -[X^0 Y X Y^0]$.

Let us express the Hausdorff series $H = \ln(e^X e^Y)$ via associative monomials in X, Y, where the exponents $s_1, \ldots, s_m$ are positive:

$$H = \sum c_x(s_1, \ldots, s_m)X^{s_1}Y^{s_2}\cdots(X \vee Y)^{s_m}+$$

$$+\sum c_y(s_1, \ldots, s_m)Y^{s_1}X^{s_2}\cdots(Y \vee X)^{s_m}, \text{ where}$$

$(X \vee Y)^{s_m}$ is X^{s_m} for odd m and Y^{s_m} for even m. Put $m' = [m/2]$.

Theorem 2.6. *[7] The generating function for the coefficients* c_x *is*

$$\sum_{s_1,\ldots,s_m} c_x(s_1, \ldots, s_m)Z_1^{s_1} \cdots Z_m^{s_m} = \sum_{i=1}^{m} Z_i e^{m'Z_i} \prod_{j \neq i} \frac{e^{Z_j} - 1}{e^{Z_i} - e^{Z_j}}.$$

One has $c_y(s_1, \ldots, s_m) = (-1)^{n-1}c_x(s_1, \ldots, s_m)$, *where* $n = \sum_{i=1}^{m} s_i$.

For example, $\sum_{s=1}^{\infty} c_x(s)X^s = X$ and $\sum_{s=1}^{\infty} c_y(s)Y^s = Y$, hence the Hausdorff series starts as expected: $H = X + Y + \cdots$. Theorem 2.6 implies that $c_x(s_1, \ldots, s_m)$ is invariant under any permutation of the s_i and $c_x(s_1, \ldots, s_m) = 0$ when m is odd and n is even.

3. The metabelian Baker-Campbell-Hausdorff formula

Here we give an elementary proof of Theorem 1.1. After we interpret the resulting metabelian BCH formula as a linear part of a deeper formula via commutators of commutators, see Proposition 3.8.

3.1. A self-contained proof of the metabelian BCH formula.
Firstly we describe a linear basis of the metabelian quotient
$\bar{L} = \hat{L}/\big[[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]\big]$ of the free Lie algebra L generated by X, Y. For any word W of length ≥ 2, the Jacobi identity implies

$$\big[X, [Y, [W]]\big] - \big[Y, [X, [W]]\big] = \big[[XY], [W]\big] = 0.$$

So we can permute the letters X, Y and express any long commutator via the elements $[X^k Y^l XY]$ for $k, l \geq 0$. Lemma 3.1 states that these elements are linearly independent in the quotient $\bar{L}$.

Lemma 3.1.   *[12, Theorem 5.7], [2, Section 4.7] Let L be the free Lie algebra generated by X, Y. The metabelian quotient $\bar{L} = \hat{L}/\big[[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]\big]$ has the linear basis $X, Y, [X^k Y^l XY], \ k, l \geq 0$.*

Let us express the Hausdorff series $H = \ln(e^X e^Y)$ via associative monomials in X, Y as before Theorem 2.6. Look at the terms of the type $c_{rs} X^r Y^s$, where the letter X always precedes Y. The generating function $c(u, v) = \sum\limits_{r,s \geq 1} c_{rs} u^r v^s$ of the corresponding coefficients can be extracted from Theorem 2.6, but we prefer a simple independent proof.

Lemma 3.2.   *The generating function $c(u, v)$ of Golberg's coefficients c_{rs} in front of the terms $X^r Y^s$ in the Hausdorff series $\ln(e^X e^Y)$ equals*

$$c(u, v) = \sum_{r,s \geq 1} c_{rs} u^r v^s = u e^u \frac{e^v - 1}{e^u - e^v} + v e^v \frac{e^u - 1}{e^v - e^u}.$$

Proof.   Put $P = e^X - 1$ and $Q = e^Y - 1$. The 'X before Y' part of

$$H = \ln(e^X e^Y) = \ln(1 + P + Q + PQ) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (P + Q + PQ)^m$$

equals $\displaystyle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \Big(\sum_{\substack{r + s = m \\ r, s \geq 1}} P^r Q^s + \sum_{\substack{r + s = m - 1 \\ r, s \geq 0}} P^{r+1} Q^{s+1} \Big).$ (∗)

The above sum does not change if permute P, Q. Lemma 3.3 is a straightforward computation, multipy both sides by $P - Q$.

Lemma 3.3. *For commuting variables P, Q, we have*

(a) $$\sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{\substack{r+s=m \\ r,s \geq 1}} P^r Q^s = \frac{Q \ln(1+P) - P \ln(1+Q)}{P-Q},$$

(b) $$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{\substack{r+s=m+1 \\ r,s \geq 1}} P^r Q^s = \frac{PQ}{P-Q} \ln \frac{1+P}{1+Q}.$$

In $(*)$ replace P, Q by $e^u - 1, e^v - 1$, respectively, where u, v are commuting variables of the function $(*)$. By Lemma 3.3 we have

$$c(u,v) = \frac{Q \ln(1+P) - P \ln(1+Q)}{P-Q} + \frac{PQ}{P-Q} \ln \frac{1+P}{1+Q} =$$

$$= \frac{u(e^v - 1) - v(e^u - 1) + (u-v)(e^u - 1)(e^v - 1)}{e^u - e^v} =$$

$$= \frac{ue^u e^v - ve^u e^v - ue^u + ve^v}{e^u - e^v} = ue^u \frac{e^v - 1}{e^u - e^v} + ve^v \frac{e^u - 1}{e^v - e^u} \text{ as required. } \blacksquare$$

Lemma 3.4. *Let L be the free Lie algebra generated by X, Y. The Hausdorff series has the form*

$$\ln(e^X e^Y) = X + Y + \sum_{k,l \geq 0} h_{kl}[X^k Y^l XY] + H',$$

where $h_{kl} \in \mathbb{Q}$, $H' \in [[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]]$. The only terms contributing to $c_{rs} X^r Y^s$ are $h_{kl}[X^k Y^l XY]$. One has $c_{k+1,l+1} = (-1)^l h_{kl}$ for $k, l \geq 0$.

Proof. Let the *length* of a non-commutative monomial in X, Y be the length of the resulting word after replacing all the positive powers by 1, e.g. the length of $X^2 Y X^3$ is 3. The shortest monomials coming from a commutator $[A, B]$, $A, B \in L$, are $X^r Y^s$ and $Y^s X^r$, $r, s \geq 1$. The polynomial expression of any commutator $[[A, B], [A', B']]$ with $A', B' \in L$ starts with monomials like $X^r Y^{s+s'} X^{r'}$, $Y^s X^{r+r'} Y^{s'}$. The result follows: $[X^k Y^l XY] = (-1)^l X^{k+1} Y^{l+1} + (\text{longer monomials})$. $\blacksquare$

Now we finish the proof of Theorem 1.1.

Proof. Under $\hat{L} \to \widehat{\bar{L}}$ the Hausdorff series maps onto $\bar{H} = X + Y + \sum_{k,l \geq 0} h_{kl}[X^k Y^l XY]$ for some coefficients $h_{kl} \in \mathbb{Q}$. By Lemma 3.4 the generating function $c(u,v)$ of Goldberg's coefficients can be expressed via $h(x,y) = \sum_{k,l \geq 0} h_{kl} x^k y^l$ as $c(x,y) = xyh(x,-y)$. We consider these

commutative series as formal Laurent ones, although the results always
have non-negative powers. Lemma 3.2 implies

$$h(x,y) = -\frac{1}{xy}c(x,-y) = -\frac{1}{xy}\left(xe^x\frac{e^{-y}-1}{e^x-e^{-y}} - ye^{-y}\frac{e^x-1}{e^{-y}-e^x}\right) =$$

$$= -\frac{e^x}{y}\frac{1-e^y}{e^{x+y}-1} + \frac{1}{x}\frac{e^x-1}{1-e^{x+y}} = \frac{1}{e^{x+y}-1}\left(\frac{e^{x+y}-e^x}{y} - \frac{e^x-1}{x}\right) =$$

$$= \frac{1}{y} - \frac{e^x-1}{e^{x+y}-1}\left(\frac{1}{y}+\frac{1}{x}\right) = \frac{1}{y}\left(1 - \frac{e^x-1}{x}\frac{x+y}{e^{x+y}-1}\right) \quad \text{as required.} \quad \blacksquare$$

Example 3.5. Theorem 1.1 allows us to compute easily first terms of
the metabelian series $\bar{H}$ obtained from $\ln(e^X e^Y)$ by $\hat{L} \to \widehat{\bar{L}}$. Suffice to
expand the operator $h(x,y)$ acting on $[XY]$ in Theorem 1.1, namely
$\bar{H} = X + Y + h(x,y)[XY]$, where $h(x,y) =$

$$\frac{1}{2} + \frac{x-y}{12} - \frac{xy}{24} - \frac{x^3+4x^2y-4xy^2-y^3}{720} + \frac{x^3y+4x^2y^2+xy^3}{1440} + (\deg \geq 5).$$

The above terms agree with the results from [10, Appendix, Proposition
A.4] and coincide with the commutators of $\ln(e^X e^Y)$ up to degree 4.

3.2. A deeper BCH formula via commutators of commutators.
The original BCH formula of Theorem 2.4 gives a recursive way to
compute $\ln(e^X e^Y)$ via commutators in X,Y. The linear part of this
formula is $X + Y$. Here we interpret the metabelian BCH formula of
Theorem 1.1 as a linear part of a deeper formula for $\ln(e^X e^Y) - X - Y$
via commutators of commutators, see Proposition 3.8.

Let L be the free Lie algebra generated by X,Y. Denote by
$\{m,n\}$ the long commutator $[X^m Y^{n+1} X]$, e.g. $\{0,0\} = [YX]$. The
series H_1 of Theorem 2.4 can be rewritten as follows: $H_1 = X + \sum_{n=1}^{\infty}\frac{B_n}{n!}\{0,n-1\}$. Lemma 3.6 computes the action of x,y on long com-
mutators.

Lemma 3.6. *The operators x,y act as follows:*
$x\{m,n\} = \{m+1,n\},$
$y\{m,n\} = \{m,n+1\} + \sum_{k=1}^{m}\binom{m}{k}[\{k-1,0\},\{m-k,n\}].$

Proof. The first formula is trivial, the second one is obtained by
induction on m. The base: $y\{0,n\} = [Y^{n+2}X] = \{0,n+1\}$. The

inductive step:

$$y\{m+1,n\} = [YX^{m+1}Y^{n+1}X]$$

$$= \big[[YX],[X^mY^{n+1}X]\big] + xy[X^mY^{n+1}X]$$

$$= [\{0,0\},\{m,n\}] + x\{m,n+1\} + \sum_{k=1}^{m}\binom{m}{k}x[\{k-1,0\},\{m-k,n\}]$$

$$= \sum_{k=1}^{m}\binom{m}{k}\big([\{k,0\},\{m-k,n\}] + [\{k-1,0\},\{m-k+1,n\}]\big)+$$

$$[\{0,0\},\{m,n\}] + \{m+1,n+1\}$$

$$= \sum_{k=1}^{m+1}\left(\binom{m}{k-1} + \binom{m}{k}\right)[\{k-1,0\},\{m-k+1,n\}] + \{m+1,n+1\}$$

$$= \{m+1,n+1\} + \sum_{k=1}^{m+1}\binom{m+1}{k}[\{k-1,0\},\{m-k+1,n\}]$$

∎

Lemma 3.6 allows us to compute $x^k y^l\{m,n\}$ for all $k,l > 0$ by using the Leibnitz rule, e.g.

$$y[\{m,n\},\{r,s\}] = [y\{m,n\},\{r,s\}] + [\{m,n\},y\{r,s\}].$$

We rewrite the derivative D_Y via the operators $x = \operatorname{ad} X$, $y = \operatorname{ad} Y$.

Lemma 3.7. *The derivative D_Y mapping X and Y to 0 and H_1, respectively, can be expressed via the adjoint operators x, y as follows:*

$$D_Y\{m,0\} = -\sum_{l=1}^{\infty}\frac{B_l}{l!}\{m+1,l-1\},$$

$$D_Y\{m,n\} = -x^m y^n \sum_{l=1}^{\infty}\frac{B_l}{l!}\{1,l-1\} + x^m \sum_{k=0}^{n-1} y^k\Big(\{1,n-k-1\}$$

$$+ \sum_{l=1}^{\infty}\frac{B_l}{l!}\big[\{0,l-1\},\{0,n-k-1\}\big]\Big),\ n \geq 1.$$

Proof. We have

$$D_Y([X^m YX]) = -[X^{m+1}H_1] = -x^{m+1}\sum_{l=1}^{\infty}\frac{B_l}{l!}\{0,l-1\}.$$

The case $n = 0$ follows from Lemma 3.6. The case $n \geq 1$ follows from the Leibnitz rule:

$$D_Y([X^m Y^{n+1}X]) = x^m y^n[H_1,X] + x^m \sum_{k=0}^{n-1} y^k[H_1,[Y^{n-k-1}X]] \qquad ∎$$

Now we rewrite $\ln(e^X e^Y)$ via long commutators and commutators of commutators. Proposition 3.8 follows directly from Theorem 2.4.

Proposition 3.8. *In the graded completion of the free Lie algebra generated by* X, Y *we have*

$$\ln(e^X e^Y) = X + Y + \sum_{n=1}^{\infty} \frac{B_n}{n!}\{0, n-1\} + \sum_{m=2}^{\infty} H_{(m)}, \text{ where}$$

$$H_{(2)} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{B_k}{k!}D_Y\{0, k-1\}, \quad H_{(m+1)} = \frac{D_Y H_{(m)}}{m+1}, \ m \geq 2.$$

Theorem 1.1 states that the metabelian image of $\ln(e^X e^Y)$ is equal to $X + Y - \sum_{kl} h_{kl}\{k, l\}$, where the generating function of the coefficients h_{kl} is $\sum_{kl} h_{kl}x^k y^l = \frac{1}{y}\left(1 - \frac{e^x - 1}{x}\frac{x+y}{e^{x+y}-1}\right)$. This metabelian part can be interpreted as an infinite linear combination of long commutators. All non-linear terms in the formula of Proposition 3.8 can be rewritten via commutators of long commutators due to Lemmas 3.6 and 3.7

Proposition 3.8 and Lemmas 3.6, 3.7 give a hope to extend Theorem 1.1 to the quotient $\tilde{L} = \hat{L}/[L', [L', L']]$, $L' = [\hat{L}, \hat{L}]$. A linear basis of the graded completion of $\tilde{L}$ consists of $X, Y, \{m, n\}, [\{k, l\}, \{m, n\}]$, where either $k > m \geq 0$, $l, n \geq 0$ or $k = m$, $l > n \geq 0$, see [12, §4.1].

4.   Applications of the metabelian BCH formula

We show how powerful the metabelian BCH formula is for solving exponential equations in metabelian Lie algebras. We can rewrite both sides of a given equation in a linear basis and compare coefficients.

4.1.   The metabelian Zassenhaus formula. A linear combination C of commutators in a free Lie algebra is called a *homogeneous* Lie element of *degree* n if the length of all commutators in C is n. The standard definition is equivalent to the above one [12, § 1.3]. According to W. Magnus [11, section IV], Zassenhaus proved, but didn't publish the following remarkable result.

Theorem 4.1.   (Zassenhaus) *Let* L *be the free Lie algebra generated by* X, Y. *Then* L *has a uniquely determined homogeneous Lie element* C_n *of degree* $n = 2, 3, 4, \ldots$, *satisfying the* Zassenhaus *equation* $e^{X+Y} = e^X e^Y e^{C_2} e^{C_3} e^{C_4} \cdots$ *in the universal enveloping algebra of* L.

W. Magnus proved the above theorem and gave a recursive way to compute the elements C_n in [11, section IV]. We describe explicitly the metabelian images $\bar{C}_n$ of the elements C_n in the quotient $\bar{L}$.

Proposition 4.2.   *Let* L *be the free Lie algebra generated by* X, Y. *In the universal enveloping algebra of* $\bar{L} = \hat{L}/[[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]]$, *the solution*

to the metabelian *Zassenhaus equation* $e^{X+Y} = e^X e^Y e^{\bar{C}_2} e^{\bar{C}_3} e^{\bar{C}_4} \cdots$ *is*

$$\sum_{n=2}^{\infty} \bar{C}_n = \frac{1}{x+y} \cdot \frac{e^{-y}-1}{y} \cdot \left(1 + \frac{e^{-x}-1}{x} \cdot \frac{y}{e^y-1} \right) [XY],$$

where $\bar{C}_n \in \bar{L}$ *are homogeneous Lie elements of degree* $n = 2, 3, 4, \ldots$

Proof.   Use Theorem 1.1 in the quotient $\bar{L}$: $Z = \ln(e^{-X} e^{X+Y}) =$

$$= (-X) + (X+Y) + \frac{1}{x+y} \left(1 + \frac{e^{-x}-1}{x} \frac{y}{e^y-1} \right) [-X, X+Y] =$$

$$= Y - \frac{1}{x+y} \left(1 + \frac{e^{-x}-1}{x} \frac{y}{e^y-1} \right) [XY], \text{ where } x = \operatorname{ad} X, \; y = \operatorname{ad} Y.$$

$Z - Y$ belongs to $[L, L]$ and, for $z = \operatorname{ad} Z$, the operator $\dfrac{z-y}{e^{z-y}-1}$ acts identically on $[L, L]$ modulo commutators of commutators. We perform computations for commutative series in the algebra of Laurent series. The result will be a genuine formal series with non-negative powers:

$$\ln(e^{-Y} e^{-X} e^{X+Y}) = \ln(e^{-Y} e^Z) = -Y + Z + \frac{1}{y} \left(1 - \frac{e^{-y}-1}{-y} \right) [-Y, Z] =$$

$$\frac{-1}{x+y} \left(1 + \frac{e^{-x}-1}{x} \frac{y}{e^y-1} - \left(1 + \frac{e^{-y}-1}{y} \right) \left(1 + \frac{e^{-x}-1}{x} \frac{y}{e^y-1} \right) \right) [XY]$$

$$= \frac{1}{x+y} \cdot \frac{e^{-y}-1}{y} \cdot \left(1 + \frac{e^{-x}-1}{x} \frac{y}{e^y-1} \right) [XY].$$

It remains to notice that $\bar{C}_n$ consist of commutators only, hence

$$\ln(e^{-Y} e^{-X} e^{X+Y}) = \ln \left(\prod_{n=2}^{\infty} e^{\bar{C}_n} \right) = \sum_{n=2}^{\infty} \bar{C}_n \text{ in } \bar{L} \text{ as required.} \quad \blacksquare$$

Example 4.3.   Proposition 4.2 allows us to calculate effectively

$$\sum_{n=2}^{\infty} \bar{C}_n = \left(-\frac{1}{2} + \frac{x}{6} + \frac{y}{3} - \frac{x^2}{24} - \frac{xy}{8} - \frac{y^2}{8} \right) [XY] + (\text{degree} \geq 5), \text{ hence}$$

$$\bar{C}_2 = -\frac{[XY]}{2}, \bar{C}_3 = \frac{[X^2Y]}{6} + \frac{[YXY]}{3}, \bar{C}_4 = -\frac{[X^3Y]}{24} - \frac{[XYXY]}{8} - \frac{[Y^2XY]}{8}.$$

The elements computed above coincide with the original C_2, C_3, C_4 since the subspace of L spanned by all commutators up to degree 4 maps injectively to $\bar{L}$.

4.2. The commutator equation in the metabelian quotient. The famous Kashiwara-Vergne conjecture [9, p. 250, Proposition 5.3] involves the following *commutator* equation: $\ln(e^X e^Y) - X - Y = [X, F] + [Y, G]$ for unknown F, G in the graded completion of the free Lie algebra L generated by X, Y. M. Kashiwara and M. Vergne proved the existence in their conjecture for a soluble Lie algebra [9, Proposition 0]. Recently A. Alekseev and E. Meinrenken showed the existence of a solution for any Lie algebra [1]. The Kashiwara-Vergne solution was obtained from a differential equation not leading to a closed formula.

We solve the commutator equation completely in the graded completion of the metabelian quotient $\bar{L} = \hat{L}/\big[[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]\big]$. Denote by $\bar{H}$ the image of the Hausdorff series $\ln(e^X e^Y)$ under $\hat{L} \to \bar{L}$. The *commutator* equation

$$\bar{H} - X - Y = [X, F(X, Y)] + [Y, G(X, Y)]$$

has the *symmetry*

$$\{F(X, Y), G(X, Y)\} \leftrightarrow \{(G(-Y, -X), F(-Y, -X)\}.$$

So we can restrict our attention to the *symmetrized* equation

$$\bar{H} - X - Y = [X, F(X, Y)] + [Y, F(-Y, -X)]$$

for $F, G \in \bar{L}$. Put $x = \mathrm{ad}\, X, y = \mathrm{ad}\, Y$.

Proposition 4.4. *Any solution to the equation*

$$\bar{H} - X - Y = [X, F(X, Y)] + [Y, F(-Y, -X)]$$

in $\bar{L}$ equals $F(X, Y) = aX + \dfrac{Y}{4} + f(x, y)[XY]$, where

$$f(x, y) = \frac{1}{y(x - y)} - \frac{1}{4x} - \frac{e^x - 1}{x} \cdot \frac{x + y}{e^{x+y} - 1} \cdot \frac{(x + y)e^y + 3x - y}{4xy(x - y)} + yg(x, y),$$

$f(x, y)$ is a commutative series with non-negative powers, a is a constant, $g(x, y)$ is any genuine series satisfying $g(x, y) = -g(-y, -x)$.

Proposition 4.4 will follow from Lemmas 4.5 and 4.6.

Lemma 4.5. *Any solution to*

$$\bar{H} - X - Y = [X, F(X, Y)] + [Y, F(-Y, -X)]$$

has the form $F(X, Y) = aX + \dfrac{Y}{4} + f(x, y)[XY]$, where a is a constant and $f(x, y)$ is a commutative series with non-negative powers, satisfying

$$(4.5) \qquad xf(x, y) - yf(-y, -x) = -\frac{1}{2} + \frac{1}{y}\left(1 - \frac{e^x - 1}{x} \cdot \frac{x + y}{e^{x+y} - 1}\right).$$

Proof. Lemma 3.1 implies that any series $F \in \bar{L}$ is $F = aX + bY + f(x,y)[XY]$ for some constants a, b and a commutative series $f(x,y)$. By Theorem 1.1 the symmetrized equation is equivalent to

$$\frac{1}{y}\left(1 - \frac{e^x - 1}{x} \cdot \frac{x+y}{e^{x+y} - 1}\right) = b - (-b) + xf(x,y) - yf(-y,-x).$$

Since the left hand side starts with $\dfrac{1}{2}$, we get $b = \dfrac{1}{4}$ and (4.5) holds. ∎

Lemma 4.6. *In the ring of formal series with non-negative powers, any solution to $xf(x,y) = yf(-y,-x)$ has the form $f(x,y) = yg(x,y)$, where g is any function verifying the symmetry $g(-y,-x) = -g(x,y)$.*

Proof. The given condition implies that $f(x,y) = yg(x,y)$ for a series $g(x,y)$. The substitution gives $g(-y,-x) = -g(x,y)$. ∎

Now we finish the proof of Proposition 4.4.

Proof. Put $h(x,y) = \dfrac{1}{y}\left(1 - \dfrac{e^x - 1}{x} \cdot \dfrac{x+y}{e^{x+y} - 1}\right)$. The function $h(x,y)$ appeared in Theorem 1.1 and satisfies the important symmetry $h(x,y) = h(-y,-x)$ since $\ln(e^X e^Y) = -\ln(e^{-Y} e^{-X})$. Equation (4.5) is $xf(x,y) - yf(-y,-x) = h(x,y) - \dfrac{1}{2}$, hence

$$\begin{cases} \text{Odd}\, h(x,y) = x\text{Even}\, f(x,y) - y\text{Even}\, f(-y,-x), \\ \text{Even}\, h(x,y) - \dfrac{1}{2} = x\text{Odd}\, f(x,y) - y\text{Odd}\, f(-y,-x), \end{cases} \quad \text{where}$$

$$\text{Odd}\, h(x,y) = \frac{h(x,y) - h(-x,-y)}{2} = \frac{1}{y} - \frac{e^x - 1}{x} \cdot \frac{e^y + 1}{2y} \cdot \frac{x+y}{e^{x+y} - 1},$$

$$\text{Even}\, h(x,y) = \frac{h(x,y) + h(-x,-y)}{2} = \frac{e^x - 1}{x} \cdot \frac{e^y - 1}{2y} \cdot \frac{x+y}{e^{x+y} - 1}.$$

The property $\text{Odd}\, h(x,x) = 0$ implies that the series $\text{Odd}\, h(x,y)$ is divisible by $x - y$. The function $\text{Even}\, f(x,y) = \dfrac{\text{Odd}\, h(x,y)}{x-y}$ satisfies the first equation in the above system. Actually, we have

$$\text{Even}\, f(-y,-x) = \frac{\text{Odd}\, h(-y,-x)}{(-y)-(-x)} = \frac{\text{Odd}\, h(x,y)}{x-y} = \text{Even}\, f(x,y).$$

The function $\text{Odd}\, f(x,y) = \dfrac{1}{2x}\left(\text{Even}\, h(x,y) - \dfrac{1}{2}\right)$ verifies the second equation. The expression in the brackets is divisible by x due to $\text{Even}\, h(0,y) = \dfrac{1}{2}$. So $x\text{Odd}\, f(x,y) - y\text{Odd}\, f(-y,-x) = \text{Even}\, h(x,y) - \dfrac{1}{2}$.

The final solution is

$$f(x,y) = \frac{\text{Odd } h(x,y)}{x-y} + \frac{\text{Even } h(x,y)}{2x} - \frac{1}{4x}$$

$$= \frac{1}{y(x-y)} - \frac{1}{4x} - \frac{e^x - 1}{x} \cdot \frac{x+y}{e^{x+y}-1} \cdot \frac{(x+y)e^y + 3x - y}{4xy(x-y)}.$$

∎

We may pose the problem to describe all solutions to the symmetrized equation $\ln(e^X e^Y) - X - Y = [X, F(X,Y)] + [Y, F(-Y, -X)]$ in the graded completion $\hat{L}$ of free Lie algebra L generated by X, Y. A closed BCH formula in a linear basis of the graded completion $\hat{L}$ would help to find explicitly all solutions to the Kashiwara-Vergne conjecture.

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