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Actions of $\mathbb{R}\cdot\widetilde{\mathrm{SL}_2\mathbb{R}}$ on Laguerre Planes Related to the Moulton Planes

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Abstract. We construct flat Laguerre planes that admit the universal covering group of $\mathrm{SL}_2\mathbb{R}$, extended by a factor $\mathbb{R}$, as a group of automorphisms. These planes contain the affine Moulton planes as derived planes, and they are constructed by gluing together two copies of a Moulton plane, folded up along their fixed lines. The action of the Moulton group carries over (modulo $\mathbb{Z}_2$) to the Laguerre planes, and the circles and parallel classes are essentially orbits of 4 types of one-parameter groups of the Moulton action. These planes have group dimension 4 and also provide examples of flat Laguerre planes of Kleinewillinghöfer type II.G.1.
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1. Introduction

The analytic and group structures of the universal covering group $\tilde{\Omega}$ of $\Omega = \mathrm{PSL}_2(\mathbb{R})$ are well known; see, for example, [2], [5], [7]. However this group is most elusive as it permits no faithful linear representation and only a few geometries are known on which $\tilde{\Omega}$ acts as a group of automorphisms. Among the 2-dimensional projective planes only the Moulton planes, see [12], section 34, or the following section 2., admit $\tilde{\Omega}$ as a collineation group. The Moulton planes can be obtained from an action of a certain group of the form $\mathbb{R}\cdot\tilde{\Omega}$ on $\mathbb{R}^2$, fixing the origin. One takes as lines the closures of the orbits of the center and all orbits of all parabolic one-parameter groups of $\tilde{\Omega}$; finally one adds a line at infinity.

We shall show that the Moulton plane minus the origin may be folded up along the line at infinity in a way compatible with the action. This yields an action on a cylinder with one boundary circle. Gluing two copies of this cylinder along the two boundaries we obtain an action of the original Moulton group modulo $\mathbb{Z}_2$ on a cylinder, having 3 orbits. We shall construct a Laguerre plane from this action

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by taking as circles all orbits of compact one-parameter groups of $\mathbb{R} \cdot \tilde{\Omega}$ and of all hyperbolic and parabolic one-parameter groups of $\tilde{\Omega}$; some of the circles consist of two orbits, joined by a single point. The orbits of the center yield the parallel classes of the Laguerre plane.

Although it is not obvious from the construction, the Moulton plane reappears if we take the derived affine plane of our Laguerre plane at a point of the 1-dimensional orbit.

In [14] it was shown that a connected locally simple Lie subgroup of the automorphism group of a flat (i.e., 2-dimensional) Laguerre plane is isomorphic to either $\Omega = \mathrm{PSL}_2(\mathbb{R})$ or $\tilde{\Omega}$. The former group occurs in the classical flat Laguerre plane and also in certain semi-classical Laguerre planes, see [13], but no flat Laguerre planes were known on which the latter group acts. Our aim with the above construction is to provide examples for the latter kind of flat Laguerre planes. The planes obtained also provide examples of flat Laguerre planes of Kleinewillinghöfer type II.G.1, one of the types whose existence remained open in [11] and [17].

2. Moulton Planes and Moulton Groups

A *flat affine plane* is an affine plane whose point set is $\mathbb{R}^2$ and whose lines are closed subsets homeomorphic to $\mathbb{R}$. The Moulton planes are flat affine planes. They were introduced by Moulton in 1902, see [9], and are some of the earliest examples of non-classical flat affine planes. The most homogeneous non-classical flat projective planes are the projective extensions of these flat affine planes.

We fix a real number $k > 1$ and replace every line in the Euclidean plane with negative slope m by a line that starts out as this Euclidean line in the right half-plane and continues as a line of slope km in the left half-plane, see [12], 31.25b. This gives the following bent lines.

$$\{(x, mx + t) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(x, kmx + t) \in \mathbb{R}^2 \mid x \leq 0\},$$

where $m, t \in \mathbb{R}, m < 0$. You can also think of this plane as being glued together along the y-axis from two Euclidean halves. The number k is the 'glue' factor.

For our purpose we need lines to be glued together along the x-axis instead. This is achieved by applying the transformation $(x, y) \mapsto (y, x)$. The lines of the resulting plane $\mathcal{M}_k$ are

- the horizontal lines $y = a$ for $a \in \mathbb{R}$;
- the vertical lines $x = b$ for $b \in \mathbb{R}$;
- the Euclidean lines $y = m(x - b)$ for $b, m \in \mathbb{R},\ m > 0$; and
- the bent Euclidean lines $y = \begin{cases} m(x-b), & \text{if } x \geq b, \\ km(x-b), & \text{if } x < b \end{cases}$ for $b, m \in \mathbb{R},\ m < 0$.

The projective extensions $\overline{\mathcal{M}}_k$ of the Moulton planes $\mathcal{M}_k$ play a prominent role in the theory of flat projective planes. Two such planes $\overline{\mathcal{M}}_k$ and $\overline{\mathcal{M}}_{k'}$ are isomorphic if and only if $k' = k$, and none of these planes are Desarguesian.

It is quite cumbersome to extract information about the full automorphism group of $\overline{\mathcal{M}}_k$ from the above models. Clearly, the maps of the form

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (cx + b, ay)$$

where $a, b, c \in \mathbb{R}$, $a, c > 0$, form a 3-dimensional group of collineations of $\mathcal{M}_k$. Furthermore, the transformation

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto \begin{cases} (x, -y), & \text{for } y \geq 0, \\ (x, -ky), & \text{for } y \leq 0 \end{cases}$$

also is a collineation of $\mathcal{M}_k$.

In fact, the full automorphism group of $\overline{\mathcal{M}}_k$ is 4-dimensional and does not fix the line at infinity. The only point and line fixed by all automorphisms of are the point at infinity of the y-axis and the x-axis, respectively. Furthermore, the automorphism group of a flat projective plane has dimension 4 if and only if it is isomorphic to one of the mutually non-isomorphic projective Moulton planes $\overline{\mathcal{M}}_k$ for $k > 1$.

These facts are best seen by looking at the *radial model* $\overline{\mathcal{M}}(s)$ of the Moulton plane $\overline{\mathcal{M}}_k$, where $k = e^{2\pi s}$; see [1] or, for the version preferred here, [12], Section 34. The affine plane $\mathcal{M}(s)$ is isomorphic to the complement of the x-axis in $\overline{\mathcal{M}}_k$. Its point set is the complex number plane $\mathbb{C}$, and its lines are the ordinary lines passing through the origin and the curves of the form

$$\left\{ \frac{ce^{s\varphi}}{\cos\varphi} e^{i\varphi} \;\middle|\; -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \right\},$$

where $c \in \mathbb{C}^\times$. Obviously, this line system is invariant under rotation and under multiplication by positive real numbers. An isomorphism between $\overline{\mathcal{M}}(s)$ and $\overline{\mathcal{M}}_k$ is obtained by extending the homeomorphism

$$\mathbb{C} \setminus i\mathbb{R} \to (\mathbb{R} \times \mathbb{R} \setminus \{0\}) : re^{i\varphi} \mapsto \left(\tan\varphi, \frac{e^{s\varphi}}{r\cos\varphi} \right),$$

where $r > 0$, $\pi/2 \neq \varphi \in (-\pi/2, 3\pi/2)$. In fact, this homeomorphism uniquely extends to an isomorphism between $\overline{\mathcal{M}}(s)$ and $\overline{\mathcal{M}}_k$ that maps the line at infinity of $\mathcal{M}(s)$ to the x-axis in $\overline{\mathcal{M}}_k$.

Let Δ denote the automorphism group of $\overline{\mathcal{M}}(s)$. This group fixes (precisely) the origin and the line at infinity, hence it can be seen within the affine plane $\mathcal{M}(s)$. The group Δ is a connected 4-dimensional Lie group. We exhibit four one-parameter groups generating Δ, see [12], 34.4:

- the central one-parameter group $H = \{\eta_t \mid 0 < t \in \mathbb{R}\}$, where $\eta_t(re^{i\varphi}) = tre^{i\varphi}$;
- the rotation group $R = \{\rho_t \mid t \in \mathbb{R}\} \cong \mathrm{SO}_2\mathbb{R}$, where $\rho_t(re^{i\varphi}) = re^{i(\varphi+t)}$;
- the hyperbolic group $\Xi = \{\xi_t \mid 0 < t \in \mathbb{R}\}$ defined below; and
- the parabolic group $E = \{\varepsilon_t \mid t \in \mathbb{R}\}$ defined below.

The groups H, E, Ξ are also considered (under a similar name) in [12]. Note, however, that R does not correspond to the group of spiral rotations considered there, which is contained in the commutator subgroup $\tilde{\Omega}$ of Δ and is not compact; for us, the compact group R is more important.

The definition of ξ_t is given by $\xi_t(x) = x/t$ for $x \in i\mathbb{R}$ and

$$\xi_t(re^{i\varphi}) = \frac{r\cos\varphi}{t\cos\psi} e^{s(\psi-\varphi)} e^{i\psi},$$

where $\tan\psi = t^2 \tan\varphi$ and $\varphi, \psi \in (-\pi/2, \pi/2)$ or $\varphi, \psi \in (\pi/2, 3\pi/2)$. The map ε_t has a very similar form; note that it is written incorrectly in [12]: on $i\mathbb{R}$, the action of ε_t is the identity, and

$$\varepsilon_t(re^{i\varphi}) = \frac{r\cos\varphi}{\cos\psi} e^{s(\psi-\varphi)} e^{i\psi},$$

where $\tan\psi = \tan\varphi + t$ and $\varphi, \psi \in (-\pi/2, \pi/2)$ or $\varphi, \psi \in (\pi/2, 3\pi/2)$.

According to [12], 34.6, the group $\Delta = \Delta(s)$ generated by H, R, E, Ξ is locally isomorphic to $\mathrm{GL}_2\mathbb{R}$ and is independent of s (up to isomorphism). The commutator group Δ' is isomorphic to the universal covering group $\tilde{\Omega}$ of $\mathrm{PSL}_2\mathbb{R}$. The center of Δ is generated by H together with $\rho_\pi = -\mathrm{id}$, and it intersects the commutator group precisely in the infinite cyclic center of the latter. Although the Moulton groups $\Delta(s)$, $\Delta(s')$ determined by different choices $s, s' > 0$ are isomorphic groups, they are different as transformation groups of $\mathbb{C}^\times$, see [12], 34.7.

3. Action of the Moulton group on the Laguerre cylinder

In this section, we construct the point sets $Z = Z(s)$ of our new Laguerre planes and we describe an action of the Moulton group $\Delta(s)$ on $Z(s)$; later it will turn out that this yields the maximal connected automorphism group of those Laguerre planes. The actions of $\Delta(s)$ on $\mathbb{C}^\times$ and on $Z(s)$ will be related via an equivariant twofold covering map α from $\mathbb{C}^\times$ to the upper half of the cylinder Z. The normal subgroup $\langle\rho_\pi\rangle$ of order 2 will act trivially on Z.

We fix $k > 1$ for the remainder of this paper and let $s = \frac{\ln k}{2\pi}$ so that $k = e^{2\pi s}$. For $x \in \mathbb{R}$ let

$$\mu_0(x) = \begin{cases} 1, & \text{if } x > 0, \\ \sqrt{k}, & \text{if } x = 0, \\ k, & \text{if } x < 0. \end{cases}$$

The value of $\mu_0(0)$ does not matter too much, because in every occurrence of $\mu_0(0)$ it is multiplied by 0. The choice above is such that $\mu_0(x) = \lim_{c\to 0+} e^{2s\cot^{-1}(x/c)}$ where $\cot^{-1} : \mathbb{R} \to (0, \pi)$ is the inverse cotangent. Since $\mu_0(x)x^j$ occurs frequently for $j = 1, 2$, we introduce the abbreviation

$$\mu_j(x) = \mu_0(x)x^j = \begin{cases} x^j, & \text{if } x \geq 0, \\ kx^j, & \text{if } x < 0. \end{cases}$$

The *Laguerre cylinder* $Z = Z(s)$ is defined as

$$Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R},$$

with a topology defined as follows. We consider $\mathbb{R} \cup \{\infty\} \approx \mathbb{S}_1$ as the one-point-compactification of $\mathbb{R}$ and define the topology of Z as the inverse image of the product topology under the map $Z \to (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ given by

$$(x, y) \to \begin{cases} (x, \frac{y}{\mu_2(x)}), & \text{if } x \in \mathbb{R},\ x \geq 1 \text{ or } x \leq -1/\sqrt{k}, \\ (x, y), & \text{if } x = \infty \text{ or } -1/\sqrt{k} \leq x \leq 1. \end{cases}$$

In other words, a sequence $(x_n, y_n) \in Z$ converges to (∞, y) if, and only if, those y_n with $x_n = \infty$ converge to y, and for the remainder of the sequence we have $x_n \to \infty$ and

$$y = \lim \frac{y_n}{\mu_2(x_n)}.$$

At all points (x, y) with $x \neq \infty$, the neighborhood system is the usual one. This topology is distinct from the product topology but is, by its very definition, homeomorphic to it.

We define two subsets $Z_\pm \subseteq Z$ by the conditions $\pm y > 0$; their common boundary is the set $L_0 \approx \mathbb{S}_1$ defined by $y = 0$. We shall next describe a map $\alpha : \mathbb{C}^\times \to Z$, which will turn out to be a twofold covering of Z_+; there is a similar covering of Z_- which will not be made explicit at this point. Later we shall introduce an action of $\Delta(s)$ on Z such that α is equivariant. Note that α is closely related to a map considered in Section 2.

Proposition 3.1. *The following formula defines a continuous surjective map* $\alpha : \mathbb{C}^\times \to Z_+ \subseteq Z(s)$*:*

$$\alpha(re^{i\varphi}) = \begin{cases} (-\tan\varphi, \frac{e^{2s\varphi}}{r^2\mu_2(\cos\varphi)}), & \textit{if } \frac{\pi}{2} \neq \varphi \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \\ (\infty, \frac{1}{r^2\sqrt{k}}), & \textit{if } \varphi \in \{-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\}. \end{cases}$$

Proof. Only continuity of α at points $r_0 e^{i\varphi_0}$ with $\varphi_0 \in \{-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\}$ requires a proof. Let $r_n \to r_0$ and $\varphi_n \to \varphi_0$; we may assume that $\varphi_n \neq \varphi_0$. Then $\alpha(r_n e^{i\varphi_n})$ converges to (∞, y_0), where

$$y_0 = \lim \frac{e^{2s\varphi_n}}{r^2\mu_2(\cos\varphi_n)\mu_2(-\tan\varphi_n)},$$

provided that this limit exists. Checking the cases $\varphi_n \to -\frac{\pi}{2}+$, $\varphi_n \to \frac{\pi}{2}-$, $\varphi_n \to \frac{\pi}{2}+$, $\varphi_n \to \frac{3\pi}{2}-$ separately, it is straightforward to verify that the limit does exist and has the value $\frac{1}{r^2\sqrt{k}}$. ■

Our next aim is to use the map α in order to transfer the action of $\Delta(s)$ on $\mathbb{C}^\times$ to Z_+ (and later, to Z). This will be done by employing a general technique provided by the following lemma. We assume that we are given several things, namely,

1. a group G and a subset $H \subseteq G$ generating G;

2. sets X and Y

3. an action $G \times X \to X$, written as $(g, x) \to x^g$;

4. a map $H \times Y \to Y$, written as $(h, y) \to y^h$ such that $y \to y^h$ is bijective for fixed h;
5. a surjective map $f : X \to Y$ which is equivariant for generators, i.e., such that $f(x^h) = f(x)^h$ holds for each $h \in H$.

Lemma 3.2. *In the situation described above, there is a unique action of G on Y such that $f : X \to Y$ is G-equivariant. If X, Y are topological spaces and all maps $y \to y^h$, $h \in H$, are homeomorphisms of Y, then the same is true for the maps $y \to y^g$, $g \in G$.*

Proof. The second assertion is a trivial consequence of the first one. For the first assertion, we shall give the proof under the special assumption that every element of G is a product of some elements of H, i.e., that taking inverses is not required in the generation process. If that has been proved, then the general case will follow, because we can define the map $y \to y^{h^{-1}}$ to be the inverse of the map $y \to y^h$; then equivariance with respect to h^{-1} follows from equivariance with respect to h. Indeed, let $x \in X$ be given and define x', y' and y by $x = (x')^h$, $y' = f(x')$, and $y = (y')^h$, respectively. Then $f(x) = f((x')^h) = f(x')^h$, and hence $f(x^{h^{-1}}) = f(x') = f(x)^{h^{-1}}$, as desired. Therefore, the special case may be applied to the situation where H is replaced with $H \cup H^{-1}$, and the lemma will follow.

In the special situation, there is no other choice but to define

$$y^{(h_1 \ldots h_n)} = (\ldots (y^{h_1})^{h_2} \ldots)^{h_n}$$

for $h_i \in H$. Then we claim that $f(x)^g = f(x^g)$ holds for all $g \in G$: this follows by induction from assumption 5; the first inductive step is $f(x)^{(h_1 h_2)} = (f(x)^{h_1})^{h_2} = (f(x^{h_1}))^{h_2} = f((x^{h_1})^{h_2}) = f(x^{(h_1 h_2)})$.

It remains to be shown that we have defined an action on Y. So let $g_1, g_2 \in G$ be arbitrary elements and set $g = g_1 g_2$. We have to show that $y^g = (y^{g_1})^{g_2}$ holds for each $y \in Y$. Choose $x \in X$ such that $y = f(x)$. Then $y^g = f(x)^g = f(x^g)$ by what we have just proved. On the other hand, in the same way we obtain $(y^{g_1})^{g_2} = (f(x)^{g_1})^{g_2} = (f(x^{g_1}))^{g_2} = f((x^{g_1})^{g_2}) = f(x^{(g_1 g_2)}) = f(x^g)$, which finishes the proof. ■

We shall apply the lemma to $X = \mathbb{C}^\times$, $Y = Z_+$ and $f = \alpha$; the part of G will be played by Δ, and the generating set H will be the union of the one-parameter groups H, E, Ξ together with the set $\{\rho_t \mid 0 < t < \pi\}$ (which generates R). We would be allowed to omit one of the sets E, Ξ, but we shall need to know explicitly how each of these one-parameter groups acts on Z, and their action is easy to write down anyway. Only the action of R is a little more complicated. We need to define the action of the generators on Z_+, but actually we shall define them on all of Z at no extra cost.

For $p, q, r \in \mathbb{R}$ such that $p, r > 0$, we define a bijection $\gamma_{p,q,r} : Z \to Z$ by

$$\gamma_{p,q,r} : (x, y) \mapsto \begin{cases} (px + q, ry), & \text{if } x \in \mathbb{R} \\ (\infty, ry/p^2), & \text{if } x = \infty, \end{cases}$$

and we set $\eta_t = \gamma_{1,0,t^{-2}}$, $\varepsilon_t = \gamma_{1,-t,1}$, $\xi_t = \gamma_{t^2,0,t^2}$, so that we have

$$\eta_t(x, y) = (x, t^{-2} y), \quad \varepsilon_t(x, y) = (x - t, y), \quad \xi_t(x, y) = (t^2 x, t^2 y),$$

except for the special cases $\epsilon_t(\infty, y) = (\infty, y)$ and $\xi_t(\infty, y) = (\infty, t^{-2}y)$. It is easily seen that the maps $\gamma_{p,q,r}$ form a permutation group of Z isomorphic to the group $\mathrm{L}_2 \times \mathbb{R}$, where L_2 denotes the non-abelian 2-dimensional Lie group, but we shall obtain more information later.

Finally, let ρ_0 be the identity of Z and define $\rho_t : Z \to Z$ for $0 < t < \pi$ by

$$\rho_t : (x, y) \mapsto \begin{cases} \left(\frac{x\cos t - \sin t}{x \sin t + \cos t}, \frac{e^{2st}y}{\mu_2(x\sin t + \cos t)}\right), & \text{if } x \in \mathbb{R},\ x\sin t + \cos t \neq 0, \\ (\infty, (\sin^2 t)e^{2st}y/k), & \text{if } x \in \mathbb{R},\ x\sin t + \cos t = 0, \\ (\cot t, e^{2st}y/\sin^2 t), & \text{if } x = \infty. \end{cases}$$

We extend the definition of ρ_t to all $t \in \mathbb{R}$ by reducing modulo π.

Our next task is to show that all generating maps are homeomorphisms of Z_+; in fact, we shall show this for Z instead of Z_+.

Proposition 3.3. *The maps $\eta_t, \varepsilon_t, \xi_t$ and ρ_t defined above are homeomorphisms of the Laguerre cylinder Z.*

Proof. It suffices to show continuity of the maps, since the inverses are of the same kind; for instance, the inverse of ρ_t is $\rho_{\pi - t}$. We shall concentrate on the proof of continuity for ρ_t; the treatment of the other maps (and of the composite maps $\gamma_{p,q,r}$) is very easy and will be left to the reader. The points where continuity of ρ_t is not obvious are those where the second or third line in the definition of ρ_t applies.

To show continuity at (∞, y_0), consider a sequence (x_n, y_n) converging to this point. This means that $x_n \to \infty$ (and we may assume that $x_n \neq \infty$ for all n), and that $\frac{y_n}{\mu_2(x_n)} \to y_0$. We have to show that $\rho_t(x_n, y_n) \to \rho_t(\infty, y_0)$. We may assume that $t \in (0, \pi)$, hence $\mu_0(x_n \sin t + \cos t) = \mu_0(x_n)$ for n large, and the limit in question is equal to

$$\lim \left(\frac{x_n \cos t - \sin t}{x_n \sin t + \cos t}, \frac{e^{2st}y_n}{\mu_2(x_n \sin t + \cos t)} \right) = \left(\cot t, \frac{\lim(e^{2st}y_n\mu_2({x_n}^{-1}))}{\lim(\sin t + x_n^{-1}\cos t)^2} \right).$$

Comparing this to the definition of $\rho_t(\infty, y_0)$ establishes our claim.

Now consider a point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ such that $x_0 \sin t + \cos t = 0$. We are given a sequence (x_n, y_n) converging to this point in the usual sense, and we may assume that $x_n \sin t + \cos t$ is always nonzero. Our assumption implies that $x_0 = -\cot t$, and thus

$$x_0 \cos t - \sin t = -(\sin t)^{-1} < 0$$

(we assume again that $t \in (0, \pi)$). We have to show that $\rho_t(x_n, y_n)$ converges to the point $(\infty, (\sin^2 t)e^{2st}y_0 k^{-1})$. This means that in the first place, $h(x_n) = \frac{x_n \cos t - \sin t}{x_n \sin t + \cos t}$ tends to ∞ (which is the case) and that, moreover,

$$\frac{e^{2st}y_n}{\mu_0(x_n \sin t + \cos t)(x_n \sin t + \cos t)^2 \mu_0(h(x_n))h(x_n)^2}$$

converges to $(\sin^2 t)e^{2st}y_0k^{-1}$. Now we have seen that the numerator of $h(x_n)$ is negative, which shows that $\mu_0(h(x_n)) = \mu_0(-(x_n \sin t + \cos t))$, hence the product of the μ_0's in the denominator is k. Moreover, $x_n \cos t - \sin t \to -(\sin t)^{-1}$, and our claim follows. ∎

Next, we aim to prove equivariance of α. It would suffice to do this for a minimized system of generators. However, in most cases it is completely straightforward to obtain equivariance with respect to a one-parameter group, and this justifies the explicit definitions given above.

Proposition 3.4. *The map $\alpha : \mathbb{C}^\times \to Z$ introduced in 3.1 is equivariant with respect to all generators of Δ introduced above, that is, the maps $\eta_t, \varepsilon_t, \xi_t$, and ρ_t.*

Proof. We deal only with the hard part, ρ_t. We may assume that $t \in (0, 2\pi)$, for if we prove this much, then we know that ρ_π induces the identity on Z_+, and we know from 3.2 that the family of maps ρ_t, $t \in (0, 2\pi)$, of Z embeds in a one-parameter group. It follows then that ρ_t may be computed by reduction of t mod π. This agrees with our definition of ρ_t for general t.

We have to prove the equation $\alpha(\rho_t(re^{i\varphi})) = \rho_t(\alpha(re^{i\varphi}))$ for all $r > 0$ and all $\varphi \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$. Now the definitions of α and of ρ_t involve case distinctions. Since we know that α and ρ_t are continuous, it suffices to prove the equation in the generic cases. Thus, the action of ρ_t on Z is given as follows:

$$\rho_t(x, y) = \left(\frac{x \cos t - \sin t}{x \sin t + \cos t}, \frac{e^{2s\tilde{t}} y}{\mu_2(x \sin \tilde{t} + \cos \tilde{t})} \right),$$

where we have written $\tilde{t} \in [0, \pi)$ for t reduced mod π, and we have omitted the tilde where it makes no change. Here we ignore the exceptional cases $x = \infty$ and $x \sin t + \cos t = 0$. The formula for α may be used in the generic form valid for $\frac{\pi}{2} \neq \varphi \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Then we have

$$\alpha(\rho_t(re^{i\varphi})) = \left(-\tan(\varphi + t), \frac{e^{2s(\widehat{\varphi + t})}}{r^2 \mu_2(\cos(\varphi + t))} \right),$$

where $\widehat{\varphi + t} \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ denotes $\varphi + t$ reduced mod 2π. Now let

$$\varphi + t = \widehat{\varphi + t} + 2n\pi, \qquad t = \tilde{t} + m\pi.$$

Then we have $2s(\widehat{\varphi + t}) = 2s(\varphi + t) - 4ns\pi = 2s\varphi + 2s\tilde{t} + 2sm\pi - 4sn\pi$, hence (with $k = e^{2\pi s}$),

$$e^{2s(\widehat{\varphi + t})} = e^{2s\varphi} e^{2s\tilde{t}} k^{m-2n}.$$

On the other hand, we obtain from the definitions that

$$\rho_t(\alpha(re^{i\varphi})) = \left(\frac{-\tan \varphi \cos t - \sin t}{-\tan \varphi \sin t + \cos t}, \frac{e^{2s\tilde{t}} e^{2s\varphi}}{r^2 \mu_2(\cos \varphi)\, \mu_2(-\tan \varphi \sin \tilde{t} + \cos \tilde{t})} \right).$$

This is simplified using the identities $-\tan \varphi \cos t - \sin t = -(\cos \varphi)^{-1} \sin(\varphi + t)$ and $-\tan \varphi \sin t + \cos t = (\cos \varphi)^{-1} \cos(\varphi + t)$. Together, the equations obtained so far yield

$$\rho_t(\alpha(re^{i\varphi})) = \left(-\tan(\varphi + t), \frac{e^{2s(\widehat{\varphi + t})}}{r^2 k^{m-2n} \mu_0(\cos \varphi) \mu_0((\cos \varphi)^{-1} \cos(\varphi + \tilde{t})) \cos^2(\varphi + t)} \right).$$

Now our claim will follow from the identity

$$\mu_0(\cos(\varphi+t)) = k^{m-2n}\mu_0(\cos\varphi)\mu_0((\cos\varphi)^{-1}\cos(\varphi+\tilde{t})).$$

It suffices to prove this identity for $m \in \{0,1\}$. This is routinely checked by a case distinction depending on which of the intervals $(-\frac{\pi}{2},\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{3\pi}{2})$, $(\frac{3\pi}{2},\frac{5\pi}{2})$, or $(\frac{5\pi}{2},\frac{7\pi}{2})$ contains φ or $\varphi+t$. The details are omitted. ■

We introduce one more map acting on the Laguerre cylinder Z, namely the involution σ given by

$$\sigma : (x,y) \mapsto \begin{cases} (x,-y), & \text{for } y \geq 0 \\ (x,-ky), & \text{for } y \leq 0. \end{cases}$$

This map fixes all points of the set

$$L_0 = (\mathbb{R} \cup \{\infty\}) \times \{0\},$$

and it interchanges the complementary components Z_+, Z_- of L_0. We are now ready to state some comprehensive results.

Theorem 3.5. *1. There is an action of the Moulton group $\Delta = \Delta(s)$ on the Laguerre cylinder $Z(s)$ extending the actions of the one-parameter groups η, ε, ξ and ρ and generated by them.*

2. *The action of Δ on Z has three orbits, Z_+, Z_- and L_0.*

3. *Let $\tau : Z \to Z$ be defined by $\tau(x,y) = (x,-y)$. The maps $\alpha : \mathbb{C}^\times \to Z_+$ and $\tau \circ \alpha : \mathbb{C}^\times \to Z_-$ are twofold coverings and are equivariant with respect to Δ.*

4. *The map α has a continuous Δ-equivariant extension $\overline{\alpha}$ to the complement of the Δ-fixed point in the Moulton plane, sending the Δ-fixed line at infinity bijectively onto L_0.*

5. *The kernel of the action of Δ on Z is generated by the element ρ_π of order 2. The center of the effective factor group $\Delta_{\text{eff}} = \Delta/\langle\rho_\pi\rangle$ is the one-parameter group H (considered as a subgroup of Δ_{eff}). The simply connected cover $\tilde{\Omega}$ of $\mathrm{SL}_2\mathbb{R}$ is (isomorphic to) a subgroup of Δ_{eff}, and $\Delta_{\text{eff}} = H\tilde{\Omega}$. The intersection $H \cap \tilde{\Omega}$ is the infinite cyclic center of $\tilde{\Omega}$. In particular, Δ_{eff} is 4-dimensional.*

6. *The kernel of the action of Δ on L_0 is the center $H\langle\rho_\pi\rangle$ of Δ, and the effective action induced on L_0 is the 2-transitive standard action of $\Omega = \mathrm{PSL}_2\mathbb{R}$ on the projective line.*

7. *The map σ normalizes Δ_{eff}, in fact, $\sigma\gamma_{p,q,r}\sigma = \gamma_{p,q,kr}$ and $\sigma\rho_t\sigma = \gamma_{1,0,k}\rho_t$. In particular, Δ_{eff} has index 2 in the extension $\Gamma = \Delta_{\text{eff}}\langle\sigma\rangle$, and $\Delta_{\text{eff}} = \Gamma^1$ is the identity component of Γ.*

8. *The action of Γ on Z has two orbits, L_0 and $Z \setminus L_0$.*

It is worthwhile to picture the map $\overline{\alpha}$. What happens is that the Moebius strip $P\setminus\{0\}$ (the point set of the projective Moulton plane minus the fixed point 0) is folded along its 'middle circle' L_∞, thus producing a cylinder with one boundary curve (resulting from the folding line) and one 'open boundary'.

Proof. The maps τ defined in 3 and $\zeta : Z \to L_0$ defined by $\zeta(x,y) = (x,0)$ commute with each of the homeomorphisms $\rho_t, \eta_t, \varepsilon_t, \xi_t$ of Z, hence by 3.4 the three maps $\alpha : \mathbb{C}^\times \to Z_+$, $\tau \circ \alpha : \mathbb{C}^\times \to Z_-$ and $\zeta \circ \alpha : \mathbb{C}^\times \to L_0$ are equivariant with respect to those homeomorphisms. Therefore, assertion 1 together with the last part of 3 (equivariance with respect to all of Δ) follows from 3.2. Assertion 2 is now obvious.

For the remaining part of 3 (the covering property), it is sufficient to consider α, since τ is a homeomorphism $Z_+ \to Z_-$. The complement $Z \setminus (\{\infty\} \times \mathbb{R})$ is evenly covered by α; in fact, each of the two components of $\mathbb{C}^\times \setminus i\mathbb{R}$ is mapped homeomorphically onto this set. Transitivity of Δ on Z_+ now implies that every point of Z has an open neighborhood that is evenly covered by α, and the claim 3 follows.

Let 0 and L_∞ denote the fixed point and the fixed line of the Moulton plane $\mathcal{M}(s)$, respectively. The map α sends each affine line L passing through 0 onto some generator of the cylinder Z. We extend α by sending the point of intersection $L \wedge L_\infty$ to the point $\alpha(L) \wedge L_0$. The extension $\overline{\alpha}$ is equivariant, because Δ permutes the lines passing through 0 and also permutes the generators. Moreover, $\overline{\alpha}$ is continuous, because a sequence of points $a_n = r_n e^{i\varphi_n} \in \mathbb{C}^\times = P \setminus L_\infty$ converges to $a \in L_\infty$ if, and only if, $r_n \to \infty$ and the projections $(a_n \vee 0) \wedge L_\infty$ converge to a. If this is the case, then the second coordinates of $\alpha(a_n)$ converge to 0 and the first coordinates converge to $\overline{\alpha}(a)$, hence $\alpha(a_n) \to \overline{\alpha}(a)$ as desired.

In [12], 34.6, it is proved that the commutator group Δ' is (isomorphic to) $\tilde{\Omega}$ and that Δ is the product of its center $H\langle\rho_\pi\rangle$ with $\tilde{\Omega}$, the intersection of the two groups being the infinite cyclic center of $\tilde{\Omega}$. (Note that ρ_π has a different meaning in [12]; what we call ρ_π is denoted $-\mathbf{1}$ there.) In particular, the factor group $\Delta/(H\langle\rho_\pi\rangle)$ is the simple group Ω. Since $H\langle\rho_\pi\rangle$ acts trivially on L_0 but Δ does not, we infer that the former group is the kernel of the action on L_0 and the induced group is isomorphic to Ω. There is only one action of Ω on the circle, and claim 6 is proved. See below for a more direct proof.

Assertion 5 now follows easily; just observe that the kernel of the action on Z must be contained in the kernel on L_0, and that H acts faithfully on Z. The center of Δ_{eff} is not larger than H since the factor group Ω is simple. Assertion 7 is verified by direct computation, and 8 is obvious. ∎

As an alternative, we give a direct proof of the isomorphism of the effective action on L_0 to the action of Ω on the projective line (the set of all 1-dimensional subspaces of $\mathbb{R}^2$). Each element of Ω can be represented by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$. This matrix operates on the set of 1-dimensional subspaces of $\mathbb{R}^2$ like the fractional linear map $x \mapsto \frac{ax+b}{cx+d}$ on $\mathbb{R} \cup \{\infty\}$. Under this correspondence the standard group $\mathrm{SO}_2(\mathbb{R})$ of rotations $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in \mathbb{R}$ yields the permutations $x \mapsto \frac{x \cos t - \sin t}{x \sin t + \cos t}$ induced by ρ_t on L_0. The maps induced by $\gamma_{p,q,1}$ are obtained in a similar way from upper triangular matrices. Together, these matrices generate Ω, and the isomorphism follows.

We have avoided the question whether the action of Δ on Z is continuous, i.e., whether $\delta(z)$ depends continuously on the pair (δ, z). We do know from 3.2

that each $\delta \in \Delta$ acts as a homeomorphism of Z. This will suffice for our purposes. Continuity of the action can be deduced from continuity of the Moulton action, using the equivariant maps $\overline{\alpha}$ and $\tau \circ \overline{\alpha}$ from Theorem 3.5 and observing that these maps admit local cross sections.

We remark here that some of the results of 3.5 could have been obtained independently of the facts about Moulton actions that we took from [12]. In a first draft of this paper, we started from the definition of a Laguerre plane given in the next section and showed that the generators of Γ considered in the present section are automorphisms of that plane (by essentially the same proof as in the next section) and we used known facts about automorphism groups of Laguerre planes. We prefer the present version because it gives a more complete picture of the structure of Γ and its action.

4. The Laguerre planes

In this section we present the circles that make up our Laguerre planes. In general, a flat (or 2-dimensional) Laguerre plane $\mathcal{L} = (Z, \mathcal{C})$ is an incidence structure on the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ (where $\mathbb{S}^1$ is represented as $\mathbb{R} \cup \{\infty\}$ as before) where $\mathcal{C}$ is a collection of graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$ such that the usual axioms of joining and touching are satisfied and parallel classes of points are the verticals (or generators) on the cylinder; compare [3] and [4] or [10] Chapter 5 or [16].

For a given $k = e^{2\pi s} > 1$, we are going to construct a plane $\mathcal{L}_k$. The automorphism group of this plane will be the group Γ constructed in Section 3; see Theorem 3.5. Therefore, we shall use the Laguerre cylinder Z from Section 3 as the point set. Since the topology of Z is not the product topology in our representation of Z, our circles will be graphs of functions that are discontinuous at ∞. Nevertheless, they will be topological circles in the topology induced by Z.

The first one of our circles is the 1-dimensional Δ-orbit L_0. The shape of the other circles will be obtained from the action of certain one-parameter groups of Δ_{eff}. Since the circle space $\mathcal{C}$ of a flat Laguerre plane is a 3-manifold, it is clear that every circle C is fixed by some one-parameter group Φ, and Φ has to be distinct from H because C should meet every generator exactly once. Thus, elements of Φ have the form $\eta_{at}\psi_t$, where $a \in \mathbb{R}$ and ψ is a nontrivial one-parameter group of $\tilde{\Omega}$. Conjugation in Δ does not change the number a, but can be used to reduce ψ to one of the three types of one-parameter groups in $\tilde{\Omega}$, represented by ε, ξ, and by the spiral rotations $\tilde{\rho}_t = \eta_{e^{st}}\rho_t$ (in our notation; in [12], these maps are simply called ρ_t). All one-parameter subgroups of $\tilde{\Omega}$ are non-compact. The one-parameter groups that we actually use in order to construct circles are the Δ-conjugates of E, Ξ and of the compact group $R \cong \mathrm{SO}_2\mathbb{R}$ (which is not contained in $\tilde{\Omega}$). Notice that also the H-orbits are objects of the Laguerre plane; the parallel classes consist of two of these orbits plus one point on L_0.

The choice of an orbit of a given one-parameter group does not matter very much. If Φ fixes C, and $A = \Phi(a) \subseteq Z_+$ is a nontrivial orbit contained in C, then all H-images of A are also orbits of Φ because H is central, and they are contained in the corresponding H-images of C. Thus we may choose suitable orbits of the preferred one-parameter groups arbitrarily and combine them into circles (not so arbitrarily), and then apply the group Δ to obtain all circles.

The R-orbits in Z_+ are simply the α-images of half circles

$$\left\{re^{i\varphi} \mid -\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}\right\}.$$

They are topological circles because ρ_π acts as the identity on Z_+. By definition of α, see 3.1, they have the following form:

$$C = \left\{\left(-\tan\varphi, \frac{e^{2s\varphi}}{r^2 cos^2\varphi}\right) \;\middle|\; -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\} \cup \{(\infty, r^{-2}k^{-\frac{1}{2}})\},$$

where $0 < r \in \mathbb{R}$; observe that $\mu_0(\cos\varphi) = 1$ in the given range. If we define $x = -\tan\varphi$ for $\varphi \neq \frac{\pi}{2}$, then $\varphi = -\tan^{-1}x = \cot^{-1}x - \frac{\pi}{2}$, and $(\cos\varphi)^{-2} = x^2 + 1$. This gives

$$C = \left\{(x, ae^{2s\cot^{-1}x}(1+x^2)) \;\middle|\; x \in \mathbb{R}\right\} \cup \{(\infty, a)\},$$

where $a = r^{-2}k^{-\frac{1}{2}}$. Since R is centralized by RH and Δ is generated by this set together with $E\Xi$, we obtain all orbits in Z_+ of all conjugates of R if we apply the transformations $(x,y) \to (cx+b, y)$ to C, where $b, c \in \mathbb{R}$ and $c > 0$. The orbits in Z_- are obtained from these by the transformation τ introduced in 3.5. This leads to the circles $D_{a,b,c}$ appearing in the following list. It is clear from the construction that the set of these circles is invariant under Δ. They are precisely the circles disjoint from L_0; this corresponds to the fact that R is of elliptic type, i.e., fixes no point on L_0.

Here we have looked at all orbits of one-parameter groups conjugate to R. For the remaining groups E and Ξ, we shall be content to show that all orbits of this group itself have been used to form circles appearing in the final list. That the same is true for conjugate groups will follow once we know that the whole system of circles is Δ-invariant. Invariance under the groups H, E and Ξ is fairly obvious, compare 4.1 below, but R-invariance requires some computations, which are omitted. Since these one-parameter groups generate Δ, invariance under Δ then follows.

Consider the group E, which is of parabolic type (i.e., E fixes exactly one point on L_0, namely the point $(\infty, 0)$). The E-orbits will be used accordingly to assemble circles touching the circle L_0 in that unique fixed point. Now ε_t acts on Z by $(x,y) \to (x-t, y)$, hence the orbits have the form

$$L_c = \{(x,c) \mid x \in \mathbb{R}\} \cup \{(\infty, 0)\}.$$

Under the mapping ρ_t, the circle L_c is taken to the circle $C_{ck^{-1}e^{2st}\sin^2 t, \cot t, \cot t}$ appearing in the list below.

Similarly, the group Ξ of hyperbolic type (fixing two points on L_0) yields those circles which intersect L_0 twice, namely in the points $(0,0)$ and $(\infty, 0)$. We have $\xi_t(x,y) = (t^2x, t^2y)$, hence the orbits of Ξ are Euclidean rays starting from $(0,0)$. We use them to compose lines $L_{m,b}$ that look like lines of the affine Moulton plane $\mathcal{M}_k$; observe, however, that they have nothing to do with the lines of the plane $\mathcal{M}(s)$ that we used to cover Z_+ via the mapping α. Yet the appearance of Moulton lines is not a coincidence: it will turn out that the derivation of $\mathcal{L}_k$ in the point $(\infty, 0)$ is equal to $\mathcal{M}_k$.

Here is our definition of circles of $\mathcal{L}_k$:

$$\begin{aligned}
L_c &= \{(x,c) \mid x \in \mathbb{R}\} \cup \{(\infty,0)\} \\
&\quad \text{for } c \in \mathbb{R}; \\
L_{m,b} &= \Big\{(x, m(x-b)) \mid x \in \mathbb{R}\Big\} \cup \{(\infty,0)\} \\
&\quad \text{where } m,b \in \mathbb{R}, m > 0; \\
L_{m,b} &= \{(x, m\mu_1(x-b)) \mid x \in \mathbb{R}\} \cup \{(\infty,0)\} \\
&\quad \text{where } m,b \in \mathbb{R}, m < 0; \\
C_{a,b,c} &= \{(x, a\mu_1(x-b)(x-c)) \mid x \in \mathbb{R}\} \cup \{(\infty,a)\} \\
&\quad \text{where } a,b,c \in \mathbb{R}, a > 0, b \leq c; \\
C_{a,b,c} &= \{(x, a(x-b)\mu_1 x - c) \mid x \in \mathbb{R}\} \cup \{(\infty,a)\} \\
&\quad \text{where } a,b,c \in \mathbb{R}, a < 0, b \leq c; \\
D_{a,b,c} &= \Big\{(x, ae^{2s\cot^{-1}\frac{x-b}{c}}((x-b)^2+c^2)) \mid x \in \mathbb{R}\Big\} \cup \{(\infty,a)\} \\
&\quad \text{where } a,b,c \in \mathbb{R}, a \neq 0, c > 0.
\end{aligned}$$

The sets L_c and $L_{m,b}$ are precisely the non-vertical lines of the Moulton plane $\mathcal{M}_k$ except for the point $(\infty,0)$. The sets $C_{a,b,c}$ are composed of two branches of Euclidean parabolae. For $b \neq c$ these parabolae pass through the same two points on the x-axis; the two parts are pasted together at one of their common points on the x-axis at which both the parabolae have non-positive Euclidean slopes and such that some bent line of the Moulton plane $\mathcal{M}_k$ becomes a tangent at this point. For $b = c$ the sets $C_{a,b,b}$ have only one point in common with L_0, the point $(b,0)$, and except for this point are entirely above or below L_0 for $a > 0$ and $a < 0$, respectively.

Let

$$\begin{aligned}
\mathcal{C}_k &= \{L_c \mid c \in \mathbb{R}\} \cup \{L_{m,t} \mid m,t \in \mathbb{R}, m \neq 0\} \\
&\quad \cup \{C_{a,b,c} \mid a,b,c \in \mathbb{R}, a \neq 0, b \leq c\} \\
&\quad \cup \{D_{a,b,c} \mid a,b,c \in \mathbb{R}, a \neq 0, c > 0\}
\end{aligned}$$

We denote by $\mathcal{L}_k = (Z, \mathcal{C}_k)$ the incidence structure with point set the cylinder Z and circle set $\mathcal{C}_k$. We say that two points (x,y) and (x',y') are parallel if and only if $x = x'$; the parallel classes, that is, the maximal sets of mutually parallel points, are just the generators of Z, that is, the vertical lines in our description.

We will show in Section 5. that $\mathcal{L}_k$ is a Laguerre plane and indeed a flat Laguerre plane. In the last section we show that the full automorphism group of $\mathcal{L}_k$ is in fact the 4-dimensional group Γ introduced in 3.5, and that the planes $\mathcal{L}_k$ are mutually non-isomorphic. As we remarked above, the generating one-parameter groups of Δ_{eff} consist of automorphisms of $\mathcal{L}_k$. In fact, also the map σ introduced in 3.5 is an automorphism (but not the map τ!). We list the images of circles under the generating automorphisms (except for rotations) in the following proposition.

Proposition 4.1. *σ and each $\gamma_{p,q,r}$ for $p,q,r \in \mathbb{R}$, $p,r > 0$, is an automorphism of $\mathcal{L}_k$. More specifically,*

$$\begin{array}{rclrcl}
\gamma_{p,q,r}(L_c) & = & L_{rc}, & \sigma(L_c) & = & L_{-\mu_1(c)},\\
\gamma_{p,q,r}(L_{m,b}) & = & L_{rm/p,pc+q}, & \sigma(L_{m,b}) & = & L_{-\mu_1(m),b},\\
\gamma_{p,q,r}(C_{a,b,c}) & = & C_{ra/p^2,pb+q,pc+q}, & \sigma(C_{a,b,c}) & = & C_{-\mu_1(a),b,c},\\
\gamma_{p,q,r}(D_{a,b,c}) & = & D_{ra/p^2,pb+q,pc}, & \sigma(D_{a,b,c}) & = & D_{-\mu_1(a),b,c}.
\end{array}$$

Since the group Γ is generated by these maps together with the rotations ρ_t, see Theorem 3.5, we obtain the following corollary.

Corollary 4.2. *The group Γ introduced in 3.5 consists of automorphisms of $\mathcal{L}_k$. Moreover, $\Sigma = \{\gamma_{p,q,p^2} \mid p, q \in \mathbb{R}, p > 0\}$ is a subgroup of Γ that acts trivially on the infinite parallel class $\Pi_\infty = \{\infty\} \times \mathbb{R}$.*

5. The Laguerre axioms

The axioms of a Laguerre plane are equivalent to the condition that for each point p of a Laguerre plane $\mathcal{L}$ the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set A_p consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p is an affine plane. We call $\mathcal{A}_p$ the *derived affine plane at p*.

Since the group of automorphisms obtained in Section 4. has two orbits on the cylinder Z by Proposition 3.5, it suffices to verify that the derived incidence structures at the points $(\infty, 0)$ and $(\infty, 1)$ are affine planes in order to show that the incidence structure $\mathcal{L}_k$ from Section 4. is a Laguerre plane. For the first point we just obtain, by construction, the Moulton plane $\mathcal{M}_k$ as described in Section 2.

Proposition 5.1. *Each derived incidence structure of $\mathcal{L}_k$ at a point of L_0 is an affine plane isomorphic to the Moulton plane $\mathcal{M}_k$. In particular, these derived affine planes are non-Desarguesian.*

The verification that the derived incidence structure $\mathcal{A}$ at the second point $(\infty, 1)$ is an affine plane is done in a number of steps. Recall that $\mathcal{A}$ has point set $\mathbb{R}^2$ and lines the vertical Euclidean lines and the restrictions to $\mathbb{R}^2$ of the circles $C_{1,b,c}$ and $D_{1,b,c}$, that is, these circles minus the point $(\infty, 1)$. We refer to such a line as $C_{b,c}$ or $D_{b,c}$, respectively. For convenience we sometimes make the coordinate transformation

$$\psi : (x, y) \mapsto \begin{cases} (x, \sqrt{y}), & \text{if } y \geq 0, \\ (x, -\sqrt{-y}), & \text{if } y < 0 \end{cases}$$

to obtain an isomorphic model $\psi(\mathcal{A})$. The advantage of $\psi(\mathcal{A})$ is that

$$\psi(C_{b,b}) = \{(x, x - b) \mid x \geq b\} \cup \{(x, -\sqrt{k}(x - b) \mid x < b\}$$

is made up of halves of two Euclidean lines. In order to distinguish lines of $\psi(\mathcal{A})$ from Euclidean lines we denote the Euclidean line $y = mx + t$ by $E_{m,t}$. We further call the Euclidean lines $E_{1,-b}$ and $E_{-\sqrt{k},b\sqrt{k}}$ the *supporting Euclidean lines of $\psi(C_{b,b})$*.

Note that each automorphism γ_{p,q,p^2} in Σ from Proposition 4.2 induces a collineation $\gamma_{p,q}$ of $\mathcal{A}$. Furthermore, in the coordinates of $\psi(\mathcal{A})$ the collineation

$\gamma_{p,q}$ becomes the dilatation $\gamma^{\psi}_{p,q} : (x, y) \mapsto (px + q, py)$ of the real affine plane. In particular, Euclidean lines are mapped to parallel Euclidean lines.

We begin with some properties of the functions that describe $\psi(D_{0,1})$ and $\psi(C_{0,1})$. Note that $\psi(D_{0,1})$ is entirely above the x-axis whereas $\psi(C_{0,1})$ has two connected pieces above the x-axis; compare Figures 1 and 2 below. We call the collection of all points of $\psi(C_{0,1})$ whose abscissas are greater than 1 the right branch and those points whose abscissas are negative the left branch.

Lemma 5.2. *1. Let*

$$f(x) = e^{s \cot^{-1} x}\sqrt{x^2 + 1}$$

so that $\psi(D_{0,1})$ is the graph of f. Then the derivative $f'(x)$ is strictly increasing from $-\sqrt{k}$ to 1 and $E_{1,s}$ and $E_{-\sqrt{k},-s\sqrt{k}}$ are oblique asymptotes for f. These Euclidean lines are supporting Euclidean lines for $\psi(C_{-s,-s})$ and intersect on L_0 in the point $(-s, 0)$. Furthermore, $\psi(D_{0,1})$ is strictly above $\psi(C_{-s,-s})$. If (x_i, y_i), $i = 0, 1$, are two points on $\psi(D_{0,1})$ with $x_0 < x_1$, then $-\sqrt{k} < \frac{y_1-y_0}{x_1-x_0} < 1 < \frac{y_1+y_0/\sqrt{k}}{x_1-x_0}$.

2. Let

$$g(x) = \begin{cases} \sqrt{\mu_1(x)(x-1)}, & \text{for } x \le 0 \text{ or } x \ge 1, \\ -\sqrt{x(1-x)}, & \text{for } 0 < x < 1, \end{cases}$$

so that $\psi(C_{0,1})$ is the graph of g. Then $g'(x)$ is strictly decreasing from $-\sqrt{k}$ to $-\infty$ on the interval $(-\infty, 0)$ and from $+\infty$ to 1 on $(1, \infty)$. The Euclidean lines $E_{1,-1/2}$ and $E_{-\sqrt{k},\sqrt{k}/2}$ are oblique asymptotes for g. These Euclidean lines are supporting Euclidean lines for $\psi(C_{1/2,1/2})$ and intersect on L_0 in the point $(1/2, 0)$. Furthermore, $\psi(C_{0,1})$ is strictly below $\psi(C_{1/2,1/2})$. If (x_i, y_i), $i = 0, 1$, are two points on $\psi(C_{0,1})$ with $x_0 < x_1$, then $\frac{y_1-y_0}{x_1-x_0} \ge 1$ in case these points are on the right branch, $\frac{y_1-y_0}{x_1-x_0} \le -\sqrt{k}$ if they are on the left branch, and $-\sqrt{k} < \frac{y_1-y_0}{x_1-x_0} < 1$ but $\frac{y_1+y_0/\sqrt{k}}{x_1-x_0} \le 1$ if (x_0, y_0) is on the left branch and (x_1, y_1) is on the right branch.

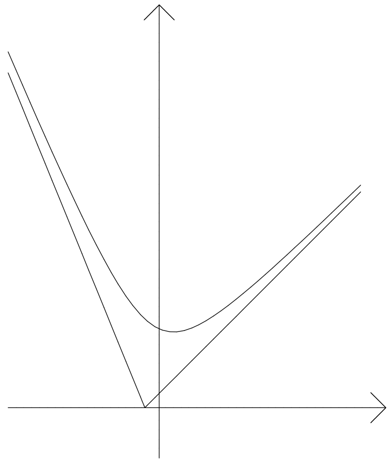

Figure 1
$\psi(D_{0,1})$ and $\psi(C_{-s,-s})$ for $k = 6$

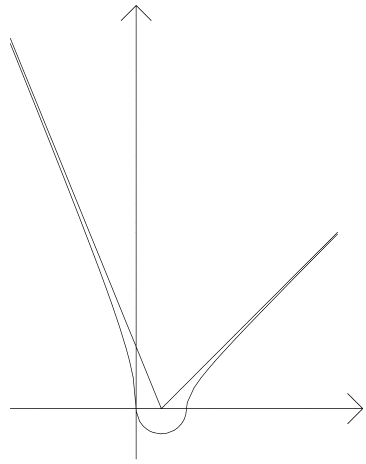

Figure 2
$\psi(C_{0,1})$ and $\psi(C_{1/2,1/2})$ for $k = 6$

Proof. 1) Differentiating f twice one finds

$$\begin{aligned} f'(x) &= e^{s\cot^{-1}x}(x-s)/\sqrt{x^2+1}, \\ f''(x) &= (s^2+1)e^{s\cot^{-1}x}/(x^2+1)^{3/2}. \end{aligned}$$

Hence $f''(x) > 0$ for all $x \in \mathbb{R}$ and f' is strictly increasing. But $\lim_{x\to\infty} f'(x) = 1$ and $\lim_{x\to-\infty} f'(x) = -\sqrt{k}$. This proves the statements on f'.

Using l'Hôpital's rule we obtain

$$\begin{aligned} \lim_{x\to\infty}(f(x)-x) &= \lim_{x\to\infty} \frac{e^{s\cot^{-1}x}\sqrt{1+x^{-2}}-1}{x^{-1}} \\ &= \lim_{x\to\infty} e^{s\cot^{-1}x}(sx+1)/\sqrt{x^2+1} \\ &= s \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to-\infty}(f(x)+\sqrt{k}x) &= \lim_{x\to-\infty} \frac{-e^{s\cot^{-1}x}\sqrt{1+x^{-2}}+\sqrt{k}}{x^{-1}} \\ &= \lim_{x\to-\infty} e^{s\cot^{-1}x}(sx+1)/\sqrt{x^2+1} \\ &= s\sqrt{k}. \end{aligned}$$

Since $f'(x) - 1 < 0$ for all $x \in \mathbb{R}$, the function $x \mapsto f(x) - (x+s)$ is strictly decreasing and $f(x) > x + s$ for all $x \in \mathbb{R}$. Similarly, $f'(x) + \sqrt{k} > 0$ for all $x \in \mathbb{R}$ so that the function $x \mapsto f(x) + \sqrt{k}(x+s)$ is strictly increasing. Therefore $f(x) > -\sqrt{k}(x+s)$ for all $x \in \mathbb{R}$.

The first two inequalities involving two points on $\psi(D_{0,1})$ follow from the mean value theorem and from the fact that f is strictly convex with derivatives between $-\sqrt{k}$ and 1. As for the third inequality, we consider the Euclidean lines of slopes $-\sqrt{k}$ and 1 through the left and right point, respectively. Each of these Euclidean lines is above the asymptote of the same slope. But the asymptotes intersect in a point on the x-axis. Therefore the point of intersection of the Euclidean lines must be above the x-axis. Explicitly, we have the Euclidean lines $y = x - x_1 + y_1$ and $y = -\sqrt{k}(x - x_0) + y_0$ which have the point

$$\left(\frac{1}{1+\sqrt{k}}(y_0 - y_1 + x_1 + x_0\sqrt{k}), \frac{\sqrt{k}}{1+\sqrt{k}}(y_1 + y_0/\sqrt{k} - x_1 + x_0)\right)$$

in common. This shows that $\frac{y_1+y_0/\sqrt{k}}{x_1-x_0} > 1$.

2) The statements follow similarly to the first part from

$$g'(x) = \mu_0(x)(x - \frac{1}{2})/g(x) \qquad \text{and} \qquad g''(x) = -\mu_0(x)^2/(4g(x)^3)$$

for $x < 0$ or $x > 1$. If (x_0, y_0) is on the left branch and (x_1, y_1) is on the right branch, then the right branch is strictly between the Euclidean lines though (x_0, y_0) of slopes 1 and $-\sqrt{k}$. This implies that $-\sqrt{k} < \frac{y_1-y_0}{x_1-x_0} < 1$. The inequality $\frac{y_1+y_0/\sqrt{k}}{x_1-x_0} \leq 1$ follows from the fact that the Euclidean lines though (x_0, y_0) of slope $-\sqrt{k}$ and through (x_1, y_1) of slope 1 must intersect below or on the x-axis. ∎

Since $\gamma_{c,b}(D_{0,1}) = D_{b,c}$ and $\gamma_{c-b,b}(C_{0,1}) = C_{b,c}$ the results of Lemma 5.2 carry over to arbitrary lines $D_{b,c}$ and $C_{b,c}$, after application of ψ. The right and left branch of $C_{b,c}$ are defined analogously to the right and left branch of $\psi(C_{0,1})$ in the obvious way.

Corollary 5.3. *Let (x_i, y_i), $i = 0, 1$, be two points with $x_0 < x_1$ and $y_0, y_1 > 0$.*

1. *If the two points are on a line $D_{b,c}$, then $-\sqrt{k} < \frac{\sqrt{y_1}-\sqrt{y_0}}{x_1-x_0} < 1 < \frac{\sqrt{y_1}+\sqrt{y_0/k}}{x_1-x_0}$.*
2. *If the two points are on a line $C_{b,c}$, then $\frac{\sqrt{y_1}-\sqrt{y_0}}{x_1-x_0} \geq 1$ in case they are on the right branch, $\frac{\sqrt{y_1}-\sqrt{y_0}}{x_1-x_0} \leq -\sqrt{k}$ if they are on the left branch, and $-\sqrt{k} < \frac{\sqrt{y_1}-\sqrt{y_0}}{x_1-x_0} < 1$ but $\frac{\sqrt{y_1}+\sqrt{y_0/k}}{x_1-x_0} \leq 1$ if (x_0, y_0) is on the left branch and (x_1, y_1) is on the right branch of $C_{b,c}$.*

Proposition 5.4. *$\mathcal{A}$ is a linear space, that is, any two points can be uniquely joined by a line.*

Proof. Let (x_i, y_i), $i = 0, 1$ be two points of $\mathbb{R}^2$. The existence and uniqueness of a joining line is obvious for $x_0 = x_1$. Using the group of collineations of $\mathcal{A}$ induced by Σ we can therefore assume that $x_0 = 0$ and $x_1 = 1$. If at least one y_i is non-positive, any joining line must come from a circle $C_{1,b,c}$. For example, in case $y_0 \leq 0$, the joining line is found from the system of equations

$$y_0 = bc \tag{1}$$
$$y_1 = (1-b)(1-c) \tag{2}$$

as in the classical flat Laguerre plane. This system of equations leads to

$$b^2 + (y_1 - y_0 - 1)b + y_0 = 0. \tag{3}$$

The discriminant of this quadratic equation in b is

$$D = (y_1 - y_0 - 1)^2 - 4y_0, \tag{4}$$

which is non-negative. Hence (3) has two solutions, the smaller one being the parameter b and the larger one being the parameter c of a joining line $C_{b,c}$.

In case $y_0 > 0 \geq y_1$ one obtains a similar system of equations which leads to essentially the same equation (3) and discriminant (4), the only difference being that y_0 is replaced by y_0/k.

We now assume that $y_1, y_2 > 0$. If not all inequalities in Corollary 5.3 are satisfied, the two points must be on a line of the form $C_{b,c}$. This leads to similar systems of equations for b and c as above, and consequently to a similar quadratic equation (3), where y_0 or y_1 may be replaced by y_0/k and y_1/k, respectively. For example, if $\sqrt{y_1} - \sqrt{y_0} \geq 1$, both points must be on the right branch of $C_{b,c}$ and we obtain exactly the same equation (3). But $\sqrt{y_1} - \sqrt{y_0} \geq 1$ implies $y_1 = (\sqrt{y_1})^2 \geq (\sqrt{y_0} + 1)^2 = y_0 + 1 + 2\sqrt{y_0}$ so that $y_1 - y_0 - 1 \geq 2\sqrt{y_0}$ and the discriminant D as in (4) is non-negative.

Likewise, if $\sqrt{y_0/k}+\sqrt{y_1} \le 1$, then $-\sqrt{k} < -\sqrt{y_0} < \sqrt{y_1}-\sqrt{y_0} < \sqrt{y_1} < 1$ so that $(0, y_0)$ must be on the left branch of $C_{b,c}$ and $(1, y_1)$ on the right branch. This means that in equation (1) and also in the subsequent equations (3) and (4) y_0 is replaced by y_0/k. Now $\sqrt{y_0/k}+\sqrt{y_1} \le 1$ implies $y_1 \le (1-\sqrt{y_0/k})^2 = 1+(y_0/k)-2\sqrt{y_0/k}$ so that $2\sqrt{y_0/k} \le 1+(y_0/k)-y_1$ and the discriminant D is again non-negative.

Note that in any of the cases above, the joining line $C_{b,c}$ is unique.

We finally assume that all inequalities in Corollary 5.3(1) are satisfied. A joining line cannot be of the form $C_{b,c}$ in this case and we have to show that there is a unique line of the form $D_{b,c}$ though these points.

The inequalities in Corollary 5.3(1) imply that there is an m such that both points $(0, y_0)$ and $(1, y_1)$ are above $C_{m,m}$. More precisely, one finds that $1-\sqrt{y_1} < m < \sqrt{y_0/k}$. (One only has to consider the lines of the form $C_{m,m}$ through each of these points and below the other point.) Furthermore, for each such m there is precisely one $c > 0$ such that $D_{m+sc,c}$ passes through $(0, y_0)$. ($\psi(D_{m+sc,c})$ has the supporting Euclidean lines of $\psi(C_{m,m})$ as asymptotes; compare Lemma 5.2(1).) In order to see this consider the function $f_m(c) = e^{2s\cot^{-1}(-s-m/c)}((m+sc)^2+c^2)$. Its derivative is $f'_m(c) = 2c(s^2+1)e^{2s\cot^{-1}(-s-m/c)} > 0$ so that f_m is strictly increasing in c. But $\lim_{c\to\infty} f_m(c) = \infty$ and $\lim_{c\to 0} f_m(c) = \mu_2(-m)$. If $m \ge 0$, then $\mu_2(-m) = km^2 < y_0/k < y_0$, because $k > 1$. For $m < 0$ we have $\mu_2(-m) = m^2 < (1-\sqrt{y_1})^2 < y_0$ by one of the inequalities in Corollary 5.3(1). Hence, in any case $\lim_{c\to 0} f_m(c) < y_0$ and by continuity there must be a c_m such that $f_m(c_m) = y_0$. But for this c_m the point $(0, y_0)$ is on D_{m+sc_m,c_m}. Moreover, c_m is uniquely determined because f_m is injective.

We differentiate $f_m(c_m)$ with respect to m. Since $f_m(c_m) = y_0$ we obtain

$$0 = \frac{d}{dm} f_m(c_m) = 2e^{2s\cot^{-1}(-s-m/c_m)}(m+2sc_m+(s^2+1)c_m c'_m)$$

where c'_m denotes the derivative of c_m. Hence

$$m+2sc_m+(s^2+1)c_m c'_m = 0. \tag{5}$$

Let $h(m) = e^{2s\cot^{-1}(-s+(1-m)/c_m)}((1-m-sc)^2+c^2)$ so that $(1, h(m))$ is the point of intersection of D_{m+sc_m,c_m} with the vertical line $x = 1$. For the derivative of h one finds

$$\begin{aligned} h'(m) &= 2e^{2s\cot^{-1}(-s+(1-m)/c_m)}(m-1+2sc_m+(s^2+1)c_m c'_m) \\ &= -2e^{2s\cot^{-1}(-s+(1-m)/c_m)} < 0 \end{aligned}$$

by (5). This shows that h is strictly decreasing. For m close to $1-\sqrt{y_1}$ one finds $h(m) > y_1$ (D_{m+sc_m,c_m} intersects $x = 1$ above the other point $(1, y_1)$) and for m close to $\sqrt{y_0/k}$ one has $h(m) < y_1$ (D_{m+sc_m,c_m} intersects $x = 1$ close to $(1, \mu_2(1-\sqrt{y_0/k}))$, the point of intersection of $C_{\sqrt{y_0/k},\sqrt{y_0/k}}$ with $x = 1$). However, $\mu_2(1-\sqrt{y_0/k}) < y_0$ so that by continuity there is an m such that $h(m) = y_1$), that is, D_{m+sc_m,c_m} passes through both points $(0, y_0)$ and $(1, y_1)$. Moreover, m is uniquely determined because h is injective.

Note that for a line $D_{b,c}$ through both points there is a unique line $C_{m,m}$ such that the supporting Euclidean lines of $\psi(C_{m,m})$ are asymptotes of $\psi(D_{b,c})$. From what we found above we know that $D_{b,c}$ then must be the line D_{m+sc_m,c_m}. Thus the line joining both points is unique in this case too. ■

We now deal with the parallel axiom in $\mathcal{A}$ and first establish criteria for two lines to be parallel.

Lemma 5.5. *Two lines $C_{b,c}$ and $C_{b',c'}$ are parallel if and only if $b+c=b'+c'$.*

Proof. Suppose that $b+c=b'+c'$ and, without loss of generality, that $b<b'$. From

$$\begin{aligned}(x-b')(x-c') &= x^2-(b'+c')x+b'c'\\ &= x^2-(b+c)x+bc+b'c'-bc\\ &= (x-b)(x-c)+b'c'-bc\end{aligned}$$

we see that $\mu_1(x-b)(x-c)\neq\mu_1(x-b')(x-c')$ for $x\leq b$ or $x\geq b'$. But for $b<x<b'$ we have $\mu_1(x-b)(x-c)<0$ and $\mu_1(x-b')(x-c')>0$ so that $\mu_1(x-b)(x-c)\neq\mu_1(x-b')(x-c')$ for all $x\in\mathbb{R}$. This shows that $C_{b,c}$ and $C_{b',c'}$ are parallel.

Conversely, assume that $b+c\neq b'+c'$. If $u\in\{b,c\}\cap\{b',c'\}$, then $((u,0)$ is a finite point on both $C_{b,c}$ and $C_{b',c'}$. We thus assume that $\{b,c\}\cap\{b',c'\}=\emptyset$, and, by symmetry, that $b<b'$.

Let $f(x)=\mu_1(x-b')(x-c')-\mu_1(x-b)(x-c)$. From $\mu_0(x-b)=\mu_0(x-b')$ for $x<b$ or $x>b'$ we find that

$$x=\frac{bc-b'c'}{b+c-b'-c'}$$

is a zero of f unless this value is between b and b'. But then

$$\frac{(c'-b)(b'-b)}{b+c-b'-c'}=b-\frac{bc-b'c'}{b+c-b'-c'}<0<b'-\frac{bc-b'c'}{b+c-b'-c'}=\frac{(c-b')(b'-b)}{b+c-b'-c'}.$$

Since $b<b'\leq c'$, the inequalities above imply that $b\leq c<b'\leq c'$. But then

$$\begin{aligned}f(c) &= k(c-b')(c-c')>0,\\ f(b') &= -(b'-b)(b'-c)<0.\end{aligned}$$

Hence, by continuity, f has a zero between c and b'. This shows that in any case f has a zero and thus $C_{b,c}$ and $C_{b',c'}$ have a finite point of intersection. ■

Recall that by Lemma 5.2 the lines $D_{0,1}$ and $C_{-s,-s}$ are parallel. This yields a clue as to when lines of $\mathcal{A}$ of different types are parallel.

Lemma 5.6. *Two lines $D_{b,c}$ and $C_{b',c'}$ are parallel if and only if $b-sc=(b'+c')/2$.*

Proof. Suppose that $b - sc = (b' + c')/2 = m$. From Lemma 5.5 we know that $C_{b',c'}$ and $C_{m,m}$ are parallel. Furthermore, because $\mu_1(m - b')(m - c') = -(c' - b')^2/4 \leq 0$, we see that $C_{b',c'}$ lies strictly below $C_{m,m}$ unless $b' = c' = m$. Lemma 5.2 shows that $D_{0,1}$ is strictly above $C_{-s,-s}$. Hence $D_{b,c} = \gamma_{c,b}(D_{0,1})$ is strictly above $\gamma_{c,b}(C_{-s,-s}) = C_{m,m}$. This shows that $D_{b,c}$ and $C_{b',c'}$ are parallel.

Conversely, let $b - sc = m$ but $b' + c' \neq 2m$. From what we have already seen we know that $D_{b,c}$ and $C_{m,m}$ are parallel and that $D_{b,c}$ lies above $C_{m,m}$. Moreover, the supporting Euclidean lines of $\psi(C_{m,m})$ are asymptotes for $\psi(D_{b,c})$.

By Lemma 5.5 the lines $C_{m,m}$ and $C_{b',c'}$ intersect and, depending on the sign of $b' + c' - 2m$, $\lim_{x\to\pm\infty}(\mu_1(x - b')(x - c') - \mu_2(x - m)) = \pm\infty$ or $\mp\infty$, that is, $\psi(C_{b,c})$ is unboundedly above $\psi(C_{m,m})$ sufficiently far to the right and $\psi(C_{b,c})$ is unboundedly below $\psi(C_{m,m})$ sufficiently far to the left or the other way around. But $\psi(D_{b,c})$ behaves asymptotically like $\psi(C_{m,m})$ so that we obtain that $\psi(C_{b,c})$ is unboundedly above $\psi(D_{b,c})$ sufficiently far to the right and $\psi(C_{b,c})$ is unboundedly below $\psi(D_{b,c})$ sufficiently far to the left or the other way around. In any case, connectedness implies that $\psi(D_{b,c})$ and $\psi(C_{b',c'})$, and thus $D_{b,c}$ and $C_{b',c'}$, have a point of intersection. ■

Lemma 5.7. *Two lines $D_{b,c}$ and $D_{b',c'}$ are parallel if and only if $b-sc = b'-sc'$.*

Proof. Suppose that $b - sc = b' - sc' = m$ and, without loss of generality, that $c < c'$. Let $f(t,x) = e^{2s\cot^{-1}\frac{x-m-st}{t}}((x - m - st)^2 + t^2)$ so that $y = f(c,x)$ and $y = f(c',x)$ describe the lines $D_{b,c}$ and $D_{b',c'}$. For the partial derivatives $\frac{\partial f}{\partial t}(t,x)$ one finds $\frac{\partial f}{\partial t}(t,x) = 2e^{2s\cot^{-1}\frac{x-m-st}{t}}(s^2 + 1)t$. Hence $\frac{\partial f}{\partial t}(t,x) > 0$ and $t \mapsto f(t,x)$ is strictly increasing. In particular, $f(c,x) < f(c',x)$ and $D_{b,c}$ and $D_{b',c'}$ have no finite point of intersection. This shows that $D_{b,c}$ and $D_{b',c'}$ are parallel.

Conversely, let $b-sc = m \neq m' = b'-sc'$. From Lemma 5.6 we already know that $D_{b,c}$ and $C_{m,m}$ are parallel and that $C_{m',m'}$ and $D_{b',c'}$ are parallel. By Lemma 5.5 the lines $C_{m,m}$ and $C_{m',m'}$ intersect so that the difference between the respective describing functions is unbounded and takes on opposite signs for large positive x or large negative x. However, $\psi(D_{b,c})$ and $\psi(D_{b',c'})$ behave asymptotically like $\psi(C_{m,m})$ and $\psi(C_{m',m'})$, respectively, so that, as in the proof of Lemma 5.6, $D_{b,c}$ and $C_{b',c'}$ have a point of intersection by connectedness. ■

Proposition 5.8. *$\mathcal{A}$ is an affine plane.*

Proof. ¿From Proposition 5.4 we already know that $\mathcal{A}$ is a linear space, so it only remains to verify the parallel axiom in $\mathcal{A}$. This axiom is clearly satisfied for vertical lines. By Lemmas 5.5, 5.7 and 5.6 each non-vertical line of $\mathcal{A}$ is characterised by an $m \in \mathbb{R}$. Let $\mathcal{B}_m$ be the collection of all lines $C_{m+c,m-c}$ for $c \leq 0$ and $D_{m+sc,c}$ for $c > 0$. Any two lines in $\mathcal{B}_m$ are parallel by Lemmas 5.5, 5.7 and 5.6. Let

$$f(c,x) = \begin{cases} \mu_0(x - m - c)((x - m)^2 - c^2), & \text{for } c \leq 0, \\ e^{2c\cot^{-1}(\frac{x-m}{c} - s)}((x - m - sc)^2 + c^2), & \text{for } c > 0. \end{cases}$$

(Thus $y = f(c, x)$ describes the line with parameter c in $\mathcal{B}_m$.) It readily follows that for given $x \in \mathbb{R}$ the function $c \mapsto f(c, x)$ is continuous in c and that $\lim_{c\to\pm\infty} f(c, x) = \pm\infty$. Hence every $y \in \mathbb{R}$ occurs for precisely one $c \in \mathbb{R}$ as $y = f(c, x)$. The line corresponding to this c is the parallel we are looking for. ∎

Theorem 5.9. *$\mathcal{L}_k$ is a flat Laguerre plane. Furthermore, $\mathcal{L}_k$ is not ovoidal.*

Proof. As mentioned at the beginning of this section, the verification that $\mathcal{L}_k$ is a Laguerre plane is equivalent to showing that the derived incidence structures at $(\infty, 0)$ and $(\infty, 1)$ are affine planes. This has been done in Propositions 5.1 and 5.8. The Laguerre cylinder L is homeomorphic to $\mathbb{S}_1 \times \mathbb{R}$ by construction, see section 3.. As we remarked in section 4., each circle of $\mathcal{L}_k$ is homeomorphic to $\mathbb{S}_1$. Therefore, $\mathcal{L}_k$ is in fact a topological Laguerre plane, see [4], 3.10.

Since each derived affine plane of an ovoidal flat Laguerre plane is Desarguesian, we obtain from Proposition 5.1 that $\mathcal{L}_k$ cannot be ovoidal. ∎

Note that circles can be defined in exactly the same way as in section 4. for arbitrary $k > 0$, not just for $k > 1$. The resulting incidence structure is again a flat Laguerre plane. For $k = 1$ one obtains the classical real Laguerre plane. We did not include this special case for simplicity as the automorphism group of the classical real Laguerre has rather different properties from the automorphism groups of the planes $\mathcal{L}_k$, see the following section. Furthermore, $\mathcal{L}_k$ and $\mathcal{L}_{1/k}$ are isomorphic via the mapping $(x, y) \to (-x, -y)$, so that one obtains the full picture by restricting oneself to $k > 1$.

6. Automorphism group and Kleinewillinghöfer type

Every automorphism of a flat Laguerre plane is continuous and thus a homeomorphism of Z. The collection of all automorphisms of a flat Laguerre plane $\mathcal{L}$ forms a group with respect to composition, the automorphism group of $\mathcal{L}$. This group is a Lie group of dimension at most 7 with respect to the compact-open topology, see [14].

We say that a flat Laguerre plane has *group dimension* n if its automorphism group is n-dimensional. All flat Laguerre planes of group dimension at least 5 have been classified; see [8]. The classical flat Laguerre plane is the only flat Laguerre plane of group dimension at least 6 (and indeed has group dimension 7) and the flat Laguerre planes of group dimension 5 are precisely the ovoidal Laguerre planes over proper skew parabolae, cf. [8], Theorem 1.

Furthermore, in the same paper [8], in Theorem 2 all flat Laguerre planes whose automorphism groups are 4-dimensional and fix one point were classified. There are two families, one consisting of planes of shear type over a pair of two different skew parabolae and the other consisting of the non-classical planes of translation type over a pair of skew parabolae; the former planes contain two fixed parallel classes, the latter only one. All flat Laguerre planes of group dimension 4 whose automorphism group fixes a parallel class were determined by the second author. Our planes provide examples of flat Laguerre planes of group dimension 4 such that neither is a parallel class fixed by the connected component of the automorphism group containing the identity nor is the group transitive on the cylinder. The other known flat Laguerre planes of group dimension 4 with this kind of orbit structure are certain semi-classical Laguerre planes obtained by pasting along a circle, see [13]. (The planes $\mathcal{L}(\varphi, \mathrm{id})$, in the notation of [13], with one of the describing homeomorphisms being the identity and the other one, φ, being multiplicative.)

Theorem 6.1. *The 4-dimensional group Γ described in 3.5 is the full group of automorphisms of $\mathcal{L}_k$.*

Proof. Since, on the one hand, the group Γ consists of automorphisms of $\mathcal{L}_k$ and has dimension 4, the automorphism group of $\mathcal{L}_k$ must be at least 4-dimensional. On the other hand, a flat Laguerre plane of group dimension at least 5 is ovoidal, see [8], Theorem 1, hence by Theorem 5.9, $\mathcal{L}_k$ has group dimension 4.

Suppose that there is an automorphism of $\mathcal{L}_k$ that does not fix L_0. Then Γ and thus also Γ^1 is transitive on Z. Hence $\mathcal{L}_k$ must be classical by [15] and thus each derived affine plane must be Desarguesian—a contradiction to Proposition 5.1.

Let γ be an automorphism of $\mathcal{L}_k$. Using a rotation ρ_t, if necessary, we may assume that γ fixes the point $(\infty, 0)$. But then γ induces a collineation of the derived affine plane at $(\infty, 0)$, that is, a collineation $\tilde{\gamma}$ of $\mathcal{M}_k$. In $\mathcal{M}_k$ it now follows that $\tilde{\gamma}$ is a composition of collineations of the form $\gamma_{a,b,c}$ and σ. ■

Any isomorphism between Laguerre planes $\mathcal{L}_k$ and $\mathcal{L}_{k'}$ induces an isomorphism of the corresponding group actions. Being the unique 1-dimensional orbit, the distinguished circle L_0 must therefore be mapped to L_0 under any isomorphism. Since the Moulton planes $\mathcal{M}_k$ are mutually non-isomorphic, we obtain the following corollary by passing to derived affine planes at points of L_0.

Corollary 6.2. *The flat Laguerre planes $\mathcal{L}_k$ are mutually non-isomorphic.*

Kleinewillinghöfer [6] classified Laguerre planes with respect to *central automorphisms*, that is, automorphisms that fix at least one point such that central collineations are induced in the derived projective plane at one of the fixed points. In [11] and [17] flat Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types with respect to the full automorphism group. In particular, all possible types of flat Laguerre planes with respect to Laguerre translations and Laguerre homotheties were completely determined. Examples for some of the possible Kleinewillinghöfer types of flat Laguerre planes can be found in [11] section 6 and [17] but there are still a few open cases. One of them is Kleinwillinghöfer type II.G.1 in which there is a circle C such that the group of all C-homologies and the groups of all $(|p|, B(p,C))$-translations for $p \in C$ are linearly transitive; see [11] or [17] for a definition of these kinds of automorphisms. (Here $B(p,C)$ denotes the tangent bundle of all circles that touch the circle C at the point p.)

Theorem 6.3. *$\mathcal{L}_k$ is of Kleinewillinghöfer type II.G.1.*

Proof. The maps $\gamma_{a,0,1}$ for $a > 0$ and σ from Section 3. are automorphisms, see Proposition 4.1. They fix exactly the points of the circle L_0, that is, each is an L_0-homology. Furthermore the group

$$\{\gamma_{a,0,1} \mid a > 0\} \cup \{\gamma_{a,0,1}\sigma \mid a > 0\}$$

is transitive on each parallel class minus its point of intersection with L_0. This shows that the set of circles for which the automorphism group of $\mathcal{L}_k$ is linearly transitive with respect to Laguerre homologies contains L_0.

Since L_0 is fixed by every automorphism of $\mathcal{L}_k$, one sees that L_0 is the only circle for which Γ can be linearly transitive with respect to Laguerre homologies. This shows that $\mathcal{L}_k$ must have type II with respect to Laguerre homologies.

Likewise, the automorphisms $\gamma_{1,b,1}$ for $b \in \mathbb{R}$ fix each circle L_c and each point on the infinite parallel class $\Pi_\infty = \{\infty\} \times \mathbb{R}$, that is, $\gamma_{1,b,1}$ is a $(\Pi_\infty, B(\infty, L_0))$-translation in the notation of [11]. The group $\{\gamma_{1,b,1} \mid b \in \mathbb{R}\}$ is transitive on each circle L_c minus the point $(\infty, 0)$. Applying the rotations ρ_t this therefore shows that the set of all tangent bundles for which the automorphism group of $\mathcal{L}_k$ is linearly transitive with respect to Laguerre translations contains each bundle $B(p, L_0)$ for $p \in L_0$.

L_0 being fixed by every automorphism implies that $B(p, L_0)$ for each $p \in L_0$ are the only tangent bundles for which Γ can be linearly transitive with respect to Laguerre translations and there cannot be any G-translations. This shows that $\mathcal{L}_k$ must have type G with respect to Laguerre translations.

Finally, assume that Γ is linearly transitive with respect to some Laguerre homotheties. Then the centres of the Laguerre homotheties must be on L_0 and Γ must be (p, q)-transitive for all distinct $p, q \in L_0$. But this implies that the derived affine plane at any point of L_0 is Desarguesian—a contradiction to Proposition 5.1. This shows that $\mathcal{L}_k$ must have type 1 with respect to Laguerre homotheties. ■

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