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Period polynomials and Ihara brackets

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Abstract. Schneps [J. Lie Theory **16** (2006), 19–37] has found surprising links between Ihara brackets and even period polynomials. These results can be recovered and generalized by considering some identities relating Ihara brackets and classical Lie brackets. The period polynomials generated by this method are found to be essentially the Kohnen-Zagier polynomials.
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1. Introduction

The Ihara bracket is defined on a subspace of the free Lie algebra $\mathcal{L}$ on two generators, whose elements are intepreted as special derivations of $\mathcal{L}$. One can find relations between the two brackets. This gives rise to interesting identities, which surprisingly, turn out to be related to period polynomials [12].

The *period polynomials* $r^{\pm}(f)$ of a modular form $f \in M_k = M_k(SL_2(\mathbb{Z}))$ are defined by

$$r(f)(t) = \int_0^{i\infty} f(z)(t-z)^w dz \quad (w = k-2)\,, \tag{1}$$

and $r = r^+ + ir^-$. These polynomials satisfy the *period relations*

$$P(t) + t^w P\left(\frac{-1}{t}\right) = 0\,, \tag{2}$$

$$P(t) + t^w P\left(1 - \frac{1}{t}\right) + (t-1)^w P\left(\frac{1}{1-t}\right) = 0\,. \tag{3}$$

The solutions of this system are in one-to-one correspondence with cusp forms (except for the $t^w - 1$ which are obtained from Eisenstein series). This is the Eichler-Shimura correspondence (see, *e.g.*, [9]).

Writing

$$r(f)(t) = \sum_{n=0}^{w} i^{-n+1} \binom{w}{n} r_n(f)\,, \tag{4}$$

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one defines linear forms r_n on M_k. Since this is an inner product space for the Petersson scalar product, there exist modular forms $R_{n;k}$ such that for all $f \in S_k$,

$$r_n(f) = (f, R_{n;k})\,. \tag{5}$$

The period polynomials of the $R_{n;k}$ have been explicitly computed by Kohnen and Zagier [9, Theorem 1, p. 208]. Their result can be stated as follows: let $B_n(t)$ be the Bernoulli polynomials and b_n be the Bernoulli numbers defined by

$$\frac{ue^{tu}}{e^u - 1} =: \sum_{n\geq 0} B_n(t)\frac{u^n}{n!} \qquad \text{and} \qquad B_n(t) =: \sum_{i=0}^{n} \binom{n}{i} b_i t^{n-i}. \tag{6}$$

and let

$$\begin{aligned} P^{\pm}_{n;k}(t) := \pm\, & \frac{1}{n+1}\left[B_{n+1}(t) \mp t^w B_{n+1}\left(\frac{1}{t}\right)\right] \\ & + \frac{1}{p+1}\left[B_{p+1}(t) \mp t^w B_{p+1}\left(\frac{1}{t}\right)\right]\,, \end{aligned} \tag{7}$$

where $w = k - 2 = n + p$. Then [9],

$$P^{\pm}_{n;k}(t) = r^{\pm}(\lambda^{\pm}_{n;k} R_{n;k} + \mu^{\pm}_{n;k} G_k) \tag{8}$$

where G_k is the Eisenstein series and $\lambda^{\pm}_{n;k}, \mu^{\pm}_{n;k}$ are explicit constants that we shall not need.

Hence, any linear combination of the $P^{\pm}_{n;k}(t)$ for a given k is a period polynomial. Such combinations arise naturally from the Ihara bracket. In [12], Schneps obtained a characterization of even period polynomials in terms of linear relations between certain brackets. In this note, we give a new proof of this result and obtain a similar characterization of odd period polynomials. We also give a simpler way to generate these polynomials.

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2. Ihara brackets and the stable derivation algebra

The stable derivation algebra $\mathfrak{F}$ appears in works by Ihara on Galois representations on $\pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$ [3, 4, 5, 6, 7, 8]. It plays also a crucial rôle in recent works on multiple zeta values [1, 2, 10].

The underlying vector space of $\mathfrak{F}$ can be identified with a subalgebra of the free Lie algebra over two letters [8]. Let $\mathcal{L}$ be the free $\mathbb{Q}$-Lie algebra over two letters a, b. By Lazard elimination [11], the Lie algebra decomposes as

$$\mathcal{L} = \mathbb{Q}a \oplus \mathcal{F}^1\mathcal{L} \tag{9}$$

where the Lie algebra $\mathcal{F}^1\mathcal{L}$ is free over the infinite sequence

$$\phi_{n+1} = \frac{1}{n!}\mathrm{ad}_a^n(b) \qquad (n \geq 0), \tag{10}$$

where ad denotes the adjoint representation.

Let V be any vector space and $\mathrm{Der}(\mathcal{L}(V))$ be the Lie algebra of derivations of the free Lie algebra $\mathcal{L}(V)$. The commutator of two inner derivations satisfies

$$[\mathrm{ad}_f, \mathrm{ad}_g] = \mathrm{ad}_{[f,g]} \tag{11}$$

where $f, g \in \mathcal{L}$ and $[f, g]$ is the bracket in $\mathcal{L}$. More generally, if $\pi : V \to V$ is a projector, one can associate with any element $f \in \mathcal{L}(V)$ a derivation D_f defined by

$$D_f(v) = [\pi(v), f]\,, \tag{12}$$

which satisfies

$$[D_f, D_g] = D_{\{f,g\}} \quad \text{where} \quad \{f, g\} := [f, g] + D_g(f) - D_f(g)\,. \tag{13}$$

In the special case where $V = \mathbb{Q}a \oplus \mathbb{Q}b$ and $\pi(a) = a,\ \pi(b) = 0$, the operation $\{f, g\}$ is called the Ihara bracket. Let $\mathcal{G}$ be the Lie subalgebra of $\mathrm{Der}(\mathcal{L})$ generated by the special derivations $D_f\,,\ f \in \mathcal{L}$. It can be identified with the Ihara subalgebra $\mathcal{F}^1\mathcal{L}$ of $\mathcal{L}$ generated by the ϕ_n, so that it is also a Lie subalgebra of $\mathcal{L}$ for the ordinary bracket. The stable derivation algebra $\mathfrak{F}$ is a subalgebra of $\mathcal{G}$, spanned by the D_f such that f satisfies certain identities ([8], see also [1]).

In [1, 6], Furusho and Ihara consider the filtration

$$\mathcal{L} = \mathcal{F}^0\mathcal{L} \supset \mathcal{F}^1\mathcal{L} \supset \cdots \supset \mathcal{F}^n\mathcal{L} \supset \cdots \tag{14}$$

where

$$\mathcal{F}^0\mathcal{L} = \mathcal{L} \text{ and } \mathcal{F}^n\mathcal{L} = [\mathcal{F}^1\mathcal{L}, \mathcal{F}^{n-1}\mathcal{L}] \text{ for } n \geq 2. \tag{15}$$

This induces a filtration on $\mathfrak{F}$

$$\mathfrak{F} = \mathcal{F}^1\mathfrak{F} \supseteq \mathcal{F}^2\mathfrak{F} \supseteq \cdots \supseteq \mathcal{F}^n\mathfrak{F} \supseteq \cdots\,, \tag{16}$$

and the compatibility of the Ihara bracket with this filtration implies that the Lie algebra $\mathfrak{F}$ is generated by polynomials which do not belong to $\mathcal{F}^2\mathfrak{F}$. From [8], $\dim_{\mathbb{Q}} \mathcal{F}^1\mathfrak{F}_n/\mathcal{F}^2\mathfrak{F}_n = 1$ if n is odd and greater than 1 and 0 otherwise. Hence, the Lie algebra $\mathfrak{F}$ is generated by a set

$$\{f_{2n+1} \in \mathfrak{F}_{2n+1} \,|\, n \geq 1\}. \tag{17}$$

such that

$$f_{2n+1} \equiv (2n)!\,\phi_{2n+1} \bmod \mathcal{F}^2\mathfrak{F} \tag{18}$$

Therefore,

$$\{f_{2n+1}, f_{2n'+1}\} \equiv (2n)!\,(2n')!\,\{\phi_{2n+1}, \phi_{2n'+1}\} \mod \mathcal{F}^3\mathfrak{F} \tag{19}$$

which implies the equivalence between equalities of the form

$$\sum_{\substack{i+j=n \\ i,j\leq 3 \text{ odd}}} a_{ij}\,\{f_i, f_j\} \equiv 0 \bmod \mathcal{F}^3\mathfrak{F} \tag{20}$$

and

$$\sum_{\substack{i+j=n \\ i,j\leq 3 \text{ odd}}} a_{ij}\,(i-1)!\,(j-1)!\,\{\phi_i, \phi_j\} = 0\,. \tag{21}$$

This is precisely the kind of linear relations giving rise to period polynomials. Those found by Schneps [12] are obtained by substituting

$$\{\phi_i, \phi_j\} \to \frac{1}{(i-1)!(j-1)!}(t^{i-1} - t^{j-1}) \tag{22}$$

in the left-hand side of relations like (21). Let us note that if we set $\varphi_i(t) = \frac{1}{(i-1)!}t^{i-1}$ and $\langle f(t), g(t)\rangle = f(t)g(1) - f(1)g(t)$ (this is a Lie bracket), then (22) amounts to replacing $\{\phi_i, \phi_j\}$ by $\langle \varphi_i(t), \varphi_j(t)\rangle$.

All the calculations presented in this paper aim at finding explicit linear relations between the $[\phi_m, \phi_n]$, $\{\phi_m, \phi_n\}$ and $D_{\phi_m}(\phi_n)$. As a general rule, the coefficients are expressed in terms of Bernoulli numbers and can be related to the Kohnen-Zagier polynomials.

3. Computation of Ihara brackets

The aim of this section is to express the Ihara bracket in terms of classical brackets. Our first step is to compute the action of the special derivations on the generators ϕ_n. Let us start with the simple equality

$$D_{\phi_n} a = -n\phi_{n+1}\,. \tag{23}$$

Since D_{ϕ_n} is a derivation, it follows from (23) that

$$D_{\phi_n}(\phi_p) = -\frac{n}{(p-1)!}\sum_{i=1}^{p-1}(p-i-1)!\,\mathrm{ad}_a^{i-1}\,[\phi_{n+1}, \phi_{p-i}]. \tag{24}$$

The Leibniz formula gives

$$\mathrm{ad}_a^{i-1}[\phi_{n+1}, \phi_{p-i}] = \sum_{j=0}^{i-1}\binom{i-1}{j}\frac{(n+j)!(p-j-2)!}{n!(p-i-1)!}\,[\phi_{n+j+1}, \phi_{p-j-1}]\,. \tag{25}$$

Substituting in (24) and rearranging the sums, one gets, for all $n, p \geq 1$

$$\begin{aligned} D_{\phi_n}(\phi_p) &= \sum_{j=1}^{p-1}\binom{n+j-1}{j}[\phi_{p-j}, \phi_{n+j}] \\ &= \sum_{k=1}^{p-1}\binom{n+p-1-k}{p-k}[\phi_k, \phi_{p+n-k}]\,. \end{aligned} \tag{26}$$

Hence, from the definition of Ihara bracket, one has, for all $n, p \geq 1$

$$\begin{aligned} \{\phi_n, \phi_p\} = {} & [\phi_n, \phi_p] \\ & + \sum_{k=1}^{n-1}\binom{n+p-1-k}{n-k}[\phi_k, \phi_{n+p-k}] \\ & - \sum_{k=1}^{p-1}\binom{n+p-1-k}{p-k}[\phi_k, \phi_{n+p-k}]\,. \end{aligned} \tag{27}$$

This equation can be rewritten as

$$\{\phi_n, \phi_p\} = \sum_{k=1}^{\max(n-1,p-1)} \left(\binom{n+p-1-k}{n-k} - \binom{n+p-1-k}{p-k} \right) [\phi_k, \phi_{n+p-k}] , \tag{28}$$

for all $n, p \geq 1$ as one can check on all cases $n < p$, $n = p$, and $n > p$.

In terms of generating series, both Equations (26) and (28) have simple expressions. Set

$$\Phi(x) := \sum_{n\geq 1} \phi_n x^{n-1} . \tag{29}$$

Then,

$$\begin{aligned}
D_{\Phi(x)}\Phi(y) &:= \sum_{n,p\geq 1} D_{\phi_n}(\phi_p)\, x^{n-1} y^{p-1} \\
&= \sum_{n,p\geq 1} \sum_{k=1}^{p-1} \binom{n+p-1-k}{p-k} [\phi_k\, y^{k-1}, \phi_{n+p-k}\, x^{n-1} y^{p-k}] \\
&= \left[\sum_{k\geq 1} \phi_k\, y^{k-1}, \ \ \sum_{s\geq 1} \phi_s \sum_{n=1}^{s-1} \binom{s-1}{n-1} x^{n-1} y^{s-n} \right] \\
&= \left[\Phi(y), \sum_{s\geq 1} \phi_s ((x+y)^{s-1} - x^{s-1}) \right] \\
&= [\Phi(y), \Phi(x+y) - \Phi(x)] \\
&= [\Phi(x), \Phi(y)] + [\Phi(y), \Phi(x+y)].
\end{aligned} \tag{30}$$

Hence,

$$D_{\Phi(x)}\Phi(y) = [\Phi(x), \Phi(y)] + [\Phi(y), \Phi(x+y)] \tag{31}$$

and

$$\{\Phi(x), \Phi(y)\} = [\Phi(y), \Phi(x)] + [\Phi(x) - \Phi(y), \Phi(x+y))] . \tag{32}$$

4. Inversion of Equation (31) and period polynomials

4.1. Generic inversion of (31).

Let $(F_{i,j})_{i,j\geq 1}$ and $(G_{i,j})_{i,j\geq 1}$ be two bi-indexed sequences of elements of some vector space whose generating series

$$F(x,y) = \sum_{i,j\geq 1} F_{i,j}\, x^{i-1} y^{j-1}, \qquad G(x,y) = \sum_{i,j\geq 1} G_{i,j}\, x^{i-1} y^{j-1}, \tag{33}$$

satisfy

$$F(x,y) = G(x,y) - G(x+y,y) = (1 - e^{y\partial_x})\, G(x,y). \tag{34}$$

Thanks to Equation (31), this is the case of

$$F(x,y) = D_{\Phi(x)}\Phi(y) \quad \text{and} \quad G(x,y) = [\Phi(x), \Phi(y)]. \tag{35}$$

If $F(x,y)$ is given, this formula determines $G(x,y)$ up to a function of y:

$$\begin{aligned}
y\partial_x G(x,y) =& \frac{y\partial_x}{1-e^{y\partial_x}} F(x,y) \\
=& -\sum_{k\geq 0} \frac{b_k}{k!} y^k \partial_x^k \sum_{i,j\geq 1} F_{i,j}\, x^{i-1} y^{j-1} \\
=& -\sum_{k\geq 0} \frac{b_k}{k!} \sum_{i\geq k+1;\, j\geq 1} (i-1)\cdots(i-k) F_{i,j}\, x^{i-1-k} y^{j-1+k} \\
=& -\sum_{k\geq 0} \sum_{i\geq k+1;\, j\geq 1} \binom{i-1}{k} b_k\, F_{i,j}\, x^{i-1-k} y^{j-1+k} \\
=& -\sum_{i\geq 1} \sum_{k=0}^{i-1} \binom{i-1}{k} b_k\, x^{i-1-k} y^k \sum_{j\geq 1} F_{i,j}\, y^{j-1} \\
=& -\sum_{i\geq 0} \sum_{k=0}^{i} \binom{i}{k} b_k \left(\frac{x}{y}\right)^{i-k} y^i \sum_{j\geq 1} F_{i+1,j}\, y^{j-1}
\end{aligned} \tag{36}$$

so that, finally

$$y\partial_x G(x,y) = \sum_{i\geq 0} -B_i\left(\frac{x}{y}\right) y^i \sum_{j\geq 1} F_{i+1,j}\, y^{j-1}. \tag{37}$$

Then, comparing the coefficients of $x^n y^p$ (with $n\geq 0$ and $p\geq 1$) on both sides of Equation (37), one obtains

$$\begin{aligned}
(n+1)G_{n+2,p} &= -\sum_{i=n}^{n+p} \binom{i}{i-n} b_{i-n}\, F_{i+1,n+p+1-i} \\
&= -\sum_{i=0}^{p} \binom{n+i}{i} b_i\, F_{n+1+i,p+1-i}\,,
\end{aligned} \tag{38}$$

so that

Proposition 4.1. *For all $n\geq 2$ and $p\geq 1$,*

$$G_{n,p} = -\frac{1}{n-1}\sum_{i=0}^{p}\binom{n-2+i}{n-2} b_i\, F_{n-1+i,p+1-i}\,. \tag{39}$$

■

Now, assume that $G_{n,p} = [\phi_n, \phi_p]$, so that $F_{n,p} = D_{\phi_n}(\phi_p)$.

Proposition 4.2. *For all $n\geq 2$ and $p\geq 1$,*

$$\begin{aligned}
[\phi_n,\phi_p] &= -\frac{1}{n-1}\sum_{i=0}^{p}\binom{n-2+i}{n-2} b_i\, D_{\phi_{n-1+i}}(\phi_{p+1-i}) \\
&= -\sum_{i=0}^{p}\binom{n-1+i}{n-1}\frac{b_i}{n-1+i} D_{\phi_{n-1+i}}(\phi_{p+1-i}).
\end{aligned} \tag{40}$$

■

Note that in the first equation, the summation can be taken up to $p-1$ since $D_{\phi_i}(\phi_1) = D_{\phi_i}(b) = 0$ for all i. Now, since the Lie bracket is antisymmetric, one gets the following relations between the $D_{\phi_i}(\phi_j)$:

Corollary 4.3. (i) *For any $n > 1$,*

$$\sum_{i=0}^{n} \binom{n-2+i}{n-2} b_i \, D_{\phi_{n-1+i}}(\phi_{n+1-i}) = 0, \tag{41}$$

(ii) *Setting $b_i = 0$ if $i < 0$, one has, for all $n, p \geq 2$*

$$\sum_{i=1}^{n+p-1} \left(\binom{i-1}{i-p+1} \frac{b_{i-p+1}}{p-1} + \binom{i-1}{i-n+1} \frac{b_{i-n+1}}{n-1} \right) D_{\phi_i}(\phi_{n+p-i}) = 0. \tag{42}$$

■

4.2. Period polynomials.

Let us now consider the specialization

$$F_{i,j} = \frac{1}{(i-1)!} \frac{1}{(j-1)!} (t^{i-1} - \epsilon t^{j-1}), \tag{43}$$

where $\epsilon = \pm 1$. Then, Equation (39) gives

$$\begin{aligned} G_{n,p} =& \frac{-1}{n-1} \sum_{i=0}^{p} \binom{n-2+i}{n-2} b_i \frac{t^{n-2+i} - \epsilon t^{p-i}}{(n-2-i)!(p-i)!} \\ =& \frac{-1}{(n-1)!p!} \sum_{i=0}^{p} \binom{p}{i} b_i (t^{n-2+i} - \epsilon t^{p-i}) \\ =& \frac{-1}{(n-1)!p!} \left(t^{n+p-2} B_p\left(\frac{1}{t}\right) - \epsilon B_p(t) \right) \\ =& \frac{1}{(n-1)!(p-1)!} \left(\frac{1}{p} \left(\epsilon B_p(t) - t^{n+p-2} B_p\left(\frac{1}{t}\right) \right) \right). \end{aligned} \tag{44}$$

Note that in terms of generating series, the expressions of F and G are simple:

$$F(x,y) = F(x,y) = e^{tx+y} - \epsilon e^{x+ty} \text{ and } G(x,y) = f(y,t) + \frac{e^{tx+y}}{1-e^{ty}} - \epsilon \frac{e^{x+ty}}{1-e^{y}}. \tag{45}$$

One recognizes in the coefficients $G_{n,p}$ the building blocks of the period polynomials introduced by Kohnen and Zagier ([9], Theorem 1). It follows from their results that,

$$n!\,p!\,(G_{n+1,p+1} + \epsilon G_{p+1,n+1}) = P^{+}_{n,n+p+2}(t) \tag{46}$$

is an even period polynomial for n, p even and $\epsilon = 1$, and

$$n!\,p!\,(G_{n+1,p+1} + \epsilon G_{p+1,n+1}) = P^{-}_{n,n+p+2}(t) \tag{47}$$

is an odd period polynomial for n, p odd and $\epsilon = -1$.

Note that in the case of even period polynomials, this amounts to substituting

$$D_{\phi_i}(\phi_j) \mapsto \frac{1}{(i-1)!(j-1)!} (t^{i-1} - t^{j-1}) \tag{48}$$

in the left-hand sides of the linear relations (41) and (42), which is analogous to the result of Schneps [12].

5. Ordinary brackets in terms of Ihara brackets

We shall now give a partial inversion of Equation (28), *i.e.*, express the Lie brackets $[\phi_{2n}, \phi_k]$ as a linear combination of Ihara brackets. Here are some examples:

$$[\phi_2, \phi_k] = \tfrac{1}{k-1}\{\phi_3, \phi_{k-1}\} - \tfrac{1}{2}\{\phi_2, \phi_k\}. \tag{49}$$

$$[\phi_4, \phi_k] = \tfrac{k}{12}\{\phi_3, \phi_{k+1}\} - \tfrac{1}{2}\{\phi_4, \phi_k\} + \tfrac{1}{k-1}\{\phi_5, \phi_{k-1}\}. \tag{50}$$

$$[\phi_6, \phi_k] = -\tfrac{(k+2)(k+1)k}{720}\{\phi_3, \phi_{k+3}\} + \tfrac{k}{12}\{\phi_5, \phi_{k+1}\} - \tfrac{1}{2}\{\phi_6, \phi_k\} + \tfrac{1}{k-1}\{\phi_7, \phi_{k-1}\}. \tag{51}$$

$$\begin{aligned}[\phi_8, \phi_k] =& \tfrac{(k+4)(k+3)(k+2)(k+1)k}{30240}\{\phi_3, \phi_{k+5}\} - \tfrac{(k+2)(k+1)k}{720}\{\phi_5, \phi_{k+3}\} \\ &+ \tfrac{k}{12}\{\phi_7, \phi_{k+1}\} - \tfrac{1}{2}\{\phi_8, \phi_k\} + \tfrac{1}{k-1}\{\phi_9, \phi_{k-1}\}.\end{aligned} \tag{52}$$

The general formula is as follows.

Proposition 5.1. *For each $k \geq 2$ and $n \geq 1$, one has*

$$[\phi_{2n}, \phi_k] = \sum_{i=0}^{2n} \binom{k-1+i}{k-1} \frac{b_i}{k-1+i} \{\phi_{2n-i+1}, \phi_{k+i-1}\}. \tag{53}$$

Proof. The generating series of the right-hand side of (53) is

$$S(x,y) = \sum_{n\geq 1}\sum_{k\geq 2} x^{2n-1} y^{k-1} \sum_{i=0}^{2n-1} \frac{(k+i-2)!}{(k-1)!} \frac{b_i}{i!} \{\phi_{2n-i+1}, \phi_{k+i-1}\}. \tag{54}$$

Rearranging the sum, one obtains

$$\begin{aligned}
x\partial_y S(x,y) =& \sum_{n\geq 1}\sum_{k\geq 2} x^{2n}(k-1) y^{k-2} \sum_{i=0}^{2n} \frac{(k+i-2)!}{(k-1)!} \frac{b_i}{i!} \{\phi_{2n-i+1}, \phi_{k+i-1}\} \\
=& \sum_{n\geq 1}\sum_{k\geq 2}\sum_{i=0}^{2n} \frac{(k+i-2)!}{(k-2)!} \frac{b_i}{i!} \{\phi_{2n-i+1} x^{2n}, \phi_{k+i-1} y^{k-2}\} \\
=& \sum_{i\geq 0} x^i \frac{b_i}{i!} \left\{ \sum_{n\geq \lceil i/2 \rceil} \phi_{2n-i+1} x^{2n-i}, \sum_{k\geq 2} \frac{(k+i-2)!}{(k-2)!} y^{k-2} \phi_{k+i-1} \right\} \\
=& \sum_{i\geq 0} x^i \frac{b_i}{i!} \left\{ \frac{1}{2}(\Phi(x) + (-1)^i \Phi(-x)), \sum_{k\geq 2} \partial_y^i \, y^{k+i-2} \phi_{k+i-1} \right\} \\
=& \frac{1}{2} \sum_{i\geq 0} (x\partial_y)^i \frac{b_i}{i!} \{\Phi(x) + (-1)^i \Phi(-x), \Phi(y)\},
\end{aligned} \tag{55}$$

so that

$$2x\partial_y S(x,y) = \frac{x\partial_y}{e^{x\partial_y} - 1}\{\Phi(x), \Phi(y)\} + \frac{-x\partial_y}{e^{-x\partial_y} - 1}\{\Phi(-x), \Phi(y)\}. \tag{56}$$

Equation (32) gives

$$\{\Phi(x), \Phi(y)\} = \left(e^{x\partial_y} + e^{y\partial_x} - 1\right) [\Phi(x), \Phi(y)]. \tag{57}$$

Substituting this expression in (56), one gets

$$
\begin{aligned}
2x\partial_y S(x,y) &= x\partial_y\left(1+\frac{1}{e^{x\partial_y}-1}e^{y\partial_x}\right)[\Phi(x),\Phi(y)]\\
&\quad - x\partial_y\left(1+\frac{1}{e^{-x\partial_y}-1}e^{-y\partial_x}\right)[\Phi(-x),\Phi(y)].\\
&= x\partial_y\Big([\Phi(x)-\Phi(-x),\Phi(y)]\\
&\qquad + \frac{1}{e^{x\partial_y}-1}[\Phi(x+y),\Phi(y)] - \frac{1}{e^{-x\partial_y}-1}[\Phi(-x+y),\Phi(y)]\Big)\\
&= x\partial_y\Big([\Phi(x)-\Phi(-x),\Phi(y)]\\
&\qquad + \frac{1}{e^{x\partial_y}-1}([\Phi(x+y),\Phi(y)] + e^{x\partial_y}[\Phi(-x+y),\Phi(y)]\Big)\\
&= x\partial_y\left([\Phi(x)-\Phi(-x),\Phi(y)]\right).
\end{aligned}
\tag{58}
$$

Hence,

$$
\begin{aligned}
x\partial_y S(x,y) =& \frac{1}{2}\,x\partial_y\,[\Phi(x)-\Phi(-x),\Phi(y)]\\
=& \sum_{n\geq 1}\sum_{k\geq 2}(k-1)\,[\phi_{2n},\phi_k]\,x^{2n}y^{k-2}.
\end{aligned}
\tag{59}
$$

Comparing the coefficients of $x^{2n-1}y^{k-1}$ in (54) and of $x^{2n}y^{k-2}$ in (59) for $n\geq 1$ and $k\geq 2$, one obtains (53). ■

Note that Formula (53) is very similar to Formula (40) when substituting $n=k$ and $p=2n$:

$$
[\phi_{2n},\phi_k] = \sum_{i=0}^{2n}\binom{k-1+i}{k-1}\frac{b_i}{k-1+i}\{\phi_{2n-i+1},\phi_{k+i-1}\}. \tag{60}
$$

$$
[\phi_{2n},\phi_k] = \sum_{i=0}^{2n}\binom{k-1+i}{k-1}\frac{b_i}{k-1+i}D_{\phi_{k-1+i}}(\phi_{2n+1-i}). \tag{61}
$$

Thus, we obtain without further calculations the following analogs of (41) and (42):

Corollary 5.2. *For all $n\geq 1$,*

$$
\sum_{i=0}^{2n}\binom{2n-2+i}{2n-2}b_i\{\phi_{2n-i+1},\phi_{2n+i-1}\} = 0. \tag{62}
$$

For all $n\geq 1$ and $p\geq 1$,

$$
\sum_{i=1}^{2n+2p-1}\left(\binom{i-1}{i-2p+1}\frac{b_{i-2p+1}}{2p-1} + \binom{i-1}{i-2n+1}\frac{b_{i-2n+1}}{2n-1}\right)\{\phi_{2n+2p-i},\phi_i\} = 0. \tag{63}
$$

Note that the left-hand sides of Equations (62) and (63) are only composed of brackets of odd ϕ. For example, the first equation gives

$$\begin{aligned}
0 &= 9\,\{\phi_5,\phi_7\} - 14\{\phi_3,\phi_9\},\\
0 &= 11\,\{\phi_7,\phi_9\} - 21\,\{\phi_5,\phi_{11}\} + 66\,\{\phi_3,\phi_{13}\},\\
0 &= 13\,\{\phi_9,\phi_{11}\} - 33\,\{\phi_7,\phi_{13}\} + 143\,\{\phi_5,\phi_{15}\} - 858\,\{\phi_3,\phi_{17}\},\\
0 &= 300\,\{\phi_{11},\phi_{13}\} - 1001\,\{\phi_9,\phi_{15}\} + 5720\,\{\phi_7,\phi_{17}\} - 43758\,\{\phi_5,\phi_{19}\}\\
&\quad + 419900\,\{\phi_3,\phi_{21}\}.
\end{aligned} \tag{64}$$

whereas the second one gives

$$\begin{aligned}
0 &= 195\,\{\phi_{11},\phi_7\} - 825\,\{\phi_{13},\phi_5\} - 4004\,\{\phi_{15},\phi_3\},\\
0 &= 85\,\{\phi_{13},\phi_9\} - 442\,\{\phi_{15},\phi_7\} + 2730\,\{\phi_{17},\phi_5\} - 21216\,\{\phi_{19},\phi_3\},\\
0 &= 2193\,\{\phi_{13},\phi_{11}\} - 7973\,\{\phi_{15},\phi_9\} + 47213\,\{\phi_{17},\phi_7\} - 364803\,\{\phi_{19},\phi_5\}\\
&\quad + 3509718\,\{\phi_{21},\phi_3\}.
\end{aligned} \tag{65}$$

5.1. Period polynomials.

It follows from the discussion of Section 4. that if one writes (62) and (63) as

$$\sum_{i,j\ \text{odd}} a_{i,j}\{\phi_i,\phi_j\} = 0\,, \tag{66}$$

then

$$\sum_{i,j\ \text{odd}} a_{i,j}\frac{1}{(i-1)!(j-1)!}\left(t^{i-1}-t^{j-1}\right) \tag{67}$$

is a period polynomial as first shown by Schneps [12]. Actually, Schneps has shown that a relation of the type (66) holds iff (67) is a period polynomial. Let us recall the explanation: Ihara and Takao [8] have proved that the space of linear relations of the form (67) has the same dimension as the space of cusp forms $S_n(SL_2(\mathbb{Z}))$. We have seen that all the even period polynomials $P^+_{2n,2m}(t)$ can be obtained in this way. Hence, all linear relations between the $\{\phi_i,\phi_j\}$ (with i,j odd) are consequences of (66).

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