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# Lie Elements in pre-Lie Algebras, Trees and Cohomology Operations

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**Abstract.** We give a simple characterization of Lie elements in free pre-Lie algebras as elements of the kernel of a map between spaces of trees. We explain how this result is related to natural operations on the Chevalley-Eilenberg complex of a Lie algebra. We also indicate a possible relation to Loday's theory of triplettes.
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## 1. Main results, motivations and generalizations

All algebraic objects in this note will be defined over a field $\mathbf{k}$ of characteristic zero and $V$ will always denote a $\mathbf{k}$-vector space. We will sometimes use the formalism of operads explained, for example, in [14]. Sections 2, 3 and 4 containing the main results, however, do not rely on this language.

Let $\mathsf{pL}(V)$ denote the free pre-Lie algebra generated by $V$ and $\mathsf{pL}(V)_L$ the associated Lie algebra. We will focus on the Lie algebra $\mathsf{L}(V) \subset \mathsf{pL}(V)$ generated in $\mathsf{pL}(V)_L$ by $V$, called the *subalgebra of Lie elements* in $\mathsf{pL}(V)$. It is known [3] that $\mathsf{L}(V)$ is (isomorphic to) the free Lie algebra generated by $V$; we will give a new short proof of this statement in Section 3. Our main result, Theorem 3.3, describes $\mathsf{L}(V)$ as the kernel of a map

$$d : \mathsf{pL}(V) \to \mathsf{pL}^1(V), \tag{1}$$

where $\mathsf{pL}^1(V)$ is the subspace of degree $+1$ elements in the free graded pre-Lie algebra $\mathsf{pL}^*(V, \circ)$ generated by $V$ and a degree $+1$ 'dummy' variable $\circ$. The map (1) is later in the paper identified with a very simple map between spaces of trees, see Proposition 4.8 and Corollary 4.9.

Theorem 3.3 and Corollary 4.9 have immediate applications to the analysis of natural operations on the Chevalley-Eilenberg complex of a Lie algebra. In

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a future work we also plan to prove that Theorem 3.3 implies that the only natural multilinear operations on vector fields on smooth manifolds are, in stable dimensions, iterations of the standard Jacobi bracket. There is also a possible relation of the results of this paper with Loday's theory of triplettes. In the rest of this introductory section, we discuss some of these applications and motivations in more detail.

**1.1.** *Motivations.* In [12] we studied, among other things, the differential graded (dg-) operad $\mathcal{B}^*_{Lie}$ of natural operations on the Chevalley-Eilenberg complex of a Lie algebra with coefficients in itself, along with its homotopy version $\mathcal{B}^*_{L_\infty}$, the operad of natural operations on the Chevalley-Eilenberg complex of an $L_\infty$-algebra (= strongly homotopy Lie algebra, see [8]). We proposed:

**Problem 1.2.** Describe the homotopy types, in the non-abelian derived category, of the dg-operads $\mathcal{B}^*_{Lie}$ and $\mathcal{B}^*_{L_\infty}$ of natural operations on the Chevalley-Eilenberg complex.

The following conjecture was proposed by D. Tamarkin.

**Conjecture 1.3.** The operad $\mathcal{B}^*_{Lie}$ has the homotopy type of the operad $\mathcal{L}ie$ for Lie algebras.

It turns out that $\mathcal{B}^*_{L_\infty}$, which is tied to $\mathcal{B}^*_{Lie}$ by the 'forgetful' map $c : \mathcal{B}^*_{L_\infty} \to \mathcal{B}^*_{Lie}$, contains a dg-sub-operad $\mathsf{rpL}^* = (\mathsf{rpL}^*, d)$ generated by symmetric braces [9] such that $\mathsf{rpL}^0$ (the sub-operad of degree 0 elements) is the operad $p\mathcal{L}ie$ governing pre-Lie algebras. Moreover, both $\mathsf{rpL}^0$, $\mathsf{rpL}^1$ and the differential $d : \mathsf{rpL}^0 \to \mathsf{rpL}^1$ have very explicit descriptions in terms of planar trees. Our conviction in Conjecture 1.3 made us believe that the sub-operad

$$H^0(\mathsf{rpL}^*) = Ker\left(d : \mathsf{rpL}^0 \to \mathsf{rpL}^1\right)$$

of $\mathsf{rpL}^0 \cong p\mathcal{L}ie$ equals the operad $\mathcal{L}ie$,

$$H^0(\mathsf{rpL}^*) \cong \mathcal{L}ie. \tag{2}$$

The main result of this paper, equivalent to isomorphism (2), is therefore a step towards a solution of Problem 1.2.

**1.4.** *Generalizations.* Let us slightly reformulate the above reflections and indicate possible generalizations. Let $\mathcal{P}$ be a quadratic Koszul operad [14, Section II.3.3] and $\mathcal{B}_{\mathcal{P}_\infty} = (\mathcal{B}_{\mathcal{P}_\infty}, d)$ the dg-operad of natural operations on the complex defining the operadic cohomology of $\mathcal{P}_\infty$ (= strongly homotopy $\mathcal{P}$-algebras [14, Definition II.3.128]) with coefficients in itself. In [12] we conjectured that

$$H^0(\mathcal{B}^*_{\mathcal{P}_\infty}) \cong \mathcal{L}ie \tag{3}$$

for each quadratic Koszul operad $\mathcal{P}$.

The operad $\mathcal{B}^*_{\mathcal{P}_\infty}$ has a suboperad $\mathcal{S}^*_{\mathcal{P}_\infty}$ generated by a restricted class of operations which generalize the braces on the Hochschild cohomology complex of an associative algebra [5]. The operad $\mathcal{S}^0_{\mathcal{P}_\infty}$ of degree 0 elements in $\mathcal{S}^*_{\mathcal{P}_\infty}$ always contains the operad $\mathcal{L}ie$ for Lie algebras that represents the intrinsic brackets. The conjectural isomorphism (3) would therefore imply:

**Conjecture 1.5.** For each quadratic Koszul operad $\mathcal{P}$,

$$\mathcal{L}ie \cong Ker\left(d : \mathcal{S}^0_{\mathcal{P}_\infty} \to \mathcal{S}^1_{\mathcal{P}_\infty}\right).$$

Moving from operads to free algebras [14, Section II.1.4], an affirmative solution of this conjecture for a particular operad $\mathcal{P}$ would immediately give a characterization of Lie elements in free $\mathcal{S}^0_{\mathcal{P}_\infty}$-algebras.

From this point of view, the main result of this paper (Theorem 3.3) is a combination of a solution of Conjecture 1.5 for $\mathcal{P} = \mathcal{L}ie$ with the identification of $\mathcal{S}^0_{\mathcal{L}ie_\infty} \cong p\mathcal{L}ie$ which expresses the equivalence between symmetric brace algebras and pre-Lie algebras [6, 9]. Conjecture 1.5 holds also for $\mathcal{P} = \mathcal{A}ss$, the operad for associative algebras, as we know from the Deligne conjecture in the form proved in [7]. Since $\mathcal{S}^0_{\mathcal{A}ss_\infty}$ is the operad for (ordinary, non-symmetric) braces [5], one can obtain a description of *Lie elements* in *free brace algebras*.

**1.6.** *Loday's triplettes.* Theorem 3.3 can also be viewed as an analog of the characterization of Lie elements in the tensor algebra $T(V)$ as primitives of the bialgebra $\mathcal{H} = (T(V), \otimes, \Delta)$ with $\Delta$ the shuffle diagonal; we recall this classical result as Theorem 2.1 of Section 2. The bialgebra $\mathcal{H}$ is associative, coassociative cocommutative and its primitives $Prim(\mathcal{H})$ form a Lie algebra. To formalize such situations, J.-L. Loday introduced in [10] the notion of a *triplette* $(\mathcal{C}, \mathfrak{I}, \mathcal{A}\texttt{-alg} \xrightarrow{F} \mathcal{P}\texttt{-alg})$, abbreviated $(\mathcal{C}, \mathcal{A}, \mathcal{P})$, consisting of operads $\mathcal{C}$ and $\mathcal{A}$, 'spin' relations $\mathfrak{I}$ between $\mathcal{C}$-coalgebras and $\mathcal{A}$-algebras defining $(\mathcal{C}, \mathfrak{I}, \mathcal{A})$-bialgebras, an operad $\mathcal{P}$ describing the algebraic structure of the primitives, and a forgetful functor $F : \mathcal{A}\texttt{-alg} \to \mathcal{P}\texttt{-alg}$, see Definition 7.2 in Subsection 7.1.

The nature of associative, cocommutative coassociative bialgebras and their primitives is captured by the triplette $(\mathcal{C}om, \mathcal{A}ss, \mathcal{L}ie)$. The classical Theorem 2.1 then follows from the fact that the triplette $(\mathcal{C}om, \mathcal{A}ss, \mathcal{L}ie)$ is *good*, in the sense which we also recall in Subsection 7.1. An interesting question is whether the case of Lie elements in pre-Lie algebras considered in this paper is governed by a good triplette in which $\mathcal{A} = p\mathcal{L}ie$ and $\mathcal{P} = \mathcal{L}ie$. See Subsection 7.1 for more detail.

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## 2. Classical results revisited

In this section we recall some classical results about Lie elements in free associative algebras in a language suitable for the purposes of this paper. Let $\mathsf{T}(V)$ be the tensor algebra generated by a vector space $V$,

$$\mathsf{T}(V) = \mathbf{k} \oplus \bigoplus_{n=1}^{\infty} \mathsf{T}^n(V),$$

where $\mathsf{T}^n(V)$ is the $n$-th tensor power $\bigotimes^n(V)$ of the space $V$. Let $\mathsf{T}(V)_L$ denote the space $\mathsf{T}(V)$ considered as a Lie algebra with the commutator bracket

$$[x, y] := x \otimes y - y \otimes x, \quad x, y \in \mathsf{T}(V),$$

and let $\mathsf{L}(V) \subset \mathsf{T}(V)$ be the Lie sub-algebra of $\mathsf{T}(V)_L$ generated by $V$. It is well-known that $\mathsf{L}(V)$ is (isomorphic to) the free Lie algebra generated by $V$ [16, §4, Theorem 2].

There are several characterizations of the subspace $\mathsf{L}(V) \subset \mathsf{T}(V)$ [15, 16]. Let us recall the one which uses the *shuffle diagonal* $\Delta : \mathsf{T}(V) \to \mathsf{T}(V) \otimes \mathsf{T}(V)$ given, for $v_1 \otimes \cdots \otimes v_n \in \mathsf{T}^n(V)$, by

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{i=0}^{n} \sum_{\sigma \in Sh(i,n-i)} [v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}] \otimes [v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(n)}], \quad (4)$$

where $Sh(i, n-i)$ denotes the set of all $(i, n-i)$-shuffles, i.e. permutations $\sigma \in \Sigma_n$ such that

$$\sigma(1) < \cdots < \sigma(i) \ \text{ and } \sigma(i+1) < \cdots < \sigma(n).$$

Notice that, in the right hand side of (4), the symbol $\otimes$ has two different meanings, the one inside the brackets denotes the tensor product in $\mathsf{T}(V)$, the middle one the tensor product of two copies of $\mathsf{T}(V)$. To avoid this ambiguity, we denote the product in $\mathsf{T}(V)$ by the dot $\bullet$ , (4) will then read as

$$\Delta(v_1 \bullet \cdots \bullet v_n) := \sum_{i=0}^{n} \sum_{\sigma \in Sh(i,n-i)} [v_{\sigma(1)} \bullet \cdots \bullet v_{\sigma(i)}] \otimes [v_{\sigma(i+1)} \bullet \cdots \bullet v_{\sigma(n)}].$$

The triple $(\mathsf{T}(V), \bullet, \Delta)$ is a standard example of a unital counital associative coassociative cocommutative Hopf algebra. We will need also the *augmentation ideal* $\overline{\mathsf{T}}(V) \subset \mathsf{T}(V)$ which equals $\mathsf{T}(V)$ minus the ground field,

$$\overline{\mathsf{T}}(V) = \bigoplus_{n=1}^{\infty} \mathsf{T}^n(V),$$

and the *reduced* diagonal $\overline{\Delta} : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ defined as

$$\overline{\Delta}(x) := \Delta(x) - 1 \otimes x - x \otimes 1, \quad \text{for } x \in \overline{\mathsf{T}}(V),$$

or, more explicitly,

$$\overline{\Delta}(v_1 \bullet \cdots \bullet v_n) := \sum_{i=1}^{n-1} \sum_{\sigma \in Sh(i,n-i)} [v_{\sigma(1)} \bullet \cdots \bullet v_{\sigma(i)}] \otimes [v_{\sigma(i+1)} \bullet \cdots \bullet v_{\sigma(n)}],$$

for $v_1, \ldots, v_n \in V$ and $n \geq 1$. Clearly $\mathsf{L}(V) \subset \overline{\mathsf{T}}(V)$. The following theorem is classical [16].

**Theorem 2.1.** *The subspace* $\mathsf{L}(V) \subset \overline{\mathsf{T}}(V)$ *equals the subspace of primitive elements,*

$$\mathsf{L}(V) = Ker\left(\overline{\Delta} : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)\right).$$

The diagonal $\Delta : \mathsf{T}(V) \to \mathsf{T}(V) \otimes \mathsf{T}(V)$ is a homomorphism of associative algebras, that is

$$\Delta(x \bullet y) = \Delta(x) \bullet \Delta(y), \quad \text{for } x \in \mathsf{T}(V), \quad (5)$$

where the same $\bullet$ denotes both the multiplication in $\mathsf{T}(V)$ in the left hand side and the induced multiplication of $\mathsf{T}(V) \otimes \mathsf{T}(V)$ in the right hand side. The reduced diagonal $\overline{\Delta} : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ is, however, of a different nature:

**Proposition 2.2.** *For each* $x, y \in \overline{\mathsf{T}}(V)$,

$$\overline{\Delta}(x \bullet y) = \Delta(x) \bullet \overline{\Delta}(y) + \overline{\Delta}(x) \bullet (y \otimes 1 + 1 \otimes y) + (x \otimes y + y \otimes x). \tag{6}$$

The proof is a direct verification which we leave for the reader. We are going to reformulate (6) using an action of $\overline{\mathsf{T}}(V)$ on $\overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ defined as follows. For $\xi \in \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ and $x \in \overline{\mathsf{T}}(V)$, let

$$\begin{aligned} x * \xi &:= \Delta(x) \bullet \xi \in \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V), \quad \text{and} \\ \xi * x &:= \xi \bullet (1 \otimes x + x \otimes 1) \in \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V), \end{aligned} \tag{7}$$

where $\bullet$ denotes, as before, the tensor multiplication in $\overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$. Observe that, while

$$(x \bullet y) * \xi = x * (y * \xi) \ \text{ and } \ (x * \xi) * y = x * (\xi * y), \tag{8}$$

$(\xi * x) * y \neq \xi * (x \bullet y)$, therefore the action (7) *does not* make $\overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ a bimodule over the associative algebra $(\overline{\mathsf{T}}(V), \bullet)$. To understand the algebraic properties of the above action better, we need to recall the following important

**Definition 2.3.** ([4]) A *pre-Lie* algebra is a vector space $X$ with a bilinear product $\star : X \otimes X \to X$ such that the associator $\Phi : X^{\otimes 3} \to X$ defined by

$$\Phi(x, y, z) := (x \star y) \star z - x \star (y \star z), \ \text{ for } x, y, z \in X, \tag{9}$$

is symmetric in the last two variables, $\Phi(x, y, z) = \Phi(x, z, y)$. Explicitly,

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y) \ \text{ for each } x, y, z \in X. \tag{10}$$

There is an obvious graded version of this definition. Pre-Lie algebras are known also under different names, such as right-symmetric algebras, Vinberg algebras, etc. Pre-Lie algebras are particular examples of *Lie-admissible* algebras [13], which means that the object $X_L := (X, [-,-])$ with $[-,-]$ the commutator of $\star$, is a Lie algebra. Each associative algebra is clearly pre-Lie. In the following proposition, $\overline{\mathsf{T}}(V)_{pL}$ denotes the augmentation ideal $\overline{\mathsf{T}}(V)$ of the associative algebra $\mathsf{T}(V)$ considered as a pre-Lie algebra.

**Proposition 2.4.** *Formulas (7) define on* $\overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ *a structure of a bimodule over the pre-Lie algebra* $\overline{\mathsf{T}}(V)_{pL}$. *This means that*

$$(\xi * x) * y - \xi * (x \bullet y) = (\xi * y) * x - \xi * (y \bullet x)$$

*and*

$$(x \bullet y) * \xi - x * (y * \xi) = (x * \xi) * y - x * (\xi * y),$$

*for each* $x, y \in \overline{\mathsf{T}}(V)$ *and* $\xi \in \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$. *In particular,* $\overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ *is a module over the Lie algebra* $\overline{\mathsf{T}}(V)_L$.

**Proof.** To prove the first equality, notice that

$$(\xi * x) * y - \xi * (x \bullet y) = \xi \bullet (x \otimes y + y \otimes x) = (\xi * y) * x - \xi * (y \bullet x).$$

The second one immediately follows from (8). ■

Using action (7), rule (6) can be rewritten as

$$\overline{\Delta}(x \bullet y) = \overline{\Delta}(x) * y + x * \overline{\Delta}(y) + R(x,y),\ \ x, y \in \overline{\mathsf{T}}(V), \tag{11}$$

where the symmetric bilinear form $R(x,y) := x \otimes y + y \otimes x$ measures the deviation of $\overline{\Delta}$ from being a pre-Lie algebra derivation in

$$\mathit{Der}_{\mathit{pre-Lie}}\left(\overline{\mathsf{T}}(V)_{pL}, \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)\right).$$

On the other hand, since $R : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ is symmetric, $\overline{\Delta}$ is a derivation of the associated Lie algebra $\overline{\mathsf{T}}(V)_L$,

$$\overline{\Delta} \in \mathit{Der}_{\mathit{Lie}}\left(\overline{\mathsf{T}}(V)_L, \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)\right),$$

which implies that $\mathsf{L}(V) \subset \mathit{Ker}(\overline{\Delta})$. The following statement is completely obvious and we formulate it only to motivate Proposition 3.2 of Section 3.

**Proposition 2.5.** *The map $\overline{\Delta} : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ is uniquely determined by the rule (11) together with the requirement that $\overline{\Delta}(v) = 0$ for $v \in V$.*

Observe that the reduced diagonal $\overline{\Delta} : \overline{\mathsf{T}}(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ is the initial differential of the cobar construction

$$\mathit{Cob}(\overline{\mathsf{T}}(V), \overline{\Delta}) : \quad \overline{\mathsf{T}}(V) \xrightarrow{d} \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V) \xrightarrow{d} \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V) \xrightarrow{d} \cdots \tag{12}$$

of the coassociative coalgebra $(\overline{\mathsf{T}}(V), \overline{\Delta})$. Complex (12) calculates the cohomology

$$H^*\left(\mathsf{T}(V), \Delta\right) = \mathit{Cotor}^{*+1}_{(\mathsf{T}(V),\Delta)}(\mathbf{k}, \mathbf{k}) \tag{13}$$

of the shuffle coalgebra.

On the other hand, by the Poincaré-Birkhoff-Witt theorem, there is an isomorphism of coalgebras

$$(\mathsf{T}(V), \Delta) \cong (\mathbf{k}[\mathsf{L}(V)], \nabla),$$

where the polynomial ring $\mathbf{k}[\mathsf{L}(V)]$ in the right hand side is equipped with the standard cocommutative comultiplication $\nabla$. Dualizing the proof of the classical [11, Theorem VII.2.2], one obtains the isomorphism

$$\mathit{Cotor}^*_{(\mathbf{k}[\mathsf{L}(V)],\nabla)}(\mathbf{k}, \mathbf{k}) \cong \wedge^*(\mathsf{L}(V))$$

where $\wedge^*(-)$ denotes the exterior algebra functor. We conclude that

$$H^*\left(\mathsf{T}(V), \Delta\right) \cong \wedge^{*+1}(\mathsf{L}(V)).$$

## 3. Lie elements in the free pre-Lie algebra

In this section we show that the results reviewed in Section 2 translate to pre-Lie algebras. Let $\mathsf{pL}(V) = (\mathsf{pL}(V), \star)$ denote the free pre-Lie algebra generated by a vector space $V$ and let $\mathsf{pL}(V)_L$ be the associated Lie algebra. The following proposition is proved in [3], but we will give a shorter and more direct proof, which was kindly suggested to us by M. Livernet.

**Proposition 3.1.** *The subspace $\mathsf{L}(V) \subset \mathsf{pL}(V)_L$ generated by $V$ is isomorphic to the free Lie algebra on $V$.*

**Proof** (due to M. Livernet). Let us denote in this proof by $\mathsf{L}'(V)$ the Lie subalgebra of $\mathsf{pL}(V)_L$ generated by $V \subset \mathsf{pL}(V)$ and by $\mathsf{L}''(V)$ the Lie subalgebra of $\mathsf{T}(V)_L$ generated by $V \subset \mathsf{T}(V)$. The canonical map $\mathsf{pL}(V) \to \mathsf{T}(V)_{pL}$ clearly induces a map $\mathsf{pL}(V)_L \to \mathsf{T}(V)_L$ which restricts to a Lie algebra homomorphism $\alpha : \mathsf{L}'(V) \to \mathsf{L}''(V)$.

Let $\mathsf{L}(V)$ be, as before, the free Lie algebra generated by $V$. Since $\mathsf{L}'(V)$ is also generated by $V$, the canonical map $\beta : \mathsf{L}(V) \to \mathsf{L}'(V)$ is an epimorphism. To prove that it is a monomorphism, observe that the composition $\alpha\beta : \mathsf{L}(V) \to \mathsf{L}''(V)$ coincides with the canonical map induced by the inclusion $V \hookrightarrow \mathsf{L}''(V)$. Since $\mathsf{L}''(V)$ is isomorphic to the free Lie algebra generated by $V$ [16, §4, Theorem 2], the composition $\alpha\beta : \mathsf{L}(V) \to \mathsf{L}''(V)$ is an isomorphism, therefore $\beta$ must be monic. We conclude that the canonical map $\beta : \mathsf{L}(V) \to \mathsf{L}'(V)$ is an isomorphism, which finishes the proof. ■

Consider the free graded pre-Lie algebra $\mathsf{pL}(V, \circ)$ generated by $V$ and one 'dummy' variable $\circ$ placed in degree $+1$. Observe that

$$\mathsf{pL}^*(V, \circ) = \mathsf{pL}(V) \oplus \bigoplus_{n \geq 1} \mathsf{pL}^n(V), \tag{14}$$

where $\mathsf{pL}^n(V)$ is the subset of $\mathsf{pL}(V)$ spanned by monomials with exactly $n$ occurrences of the dummy variable $\circ$.

We need to consider also the graded pre-Lie algebra $\mathsf{rpL}(V)$ ("r" for "reduced") defined as the quotient

$$\mathsf{rpL}(V) := \mathsf{pL}(V, \circ)/(\circ \star \circ)$$

of the free pre-Lie algebra $\mathsf{pL}(V, \circ)$ by the ideal $(\circ \star \circ)$ generated by $\circ \star \circ$. The grading (14) clearly induces a grading of $\mathsf{rpL}(V)$ such that $\mathsf{rpL}^0(V) = \mathsf{pL}(V)$ and $\mathsf{rpL}^1(V) = \mathsf{pL}^1(V)$,

$$\mathsf{rpL}^*(V) = \mathsf{pL}(V) \oplus \mathsf{pL}^1(V) \oplus \bigoplus_{n \geq 2} \mathsf{rpL}^n(V). \tag{15}$$

The following statement, in which $\Phi$ is the associator (9), is an analog of Proposition 2.5.

**Proposition 3.2.** *There exists precisely one degree $+1$ map $d : \mathsf{rpL}^*(V) \to \mathsf{rpL}^{*+1}(V)$ such that $d(v) = 0$ for $v \in V$, $d(\circ) = 0$ and*

$$d(a \star b) = d(a) \star b + (-1)^{|a|} a \star d(b) + Q(a, b), \tag{16}$$

*where*

$$Q(a, b) := (\circ \star a) \star b - \circ \star (a \star b) = \Phi(\circ, a, b), \tag{17}$$

*for $a, b \in \mathsf{rpL}^*(V)$. Moreover, $d^2 = 0$.*

**Proof.** The uniqueness of the map $d$ with the properties stated in the proposition is clear. To prove that such a map exists, we show first that there exists a degree one map $\tilde{d} : \mathsf{pL}(V, \circ) \to \mathsf{pL}(V, \circ)$ of graded free pre-Lie algebras such that $\tilde{d}(v) = 0$ for $v \in V$, $\tilde{d}(\circ) = 0$ and

$$\tilde{d}(x \star y) = \tilde{d}(x) \star y + (-1)^{|x|} x \star \tilde{d}(y) + Q(x, y), \tag{18}$$

where $Q(x, y) := \Phi(\circ, x, y)$ for $x, y \in \mathsf{pL}(V, \circ)$. Let us verify that the above rule is compatible with the axiom $\Phi(x, y, z) = (-1)^{|z||y|}\Phi(x, z, y)$ of graded pre-Lie algebras. Applying (18) twice, we obtain

$$\begin{aligned} \tilde{d}\Phi(x, y, z) \;=\; & \Phi(\tilde{d}x, y, z) + (-1)^{|x|}\Phi(x, \tilde{d}y, z) + (-1)^{|x|+|y|}\Phi(x, y, \tilde{d}z) \\ & -(-1)^{|x|} x \star Q(y, z) + Q(x \star y, z) + Q(x, y) \star z - Q(x, y \star z), \end{aligned} \tag{19}$$

for arbitrary $x, y, z \in \mathsf{pL}(V, \circ)$.

Let us make a small digression and observe that the associator $\Phi$ behaves as a Hochschild cochain, that is

$$\circ \star \Phi(x, y, z) - \Phi(\circ \star x, y, z) + \Phi(\circ, x \star y, z) - \Phi(\circ, x, y \star z) + \Phi(\circ, x, y) \star z = 0.$$

It follows from the definition of the form $Q$ and the above equation that the last three terms of (19) equal $\Phi(\circ\star x, y, z) - \circ\star\Phi(x, y, z)$, therefore (19) can be rewritten as

$$\begin{aligned} \tilde{d}\Phi(x, y, z) \;=\; & \Phi(\tilde{d}x, y, z) + (-1)^{|x|}\Phi(x, \tilde{d}y, z) + (-1)^{|x|+|y|}\Phi(x, y, \tilde{d}z) \\ & -(-1)^{|x|} x \star Q(y, z) + \Phi(\circ \star x, y, z) - \circ \star \Phi(x, y, z). \end{aligned}$$

Since the right hand side of the above equality is graded symmetric in $y$ and $z$, we conclude that

$$\tilde{d}\left(\Phi(x, y, z) - (-1)^{|z||y|}\Phi(x, z, y)\right) = 0,$$

which implies the existence of $\tilde{d} : \mathsf{pL}(V, \circ) \to \mathsf{pL}(V, \circ)$ with the properties stated above. It is easy to verify, using (18) and the assumption $\tilde{d}(\circ) = 0$, that

$$\tilde{d}(\circ \star \circ) = \Phi(\circ, \circ, \circ) \tag{20}$$

and that

$$\tilde{d}^2(x \star y) = \tilde{d}^2(x) \star y + x \star \tilde{d}^2(y) + Q(\tilde{d}x, y) + (-1)^{|x|} Q(x, \tilde{d}y) + \Phi(\circ \star \circ, x, y) \tag{21}$$

for arbitrary $x, y \in \mathsf{pL}(V, \circ)$.

A simple induction on the number of generators based on (20) together with the rule (18) shows that $\tilde{d}$ preserves the ideal generated by $\circ \star \circ$. An equally simple induction based on (21) and (18) shows that $Im(\tilde{d}^2)$ is a subspace of the same ideal. We easily conclude from the above facts that $\tilde{d}$ induces a map $d : \mathsf{rpL}^*(V) \to \mathsf{rpL}^{*+1}(V)$ required by the proposition. ∎

Let us remark that each pre-Lie algebra $(X, \star)$ determines a unique *symmetric brace algebra* $(X, -\langle -, \ldots, - \rangle)$ with $x\langle y \rangle = x \star y$ for $x, y \in X$ [9, 6]. The bilinear form $Q$ in (16) then can be written as

$$Q(a, b) = \circ\langle a, b \rangle, \quad \text{for } a, b \in \mathsf{rpL}(V).$$

The complex

$$\mathsf{pL}(V) \xrightarrow{d} \mathsf{pL}^1(V) \xrightarrow{d} \mathrm{r}\mathsf{pL}^2(V) \xrightarrow{d} \cdots \tag{22}$$

should be viewed as an analog of the cobar construction (12). We will see in Section 5 that it describes natural operations on the Chevalley-Eilenberg cohomology of a Lie algebra. The main result of this paper reads:

**Theorem 3.3.** *The subspace* $\mathsf{L}(V) \subset \mathsf{pL}(V)$ *equals the kernel of the map* $d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)$,

$$\mathsf{L}(V) = Ker\left(d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)\right).$$

In Section 4 we describe the spaces $\mathsf{pL}(V)$, $\mathsf{pL}^1(V)$ and the map $d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)$ in terms of trees. Theorem 3.3 will be proved in Section 6.

## 4. Trees

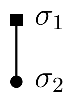
We begin by recalling a tree description of free pre-Lie algebras due to F. Chapoton and M. Livernet [1]. By a *tree* we understand a finite connected simply connected graph without loops and multiple edges. We will always assume that our trees are *rooted* which, by definition, means that one of the vertices, called the *root*, is marked and all edges are oriented, pointing to the root.

Let us denote by $\mathrm{Tr}_n$ the set of all trees with $n$ vertices numbered $1, \ldots, n$. The symmetric group $\Sigma_n$ act on $\mathrm{Tr}_n$ by relabeling the vertices. We define

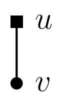
$$\mathrm{Tr}_n(V) := \mathrm{Span}_{\mathbf{k}}(\mathrm{Tr}_n) \otimes_{\Sigma_n} V^{\otimes n},\ n \geq 1,$$

where $\mathrm{Span}_{\mathbf{k}}(\mathrm{Tr}_n)$ denotes the $\mathbf{k}$-vector space spanned by $\mathrm{Tr}_n$ with the induced $\Sigma_n$-action and where $\Sigma_n$ acts on $V^{\otimes n}$ by permuting the factors. Therefore $\mathrm{Tr}_n(V)$ is the set of trees with $n$ vertices decorated by elements of $V$.

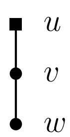
**Example 4.1.** The set $\mathrm{Tr}_1$ consists of a single tree $\bullet$ with one vertex (which is also the root) and no edges, thus $\mathrm{Tr}_1(V) \cong V$. The set $\mathrm{Tr}_2$ consists of labelled trees



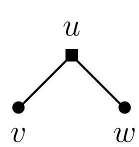

where $\blacksquare$ denotes the root and $\sigma \in \Sigma_2$. This means that $V$-decorated trees from $\mathrm{Tr}_2(V)$ look as




where $u, v \in V$, therefore $\mathrm{Tr}_2(V) \cong V^{\otimes 2}$. Similarly, $\mathrm{Tr}_3(V) \cong V^{\otimes 3} \oplus (V \otimes S^2(V))$, where $S^2(V)$ denotes the second symmetric power of $V$. The corresponding decorated trees are



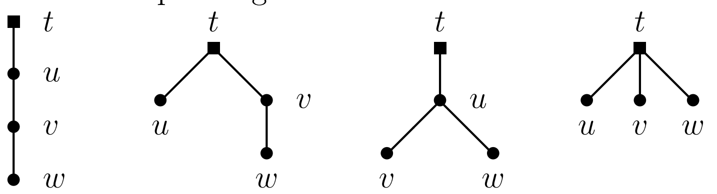

and




for $u, v, w \in V$. Finally,

$$\mathrm{Tr}_4(V) \cong V^{\otimes 4} \oplus V^{\otimes 4} \oplus (V^{\otimes 2} \otimes S^2(V)) \oplus (V \otimes S^3(V)),$$

with the summands corresponding to the decorated trees




(23)

with $t, u, v, w \in V$.

**Theorem 4.2.** (Chapoton-Livernet [1]) *Let* $\mathrm{Tr}(V) := \bigoplus_{n\geq 1} \mathrm{Tr}_n(V)$. *Then there is a natural isomorphism*

$$\mathsf{pL}(V) \cong \mathrm{Tr}(V). \tag{24}$$

The pre-Lie multiplication in the left hand side of (24) translates to the vertex insertion of decorated trees in the right hand side, see [1] for details.

**Example 4.3.** The most efficient way to identify decorated trees with elements of free pre-Lie algebras is to use the formalism of symmetric brace algebras [9]. The trees in (23) then represent the following elements of $\mathsf{pL}(V)$:

$$t\langle u\langle v\langle w\rangle\rangle\rangle,\ t\langle u, v\langle w\rangle\rangle,\ t\langle u\langle v, w\rangle\rangle \ \text{ and } \ t\langle u, v, w\rangle.$$

Using the same tree description [1] of the free graded pre-Lie algebra $\mathsf{pL}(V, \circ)$, one can easily get a natural isomorphism

$$\mathsf{pL}^1(V) \cong \mathrm{Tr}^1(V) := \bigoplus_{n\geq 0} \mathrm{Tr}^1_n(V), \tag{25}$$

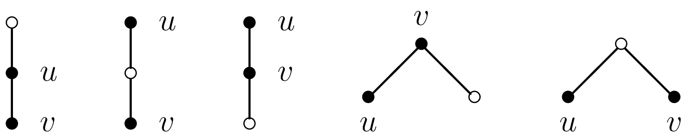
where $\mathrm{Tr}^1_n(V)$ is the set of all trees with $n$ vertices decorated by elements of $V$ and one vertex decorated by the dummy variable $\circ$. We call the vertex decorated by $\circ$ the *special vertex*.

**Example 4.4.** Clearly $\mathrm{Tr}^1_0(V) \cong \mathbf{k}$ while $\mathrm{Tr}^1_1(V) \cong V \oplus V$ with the corresponding decorated trees


where $u \in V$. Similarly,

$$\mathrm{Tr}^1_2(V) \cong V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} \oplus S^2(V)$$

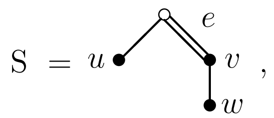
with the corresponding trees



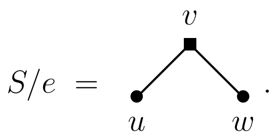

for $u, v \in V$. In the above pictures we always placed the root on the top. Some examples of decorated trees from $\mathrm{Tr}^1_n(V)$, $n \geq 3$, can also be found in Examples 4.5, 4.6, 4.7 and 4.10.

Let us describe the map $d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)$ of Theorem 3.3 in terms of decorated trees. We say that an edge $e$ of a decorated tree $S \in \mathrm{Tr}^1_n(V)$ is *special* if it is adjacent to the special vertex of $S$. Given such an edge $e$, we define the quotient $S/e \in \mathrm{Tr}_n(V)$ by contracting the special edge of $S$ into a vertex and decorating this vertex by the label of the (unique) endpoint of $e$ different from the special vertex. In the following examples, the special edge will be marked by the double line.

**Example 4.5.** If $S \in \mathrm{Tr}^1_3(V)$ is the tree

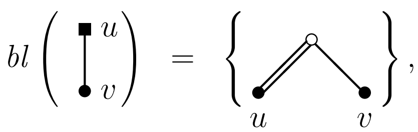



$u, v, w \in V$, then

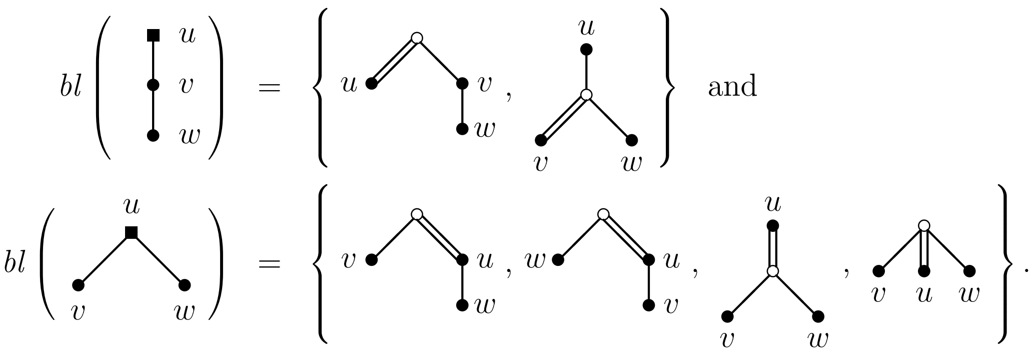



Let $T \in \mathrm{Tr}_n(V)$. We call a couple $(S, e)$, where $S \in \mathrm{Tr}^1_n(V)$ and $e$ a special edge of $S$, a *blow-up* of $T$ if $S/e \cong T$ and if the arity (= the number of incoming edges) of the special vertex of $S$ is $\geq 2$. We denote by $bl(T)$ the set of all blow-ups of $T$.

**Example 4.6.** The set $bl(\blacksquare)$ is empty. The simplest nontrivial example of a blow-up is




where the double line denotes, as in Example 4.5, the special edge. Let us give two more examples where $u$, $v$ and $w$ are elements of $V$:




The last thing we need is to introduce, for $(S, e) \in bl(T)$, the sign $\epsilon_{(S,e)} \in \{-1, +1\}$ as

$$\epsilon_{(S,e)} := \begin{cases} +1, & \text{if } e \text{ is an incoming edge of the special vertex, and} \\ -1, & \text{if } e \text{ is the outgoing edge of the special vertex.} \end{cases}$$

Finally, define the map

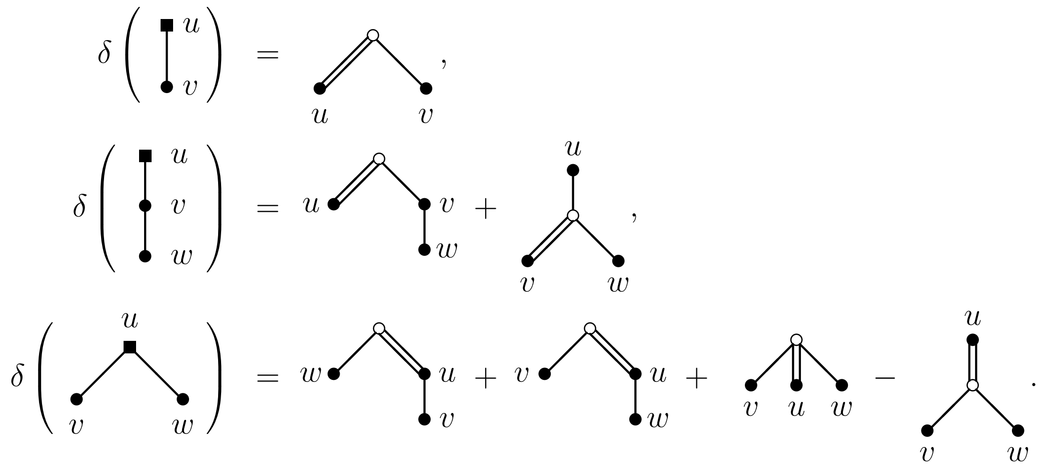
$$\delta : \mathrm{Tr}(V) \to \mathrm{Tr}^1(V) \tag{26}$$

by

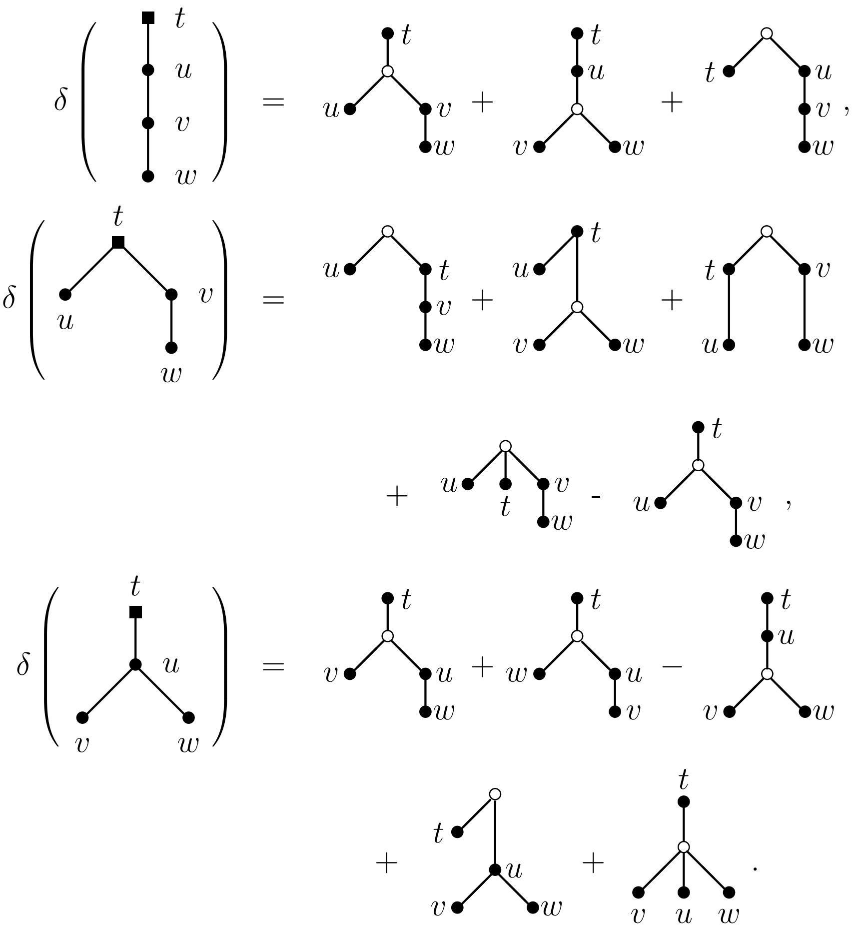
$$\delta(T) := \sum_{(S,e)\in bl(T)} \epsilon_{(S,e)} S.$$

**Example 4.7.** In this example, $t, u, v$ and $w$ are arbitrary elements of $V$. We stick to our convention that the root is placed on the top. Let us give first some examples of the map $\delta : \mathrm{Tr}(V) \to \mathrm{Tr}^1(V)$ that follow immediately from the calculations in Example 4.6. We keep the double lines indicating which edges has been blown-up:

$$\delta(\blacksquare) \;=\; 0,$$



Let us give some more formulas, this time without indicating the blown-up edges:



The proof of the following proposition is a direct verification based on the induction on the number of vertices and formula (16).

**Proposition 4.8.** *The diagram*

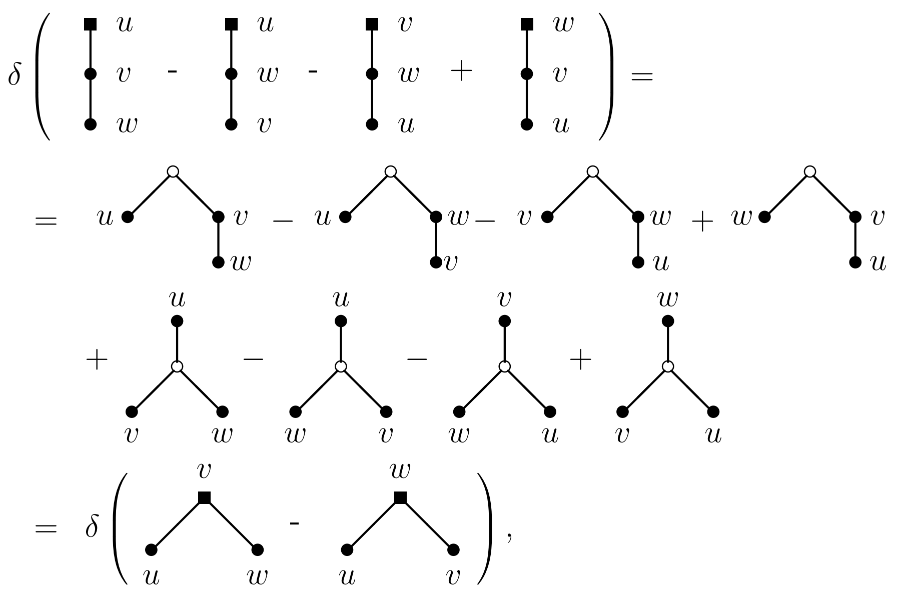
$$\begin{tikzcd} \mathsf{pL}(V) \arrow[r, "d"] \arrow[d, "\cong"] & \mathsf{pL}^1(V) \arrow[d, "\cong"] \\ \mathrm{Tr}(V) \arrow[r, "\delta"] & \mathrm{Tr}^1(V) \end{tikzcd}$$

*in which the vertical maps are isomorphism (24) and (25), is commutative.*

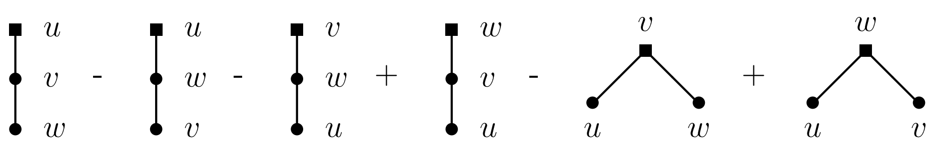
**Corollary 4.9.** *There is a natural isomorphism*

$$\mathsf{L}(V) \cong Ker(\delta : \mathrm{Tr}(V) \to \mathrm{Tr}^1(V)).$$

**Example 4.10.** It follows from the formulas given in Example 4.7 that, for each $u, v, w \in V$,



therefore the combination

$\xi_{u,v,w} :=$ 

belongs to the kernel of $\delta : \mathrm{Tr}_3(V) \to \mathrm{Tr}^1_3(V)$. It is easy to see that elements of this form in fact span this kernel and that the correspondence $\xi_{u,v,w} \mapsto [u,[v,w]]$ defines an isomorphism

$$Ker\Big(\mathrm{Tr}_3(V) \to \mathrm{Tr}^1_3(V)\Big) \cong \mathsf{L}_3(V),$$

where $\mathsf{L}_3(V) \subset \mathsf{L}(V)$ denotes the subspace of elements of monomial length 3.

## 5. Cohomology operations

In this section we show how an object closely related to the cochain complex $\mathsf{rpL}^*(V) = (\mathsf{rpL}^*(V), d)$ of (15), considered in Proposition 3.2, naturally acts on the Chevalley-Eilenberg complex of a Lie algebra with coefficients in itself. For $n \geq 1$, let $\mathbf{k}^n := \mathit{Span}_{\mathbf{k}}(e_1, \ldots, e_n)$ and let $\mathsf{rpL}^*(n)$ denote the subspace of the graded vector space $\mathsf{rpL}^*(\mathbf{k}^n)$ spanned by monomials which contain each basic element $e_1, \ldots, e_n$ exactly once.

More formally, given an $n$-tuple $t_1, \ldots, t_n \in \mathbf{k}$, consider the map $\varphi_{t_1,\ldots,t_n} : \mathbf{k}^n \to \mathbf{k}^n$ defined by

$$\varphi_{t_1,\ldots,t_n}(e_i) := t_i e_i,\ 1 \leq i \leq n.$$

Let us denote by the same symbol also the induced map $\varphi_{t_1,\ldots,t_n} : \mathsf{rpL}^*(\mathbf{k}^n) \to \mathsf{rpL}^*(\mathbf{k}^n)$. Then

$$\mathsf{rpL}^*(n) := \Big\{x \in \mathsf{rpL}^*(\mathbf{k}^n);\ \varphi_{t_1,\ldots,t_n}(x) = t_1 \cdots t_n x \ \text{ for each } t_1, \ldots, t_n \in \mathbf{k}\Big\}.$$

The above description immediately implies that $\mathsf{rpL}^*(n)$ is a $d$-stable subspace of $\mathsf{rpL}^*(\mathbf{k}^n)$, therefore $\mathsf{rpL}^*(n) = (\mathsf{rpL}^*(n), d)$ is a chain complex for each $n \geq 1$. Clearly $\mathsf{pL}(n) \cong \mathit{Span}_{\mathbf{k}}(\mathrm{Tr}_n)$ and $\mathsf{pL}^1(n) \cong \mathit{Span}_{\mathbf{k}}(\mathrm{Tr}^1_n)$. Observe that the above reduction does not erase any information, because $\mathsf{rpL}^*(V)$ can be reconstructed as

$$\mathsf{rpL}^*(V) \cong \bigoplus_{n \geq 1} \mathsf{rpL}^*(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Let us explain how each $U \in \mathsf{rpL}^d(n)$ determines an $n$-multilinear degree $d$ operation on the Chevalley-Eilenberg complex $C^*_{CE}(L; L)$ of a Lie algebra $L$ with coefficients in itself [2]. We will use the standard identification [14, Definition II.3.99]

$$C^*_{CE}(L; L) \cong \mathit{Coder}^*(\mathsf{L}^c(\downarrow L)) \tag{27}$$

where $\mathsf{L}^c(\downarrow L)$ denotes the cofree conilpotent Lie coalgebra [14] cogenerated by the desuspension $\downarrow L$ of the vector space $L$. Let $\lambda \in \mathit{Coder}^1(\mathsf{L}^c(\downarrow L))$ be the co-extension of the desuspended Lie algebra bracket

$$\downarrow \circ [-,-] \circ (\uparrow \wedge \uparrow) : \downarrow L \wedge \downarrow L \to \downarrow L$$

into a coderivation. Then $\lambda^2 = 0$ and (27) translates the Chevalley-Eilenberg differential $d_{CE}$ into the commutator with $\lambda$.

The above construction can be easily generalized to the case when $L$ is an $L_\infty$-algebra, $L = (L, l_1, l_2, l_3, \ldots)$ [8]. The structure operations $(l_1, l_2, l_3, \ldots)$ assemble again into a coderivation $\lambda \in \mathit{Coder}^1(\mathsf{L}^c(\downarrow L))$ with $\lambda^2 = 0$ [8, Theorem 2.3], and (27) can be taken for a definition of the (Chevalley-Eilenberg) cohomology of $L_\infty$-algebras with coefficients in itself.

The last fact we need to recall here is that $\mathit{Coder}^*(\mathsf{L}^c(\downarrow L))$ is a natural pre-Lie algebra, with the product $\star$ defined as follows [14, Section II.3.9]. Let $\Theta, \Omega \in \mathit{Coder}^*(\mathsf{L}^c(\downarrow L))$ and denote by $\overline{\Omega} : \mathsf{L}^c(\downarrow L) \to \downarrow L$ the corestriction of $\Omega$. The pre-Lie product $\Theta \star \Omega$ is then defined as the coextension of the composition

$$(-1)^{|\Theta||\Omega|} \cdot \overline{\Omega} \circ \Theta : \mathsf{L}^c(\downarrow L) \to \downarrow L,$$

see [14, Section II.3.9] for details.

By the freeness of the pre-Lie algebra $\mathsf{pL}^*(\mathbf{k}^n, \circ)$, each choice $f_1, \ldots, f_n \in \mathit{Coder}(\mathsf{L}^c(\downarrow L))$ determines a unique pre-Lie algebra homomorphism

$$\Psi_{f_1,\ldots,f_n} : \mathsf{pL}^*(\mathbf{k}^n, \circ) \to \mathit{Coder}^*(\mathsf{L}^c(\downarrow L))$$

such that $\Psi_{f_1,\ldots,f_n}(e_i) := f_i$ for each $1 \leq i \leq n$, and $\Psi_{f_1,\ldots,f_n}(\circ) := \lambda$. Because $\Psi_{f_1,\ldots,f_n}(\circ\star\circ) = \lambda^2 = 0$, the map $\Psi_{f_1,\ldots,f_n}$ induces a map of the quotient $\mathsf{rpL}^*(\mathbf{k}^n) = \mathsf{pL}^*(\mathbf{k}^n, \circ)/(\circ \star \circ)$

$$\mathrm{r}\Psi_{f_1,\ldots,f_n} : \mathsf{rpL}^*(\mathbf{k}^n) \to \mathit{Coder}^*(\mathsf{L}^c(\downarrow L))$$

Define finally $U(f_1, \ldots, f_n) \in C_{CE}^{d+|f_1|+\cdots+|f_n|}(L; L)$ by

$$U(f_1, \ldots, f_n) := \mathrm{r}\Psi_{f_1,\ldots,f_n}(U). \tag{28}$$

One can easily verify the following formula that relates the Chevalley-Eilenberg differential $d_{CE}$ with the differential $d$ in $\mathsf{rpL}^*(n)$:

$$\begin{aligned} d(U)(f_1, \ldots, f_n) &= d_{CE}(U(f_1, \ldots, f_n)) \\ &\quad - (-1)^{|U|} \sum_{1\leq i\leq n} (-1)^{|f_1|+\cdots+|f_{i-1}|} \cdot U(f_1, \ldots, d_{CE}(f_i), \ldots, f_n), \end{aligned}$$

for each $U \in \mathsf{rpL}^*(n)$ and $f_1, \ldots, f_n \in C^*_{CE}(L; L)$.

**Proposition 5.1.** *The collection* $\mathsf{rpL}^* := \{\mathsf{rpL}^*(n)\}_{n\geq 1}$ *forms an operad in the category of dg-vector spaces. Formula (28) determines an action that makes* $C^*_{CE}(L; L)$ *a differential graded* $\mathsf{rpL}^*$*-algebra. Consequently, the cohomology operad* $H^*(\mathsf{rpL})$ *naturally acts on the Chevalley-Eilenberg cohomology* $H^*_{CE}(L; L)$ *of an arbitrary Lie or* $L_\infty$ *algebra.*

**Proof.** The symmetric group $\Sigma_n$ acts on $\mathsf{rpL}^*(n)$ by permuting the basis $e_1, \ldots, e_n$ of $\mathbf{k}^n$. The operadic composition, induced by the vertex insertion of decorated trees representing elements of $\mathsf{pL}^*(\mathbf{k}^n)$, is constructed by exactly the same method as the one used in the proof of [14, Proposition II.1.27]. The verification that $U$ defines an operadic action is easy. ■

Let $\mathcal{B}^*_{Lie}$ denote, as in Section 1, the dg-operad of natural operations on the Chevalley-Eilenberg complex of a Lie algebra with coefficients in itself and $\mathcal{B}^*_{L_\infty}$ an analog of this operad for $L_\infty$-algebras. Because each Lie algebra is also an $L_\infty$-algebra, there exists an obvious 'forgetful' homomorphism $c : \mathcal{B}^*_{L_\infty} \to \mathcal{B}^*_{Lie}$. By Proposition 5.1, formula (28) defines maps $t : \mathsf{rpL}^* \to \mathcal{B}^*_{Lie}$ and $\widetilde{t} : \mathsf{rpL}^* \to \mathcal{B}^*_{L_\infty}$. The diagram

$$\begin{tikzcd} & \mathcal{B}^*_{L_\infty} \arrow[d, "c"] \\ \mathsf{rpL}^* \arrow[ur, "\widetilde{t}"] \arrow[r, "t"'] & \mathcal{B}^*_{Lie} \end{tikzcd}$$

is clearly commutative and $\widetilde{t} : \mathsf{rpL}^* \to \mathcal{B}^*_{L_\infty}$ is in fact an *inclusion* of dg-operads, compare the remarks in Subsection 1.1.

## 6. Proof of Theorem 3.3

For the purposes of the proof of Theorem 3.3, it will be convenient to reduce the complex $\mathsf{rpL}^*(V) = (\mathsf{rpL}^*(V), d)$ constructed in Proposition 3.2 as follows. Since the construction of $\mathsf{rpL}^*(V)$ is functorial in $V$, one may consider the map $\mathsf{rpL}^*(V) \to \mathsf{rpL}^*(0)$ induced by the map $V \to 0$ from $V$ to the trivial vector space $0$. The kernel $\overline{\mathsf{rpL}}^*(V) := \mathit{Ker}(\mathsf{rpL}^*(V) \to \mathsf{rpL}^*(0))$ is clearly a subcomplex of $\mathsf{rpL}^*(V)$. Since

$$\mathsf{rpL}^n(0) = \begin{cases} \mathit{Span}_{\mathbf{k}}(\circ), & \text{for } n = 1 \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

the complexes $\overline{\mathsf{rpL}}^*(V)$ and $\mathsf{rpL}^*(V)$ differ only at the second term, and, under the isomorphism (25),

$$\overline{\mathsf{rpL}}^1(V) \cong \bigoplus_{n\geq 1} \mathrm{Tr}^1_n(V).$$

It is also obvious that

$$\mathit{Ker}\left(d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)\right) = \mathit{Ker}\left(d : \mathsf{pL}(V) \to \overline{\mathsf{rpL}}^1(V)\right).$$

The central object of this section is the commutative diagram:

$$\begin{tikzcd} \mathsf{L}(V) \arrow[d, hook, "i"'] & \\ \mathsf{pL}(V) \arrow[r, "d"] \arrow[d, "p"'] & \overline{\mathsf{rpL}}^1(V) \arrow[d, "p^1"] \\ \overline{\mathsf{T}}(V) \arrow[r, "\overline{\Delta}"] & \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V) \end{tikzcd} \tag{29}$$

in which $i : \mathsf{L}(V) \hookrightarrow \mathsf{pL}(V)$ is the inclusion and $p : \mathsf{pL}(V) \to \overline{\mathsf{T}}(V)_{pL} = \overline{\mathsf{T}}(V)$ the canonical map of pre-Lie algebras induced by the inclusion $V \hookrightarrow \overline{\mathsf{T}}(V)$. The definition of $p^1 : \overline{\mathsf{rpL}}^1(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$ will use the following simple facts.

*Fact 1.* The graded pre-Lie algebra structure of $\mathsf{rpL}^*(V)$ induces on $\mathsf{rpL}^1(V) = \mathsf{pL}^1(V)$ a structure of a $\mathsf{pL}(V)$-bimodule.

*Fact 2.* With the structure above, $\mathsf{rpL}^1(V)$ is the free $\mathsf{pL}(V)$-bimodule generated by the dummy $\circ$.

*Fact 3.* The $*$-action (7) makes $\mathsf{T}(V) \otimes \mathsf{T}(V)$ a bimodule over the pre-Lie algebra $\mathsf{T}(V)_{pL}$. Therefore $\mathsf{T}(V) \otimes \mathsf{T}(V)$ is a $\mathsf{pL}(V)$-bimodule, via the canonical map $p : \mathsf{pL}(V) \to \mathsf{T}(V)_{pL}$.

The above facts imply that one can define a map $\widehat{p}^1 : \mathsf{rpL}(V)^1 \to \mathsf{T}(V) \otimes \mathsf{T}(V)$ by requiring that it is a $\mathsf{pL}(V)$-bimodule homomorphism satisfying

$$\widehat{p}^1(\circ) := 1 \otimes 1 \in \mathsf{T}(V) \otimes \mathsf{T}(V).$$

It is clear that this $\widehat{p}^1$ restricts to the requisite map $p^1 : \overline{\mathsf{rpL}}^1(V) \to \overline{\mathsf{T}}(V) \otimes \overline{\mathsf{T}}(V)$.

To prove that the bottom square of (29) commutes, we notice that both compositions $\overline{\Delta}p$ and $p^1d$ behave in the same way with respect to the pre-Lie multiplication $\star$ on $\mathsf{pL}(V)$. Indeed, for $a,b \in \mathsf{pL}(V))$, by (11)

$$\overline{\Delta}p(a\star b) = \overline{\Delta}(p(a)\bullet p(b)) = \overline{\Delta}p(a) * p(b) + p(a) * \overline{\Delta}p(b) + R(p(a),p(b)).$$

Similarly, by (16) and the definition of $p^1$,

$$\begin{aligned} p^1d(a\star b) &= p^1(d(a)\star b) + p^1(a\star d(b)) + p^1(Q(a,b)) \\ &= p^1d(a) * p(b) + p(a) * p^1d(b) + p^1(Q(a,b)). \end{aligned}$$

It remains to verify that $p^1(Q(a,b)) = R(p(a),p(b))$. By the definitions of $p^1$, $Q$, $R$ and the $*$-action (7),

$$\begin{aligned} p^1(Q(a,b)) &= p^1((\circ\star a)\star b) - p^1(\circ\star(a\star b)) = p^1(\circ\star a) * p(b) - p^1(\circ) * p(a\star b) \\ &= (p^1(\circ) * p(a)) * p(b) - p^1(\circ) * (p(a)\bullet p(b)) \\ &= ((1\otimes 1) * p(a)) * p(b) - (1\otimes 1) * (p(a)\bullet p(b)) \\ &= p(a)\otimes p(b) + p(b)\otimes p(a) = R(p(a),p(b)). \end{aligned}$$

Observe finally that $\overline{\Delta}p(v) = p^1dv = 0$ for $v \in V$. The commutativity $\overline{\Delta}p = p^1d$ of the bottom square of (29) then follows from the following lemma.

**Lemma 6.1.** *Let $S : \mathsf{pL}(V)\otimes\mathsf{pL}(V) \to \overline{\mathsf{T}}(V)\otimes\overline{\mathsf{T}}(V)$ be a symmetric linear map such that the expression*

$$S(a,b) * p(c) + S(a\star b,c) - S(a,b\star c),\ \ a,b,c \in \mathsf{pL}(V), \tag{30}$$

*is symmetric in $b$ and $c$. Then there exists precisely one linear map $F : \mathsf{pL}(V) \to \overline{\mathsf{T}}(V)\otimes\overline{\mathsf{T}}(V)$ such that*

*(i)* $F(a\star b) = F(a) * p(b) + p(a) * F(b) + S(a,b)$ *for each $a,b \in \mathsf{pL}(V)$, and*

*(ii)* $F(v) = 0$ *for each $v \in V$.*

**Proof.** The map $F$ is constructed by the induction on the monomial length of elements of $\mathsf{pL}(V)$, its uniqueness is obvious. The symmetry of the form in (30) in $b$ and $c$ is necessary for the compatibility of the rule (i) with the axiom (10). ■

We claim that Proposition 3.3 follows from the following

**Lemma 6.2.** *In diagram (29),*

*(i)* $di = 0$,

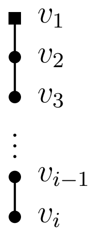
*(ii)* $Ker(d)\cap Ker(p) = 0$ *and*

*(iii)* $p(Ker(d)) \subset \mathsf{L}(V)$.

Indeed, (i) implies that $\mathsf{L}(V) \subset Ker(d)$ while (ii) and (iii) together imply that $p$ maps $Ker(d)$ monomorphically to $\mathsf{L}(V)$. Since all these spaces are graded of finite type and their maps preserve the gradings, one concludes that $\mathsf{L}(V) = Ker(d)$.

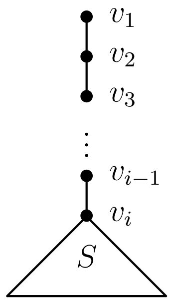
**Proof of Lemma 6.2.** The symmetry of $Q$ in (16) implies that $d$ is a derivation of the Lie algebra $\mathsf{pL}(V)_L$ associated to $\mathsf{pL}(V)$. This fact, together with $d(V) = 0$, readily implies that $d$ annihilates Lie elements in $\mathsf{pL}(V)$, which is (i).

Our proof of (ii) relies on the tree language introduced in Section 4. We will use the following terminology. A decorated tree $T \in \mathrm{Tr}(V)$ is *linear* if all its vertices are of arity $\leq 1$. Such a tree $T$ is of the form




(31)

with some $v_1, \ldots, v_i \in V$, $i \geq 1$. Each non-linear tree $T \in \mathrm{Tr}(V)$ necessarily looks as




(32)

where $S$ is a tree whose root vertex $v_i$ has arity $\geq 2$. We say that such a decorated tree has *tail of length* $i$. These notions translate to decorated trees from $\mathrm{Tr}^1(V)$ in the obvious manner.

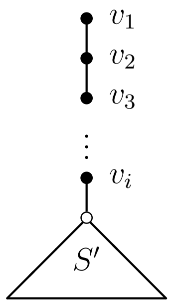
We leave to the reader to verify that, under identification (24), the map $p : \mathsf{pL}(V) \to \overline{\mathsf{T}}(V)$ is described as

$$p(T) = \begin{cases} v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes v_i, & \text{if } T \text{ is linear as in (31), and} \\ 0, & \text{if } T \text{ is non-linear.} \end{cases}$$

Therefore $Ker(p)$ consists of linear combinations of non-linear trees. Before going further, we need to inspect how the map $\delta : \mathrm{Tr}(V) \to \mathrm{Tr}^1(V)$ of (26), which is the differential $d : \mathsf{pL}(V) \to \mathsf{pL}^1(V)$ written in terms of trees, acts on non-linear trees. If $T$ is the decorated tree (32), then it immediately follows from the definition (26) of $\delta$ that

$$\delta(T) = -T' + \text{trees with tails of length} \leq i, \tag{33}$$

where $T'$ is the following decorated tree with tail of length $i+1$



in which $S'$ is the tree obtained from $S$ by replacing the root vertex decorated by $v_i$ by the special one. The map $\delta' : \mathrm{Tr}(V) \to \mathrm{Tr}^1(V)$ given by $\delta'(T) := -T'$ is a *monomorphism.*

Let $x$ be a linear combination of non-linear trees and assume $\delta(x) = 0$. We must prove that then $x = 0$. Assume $x \neq 0$ and decompose $x = x_s + x_{s-1} + \cdots + x_1$, where $x_i$ is, for $1 \leq i \leq s$, a linear combination of decorated trees with tails of length $i$, and $x_s \neq 0$. By (33), the only trees with tails of length $s+1$ in $\delta(x)$ are those spanning $\delta'(x_s)$, therefore $\delta(x) = 0$ implies $\delta'(x_s) = 0$ which in turn implies that $x_s = 0$, because $\delta'$ is monic. This is a contradiction, therefore $x = 0$ which proves (ii).

To verify (iii), notice that, by the commutativity of the bottom square of (32), $p(Ker(\delta)) \subset Ker(\overline{\Delta}))$ while $Ker(\overline{\Delta}) = \mathsf{L}(V)$ by Theorem 2.1. This finishes the proof of the lemma. ■

## 7. Some open questions and ramifications

**7.1.** *Triplettes of operads (after J.-L. Loday).* The following notion was introduced in [10].

**Definition 7.2.** The data $(\mathcal{C}, \between, \mathcal{A}\texttt{-alg} \xrightarrow{F} \mathcal{P}\texttt{-alg})$, where

(i) $\mathcal{C}$ and $\mathcal{A}$ are operads,

(ii) $\between$ are 'spin' relations intertwining $\mathcal{C}$-co-operations and $\mathcal{A}$-operations, so that $(\mathcal{C}, \between, \mathcal{A})$ determines a class of bialgebras,

(iii) the operad $\mathcal{P}$ governs the algebra structure of the primitive part $Prim(\mathcal{H})$ of $(\mathcal{C}, \between, \mathcal{A})$-bialgebras, and

(iv) $F$ is a forgetful functor functor from the category of $\mathcal{A}$-algebras to the category of $\mathcal{P}$-algebras such that the inclusion $Prim(\mathcal{H}) \subset F(\mathcal{H})$ is a morphism of $\mathcal{P}$-algebras,

is called a *triplette* of operads.

An example is $(\mathcal{C}om, \between, \mathcal{A}ss, \mathcal{L}ie)$, with $\between$ the usual bialgebra relation recalled in (5). Let $U$ be a left adjoint to $F$. A triplette in Definition 7.2 is *good* [10], if the following three conditions are equivalent:

(i) a $(\mathcal{C}, \mathfrak{I}, \mathcal{A})$-bialgebra $\mathcal{H}$ is connected,

(ii) $\mathcal{H} \cong U(Prim(\mathcal{H}))$, and

(iii) $\mathcal{H}$ is cofree among connected $\mathcal{C}$-coalgebras.

Let $\mathcal{A}(V)$ (resp. $\mathcal{P}(V)$) denote the free $\mathcal{A}$- (resp. $\mathcal{P}$-)algebra on $V$. As observed in [10], for good triplettes

$$Prim(\mathcal{A}(V)) \cong \mathcal{P}(V). \tag{34}$$

The classical Theorem 2.1 in Section 2 is a consequence of the goodness of the triplette $(\mathcal{C}om, \mathfrak{I}, \mathcal{A}ss, \mathcal{L}ie)$ mentioned above, because (34) in this case says that $Prim(T(V)) \cong \mathsf{L}(V)$. Other, in some cases very surprising, good triplettes can be found in [10]. The following problem was suggested by J.-L. Loday:

**Problem 7.3.** Are there an operad $\mathcal{C}$ and spin relations $\mathfrak{I}$ with the property that $(\mathcal{C}, \mathfrak{I}, p\mathcal{L}ie, \mathcal{L}ie)$ is a good triplette?

As we remarked in Subsection 1.4, the affirmative answer to the Deligne conjecture given in [7] implies that there exist a characterization of Lie elements in brace algebras [5] similar to our Theorem 3.3. This suggests formulating the following version of Problem 7.3 in which $\mathcal{B}race$ is the operad for brace algebras.

**Problem 7.4.** Are there an operad $\mathcal{C}$ and spin relations $\mathfrak{I}$ with the property that $(\mathcal{C}, \mathfrak{I}, \mathcal{B}race, \mathcal{L}ie)$ is a good triplette?

**7.5.** *Lie elements and cobar constructions.* In Section 2 we calculated the cohomology of the cobar construction (12) of the shuffle coalgebra and observed that $H^0(\mathsf{T}(V), \Delta)$ is isomorphic to the free Lie algebra $\mathsf{L}(V)$. In our characterization of Lie elements in pre-Lie algebras, the role of (12) is played by complex (22). This leads to the following problem, which may or may not be related to Problem 7.3,

**Problem 7.6.** Calculate the cohomology of (22). Is this complex the cobar construction of some coalgebra?

As D. Tamarkin recently informed us, methods proposed in an enlarged unfinished, unpublished version of [17] may imply that the complex (22) is acyclic in positive dimensions, as envisaged also by some conjectures formulated in [12].

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