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Lattices in Symplectic Lie Groups

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Abstract. A Lie group G equipped with a left invariant symplectic form ω^+ is called a symplectic Lie group and the pair $(\mathfrak{g}, \omega)$, where $\mathfrak{g}$ is its Lie algebra, the tangent space to G at the unit ε, is said a symplectic Lie algebra. Among others things, we determine connected and simply connected symplectic Lie groups of dimension four which have discrete cocompact subgroups, that is, uniform lattices. We describe in the solvable non nilpotent case, all isomorphy classes of lattices Γ and in this fashion obtain an infinity of nonhomeomorphic compact symplectic solvmanifolds. Finally we show that these four dimensional symplectic Lie groups have left invariant symplectic affine structures, that is, left invariant flat and torsion free symplectic connexions.
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1. Introduction and main results

The procedure of symplectic Lie double extension of a symplectic Lie algebra by a 1-dimensional Lie algebra, introduced and studied in [8] and [1] enables to construct inductively all nilpotent symplectic Lie algebras starting from the zero algebra. So, it is an essential tool to get symplectic compact nilmanifolds, together with classical results from Malcev about lattices in nilpotent Lie groups. For example, up to finite coverings we can obtain the symplectic nilmanifolds of dimension ≤ 6 using the list of symplectic nilpotent Lie algebras given in [4] and Morozov's classification of nilpotent $\mathbb{Q}$-Lie algebras for these dimensions.

Recall that a Lie group admitting a lattice is unimodular, and according to a result of Lichnerowicz-Medina [6] an unimodular symplectic Lie group is solvable; so we shall consider only solvable groups. We will prove among other results:

Theorem 1.1. *The Lie algebra of a four dimensional symplectic Lie group having lattices is a symplectic double extension of the abelian two-dimensional symplectic Lie algebra by a* 1*-dimensional Lie algebra. Up to isomorphism, there are five such algebras.*

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Among the algebras in Theorem 1.1, two are non nilpotent. The most interesting one for lattices corresponds to the direct product $G = \mathbb{R} \times G_1$ where G_1 denotes $\mathbb{R}^3$ with the product given by

$$(x, y, t) \cdot (x', y', t') = \left(x + e^t x', y + e^{-t} y', t + t'\right).$$

Then we have

Theorem 1.2. *The isomorphism classes of lattices in G_1 correspond bijectively with the classes of pairs (A, I) where A is a ring of integers in a real quadratic number field and I an ideal in A (i.e a fractionnal ideal) determined up to isomorphism. Each lattice of G_1 is contained in a finite number of non isomorphic lattices of G.*

To complete and make this result more explicit , we have

Theorem 1.3. *a) The rings involved in* Theorem 1.2 *are*
$A_n = \mathbb{Z}[X] / (X^2 - nX + 1)$ *where n is a positive integer ≥ 3, whose quotient field is $\mathbb{Q}\left(\sqrt{d_n}\right)$ and d_n is the squarefree part of $n^2 - 4$.*
b) Lattices of G_1 corresponding to pairs (A, I) and (B, J) are strictly commensurable in G_1 if and only if A and B have the same fraction field.

Recall that Γ_1 and Γ_2 are commensurable if there exists φ in $\mathrm{Aut}G_1$ such that $\Gamma_1 \cap \varphi(\Gamma_2)$ is of finite index in both of them. If φ is the identity, they are said strictly commensurable.

Remark 1.4. If (G, ω^+) is a symplectic Lie group, the formula

$$\omega\left(\nabla_{a^+} b^+, c^+\right) = -\omega\left(b^+, \left[a^+, c^+\right]\right) \tag{1}$$

where x^+ denotes the left invariant vector field on G whose value at ε is $x \in \mathfrak{g}$ defines a left invariant affine structure on G, i.e a left invariant flat and torsion free connection. This affine structure, called associated to the symplectic structure, plays an important role in what follows.

2. Symplectic double extension

Recall the construction of symplectic double extension (see [8] and [1] for more details and generalization). Denote by $\mathbb{K}$ the field of real or complex numbers and consider a symplectic $\mathbb{K}$-Lie algebra $\left(B, \omega'\right)$ and δ a derivation of B. The formula $\left(\delta \cdot \omega'\right)(a, b) := \omega'(\delta a, b) + \omega'(a, \delta b)$ for a and $b \in B$ defines a scalar 2-cocycle over B. If we put $\delta_{(2)} \cdot \omega' := \delta \cdot \left(\delta \cdot \omega'\right)$, we have

Theorem 2.1. *Let $\left(B, \omega'\right)$ be a symplectic Lie algebra and δ a derivation such that $\delta_{(2)} \cdot \omega'$ is a coboundary. Let z in B such that $\left(\delta_{(2)} \cdot \omega'\right)(a, b) = \omega'(z, [a, b])$ for every a and b in B.*

The Lie algebra semidirect product of the central extension $\mathbb{K}e \times B$ *of* B, *by means of* $\delta \cdot \omega'$, *and* $\mathbb{K}d = \mathbb{K}$ *with* $[d, e] = 0$ *and* $[d, a] = -\omega'(z, a)\, e - \delta a$ *for* $a \in B$, *is a symplectic Lie algebra for* ω, *the orthogonal sum of* ω' *over* B *and* $\omega(e, d) = 1$ *on* $Span(e, d)$.

Definition 2.2. The symplectic Lie algebra, denoted by $B(\delta, z)$, obtained in Theorem 2.1 is called the symplectic double extension of $\left(B, \omega'\right)$ by means of (δ, z).

Note that a derivation δ of B gives rise to a double extension only if $\delta_{(2)} \cdot \omega'$ is a coboundary. The cohomology class of $\delta \cdot \omega'$ depends only on δ's class modulo inner derivations. The choice of z is determined by the coboundary $\delta_{(2)} \cdot \omega'$ modulo the orthogonal of $[B, B]$ for ω'.
We give two particular cases we will use later:

Corollary 2.3. *Suppose that the symplectic Lie algebra* $\left(B, \omega'\right)$ *is unimodular. Then the symplectic double extension* $B(\delta, z)$, *is unimodular if and only if* δ *has trace zero.*

Proof. In the previous theorem, $\mathrm{Tr}\, d = -Tr\delta$ and the corollary is clear. ■

Note that if $\delta \cdot \omega' = 0$, i.e δ is an infinitesimal symplectomorphism, $\mathrm{Tr}\,\delta = 0$ and the corollary holds.
We also note here that the symplectic reduction process, inverse of the symplectic double extension, reduces an unimodular Lie algebra into an unimodular one; nevertheless there are symplectic unimodular Lie algebras which can't be obtained by double extension, for example the following one. Let $W = (\mathbb{K}^4, \omega_1)$ a symplectic abelian Lie algebra and $d = Diag(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)$ with distinct and nonzero λ_i a diagonal endomorphism of $\mathbb{K}^4$ in the canonical basis, symplectic for ω_1. Let $V = (Span(d, d^3), \omega_2)$ the symplectic abelian Lie algebra. The semidirect product $V \rtimes W = A$, with the natural action of W on V is a symplectic unimodular Lie algebra with $\omega = \omega_1 \perp \omega_2$. Since A has a trivial center, it is not a double extension as defined above.

Corollary 2.4. *Let* $\left(B, \omega'\right)$ *an abelian symplectic Lie algebra and* δ *an invertible infinitesimal symplectomorphism of* $\left(B, \omega'\right)$. *The symplectic double extension* $A = B(\delta, z)$ *is isomorphic to the double extension* $B(\delta, 0)$, $0 \in B$.

Proof. As $\delta \cdot \omega' = 0$, the central extension $\mathbb{K}e \times B = I^{\perp}$ by means of $\delta \cdot \omega'$ is abelian. For $b \in B$, we have $d(b) = [d, b] = -\omega'(z, b)\, e - \delta(b)$; let us change B into $B' := \left\{b + \omega'(x, b)\, e; b \in B\right\}$ where $x \in B$ will be chosen later and B' is any supplementary space to $\mathbb{K}e$ in $I^{\perp}$, with ω restricted to B' non degenerate. To show that $B(\delta, z)$ is isomorphic to $B(\delta, 0)$ it is enough to find B' such that $d(B') = B'$. As $d\left(b + \omega'(x, b)\, e\right) = d(b) = -\delta(b) - \omega'(z, b)\, e$, this must be $\delta(b) - \omega'(x, -\delta(b))\, e$; so for every b in B, we must have $\omega'(x, \delta(b)) = \omega'(z, b)$ which is solved by $x = (\delta^*)^{-1}(z)$ where δ^* is the adjoint of δ with respect to ω', and the result is proved. ■

Note that the left symmetric product on $B(\delta, z)$ given by (1) satisfies dd=z. With the assumption of Corollary 2.4, we modify this product so that $d^2 = 0$ without changing the bracket on A, which is unimodular.

Proof. [**Proof of Theorem 1.1**] We first describe symplectic 4-dimensional Lie algebras obtained by double extension.

Claim. Any symplectic double extension of the non abelian two dimensional $\left(B,\omega^{'}\right)$ Lie algebra is not unimodular.
As B is non abelian, $\omega^{'}$ is a coboundary; so $\delta\cdot\omega^{'}$ and $\delta_{(2)}\cdot\omega^{'}$ too. The first exact sequence

$$0 \longrightarrow \mathbb{K}e = I \longrightarrow I^{\perp} \longrightarrow B \longrightarrow 0$$

is split and we may suppose $\delta\cdot\omega^{'}=0$. As $\dim B$ is two, $\delta\cdot\omega^{'} = \mathrm{Tr}\,\delta\cdot\omega^{'}$ so $\mathrm{Tr}\,\delta=0$. Because the derivation δ is triangular in any base $\{e_1,e_2\}$ where $\mathbb{K}e_2=[B,B]$; we have $\delta e_1=\lambda e_2$ and $\delta e_2=0$ with $\lambda\in\mathbb{K}$ and $z\in B$ which satisfy $\omega^{'}(z,e_2)=0$ so $z=\mu e_2$ with $\mu\in\mathbb{K}$. This shows that derivation d on $I^{\perp}$ is given by $de=0$, $de_1=\lambda e_2$ and $de_2=0$. Then it is clear that $A=B(\delta,z)$ is not unimodular since $\mathrm{Tr}\,(ad_A(e_1))=1$. At the opposite we can:

Claim. A symplectic double extension of a two-dimensional abelian Lie algebra is unimodular. Moreover the associated simply connected symplectic Lie group admits uniform lattices.

Since $\delta_{(2)}\cdot\omega^{'}=(Tr\delta)^2\,\omega^{'}$, it is a coboundary if and only if $\mathrm{Tr}\,\delta=0$, so $\delta\cdot\omega^{'}=0$ and $I^{\perp}$ is an abelian 3-dimensional ideal of $B(\delta,z)$. As $\mathrm{Tr}\,\delta=0$, δ^2 is an homothety and so we distinguish three cases : $\delta^2=0$, δ^2 is positive or δ^2 is negative. When $\delta^2=0$, if δ is 0, the double extension is abelian; if $\delta\neq 0$, $B(\delta,z)$ is two or three step nilpotent according to z belonging to $\mathrm{Ker}\,\delta$ or not. Then we get $F_2\times\mathbb{K}$ (direct product) or F_3; we here denote F_n the filiform Lie algebra $\mathrm{Vect}\,(d,e_1,\cdots,e_n)$ where the non-zero brackets are $[d,e_i]=e_{i+1}$ for $i<n$. The associated Lie groups have lattices because the structural constants are integers. These lattices are semidirect product of $\mathbb{Z}^3$ by an action φ of $\mathbb{Z}$ given by $\varphi(1)=Id_{\mathbb{Z}^3}+T$ where $T:\mathbb{Z}^3\longrightarrow\mathbb{Z}^3$ is nilpotent of order two or three. In the first case, $\varphi(1)$ is $\begin{pmatrix}1&k&0\\0&1&0\\0&0&1\end{pmatrix}$ in a suitable basis of $\mathbb{Z}^3$, where $|k|\in\mathbb{N}\setminus\{0\}$ determines the isomorphism class of the lattice of the Lie group $\mathbb{R}\times\mathbb{H}_3$ ($\mathbb{H}_3$ is the 3-dimensional Heisenberg group). If T is nilpotent of order 3, it has rank 3 and two numerical invariants caracterize its isomorphism class: the cardinality of the torsion subgroup of $\mathbb{Z}^3/ImT^2$ and the cardinality of the torsion subgroup of ImT/ImT^2 which divides the first one.
The set of nondegenerate scalar 2-cocycles on these nilpotent Lie algebras is an open set in the space of scalar 2-cocycles which is of dimension 5 or 4 according the different cases ([8]). Now we deal with the case of non zero δ^2: we may take $\delta^2=-Id_B$ or $\delta^2=+Id_B$ according to the sign of the homothety δ^2. Here $I^{\perp}$ is also a 3-dimensional abelian ideal of A and we study now d: its characteristic (and minimal) polynomial is X^3+X or X^3-X so that d is semi-simple and the isomorphism class of $B(d,z)$ does not depend on z. Following the notations of [8], we have the following symplectic Lie algebras: $A_{3,4}(0)\times Ab_1$ and $A_{3,2}(-1)\times Ab_1$. In the first case, the associated simply connected Lie group is the direct product of $\mathbb{R}$ by the universal covering of the group of positive isometries of the euclidean plane. This group is isomorphic to $\mathbb{C}\times\mathbb{R}$ with the product $(z,t)\cdot\left(z^{'},t^{'}\right)=\left(z+e^{2i\pi t}z^{'},t+t^{'}\right)$ and it has , for instance an abelian lattice

isomorphic to $\mathbb{Z}^3$ with $t \in \mathbb{Z}$ and z a gaussian integer (this is only one example and there are others which are non abelian). In the second case, the associated group is the direct product $\mathbb{R} \times G_1$ where G_1 is $\mathbb{R}^3$ with the already mentioned product $(x, y, t) \cdot \left(x', y', t'\right) = \left(x + e^t x', y + e^{-t} y', t + t'\right)$. It has lattices because G_1 has some as showed in [10].

Finally an examination of the exhaustive list of symplectic 4-dimensional Lie algebras given in [8] shows that these five algebras, which have lattices, are the only unimodular algebras. This proves that the double extension process gives all algebras we are looking for, but this is very particular and due to dimension 4. ■

Remark 2.5. For even dimension ≥ 6, there are unimodular symplectic Lie groups which do not have lattices, because of arithmetical reasons. Here is an example. Let $A = B\,(\delta, 0)$ be the symplectic double extension of the abelian Lie algebra $B = \left(\mathbb{R}^4, \omega'\right)$ where δ is an infinitesimal symplectomorphism of B given in a symplectic basis by the diagonal matrix $\mathrm{Diag}\,(1, -1, \lambda, -\lambda)$ with λ a non rational algebraic real number. The algebra A is symplectic and unimodular of dimension six. If the simply connected Lie group associated to A had a lattice, there would be in the one parameter subgroup $t \longmapsto \exp(td)$ a non identity matrix conjugated to a matrix in $Gl\,(4, \mathbb{Z})$. Consequently there would be a non zero $t \in \mathbb{R}$ with $a = e^t$ and $b = e^{\lambda t} = a^\lambda$ would be both algebraic. But this is not true because of a result of O. Gelfond related to Hilbert's seventh problem ([3]).

Proof. [**Proof of Theorem 1.2 and 1.3**] To determine the lattices of $\mathbb{R} \times G_1$, we begin by G_1. For this, let us recall some facts of the theory of lattices in solvable Lie groups ([9]). Let Γ be a lattice in the solvable Lie group K; if N is the nilradical of K, $\Gamma \cap N$ is a lattice of the nilpotent Lie group N. Furthermore, the image of Γ by the natural projection of K over K/N is a lattice in K/N. In the case of G_1, as we may identify $\Gamma_1 \cap N$, where Γ_1 is a lattice in G_1, to $\mathbb{Z}^2$ as an abstract group, Γ_1 is described by the exact sequence of groups $1 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_1 \longrightarrow \mathbb{Z} \longrightarrow 1$. It follows that Γ_1 is a semidirect product of $\mathbb{Z}^2$ by an action φ of $\mathbb{Z}$ on $\mathbb{Z}^2$ such that $\varphi(1) = M$ belongs to $Gl\,(2, \mathbb{Z}) = \mathrm{Aut}\mathbb{Z}^2$ and is in a one-parameter subgroup $t \longmapsto M(t)$ of $Gl\,(2, \mathbb{R})$ conjugate in it to the subgroup $t \longmapsto Diag\,(e^t, e^{-t})$. For this, it enough, and necessary too, that the integral matrix M has its real eigenvalues, distinct, positive and inverse to each other. This shows that the charateristic polynomial of M must be $X^2 - nX + 1$ where n is an integer greater than 2. In an suitable basis of $\Gamma_1 \cap N$, M acts as $A = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}$, companion matrix of the corresponding polynomial.

It follows that Γ_1 determines the subring $A_n = \mathbb{Z}\,[X]\,/\,(X^2 - nX + 1)$ of the real quadratic number field $\mathbb{Q}\left(\sqrt{n^2 - 4}\right) = \mathbb{Q}\,[X]\,/\,(X^2 - nX + 1)$ generated by $\varepsilon_n = n + \sqrt{n^2 - 4}/2$. We may consider $\Gamma_1 \cap N$ as a finitely generated A_n-module of finite type of rank 1, i.e a fractional ideal of A_n.

Conversely, suppose we are given a pair (A_n, I) where I is a fractional ideal of A_n. If I is principal, $\mathbb{Z}^3$ is a group denoted by Γ_n for the product $((u, v), w)\,((u', v'), w') = ((u, v) + (u', v')A^w, w + w')$ where the matrix A acts on the right on the row (u', v') of $\mathbb{Z}^2$. If I is not principal, there is a sublattice J of $\mathbb{Z}^2$ stable by A and isomorphic to I as A_n module. Then, in the same way that for the principal case,

we make $J \times \mathbb{Z}$ a group and in fact it is a finite index subgroup of Γ_n.
We now realize Γ_n as a discrete cocompact subgroup of G_1: we consider in $G_1 = \mathbb{R}^3$ the subset

$$\Gamma_n^{'} = \left\{(u - \varepsilon_n v, \varepsilon_n u - v, w \ln \varepsilon_n) / (u, v, w) \in \mathbb{Z}^3\right\}.$$

It is easy to convince oneself that $\Gamma_n^{'}$ is a subgroup of G_1, isomorphic to Γ_n; using $\varepsilon_n^2 = n\varepsilon_n - 1$ and $\varepsilon_n^{-1} = n - \varepsilon_n$, one computes the product in G_1

$$(0, 0, \ln \varepsilon_n)(u' - \varepsilon_n v', \varepsilon_n u' - v', 0) = (\alpha, \beta, 1)$$

where one finds $(\alpha, \beta) = (u', v') A$.
When I is not a principal ideal, it is enough to have the pair (u, v) belonging to a certain sublattice of $\mathbb{Z}^2$ on which the action of A on the right gives the suitable A_n-module structure.
We now want to compare two realizations of Γ_n as lattices in G_1, i.e two injective homomorphisms φ_1 and φ_2 of Γ_n in G_1. More precisely, we want to show that there is an automorphism φ of G_1 sending $\varphi_1(\Gamma_n)$ on $\varphi_2(\Gamma_n)$, a result known to be always true for lattices in nilpotent Lie groups, but not necessarily in the solvable case. Since G_1 is a completely solvable Lie group, the existence of the isomorphim φ is guaranted by the Saito's Theorem as quoted in [[10], page 65].

Recall that two lattices in a Lie group are said strictly commensurable if their intersection is of finite index in both of them; they are called commensurable if the first is strictly commensurable with an automorphic image of the other. We thus have showed two lattices in G_1, isomorphic as abstract groups, are commensurables. It is also the case if the lattices Γ_1 and Γ_2 correspond to the same quadratic numberfield because they will have subgroups of finite index which are isomorphic.
Conversely let Γ and Γ' two lattices of G_1 which are strictly commensurable. Their images in G_1/N by the canonical projection are strictly commensurable so $\ln \varepsilon_n$ and $\ln \varepsilon_m$ are $\mathbb{Q}$-proportionnal where ε_n (resp. ε_m) denotes the unit associated to Γ (resp. to Γ'). This means there are h and k in $\mathbb{N}^*$ such that $\varepsilon_n^h = \varepsilon_m^k$ and $\mathbb{Z}\left[\varepsilon_n^h\right] = \mathbb{Z}\left[\varepsilon_m^k\right]$ is a common subring of A_n and A_m, of rank 2, which have therefore the same fraction field.
We now have to describe lattices of $G = \mathbb{R} \times G_1$ (direct product). It is clear that the direct product of a lattice of G_1 and a lattice of $\mathbb{R}$ is a lattice in G. Such lattices are isomorphic if and only if their G_1-components are also isomorphic. We proceed now to a general study to get all the lattices of G. Such a group $\bigwedge$ is given by an exact sequence $1 \longrightarrow \mathbb{Z}^3 \longrightarrow \bigwedge \longrightarrow \mathbb{Z} \longrightarrow 1$ where $\mathbb{Z}^3$ is a lattice of the nilradical of G, abelian as for G_1 : the action of $\mathbb{Z}$ on $\mathbb{Z}^3$ is given by a matrix $A = \varphi(1)$ in $Gl(3, \mathbb{Z})$ which is conjugate in $Gl(3, \mathbb{R})$ with the diagonal matrix $\mathrm{Diag}(1, \lambda, \lambda^{-1})$. So the characteristic polynomial of A is $(X - 1)(X^2 - nX + 1)$ and λ is ε_n, the biggest root of $x^2 - nx + 1 = 0$, $n \geq 3$. In $\mathbb{Z}^3$, $\mathrm{Ker}(A - I_3)$ is a free direct factor of rank one. Changing the basis in $\mathbb{Z}^3$ we may take $A = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \multicolumn{2}{c}{B} \\ 0 & & \end{pmatrix}$ where $B \in Gl(2, \mathbb{Z})$ has $X^2 - nX + 1$ as characteristic polynomial.
Therefore $\mathbb{Z}^3 = \mathbb{Z} \oplus I$ where I, via B, is a fractional ideal of A_n. Note that in a suitable $\mathbb{Z}$-basis of I the multiplication by ε_n is given by the matrix B. If $\alpha = \beta = 0$, the lattice $\bigwedge$ associated to A is a direct product and our next task is

to look for the other isomorphism classes of lattices which are not direct product: to A is associated matrix B and I, i.e a couple (A_n, I), lattice in G_1. We now wonder when matrices A and A' do give isomorphic lattices in G: A and A' must be conjugate in $Gl\,(3,\mathbb{Z})$ and so B and B' are conjugate in $Gl\,(2,\mathbb{Z})$ and the ideal I and I' are isomorphic (that means the intersection of the two lattices with G_1 are isomorphic). Changing one if necessary, we may suppose $B = B'$. We can suppose $A' = P^{-1}AP$ where $P = \begin{pmatrix} e & l_1 & l_2 \\ 0 & & \\ 0 & U & \end{pmatrix}$ where U belongs to $Gl\,(2,\mathbb{Z})$ and $e^2 = 1$; then $P^{-1} = \begin{pmatrix} \varepsilon & l_1' & l_2' \\ 0 & & \\ 0 & U^{-1} & \end{pmatrix}$ so B and U must commute. Consider $C = \{\alpha \in \operatorname{Frac}(A_n) /\ \ \alpha I \subset I\}$; it is a subring of the ring of all integers in $\operatorname{Frac}(A_n) = \mathbb{Q}\left(\sqrt{n^2-4}\right)$ containing A_n. As $UB = BU$ and U is in $Gl\,(2,\mathbb{Z})$, U is an invertible element of C. Following Dirichlet unit theorem, the group of units of C is made of the elements $\pm\varepsilon^h$, $h \in \mathbb{Z}$, where ε is the fundamental unit of C. This implies that there are integers p and q such that $U = \pm\varepsilon^p$ et $B = \pm\varepsilon^q$. Finally, equation $A' = P^{-1}AP$ is now equivalent to:

$$\left(l_1', l_2'\right) = e\,(l_1, l_2)\,\varepsilon^p + (u, v)\,(I_2 - B)$$

where matrices act on the right. We may consider (l_1, l_2) and $\left(l_1', l_2'\right)$ as elements of the group $I^* = \operatorname{Hom}(I, \mathbb{Z})$ on which A_n and C operate on the right. The subgroup $\operatorname{Im}(I_2 - B)\,(I^*)$ is of rank two and of index $n - 2 = |det\,(I_2 - B)|$. Consider the group $H = I^*/\operatorname{Im}(I_2 - B)\,I^* = \operatorname{Coker}(I_2 - B)$: it is a finite abelian group of order $n-2$. Two rows (l_1, l_2) and $\left(l_1', l_2'\right)$ give isomorphic lattices in G if and only if their images in H correspond under the action of the automorphism group induced by

$$(\alpha, \beta) \longmapsto \pm\,(\alpha, \beta)\,\varepsilon^p.$$

The number of non isomorphic lattices obtained from a matrix B is equal to the number of orbits in H under the action of this automorphism group. This finishes the proof of theorem1.2 and 1.3. ∎

Corollary 2.6. *The symplectic Lie group $G = \mathbb{R} \times G_1$ has an infinity of non isomorphic lattices and there is an infinite number of non homeomorphic symplectic solvmanifolds $M = \Gamma\backslash G$ where Γ is a lattice in G.*

Proof. As G_1 has non isomorphic lattices Γ_n for every integer n greater than 2, it is the same for G, taking $\Gamma = \mathbb{Z}z \times \Gamma_n$, where z is a non zero real number, a direct product lattice. ∎

Remark 2.7. 1) If $n = 3$, the group H is trivial and there is just one lattice associated to B. If $n > 3$, the direct product lattice corresponds to the orbit in H formed by a single element, the zero element of H. For small values of n, a direct analysis can be done to determine the number of non isomorphic lattices.
2) Let $r \geq 1$ be a positive integer and $n = 2^r + 2$, $B = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$. Then H

is isomorphic to the cyclic group of order 2^r and there are r non isomorphic indecomposable lattices associated to B, corresponding to the elements of H of order $2^r, 2^{r-1}, \cdots, 2$.
3) Note that the number of matrices B in $Gl\,(2,\mathbb{Z})$ associated to an integer $n \geq 3$ is the (finite) class number of the quadratic ring $A_n = \mathbb{Z}\,[X]\,/(x^2 - nX + 1)$ which is an easily computed multiple of the class number of the quadratic field $\mathbb{Q}\left(\sqrt{n^2-4}\right)$.

3. Geometry of some symplectic solvmanifolds

In the following, we consider manifolds $M = \Gamma\backslash\,(G,\omega^+)$ where (G,ω^+) is a symplectic Lie group, Γ a lattice in G and ∇ is the affine structure associated to ω^+. We denote by $\pi : G \longrightarrow M$ the canonical projection. It is obvious that ω^+ induces a symplectic form $\overline{\omega}$ on M for which π is a symplectic covering. We have:

Theorem 3.1. *Let (G,ω^+) a symplectic Lie group. If G is unimodular, the left invariant affine structure on G given by the left symmetric product on $\mathfrak{g}$ defined by $\omega\,(ab,c) = -\omega\,(b,[a,c])$ is complete. Therefore G is solvable, so every manifold $M = \Gamma\backslash\,(G,\omega^+)$, as above, is a compact solvmanifold.*

Proof. We may suppose G is connected and simply connected. As ω is a scalar 2-cocycle on $\mathfrak{g}$, the mapping $x \longmapsto (i(x)\omega, (ad^*x)^*)$ is a Lie algebra representation from $\mathfrak{g}$ to $\mathrm{aff}(\mathfrak{g}^*) = \mathfrak{g}^* \rtimes gl(\mathfrak{g}^*)$. It gives a Lie group representation $\rho : G \longrightarrow \mathrm{Aff}(\mathfrak{g}^*)$: orbits of ρ are leaves of an affine Poisson structure on the manifolds $\mathfrak{g}^*$. As ω is non degenerate, there are open orbits, for instance, the orbit O of the origin of the vector space $\mathfrak{g}^*$. Let $p : G \longrightarrow O$ the associate orbital mapping. It is a local diffeomorphism and the inverse image by p of the connection on O, induced by the usual connection of $\mathfrak{g}^*$, is a left invariant connection ∇ on G which is affine, i.e with zero torsion and curvature. Thus this connection is given by the left invariant symmetric product on $\mathfrak{g}$,

$$ab = L_a b = R_b a$$

where $L_a = q^{-1} \circ ad^* a \circ q$ and $q(x) = i(x)\omega$. Connection ∇ is complete if and only if the Poisson structure has only one leaf i.e $O = \mathfrak{g}^*$. But a result of Lichnerowicz-Medina ([6]) tells that an affine Poisson structure whose linear part is a Lie algebra $\mathfrak{g}$ has only one symplectic leaf if and only if $\mathfrak{g}$ is unimodular and symplectic. As here $\mathfrak{g}$ is unimodular, ∇ is complete and p, the developping map of ∇, is a diffeomorphism. Hence G is a solvable Lie group. ■

Corollary 3.2. *Let $M = \Gamma\backslash\,(G,\omega^+)$ be as above, where G is a connected Lie group ; then ∇ induces an affine connection $\overline{\nabla}$ on M such that $\pi : G \longrightarrow M$ is an affine (symplectic) covering.*

Remark 3.3. Note that connection ∇, (and $\overline{\nabla}$), is symplectic if and only if G is abelian. This results from the formula (**1**).

Theorem 3.4. *Let be $M = \Gamma \backslash (G, \omega^+)$ as above, where G is connected. If the natural action ψ of G on M is symplectic, then G is abelian and M is a torus.*

Proof. The action ψ is given by $\psi([\tau], \sigma) = [\tau\sigma] = \psi_\sigma[\tau]$ for σ and τ in G. Let $\overline{x}$ the fondamental field for ψ associated to $x \in \mathfrak{g} = T_\varepsilon G$ where ε is the unit element of G. Let us suppose ψ symplectic , i.e $\psi_\sigma^* \overline{\omega} = \overline{\omega}$ for every σ in G. Therefore $\mathcal{L}_{\overline{x}} \overline{\omega} = 0$ where $\mathcal{L}_{\overline{x}}$ is the Lie derivative in the direction $\overline{x}$. It means that the 1-form $i(x)\overline{\omega}$ is closed, therefore $\overline{x}$ is locally hamiltonian. According to a result of Lichnerowicz on a symplectic manifolds, the bracket of two locally hamiltonian fields is an hamiltonian field. So if $z = [x, y]$ with x and y in $\mathfrak{g}$, $\overline{z}$ is an hamiltonian field and there is a smooth function $f : M \longrightarrow \mathbb{R}$ such that $df = i(\overline{z})\overline{\omega}$. As M is compact, f has critical points. Let $[\tau]$ such a point : $\overline{z}_{[\tau]} = 0 = \frac{d}{dt}|_{t=0} [\tau \exp tz] = \pi_{*,\tau}(z_\tau^+)$ where z^+ is the left invariant vector field associated to z. But , as π is a local diffeomorphism, it follows that $z_\tau^+ = 0$ and z^+ is zero, so $\mathfrak{g}$ is abelian. ∎

For our purpose we need to recall some basic notions.
Let (M, Ω) be a $2n$-dimensional symplectic manifold and $i : L \hookrightarrow M$ an immersion. We say that L is a Lagrangian immersed submanifold of M if $T_x i(T_x L)$ is a n-dimensional totally isotropic subspace of $T_{i(x)}M$ for each $x \in L$. A Lagrangian foliation is a foliation in (M, Ω) whose leaves are Lagrangian submanifolds of M. From the Frobenius' Theorem it is clear that every Lagrangian Lie subalgebra of $\mathfrak{g} = T_\varepsilon(G)$ determines a left invariant Lagrangian foliation in (G, Ω^+).

Our manifolds have often Lagrangian foliations as showed by the following result ([7]).

Theorem 3.5. *Let be $M = \Gamma \backslash (G, \omega^+)$ as above. If G is completely solvable, G contains a connected lagrangian Lie subgroup and therefore M admits Lagrangian foliations.*

4. Lagrangian foliations and symplectic connections

Recall that there are four non abelian real unimodular Lie algebras of dimension four endowed with a scalar non degenerate 2-cocycle given in terms of a basis $\{e_j, \quad 1 \leq j \leq 4\}$ whose nonzero brackets are the following :

$$
(2) \qquad \begin{array}{lll}
\mathcal{G}_1 : & [e_1, e_2] = e_3; & [e_1, e_3] = e_4 \\
\mathcal{G}_2 : & [e_1, e_2] = e_2; & [e_1, e_3] = -e_3 \\
\mathcal{G}_3 : & [e_1, e_2] = e_3; & [e_1, e_3] = -e_2 \\
\mathcal{G}_4 : & [e_1, e_2] = e_3 &
\end{array}
$$

We have :

Theorem 4.1. *Every unimodular 4-dimensional symplectic Lie group G contains Lagrangian Lie subgroups. Moreover if $Lie(G) = \mathcal{G}_i$, $i = 2$ or 4, G admits pairs of tranversal Lagrangian Lie subgroups. Hence any symplectic solvmanifold $\Gamma \backslash G$ inherits Lagrangian foliations (if $i = 2$ or 4 they have pairs of such transversal foliations).*

Proof. We must describe in $\mathcal{G}_i$ the Lagrangian Lie subalgebras. First , suppose $i \leq 3$. In this case every nondegenerate 2-cocycle on $\mathcal{G}_i$ can be written as $\alpha e_1^* \wedge (e_4^* + \beta e_2^* + \gamma e_3^*) + \delta e_2^* \wedge e_3^*$ where $\left\{e_j^*\right\}$ is the dual basis of $\{e_k, \quad 1 \leq k \leq 4\}$ and α, β, γ and δ real numbers with $\alpha\delta \neq 0$. Every 2-dimensional subalgebra of $\mathcal{G}_i$ is abelian, except Span$\{e_1, e_2\}$ and Span$\{e_1, e_3\}$ in $\mathcal{G}_2$. If $i = 2$ or 3, Ker$e_1^* = [\mathcal{G}_i, \mathcal{G}_i] + Z(\mathcal{G}_i)$, where $Z(\mathcal{G}_i)$ denotes the center of $\mathcal{G}_i$, is an abelian and characteristic ideal of $\mathcal{G}_i$. Denote by B an 2-dimensional subalgebra of $\mathcal{G}_i$. Assume that $B \subset$ Kere_1^* . If $B =$Span$\{b_1, b_2\}$ is Lagrangian we will have $e_2^* \wedge e_3^*(b_1, b_2) = 0$ and hence $B =$Span$\{\lambda e_2 + \mu e_3, e_4\}$ with $(\lambda, \mu) \neq (0, 0)$.
If B is not contained in Kere_1^* , $B =$Span$\{b_1, b_2 = ue_2 + ve_3 + we_4\}$ with $e_1^*(b_1) \neq 0$ because $B \cap$ Kere_1^* is one dimensional. We may suppose $b_1 = e_1 + b_1^{'}$ with $b_1^{'} \in$ Kere_1^* and $[b_1^{'}, b_2] = 0$. If $i = 3$, B must be abelian so $u = v = 0$ and $B =$Span$\left\{e_1 + b_1^{'}, e_4\right\}$ is not Lagrangian. This shows that $\mathcal{G}_3$ contains Lagrangian subalgebras, necessarily included in Kere_1^*, but there are no pairs of supplementary Lagrangian subalgebras in $\mathcal{G}_3$.
Suppose now $i = 2$. If $w \neq 0$ and $(u, v) \neq (0, 0)$, B would be 3-dimensional, thus $(u, v) = (0, 0)$ and $B =$Span$\left\{e_1 + b^{'}, e_4\right\}$ which is not Langrangian.
If $w = 0$ there are two possibilities: $B =$Span$\left\{e_1 + b^{'}, e_2\right\}$ or $B =$Span$\left\{e_1 + b^{'}, e_3\right\}$ In the first case, we can set $b' = \lambda e_3 + \mu e_4$ and $\omega(e_1 + b', e_2) = \alpha\beta - \delta\lambda$, so we choose $\lambda = \alpha\beta\delta^{-1}$ to make B Lagrangian. In fact $\mathcal{G}_2$ contains also Lagrangian subalgebras of type Span$\left\{e_1 + b^{'}, e_3\right\}$. Moreover , $L =$Span$\left\{\lambda^{'} e_2 + \mu^{'} e_3, e_4\right\}$ contained in Kere_1^*, is a supplementary Lagrangian subalgebra of $B =$Span$\{e_1 + \lambda e_3 + \mu e_4, e_2\}$ if and only if the 4-order determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda^{'} & 0 \\ \lambda & 0 & \mu^{'} & 0 \\ \mu & 0 & 0 & 1 \end{vmatrix}$ does not vanish, i.e $\mu^{'} \neq 0$.
Thus there are pairs of supplementary Lagrangian subalgebras in $\mathcal{G}_2$.
We now turn to the filiform Lie algebra $\mathcal{G}_1$. One can easily show that with a basis change that do not modify the brackets, any non degenerate scalar 2-cocycle can be written $e_1^* \wedge e_4^* + e_2^* \wedge e_3^*$. As every 2-dimensional subalgebra is abelian, because of nilpotency, we see that every Lagrangian subalgebra is of type Vect$\{\lambda e_2 + \mu e_3, e_4\}$ with $(\lambda, \mu) \neq (0, 0)$. Hence $\mathcal{G}_1$ contains Lagrangian subalgebras but no pairs of supplementary Lagrangian subalgebras.
We now come to $\mathcal{G}_4$. Here the space of 2-coboundary is Span$\{e_1^* \wedge e_2^*\}$, so the classes of $e_i^* \wedge e_j^*$, where i belongs to $\{1, 2\}$ and j belongs to $\{3, 4\}$, generate $H^2(\mathcal{G}_4, \mathbb{R})$. Moreover every 2-cocyle is cohomologous to an $\omega = f \wedge e_3^* + g \wedge e_4^*$ with f and g in Span$\{e_1^*, e_2^*\}$. Then ω is non degenerate if $f \wedge g$ is nonzero. As Ker$f$ and Ker$g$ contain $Z(\mathcal{G}_4) =$Span$\{e_3, e_4\}$, we can find $e_1^{'}$ and $e_2^{'}$ in $\mathcal{G}_4$ such that Span$\left\{e_1^{'}, e_2^{'}, e_3, e_4\right\}$ is a basis of $\mathcal{G}_4$, $f = e_1^{'*}$, $g = e_2^{'*}$ and $\omega = e_1^{'*} \wedge e_3^* + e_2^{'*} \wedge e_4^*$. Then $[e_1^{'}, e_2^{'}] = \lambda e_3$ with nonzero λ, and multiplying e_3 and ω by a scalar, we may write $\omega = e_1^{'*} \wedge e_3^{'*} + e_2^{'*} \wedge e_4^{'*}$ with $[e_1^{'}, e_2^{'}] = e_3^{'}$ and then we omit the primes. Obviously $L_1 =$Span$\{e_1, e_3\}$ and $L_2 =$Span$\{e_2, e_4\}$ are supplementary Lagrangian subalgebras in $\mathcal{G}_4$. ■

Remark 4.2. Let be (G, ω^+) the connected and simply connected symplectic Lie group whose Lie algebra is $\mathcal{G}_4$, Γ a cocompact subgroup of G and $M = (\Gamma \backslash G, \bar{\omega})$ the associated symplectic nilmanifold. A direct computation shows that $b_2(M) = 2$, so, by a result of Taubes (see [2] for more information), Gromov invariant and Seiberg-Witten invariant of M coincide.

Here is a result useful in what follows (to compare with Theorem 7.7 of [11]).

Lemma 4.3. *Let (G, ω^+) be a symplectic Lie group with the left invariant affine structure given by*

$$\omega^+(\nabla^+_{x^+} y^+, z^+) = -\omega^+(y^+, [x^+, z^+]).$$

If H is a Lagrangian Lie subgroup then the connection ∇ induces on H a left invariant affine structure . Moreover if G is unimodular, both connections are complete.

Proof. For every $x, y, z \in \mathcal{G}$ =Lie$G$, we have $\omega(xy, z) = -\omega(y, [x, z])$; taking x and y in Lie(H), we see that xy is still in Lie(H), since the second terme is zero for every z in Lie(H). If G is unimodular, ∇ is complete (see [6]) that is $TrR_x = 0$ for every $x \in \mathcal{G}$. The same is true for the left symmetric product of Lie(H). ∎

Note that the connection associated to ω^+ is symplectic if and only if G is abelian. However we have:

Theorem 4.4. **[5]** *Let (M, Ω) be a symplectic manifold. We endow $\chi(M)$ with a product XY given by the formula*

$$\Omega(XY, Z) = -\Omega(Y, [X, Z]).$$

If there are on M two Lagrangian transversal foliations $\mathcal{F}_i$, $i = 1, 2$ then (M, Ω) is endowed a unique torsion free symplectic connection ∇ satisfying $\nabla\mathcal{F}_i \subset \mathcal{F}_i$ which is given by:

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) := (X_1Y_1 + [X_2, Y_1]_1, X_2Y_2 + [X_1, Y_2]_2) \tag{3}$$

where (X_1, X_2) and (Y_1, Y_2) are in $TM = F_1 \oplus F_2$, F_i denoting the subbundle of TM corresponding to $\mathcal{F}_i$.

According to our study and Hess's result we have the following:

Proposition 4.5. *Let (G, ω^+) be a unimodular symplectic 4-dimensional Lie group. If Lie(G)= $\mathcal{G}_i$, $i = 2$ ou 4 then G has a unique left invariant torsion free symplectic connection ∇ such that $\nabla\mathcal{F}_j \subset \mathcal{F}_j$, $j = 1, 2$ where $\mathcal{F}_j$ are the two subbundles of TG associated to a pair of transversal Lagrangian subgroups of G.*

In fact we have the following stronger result:

Theorem 4.6. *If (G, ω^+) is a unimodular symplectic 4-dimensional Lie group. Then G admits a complete and left invariant symplectic affine structure. Consequently every symplectic solvmanifold associated to G inherits a flat and torsion free symplectic connection.*

Proof. We must find in $\mathcal{G} = Lie(G)$ a left symmetric product $ab = L_a(b)$ satisfying $ab - ba = [a,b]_{\mathcal{G}}$ and $\omega(L_a(x), y) + \omega(x, L_a(y)) = 0$ for all a, b, x and $y \in \mathcal{G}$. We successively study the four different cases. Suppose $\mathcal{G} = \mathcal{G}_1$. Every scalar non degenerate 2-cocycle over $\mathcal{G}$ is equivalent to $\omega = e_1^* \wedge e_4^* + e_2^* \wedge e_3^*$. For the sake of simplicity, we choose $L_{e_4} = 0$. A straight computation in the basis $\{e_1, e_2, e_3, e_4\}$ shows that $L_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$,

$L_{e_2} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta-1 & 0 & 0 & 0 \\ 0 & \beta-1 & 0 & 0 \end{pmatrix}$, where α and β are real numbers related by $2\alpha(\beta-1) = \alpha+1$ gives a family of invariant connections over G satisfying the requirement of the theorem.

Suppose now that $\mathcal{G}$ is not nilpotent. There are two such real Lie algebras :$\mathcal{G}_2$ and $\mathcal{G}_3$ which are $\mathbb{C}$-isomorphic. So it is enough to study one of them, for example $\mathcal{G}_2$ endowed with $\omega = e_1^* \wedge e_4^* + e_2^* \wedge e_3^*$. If we set $L_{e_i} = 0$ for $i = 2, 3, 4$ and $L_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$, where α is any real number, we get a suitable family of left invariant connections over G with Lie(G)= $\mathcal{G}_2$.

In the last case , $\mathcal{G} = \mathcal{G}_4$ and $\omega = e_1^* \wedge e_4^* + e_2^* \wedge e_3^*$. We set $L_{e_3} = L_{e_4} = 0$, thus L_{e_1} and L_{e_2} can be written in the following block form: $L_{e_1} = \begin{pmatrix} O_2 & O_2 \\ A_1 & O_2 \end{pmatrix}$ and $L_{e_2} = \begin{pmatrix} O_2 & O_2 \\ A_2 & O_2 \end{pmatrix}$ so we have $[L_{e_1}, L_{e_2}] = 0 = L_{e_3}$. A direct computation shows that, taking $A_1 = \begin{pmatrix} \beta & \beta' \\ \gamma & \beta \end{pmatrix}$, L_{e_1} is symplectic. Then $L_{e_1}e_2 - L_{e_2}e_1 = e_3$ and L_{e_2} symplectic implies $A_2 = \begin{pmatrix} \beta'-1 & \beta \\ \beta & \beta'-1 \end{pmatrix}$ where the coefficients of A_1 are any real numbers. ∎

Remark 4.7. Let (G, ω^+) be as in theorem 4.6. with G simply connected. Integrating the representation $x \longmapsto (x, L_x)$, $x \in \mathcal{G}$, we can consider G as a transformation group of the affine space $\mathcal{G}$, the elements of G having a linear part in $Sp(\mathcal{G}, \omega)$. The particular representations exhibited in the proof of Theorem 4.6 show that this group contains non trivial one parameter subgroups of translations.

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