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An Exposition of the $sl(N,\mathbb{C})$–Weight System

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Abstract. In this paper, we show substantial formulas of the $sl(N,\mathbb{C})$-weight system from a representation theoretical viewpoint. Although some of them can be essentially recovered by the universal $sl(N,\mathbb{C})$-weight system formulated by the first author in [12], they have more efficient descriptions than the ones recovered by the universal weight system in the sense that the dimension of the representation space of $sl(N,\mathbb{C})$ used in this paper is taken to be minimal. In addition, in this paper, we also have the aim of making an exposition of the $sl(N,\mathbb{C})$-weight system: we show some elementary facts which are not easy to calculate concretely.
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1. Background of the weight system

For a quantum group $\mathcal{U}_q(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ and an arbitrary representation ρ of $\mathcal{U}_q(\mathfrak{g})$, the quantum $(\mathfrak{g},\rho)$-invariant $Q_{\mathfrak{g},\rho}$ for framed links can be defined. It takes values in the ring $\mathbb{C}[[h]]$ of the formal power series in h. Regarding expressions of the quantum invariant, it is in fact very hard to present or formulate it concretely since the structure of the quantum group and its representation are usually quite complicated.

On the other hand, in the 1990's Kontsevich constructed an invariant for knots taking values in a certain linear space $\mathcal{A}(S^1)$. This so-called Kontsevich invariant is reviewed in Section 2. by using Drinfeld's work on the universal solution of the Kniznik-Zamolodchikov equation ([4], [7]). Based on the Kontsevich invariant, Le and Murakami constructed an invariant of framed links which takes its values in the space $\mathcal{A}(\amalg S^1)$ by using a combinatorial method ([8], [9], [10]). This is called the modified Kontsevich invariant and denoted by $\widehat{Z}$ (refer to Chapter 6 in [15], for example).

Now, we have a special linear map from the algebra $\mathcal{A}(\amalg S^1)$ to $\mathbb{C}$, called the $(\mathfrak{g},\rho)$-weight system $W_{\mathfrak{g},\rho}$ defined in Section 6.6 of [15] in detail. Note that there exist two definitions of the weight system. The definition in [15] uses the representations of the Lie algebra. The other definition in [1] uses "state". These

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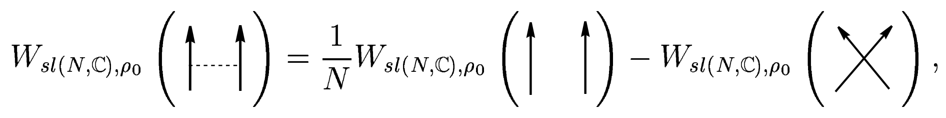
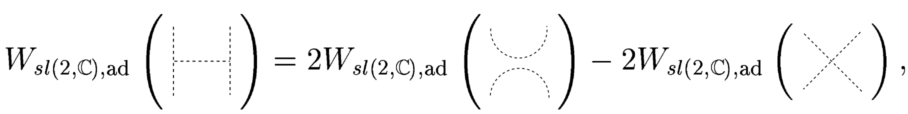
are indeed equivalent, however we will follow the definition in [15] and make a representation theoretic approach to the weight system in this paper. The graded weight system $\widehat{W}_{\mathfrak{g},\rho}$ is a linear map from $\mathcal{A}(\amalg S^1)$ to $\mathbb{C}[[h]]$ defined in Theorem 2.1. It is well known that the quantum $(\mathfrak{g},\rho)$-invariant can be derived from the modified Kontsevich invariant $\widehat{Z}$ via the graded weight system $\widehat{W}_{\mathfrak{g},\rho}$ (see Theorem 2.1). Namely, $\widehat{Z}$ is a lift of $Q_{\mathfrak{g},\rho}$ to $\mathcal{A}(\amalg S^1)$ and the graded weight system $\widehat{W}_{\mathfrak{g},\rho}$ is the projection of $\widehat{Z}$ to $Q_{\mathfrak{g},\rho}$. Note that the modified Kontsevich invariant does not depend on $\mathfrak{g}$ and ρ, while the weight system does. Thus the quantum $(\mathfrak{g},\rho)$-invariant can be reconstructed as the composition of $\widehat{Z}$ with $\widehat{W}_{\mathfrak{g},\rho}$ without $\mathcal{U}_q(\mathfrak{g})$ and its representation, but just by using $\mathfrak{g}$ and ρ. Though this process makes the calculation of the quantum invariants somewhat simpler as above, it is still very hard to calculate the weight system (and the modified Kontsevich invariant), because of the representation theoretical complexity. However, to avoid the complexity, we can in fact use the formula introduced by Le and Murakami for the calculations of the weight system $W_{sl(N,\mathbb{C}),\rho_0}$:

Theorem 1.1. **(Le and Murakami [8])**



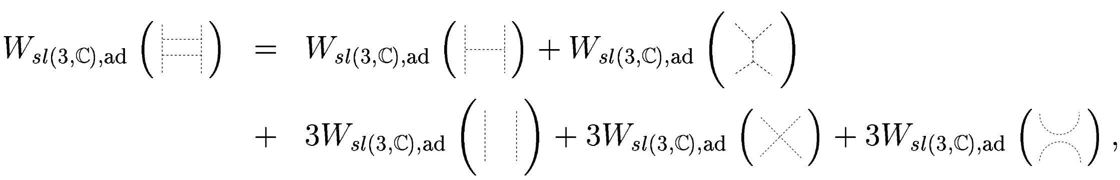
where ρ_0 is the fundamental representation.
After that, a formula for $(sl(2,\mathbb{C}),\mathrm{ad})$ was given by Chmutov and Varchenko:

Theorem 1.2. **(Chmutov and Varchenko [3])**



where ad *is the adjoint representation.*
Furthermore, Kuga and Yoshizumi studied the case of $(\mathfrak{g},\rho) = (sl(3,\mathbb{C}),\mathrm{ad})$:

Theorem 1.3. **(Kuga and Yoshizumi [16])**



where ad *is the adjoint representation.*

Remark 1.4. Let μ be the trivial representation of $sl(N,\mathbb{C})$. Then note that the direct sum $\mathrm{ad}\oplus\mu$ is isomorphic to the tensor representation $\rho_0^*\otimes\rho_0$, where ρ_0^* is the dual representation of ρ_0. Let I and E_{ij} be the unit matrix and the (i,j)-matrix unit of $gl(N,\mathbb{C})$ respectively. Considering that the representation spaces of $\mathrm{ad}\oplus\mu$ and $\rho_0^*\otimes\rho_0$ are $sl(N,\mathbb{C})\oplus\mathbb{C}I = gl(N,\mathbb{C})$ and $(\mathbb{C}^N)^*\otimes\mathbb{C}^N$ respectively, we fix a linear map ι,

$$\iota : \mathrm{ad}\oplus\mu \to \rho_0^*\otimes\rho_0,\ \iota(E_{ij}) = -\widetilde{e}_j\otimes e_i,\ (1\le i,j\le N),$$

where $\{e_i\}_{i=1}^N$ and $\{\widetilde{e}_i\}_{i=1}^N$ are the canonical basis of $\mathbb{C}^N$ and its dual respectively. We can easily check that ι is an isomorphism. This isomorphism naturally induces an isomorphism $\iota\otimes\iota$ from $(\mathrm{ad}\oplus\mu)^{\otimes 2}$ to $(\rho_0^*\otimes\rho_0)^{\otimes 2}$. Note that the inverse $(\iota\otimes\iota)^{-1}$ is $\iota^{-1}\otimes\iota^{-1}$.

Remark 1.5. Let ι and $\iota\otimes\iota$ be the isomorphisms in Remark 1.4. Then they naturally induce isomorphisms of algebras,

$$\bar{\iota} : \mathrm{End}_{sl(N,\mathbb{C})}(\mathrm{ad}\oplus\mu) \to \mathrm{End}_{sl(N,\mathbb{C})}(\rho_0^*\otimes\rho_0),\ \bar{\iota}(f) = \iota\circ f\circ\iota^{-1},$$

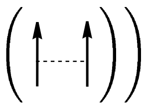
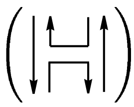
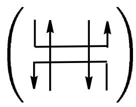
for any element f in $\mathrm{End}_{sl(N,\mathbb{C})}(\mathrm{ad}\oplus\mu)$, and

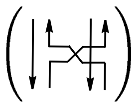
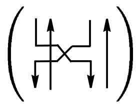
$$\overline{\iota\otimes\iota} : \mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2}) \to \mathrm{End}_{sl(N,\mathbb{C})}((\rho_0^*\otimes\rho_0)^{\otimes 2}),\ \overline{\iota\otimes\iota}(g) = (\iota\otimes\iota)\circ g\circ(\iota\otimes\iota)^{-1},$$

for any element g in $\mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2})$ respectively.

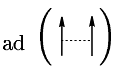
In this paper we generalize Theorems 1.2 and 1.3 as follows:

Theorem 1.6. *Let $\overline{\iota\otimes\iota}$ be the isomorphism in Remark 1.5. Then the following equation holds:*

$\overline{\iota\otimes\iota}\Big(W_{sl(N,\mathbb{C}),\mathrm{ad}}$  $\Big) = W_{sl(N,\mathbb{C}),\rho_0}$  $+\, W_{sl(N,\mathbb{C}),\rho_0}$ 

$-\, W_{sl(N,\mathbb{C}),\rho_0}$  $-\, W_{sl(N,\mathbb{C}),\rho_0}$  ,

where both sides are equal as elements in $\mathrm{End}_{sl(N,\mathbb{C})}((\rho_0^*\otimes\rho_0)^{\otimes 2})$.

Note that $W_{sl(N,\mathbb{C}),\mathrm{ad}}$ , which is an element of $\mathrm{End}_{sl(N,\mathbb{C})}(\mathrm{ad}\otimes\mathrm{ad})$, can be naturally considered as an element of $\mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2})$ via the inclusion of ad in $\mathrm{ad}\oplus\mu$. Therefore the left side of the equation in Theorem 1.6 is well-defined. We sometimes omit the isomorphism $\overline{\iota\otimes\iota}$ in the above equation for convenience.

This paper is organized as follows. In Section 2., we review some concepts related to quantum invariants and the weight system. In Section 3., we show our formulas of the weight system $W_{sl(N,\mathbb{C}),\mathrm{ad}}$. In Section 4., we calculate the weight system $W_{sl(N,\mathbb{C}),\mathrm{ad}}$ for some special cases by using the formula in Theorem 1.6. In the final section, we observe that Theorem 1.6 is a generalization of Theorems 1.2 and 1.3.

2. Short review of Jacobi diagrams, Kontsevich invariant, quantum invariants and weight systems

In this section, we review some concepts related to quantum invariants.

We first review the Jacobi diagrams, which are variously called chord, web or Feynman diagrams, and which play an important role in this paper (refer to Chapter 7 in [15] for details). Let X be a real 1-dimensional compact oriented manifold with finitely many connected components. (We assume that all components of X are ordered.) Then a Jacobi diagram with support X is defined as a trivalent graph D such that D consists of X and unoriented dashed edges satisfying the following conditions:

1. a vertex lies on the interior of X, or a vertex is a common end point of dashed edges.

2. edges adjacent to a vertex are ordered cyclically.

In this paper, we assume that the cyclic order is taken to be counterclockwise. Since a graph have no notion of knotting, we do not have to consider the sign of crossing for a Jacobi diagram. Therefore every crossing in a Jacobi diagram is drawn as a singular crossing.

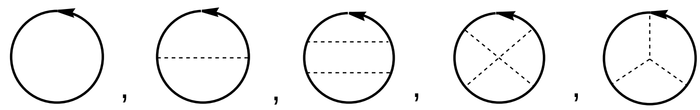


Figure 1: Jacobi diagrams with support S^1

Two Jacobi diagrams are said to be equivalent, if they are equivalent as graph with cyclic ordered trivalent vertices.

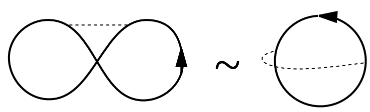


Figure 2: Two equivalent Jacobi diagrams

We next review the space $\mathcal{A}(X)$ to describe the weight system. For a 1-manifold X defined above, the space $\mathcal{A}(X)$ is a vector space over $\mathbb{C}$ spanned by Jacobi diagrams with the support X modulo the AS, the AS$'$, the IHX and the STU relations as in Figure 3. $\mathcal{A}(X)$ is naturally graded by the degree of Jacobi diagrams. The degree of a Jacobi diagram D is half the number of trivalent vertices of D. In this paper, the completion of $\mathcal{A}(X)$ in terms of the grading is also denoted by $\mathcal{A}(X)$. The vector spaces are discussed in [2] in more detail.

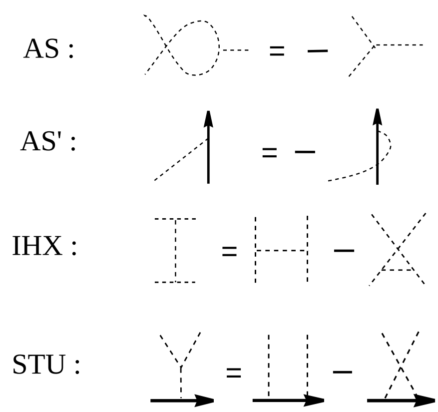


Figure 3: The AS, the AS$'$, the IHX and the STU relations

We can now review the definition of the weight system. A weight system of a Jacobi diagram can be simply thought of as a linear map between representation spaces associated with finite dimensional irreducible representations of an arbitrary simple Lie algebra such that the map commutes with the action of the Lie algebra via its representation (such a linear map is sometimes called an intertwiner). An exact definition is as follows (refer to Section 6.6 of [15] for more details). We assume that the support X of a Jacobi diagram has k connected components. For an arbitrary simple linear Lie algebra $\mathfrak{g}$, let $\rho_i : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V_i), i = 1, .., k$, be an irreducible representation of $\mathfrak{g}$. We fix the canonical basis $\{e^i_j\}, j = 1, .., \mathrm{dim}V_i$, of V_i. Let $\rho^*_i : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V^*_i)$ be the dual representation of ρ_i, where V^*_i is the dual space of V_i. Then we denote by $\widetilde{e}^i_j$ the dual base of e^i_j. Let B be the Killing form on $\mathfrak{g}$. In this paper, we define B on $sl(N, \mathbb{C})$ by $B(x, y){:=}\mathrm{tr}(xy)$ for x, y in $sl(N, \mathbb{C})$.[2] Then we fix an orthonormal basis $\{I_i\}, i = 1, .., \mathrm{dim}\mathfrak{g}$, of $\mathfrak{g}$ with respect to B. For a Jacobi diagram D with the support X, we first put $\mathfrak{g}$ on all dashed lines and V_i on the i-th component of X for all i. Next, we decompose D into the fundamental parts as in Figure 4 by using some horizontal lines (if necessary we may move some dashed lines in the process). Note that a linear map is given for each fundamental part in Figure 4. We compose the maps from bottom to top according to the decomposition of the Jacobi diagram. Then the weight system $W_{\mathfrak{g},(\rho_1,..,\rho_k)}(D)$ of the Jacobi diagram is defined as the composite map.

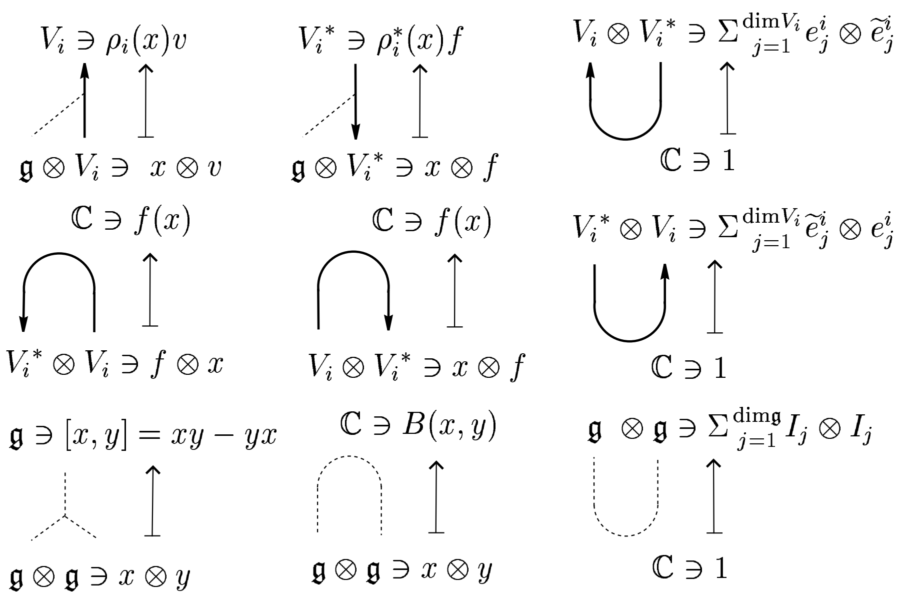


Figure 4: Fundamental parts of the Jacobi diagrams and the corresponding maps

Note that in the calculations of the $(sl(N, \mathbb{C}), \mathrm{ad})$-weight system we can consider every component of the support of a Jacobi diagram as a dashed line. For convenience, $W_{\mathfrak{g},(\rho,..,\rho)}$ is denoted by $W_{\mathfrak{g},\rho}$, and a representation ρ and its representation space V are used without distinction in the rest of this paper.

The following theorem, which is one of the most important theorems when we study quantum invariants, tells us the role of the weight system.

Theorem 2.1. **(Kassel [5], Le and Murakami [10])**
The quantum $(\mathfrak{g}, \rho)$*-invariant* $Q_{\mathfrak{g},\rho}$ *can be reconstructed by using the composition*

[2] Although B on $sl(N, \mathbb{C})$ is usually defined by $B(x, y) := 2N \cdot \mathrm{tr}(xy)$ for x, y in $sl(N, \mathbb{C})$, we use the modified version for convenience in this paper.

of the modified Kontsevich invariant $\widehat{Z}$ with the $(\mathfrak{g},\rho)$-graded weight system $\widehat{W}_{\mathfrak{g},\rho}$. Namely,

$$Q_{\mathfrak{g},\rho}(L)|_{q=e^h} = \widehat{W}_{\mathfrak{g},\rho}(\widehat{Z}(L)),$$

for an arbitrary oriented framed link L, where $\widehat{W}_{\mathfrak{g},\rho}(D) = W_{\mathfrak{g},\rho}(D)\, h^{\deg(D)}$.

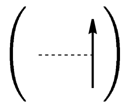
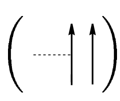
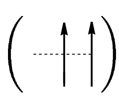
As mentioned before, according to this theorem, the graded weight system $\widehat{W}_{\mathfrak{g},\rho}$ is the projection of a lift $\widehat{Z}$ of the quantum invariant $Q_{\mathfrak{g},\rho}$ to $\mathbb{C}[[h]]$. Therefore, if we can derive a formula similar to Theorems 1.1 through 1.6 for a pair $(sl(N,\mathbb{C}), \mathrm{ad})$, and moreover we can calculate the modified Kontsevich invariant, we can essentially evaluate the quantum invariant $Q_{sl(N,\mathbb{C}),\mathrm{ad}}$ without the adjoint representation ad (recall Section 1.).

In the rest of this paper, we concentrate our interest on the formulation of the weight system for the pair $(sl(N,\mathbb{C}), \mathrm{ad})$ and its application.

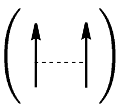
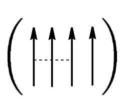
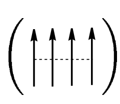
3. Representation theoretical approach to Theorem 1.6

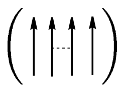
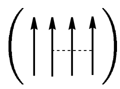
In this section, we prove Theorem 1.6. from a representation theoretical viewpoint. Note that there exists another definition of the weight system in [1] by using the "state" without representations of Lie algebra. However we follow the definition in [15] which is made by using the representations of Lie algebra. For convenience, $W_{sl(N,\mathbb{C}),\rho}$ is denoted by W_ρ from now on. The idea of the proof is to take the double of the support of the Jacobi diagram and apply Theorem 1.1 at two vertices which are connected by a dashed line.

Proof of Theorem 1.6. By using the following fact for any representations ρ and λ,

$W_{\rho\otimes\lambda}$  $= W_{(\rho,\lambda)}$  $+ W_{(\rho,\lambda)}$ ,

we first get

$W_{\rho_0{}^*\otimes\rho_0}$  $= W_{(\rho_0{}^*,\rho_0,\rho_0{}^*,\rho_0)}$  $+ W_{(\rho_0{}^*,\rho_0,\rho_0{}^*,\rho_0)}$  (1)

$+ W_{(\rho_0{}^*,\rho_0,\rho_0{}^*,\rho_0)}$  $+ W_{(\rho_0{}^*,\rho_0,\rho_0{}^*,\rho_0)}$ .

On the other hand, the representation $\rho_0^* \otimes \rho_0$ of $sl(N,\mathbb{C})$ is isomorphic to the direct sum of the adjoint and the trivial representations. Namely,

$$\rho_0^* \otimes \rho_0 = \mathrm{ad} \oplus \mu,$$

where μ is the trivial representation. Therefore the two representation spaces

$$(\mathrm{ad} \oplus \mu)^{\otimes 2} = (\mathrm{ad} \otimes \mathrm{ad}) \oplus (\mathrm{ad} \otimes \mu) \oplus (\mu \otimes \mathrm{ad}) \oplus (\mu \otimes \mu)$$

and

$$(\rho_0^* \otimes \rho_0)^{\otimes 2} = \rho_0^* \otimes \rho_0 \otimes \rho_0^* \otimes \rho_0$$

are also isomorphic. Let $\overline{\iota\otimes\iota}$ be the map from $\mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2})$ to $\mathrm{End}_{sl(N,\mathbb{C})}((\rho_0^*\otimes\rho_0)^{\otimes 2})$ as in Remark 1.5. That is, for any element g in $\mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2})$,

$$\overline{\iota\otimes\iota}(g)=(\iota\otimes\iota)\circ g\circ(\iota\otimes\iota)^{-1}.$$

Then we can easily check that $\overline{\iota\otimes\iota}$ is an isomorphism of algebras. Note that the weight system $W_{\mathrm{ad}\oplus\mu}\left(\quad\right)$ is an intertwiner on $(\mathrm{ad}\oplus\mu)^{\otimes 2}$ and so is an element of $\mathrm{End}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2})$ (refer to [15] for details on the intertwiner). Here, by the definition of $\overline{\iota\otimes\iota}$, the following equation holds:

$$\overline{\iota\otimes\iota}\left(W_{\mathrm{ad}\oplus\mu}\left(\quad\right)\right)=W_{\rho_0\otimes\rho_0^*}\left(\quad\right).$$

By using the isomorphism $\overline{\iota\otimes\iota}$ and the following relation

$$W_{\rho\oplus\lambda}\left(\quad\right)=W_{\rho}\left(\quad\right)+W_{\lambda}\left(\quad\right),$$

we get the equation below:

$$\begin{aligned}
W_{\rho_0^*\otimes\rho_0}\left(\quad\right) &= \overline{\iota\otimes\iota}\left(W_{\mathrm{ad}\oplus\mu}\left(\quad\right)\right)\\
&= \overline{\iota\otimes\iota}\left(W_{(\mathrm{ad},\mathrm{ad})}\left(\quad\right)\right)+\overline{\iota\otimes\iota}\left(W_{(\mathrm{ad},\mu)}\left(\quad\right)\right)\\
&+ \overline{\iota\otimes\iota}\left(W_{(\mu,\mathrm{ad})}\left(\quad\right)\right)+\overline{\iota\otimes\iota}\left(W_{(\mu,\mu)}\left(\quad\right)\right).
\end{aligned}$$

The Lie algebra $sl(N,\mathbb{C})$ acts on $\mathbb{C}$ as the 0-map, so we get the following conclusion:

$$W_{\rho_0^*\otimes\rho_0}\left(\quad\right)=\overline{\iota\otimes\iota}\left(W_{\mathrm{ad}}\left(\quad\right)\right). \tag{2}$$

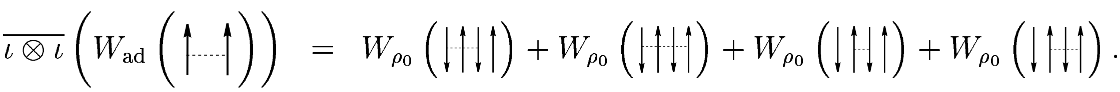
By comparing Equation (1) with Equation (2), we obtain

$$\begin{aligned}
\overline{\iota\otimes\iota}\left(W_{\mathrm{ad}}\left(\quad\right)\right) &= W_{(\rho_0^*,\rho_0,\rho_0^*,\rho_0)}\left(\quad\right)+W_{(\rho_0^*,\rho_0,\rho_0^*,\rho_0)}\left(\quad\right)\\
&+ W_{(\rho_0^*,\rho_0,\rho_0^*,\rho_0)}\left(\quad\right)+W_{(\rho_0^*,\rho_0,\rho_0^*,\rho_0)}\left(\quad\right).
\end{aligned}$$

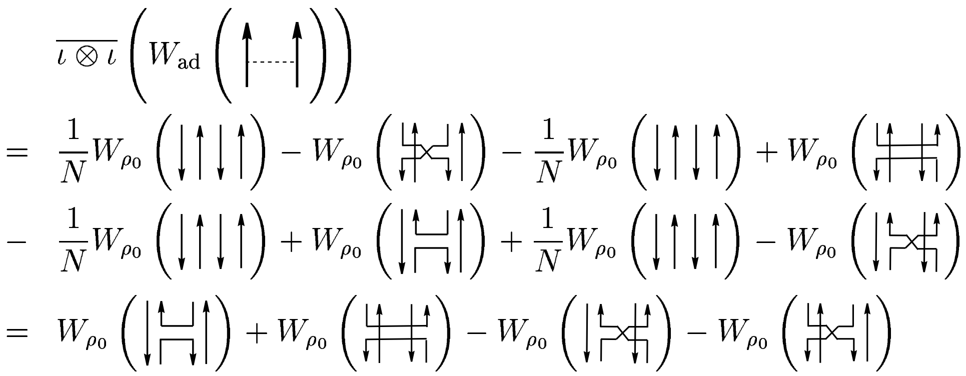
Next, since the relation below holds for any representation ρ

$$W_{\rho^*}\left(\quad\right)=W_{\rho}\left(\quad\right),$$

we have the following equation:



Finally, applying Theorem 1.1 to the equation above, we get Theorem 1.6 as follows:



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4. Demonstration

In this section we calculate the weight system of the Jacobi diagram called the wheel with $2n$ legs (refer to Figure 5).

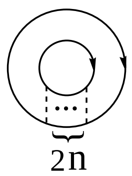
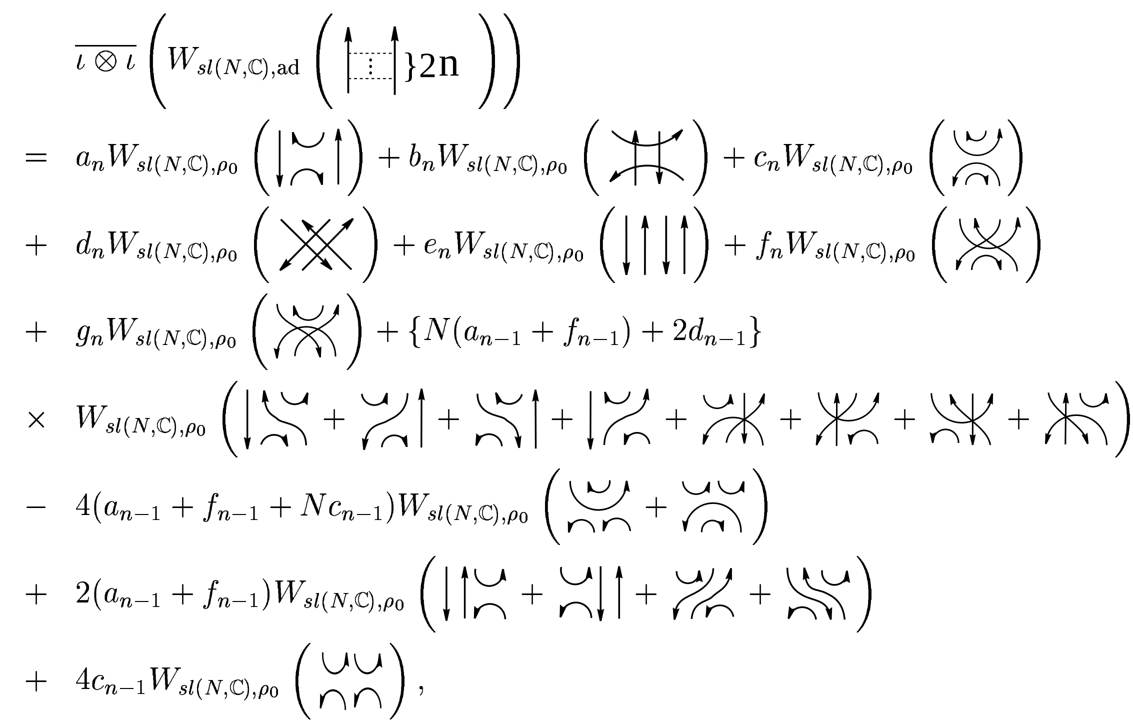


Figure 5: Wheel with $2n$ legs

Corollary 4.1. *Let* $\overline{\iota \otimes \iota}$ *be the isomorphism fixed in Remark* 1.5. *Then for*

$n \geq 1$, the following equation holds:

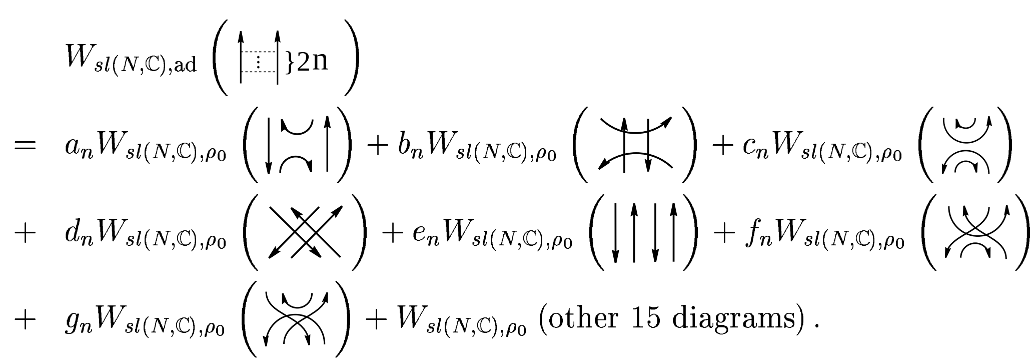


where the sequences a_n through g_n $(n \geq 1)$ are as follows:

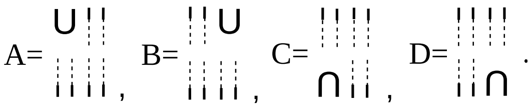
$$
\begin{aligned}
a_n &= \frac{1}{N^2-4}(N^{2n+1} - 2N^{2n-1} - 2^{2n-1}N) \\
b_n &= a_n \\
c_n &= 2^{2n-1}N^{2n-2}\left(\prod_{i=1}^{n-1}\frac{N^{2i}-2N^{2i-2}-2^{2i-1}}{N^2-4}\right) \\
&\quad + \sum_{i=1}^{n-1} N^{n-i-1}\left(\frac{6N^2(N^{2i}-2^{2i})}{N^2-4}+2^{2i+1}\right)\left(\prod_{k=i+1}^{n-1}\frac{N^{2k}-2N^{2k-2}-2^{2k-1}}{N^2-4}\right) \\
d_n &= 2^{2n-1} \\
e_n &= d_n \\
f_n &= \frac{2N^{2n-1}-2^{2n-1}N}{N^2-4} \\
g_n &= f_n.
\end{aligned}
$$

In particular, $a_0 = -f_0$, $c_0 = 0$ and $d_0 = -\frac{1}{2}$. Note that the both sides are equal as elements in $\mathrm{End}_{sl(N,\mathbb{C})}((\rho_0^* \otimes \rho_0)^{\otimes 2})$.

Proof. For convenience, $\overline{\iota \otimes \iota}$ is omitted in this proof. For $n \geq 1$, we first put



We can check that the left side of the above equation is presented by 22 diagrams on the right side of the equation in Corollary 4.1. Note that the last 15 diagrams can be ignored in the calculations to determine the coefficients of all diagrams of $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2(\mathrm{n}+1)\right)$. The reason is as follows. Each of the 15 diagrams has at least a cup $\cup$ or a cap $\cap$. In general, they can be classified into the following 4 types, **A**, **B**, **C** and **D**:



Then to calculate the weight system $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2(\mathrm{n}+1)\right)$ we consider the composition

$$W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right)\circ W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2\mathrm{n}\right)\circ W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right).$$

Remember that $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right)$ can be considered as

$$\begin{aligned} W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right) &= W_{sl(N,\mathbb{C}),\rho_0}\left(\ \right)+W_{sl(N,\mathbb{C}),\rho_0}\left(\ \right) \\ &+ W_{sl(N,\mathbb{C}),\rho_0}\left(\ \right)+W_{sl(N,\mathbb{C}),\rho_0}\left(\ \right), \end{aligned}$$

as in the proof of Theorem 1.6. Here let us denote the sum of the first and the third terms of the above equation by E and the sum of the second and the fourth term by F. Then for any Jacobi diagram D of type **A**,

$$E\circ W_{sl(N,\mathbb{C}),\rho_0}(D)=0,\ \ F\circ W_{sl(N,\mathbb{C}),\rho_0}(D)=0,$$

by the AS$'$ relation. Therefore for any Jacobi diagram D of type **A**,

$$W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right)\circ W_{sl(N,\mathbb{C}),\rho_0}(D)=(E+F)\circ W_{sl(N,\mathbb{C}),\rho_0}(D)=0.$$

Similarly we can get the same result in the case of other types, **B**, **C**, and **D**. Therefore the last 15 diagrams of $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2\mathrm{n}\right)$ are cancelled in the calculation of $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2(\mathrm{n}+1)\right)$. Moreover the coefficients of the last 15 diagrams are determined by the sequences a_n through g_n as in Corollary 4.1 by the composition above.

Considering the composition,

$$W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right)\circ W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \}2\mathrm{n}\right)\circ W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\ \right),$$

for $n \geq 1$ and the facts above, we can get the following recurrence relations:

$$\begin{aligned}
a_{n+1} &= N^2 a_n + N e_n, \\
b_{n+1} &= N^2 b_n + N e_n, \\
c_{n+1} &= 3N(a_n + b_n + f_n + g_n) + 2N(a_n + b_n)c_n + 2(d_n + e_n), \\
d_{n+1} &= 2d_n + 2e_n, \\
e_{n+1} &= 2d_n + 2e_n, \\
f_{n+1} &= N d_n + N^2 f_n, \\
g_{n+1} &= N d_n + N^2 g_n.
\end{aligned}$$

Here we can easily check that

$$a_1 = b_1 = N,\ c_1 = d_1 = e_1 = 2,\ f_1 = g_1 = 0.$$

In particular, for the formula in the case of $n = 1$, we put $a_0 = -f_0$, $c_0 = 0$ and $d_0 = -\frac{1}{2}$. Resolving the above recurrence relations we can get the sequences a_n through g_n in Corollary 4.1. ■

Corollary 4.2. *With the same notations and condition as in Corollary* 4.1*, the following equation holds:*

$$\begin{aligned}
W_{sl(N,\mathbb{C}),\mathrm{ad}}(w_{2n}) &= 2N^3 a_n + N^2 c_n + N^2(N^2+1)d_n + 2N f_n - 4N(N^2+1)a_{n-1} \\
&\quad -4N^2 c_{n-1} - 16N^2 d_{n-1} + 4N(N^2-1)f_{n-1},
\end{aligned}$$

where the Jacobi diagram w_n is the wheel with $2n$ legs.

Proof. By closing the support on both sides of Corollary 4.1, we can easily get the equation above. Here we use Lemma 4.3 below to close the support. ■

Note that in the case of the odd version w_{2n+1} we can get a similar relation to the even case.

Lemma 4.3. *Let $\iota \otimes \iota$ and $\bar{\iota}$ be the isomorphisms fixed in Remarks* 1.4 *and* 1.5 *respectively. Then for an arbitrary finite-dimensional irreducible representation ρ, the following equations hold:*

$$\bar{\iota}\left(W_{sl(N,\mathbb{C}),\rho}\left(\quad\right)\right) = W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) - \frac{1}{N} W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right),$$

$$(\iota \otimes \iota) \circ W_{sl(N,\mathbb{C}),\rho}\left(\quad\right) = W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) - \frac{1}{N} W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right),$$

$$W_{sl(N,\mathbb{C}),\rho}\left(\quad\right) \circ (\iota \otimes \iota)^{-1} = W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) - \frac{1}{N} W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right),$$

where both sides of each equation above are equal as elements in $\mathrm{End}_{sl(N,\mathbb{C})}(\rho_0^* \otimes \rho_0)$, $\mathrm{Hom}_{sl(N,\mathbb{C})}(\mu, (\rho_0^* \otimes \rho_0)^{\otimes 2})$ *and* $\mathrm{Hom}_{sl(N,\mathbb{C})}((\rho_0^* \otimes \rho_0)^{\otimes 2}, \mu)$ *respectively.*

Note that the left side of the third equation is well-defined. That is, $W_{sl(N,\mathbb{C}),\rho}(\quad)$ can be naturally considered as an element of

$$\mathrm{Hom}_{sl(N,\mathbb{C})}((\mathrm{ad} \oplus \mu)^{\otimes 2}, \mu) = \mathrm{Hom}_{sl(N,\mathbb{C})}(gl(N,\mathbb{C})^{\otimes 2}, \mathbb{C}),$$

because the Killing form B of $sl(N,\mathbb{C})$ is the restriction of that of $gl(N,\mathbb{C})$ to $sl(N,\mathbb{C})^{\otimes 2}$ (up to a constant multiple).

In the proof of Lemma 4.3, we will also show that the left side of the second equation is well-defined. We sometimes omit the isomorphisms $\bar{\iota}$, $(\iota\otimes\iota)$ and $(\iota\otimes\iota)^{-1}$ in the equations in Lemma 4.3 for convenience.

Proof of Lemma 4.3**.** Regarding the first equation, we first get the following relation by using the properties of the representation and the weight system as in the first proof of Theorem 1.6:

$$\bar{\iota}\left(W_{sl(N,\mathbb{C}),\rho}\left(\,\vdots\,\right)\right)+\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right) \tag{3}$$

$$= W_{sl(N,\mathbb{C}),\rho_0^*\otimes\rho_0}\left(\uparrow\right)=W_{sl(N,\mathbb{C}),\rho_0}\left(\downarrow\,\uparrow\right), \tag{4}$$

where the notation is the same as in the first proof of Theorem 1.6. (Note that $W_{sl(N,\mathbb{C}),\rho}(\,\vdots\,)$ is an element of $\mathrm{End}_{sl(N,\mathbb{C})}(\mathrm{ad})$. We can think of it as an element of $\mathrm{End}_{sl(N,\mathbb{C})}(\mathrm{ad}\oplus\mu)$ such that it is zero on the subspace μ and the identity map on ad. Hence the first term above is well-defined.) Thus it suffices to show the following equation:

$$\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right) = \frac{1}{N}W_{sl(N,\mathbb{C}),\rho_0}\left(\overset{\smile}{\frown}\right), \tag{5}$$

where the map $W_{sl(N,\mathbb{C}),\rho_0}(\overset{\smile}{\frown})$ is an intertwiner on $\rho_0^*\otimes\rho_0$ which is isomorphic to ad$\oplus\mu$. Note that the Jacobi diagram $\overset{\smile}{\frown}$ can be decomposed into $\frown$ and $\smile$ by cutting horizontally across the center of the diagram and so we can consider that the intertwiner $W_{sl(N,\mathbb{C}),\rho_0}(\overset{\smile}{\frown})$ is the 0-map on the subspace of $\rho_0^*\otimes\rho_0$ isomorphic to ad and a constant map $\frac{1}{N}\mathrm{Id}_\mu$ on the subspace of $\rho_0^*\otimes\rho_0$ isomorphic to μ. The reason is as follows. $f=W_{sl(N,\mathbb{C}),\rho_0}(\frown)$ is an intertwiner from $\rho_0^*\otimes\rho_0$, which is isomorphic to ad$\,\oplus\,\mu$, to μ. By the meaning of the intertwiner, f is the 0-map on the subspace of $\rho_0^*\otimes\rho_0$ isomorphic to ad and a non zero constant map $k\mathrm{Id}_\mu$ on the subspace of $\rho_0^*\otimes\rho_0$ isomorphic to μ. Similarly the intertwiner $g=W_{sl(N,\mathbb{C}),\rho_0}(\smile)$ is the embedding of μ into $\rho_0^*\otimes\rho_0$. So the composition $g\circ f$ is the 0-map on the subspace of $\rho_0^*\otimes\rho_0$ isomorphic to ad and the constant map $k\mathrm{Id}_\mu$ on the subspace isomorphic to μ. Hence we get the equation as follows:

$$\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right)=kW_{sl(N,\mathbb{C}),\rho_0}\left(\overset{\smile}{\frown}\right).$$

By comparing the right sides of the following two relations

$$\left\{\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right)\right\}^2=k^2W_{sl(N,\mathbb{C}),\rho_0}\left(\overset{\smile}{\underset{\frown}{\bigcirc}}\right)=k^2NW_{sl(N,\mathbb{C}),\rho_0}\left(\overset{\smile}{\frown}\right),$$

and

$$\left\{\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right)\right\}^2=\bar{\iota}\left(W_{sl(N,\mathbb{C}),\mu}\left(\uparrow\right)\right)=kW_{sl(N,\mathbb{C}),\rho_0}\left(\overset{\smile}{\frown}\right),$$

it follows that $k^2N=k$ and so $k=\frac{1}{N}$. (Note that k is non zero.) Thus Equation (5) holds. This completes the proof of the first equation.

Next, we prove the second equation in Lemma 4.3. By definition, we see that the weight system $(\iota \otimes \iota) \circ W_{sl(N,\mathbb{C}),\mathrm{ad}}$ () is the intertwiner from μ to $(\rho_0^* \otimes \rho_0)^{\otimes 2}$ taking $1 \in \mathbb{C}$ to the image $(\iota \otimes \iota)(T)$ of the invariant 2-tensor T for $sl(N,\mathbb{C})^{\otimes 2}$,

$$T = \sum_{i=1}^{N^2-1} I_i \otimes I_i \in sl(N,\mathbb{C})^{\otimes 2},$$

where $\{I_i\}_{i=1}^{N^2-1}$ is an orthonormal basis of $sl(N,\mathbb{C})$ with respect to the Killing form B of $sl(N,\mathbb{C})$ (for more details on the invariant 2-tensor, refer to Section 5.1 in [15] for example). Here we can easily check that the following N^2-1 elements give us an orthonormal basis of $sl(N,\mathbb{C})$ in terms of B:

$$\left\{ \frac{E_{ij}+E_{ji}}{\sqrt{2}},\ \frac{E_{ij}-E_{ji}}{\sqrt{-2}},\ \frac{1}{\sqrt{k(k+1)}} \sum_{l=1}^{k} lH_l \mid 1 \le i < j \le N,\ 1 \le k \le N-1 \right\},$$

where E_{ij} is the (i,j)-matrix unit and H_l is $E_{ll} - E_{l+1\ l+1}$ for any i, j and l. Hence the invariant 2-tensor T is given by the following:

$$\begin{aligned} &\sum_{N \ge j > i \ge 1} \frac{1}{2}(E_{ij}+E_{ji})^{\otimes 2} - \sum_{N \ge j > i \ge 1} \frac{1}{2}(E_{ij}-E_{ji})^{\otimes 2} + \sum_{i=1}^{N-1} \frac{1}{i(i+1)} \left(\sum_{j=1}^{i} jH_j \right)^{\otimes 2} \\ &= \sum_{i,j=1, i \ne j}^{N} E_{ij} \otimes E_{ji} + \sum_{i=1}^{N-1} \frac{\left(\sum_{j=1}^{i} j^2 {H_j}^{\otimes 2} + \sum_{i \ge n > m \ge 1} mn(H_m \otimes H_n + H_n \otimes H_m) \right)}{i(i+1)} \end{aligned}$$

Here the sum $T' = \sum_{i=1}^{N-1} \frac{1}{i(i+1)} (\sum_{j=1}^{i} j^2 {H_j}^{\otimes 2} + \sum_{i \ge n > m \ge 1} mn(H_m \otimes H_n + H_n \otimes H_m))$ is transformed as follows:

$$\begin{aligned} T' &= \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} \frac{j^2}{i(i+1)} {H_j}^{\otimes 2} + \sum_{N-1 \ge n > m \ge 1} \sum_{i=l}^{N-1} \frac{mn(H_m \otimes H_n + H_n \otimes H_m)}{i(i+1)} \\ &= \sum_{j=1}^{N-1} \frac{j(N-j)}{N} {H_j}^{\otimes 2} + \sum_{N-1 \ge n > m \ge 1} \frac{m(N-n)}{N} (H_m \otimes H_n + H_n \otimes H_m) \\ &= \sum_{N \ge l > k \ge 1} \sum_{j=k}^{l-1} \frac{{H_j}^{\otimes 2}}{N} + \sum_{N \ge l > k \ge 1} \sum_{l-1 \ge n > m \ge k} \frac{H_m \otimes H_n + H_n \otimes H_m}{N} \\ &= \frac{1}{N} \sum_{N \ge l > k \ge 1} \left(\sum_{j=k}^{l-1} {H_j}^{\otimes 2} + \sum_{l-1 \ge n > m \ge k} (H_m \otimes H_n + H_n \otimes H_m) \right) \\ &= \frac{1}{N} \sum_{N \ge l > k \ge 1} \left(\sum_{j=k}^{l-1} H_j \right)^{\otimes 2} = \frac{1}{N} \sum_{N \ge l > k \ge 1} (E_{kk} - E_{ll})^{\otimes 2}. \end{aligned}$$

The second equality above is shown by the following relation:

$$\sum_{i=j}^{k} \frac{1}{i(i+1)} = \frac{k+1-j}{j(k+1)},$$

which can be proved by induction. (Actually, it is easy to see the above calculation of T' from the bottom to the top.) Therefore we get an expression of T as below:

$$\begin{aligned} T &= \sum_{i,j=1,i\neq j}^{N} E_{ij}\otimes E_{ji} + \frac{1}{N}\sum_{k=1}^{N} E_{kk}\otimes \sum_{l=1}^{N}(E_{kk}-E_{ll}) \\ &= \sum_{i,j=1,i\neq j}^{N} E_{ij}\otimes E_{ji} + \sum_{k=1}^{N} E_{kk}\otimes \left(E_{kk} - \frac{1}{N}\sum_{l=1}^{N} E_{ll}\right) \\ &= \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji} - \frac{1}{N}\sum_{k,l=1}^{N} E_{kk}\otimes E_{ll}. \end{aligned}$$

Hence we get the following consequence immediately:

$$(\iota\otimes\iota)(T) = \sum_{i,j=1}^{N} \widetilde{e}_j\otimes e_i\otimes \widetilde{e}_i\otimes e_j - \frac{1}{N}\left(\sum_{k=1}^{N}\widetilde{e}_k\otimes e_k\right)\otimes\left(\sum_{l=1}^{N}\widetilde{e}_l\otimes e_l\right).$$

The right side of this equation is the same as the image of $1\in\mathbb{C}$ under the sum of weight systems

$$W_{sl(N,\mathbb{C}),\rho_0^*\otimes\rho_0}\left(\quad\right) - \frac{1}{N}W_{sl(N,\mathbb{C}),\rho_0^*\otimes\rho_0}\left(\quad\right),$$

and so this completes the proof of the second equation.

Finally, we prove the third equation. The weight system $W_{sl(N,\mathbb{C}),\rho}(\quad)$ is the Killing form B of $sl(N,\mathbb{C})$. As above, we can consider it as an element of $\mathrm{Hom}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2},\mu) = \mathrm{Hom}_{sl(N,\mathbb{C})}((gl(N,\mathbb{C}))^{\otimes 2},\mathbb{C})$. Comparing both sides of the third equation with respect to the basis $\{\widetilde{e}_i\otimes e_j\otimes\widetilde{e}_k\otimes e_l\}_{i,j,k,l=1}^{N}$ of $\rho_0^*\otimes\rho_0$, we can easily check the equality. So we omit the details. ■

Similar to Theorem 1.6, $W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\quad\right)$ can be interpreted as $W_{sl(N,\mathbb{C}),\rho_0}$ as follows.

Corollary 4.4. *Let $\overline{\iota\otimes\iota}$ be the isomorphism fixed in Remark* 1.5. *Then the following equation holds:*

$$\begin{aligned} \overline{\iota\otimes\iota}\left(W_{sl(N,\mathbb{C}),\mathrm{ad}}\left(\quad\right)\right) &= W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) + W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) \\ &\quad - W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right) - W_{sl(N,\mathbb{C}),\rho_0}\left(\quad\right), \end{aligned}$$

where both sides are equal as elements in $\mathrm{End}_{sl(N,\mathbb{C})}((\rho_0^*\otimes\rho_0)^{\otimes 2})$.

Note that the adjoint and the trivial representation of $sl(N,\mathbb{C})$ is self-dual, that is, $\mathrm{ad}^*=\mathrm{ad}$ and $\mu^*=\mu$ and so $(\mathrm{ad}\oplus\mu)\otimes(\mathrm{ad}\oplus\mu)^* = (\mathrm{ad}\oplus\mu)^{\otimes 2}$. Therefore the left side of the equation in Corollary 4.4 is well-defined.

Proof. First, transform the Jacobi diagram into $-$. Next, decompose it as in Figure 6. Then calculate the weight system of each part of the decomposed Jacobi diagram by Theorem 1.6 and Lemma 4.3 and contract them. This completes the proof. ■

We remark that formulas of the weight system for , analogous to Corollary 4.4 can be derived in the same way.

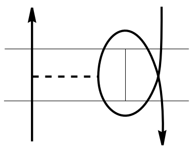


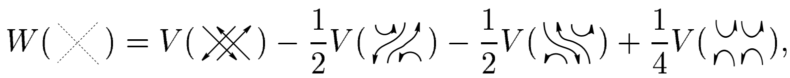
Figure 6: Decomposition using horizontal and vertical lines

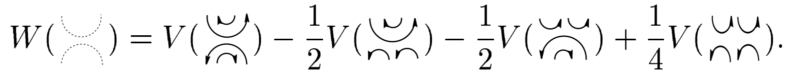
5. Reconstruction

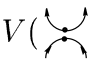
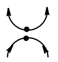
In this section, we reconstruct Theorems 1.2 and 1.3 by using Theorem 1.6. For convenience, the isomorphisms $\bar{\iota}$ and $\overline{\iota \otimes \iota}$ are omitted and the weight systems $W_{sl(N,\mathbb{C}),\mathrm{ad}}$ and $W_{sl(N,\mathbb{C}),\rho_0}$ are denoted by W and V respectively from now on. (Note that N is considered as 2 in Theorem 1.2 and as 3 in Theorem 1.3.)

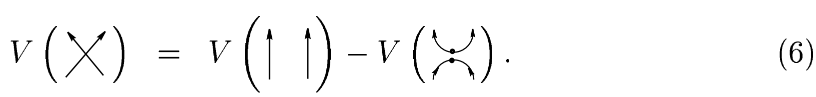
Reconstruction of Theorem 1.2**.** By Theorem 1.6, we have

The following two relations are derived from Lemma 4.3:

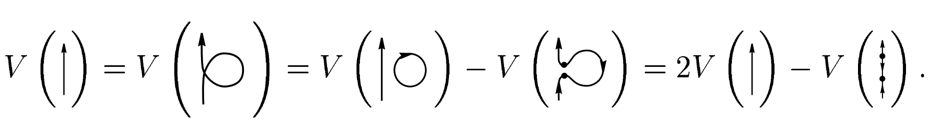




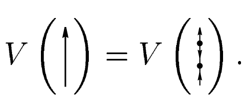
Next, a Jacobi diagram with singular points means a Jacobi diagram with bivalent vertices on the support such that the orientations of two edges at a bivalent vertex are in opposite directions. Let us define the weight system $V($$)$ of the Jacobi diagram  with two singular points as follows (this is possible in the case of $N = 2$):



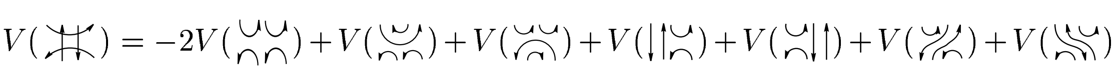
By definition, two singular points on the support can be cancelled as follows:

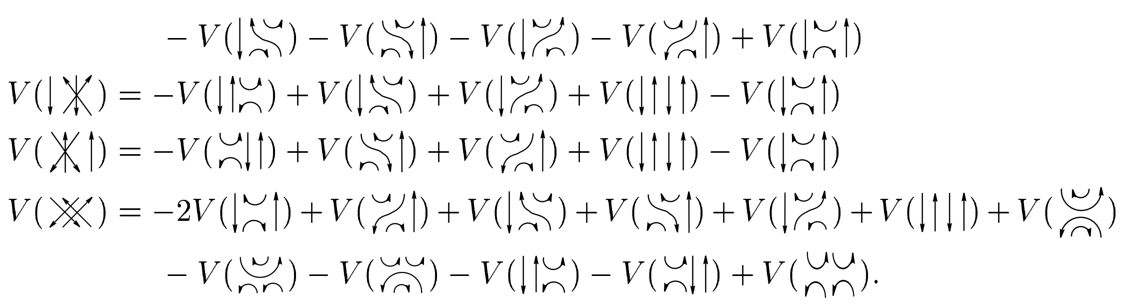


Namely, we have



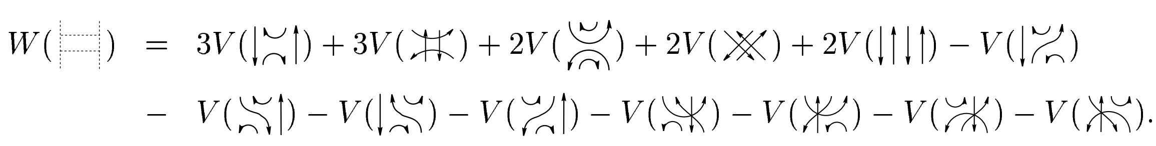
By using Equation (6) and the above cancellation of singular points, we can remove all crossings and singular points of the support. Then we get the following relations:



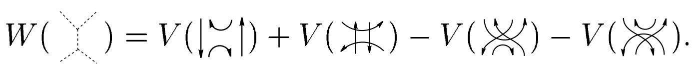


By using these relations, we can reconstruct Chmutov-Varchenko's formula. ■

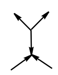
Reconstruction of Theorem 1.3. By Theorem 1.6, we get



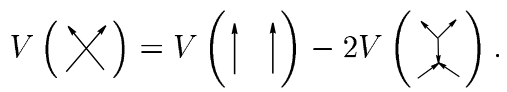
By the IHX relation, we have



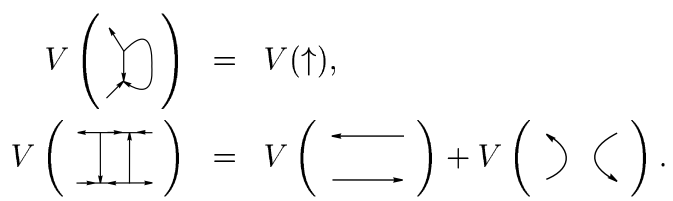
A uni-trivalent plane graph means a uni-trivalent graph which can be embedded in a plane. From now on, we suppose that every uni-trivalent plane graph is embedded in a plane. Next, for a uni-trivalent plane graph G,

$G =$  ,

we define the weight system $V(G)$ as follows:



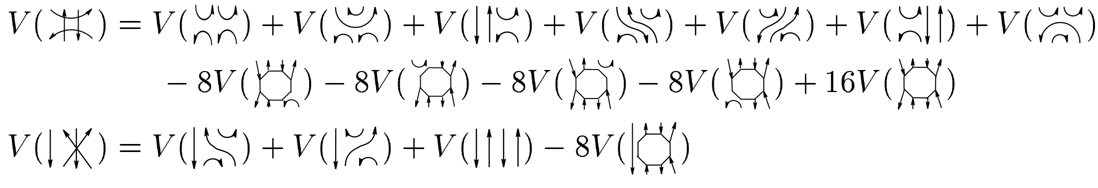
In fact, the idea of the above definition comes from Yokota's paper ([17]) and a graphical relation of the HOMFLY polynomial described by using "flow" ([11]). The following relations can be derived from the definition:

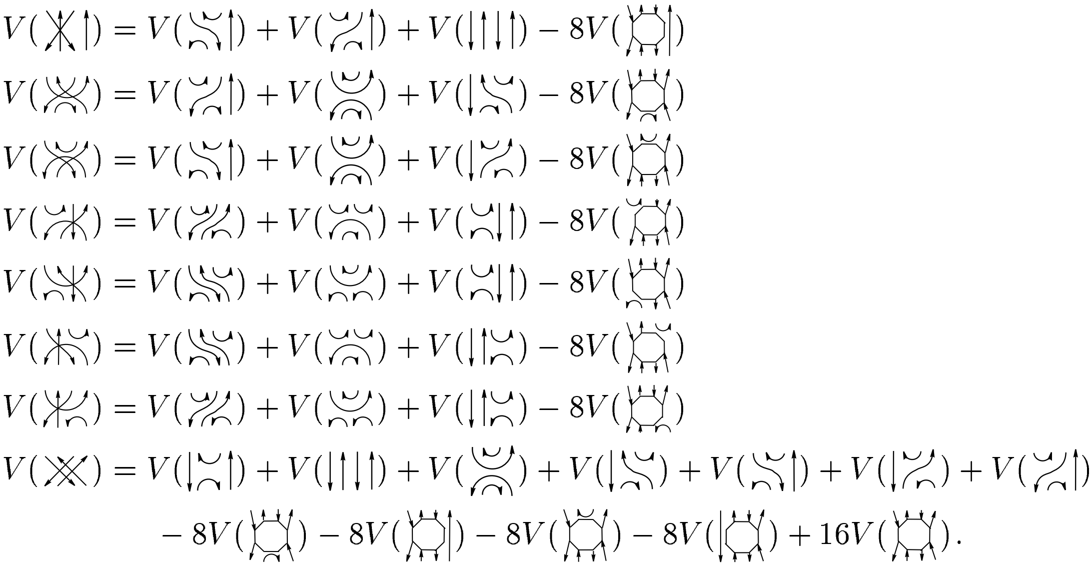


(7)

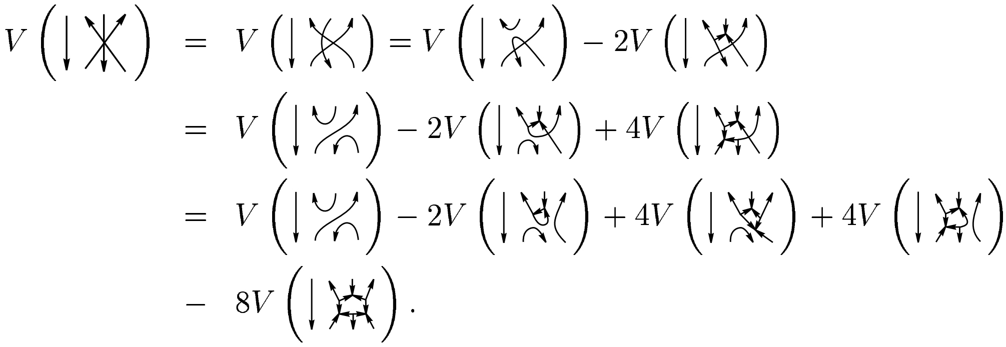
(8)

By using Equations (7) and (8) above again and again, we can get uni-trivalent plane graphs from a Jacobi diagram with crossings. Then we get the following consequence:

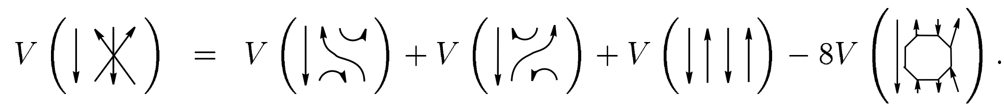




For example, the second equation is derived as follows:



By using Equations (7) and (8), we can get the following consequence:



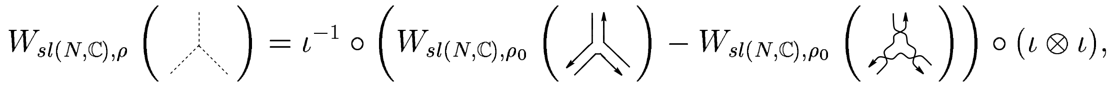
By using these relations, we can reconstruct Yoshizumi-Kuga's formula. ■

6. Remark on the formula in Theorem 1.6

Although Theorem 1.6 is generalized for the universal $sl(N,\mathbb{C})$-weight system in [12], the formula in Theorem 1.6 has a more efficient description than the formula in [12] in the sense that the dimension of the representation space of $sl(N,\mathbb{C})$ in Theorem 1.6 is taken to be minimal (refer to [12] for details).

In fact, Theorem 1.6 can be also proved by using Lemma 4.3 and the following result:

Lemma 6.1. **(Bar-Natan [1])** *Let ι and $\iota\otimes\iota$ be the isomorphisms fixed in Remark 1.4. Then for an arbitrary finite-dimensional irreducible representation ρ, the following equation holds:*

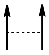


where both sides are equal as elements in $\mathrm{Hom}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2}, \mathrm{ad}\oplus\mu)$.

Note that the left side of the equation in Lemma 6.1 is well-defined. That is, the weight system $W_{sl(N,\mathbb{C}),\rho}($  $)$, which is the Lie bracket $[\cdot,\cdot]$ on $sl(N,\mathbb{C})$, can be naturally thought of as an element of

$$\mathrm{Hom}_{sl(N,\mathbb{C})}((\mathrm{ad}\oplus\mu)^{\otimes 2}, \mathrm{ad}\oplus\mu) = \mathrm{Hom}_{sl(N,\mathbb{C})}(gl(N,\mathbb{C})^{\otimes 2}, gl(N,\mathbb{C})),$$

because the Lie bracket on $sl(N,\mathbb{C})$ is the restriction of that on $gl(N,\mathbb{C})$ to $sl(N,\mathbb{C})$.

To show Theorem 1.6 by using Lemma 6.1, we first decompose the Jacobi diagram  into three parts as in Figure 7. Then we calculate the weight system of each part of the Jacobi diagram by using Lemmas 6.1 and 4.3. Composing those weight systems, we get Theorem 1.6 as in Figure 8.

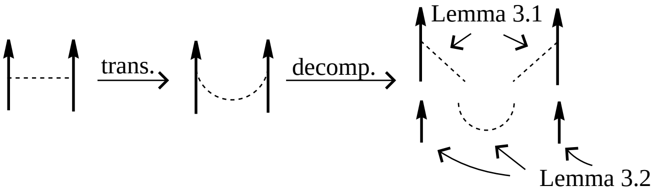


Figure 7: Transformation and decomposition

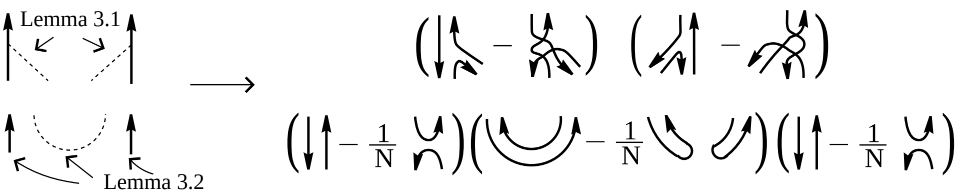


Figure 8: Evaluation using Lemmas 6.1 and 4.3

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