Journal of Lie Theory
Volume **17** (2007) 829–856

Spin Holonomy Algebras of Self-Dual 4-Forms in $\mathbb{R}^8$

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Communicated by A. Valette

Abstract. We give a complete classification of spin holonomy algebras on eight-dimensional Euclidean spaces w.r.t. a linear spin connection constructed from a self-dual 4-form T with constant coefficients. An important rôle in this classification is played by the set of spinors fixed by T, which is the algebraic model for the set of parallel spinors w.r.t. the spin connection.
Mathematics Subject Classification 2000: 53C10, 53C27, 53C29.
Key Words and Phrases: Spin connection, spin holonomy algebra.

1. Introduction

Let (M^8, g) be an eight-dimensional Riemannian spin manifold with spinor bundle denoted by $\not{S}(M)$. For any differential form T on M, not necessarily of pure degree, one can form the linear connection ∇^T on $\not{S}(M)$ by setting

$$\nabla^T_X \psi = \nabla_X \psi + (X \lrcorner T)\psi, \tag{1}$$

whenever ψ is a spinor field on M and X is in TM. We are mainly interested in parallel spinors w.r.t. the above spin connection. When T is a 3-form, the connection ∇^T on $\not{S}(M)$ is induced from a metric connection with totally skew-symmetric torsion on TM, and moreover ∇^T preserves the chirality decomposition

$$\not{S}(M) = \not{S}^+(M) \oplus \not{S}^-(M).$$

Then ∇^T-parallel spinors in $\not{S}^+(M)$ are, in the case when T is a 3-form, in one to one correspondence with unit length harmonic spinors in $\not{S}^+(M)$. Moreover, they correspond to a certain class of $Spin(7)$-structures on M (see [10] and references therein).

The next situation to look at is when T is a 4-form, when the associated connection ∇^T is no longer induced from one on TM, nor is it metric for the positive signature inner product on $\not{S}(M)$. Also the chirality decomposition fails to be preserved.

ISSN 0949–5932 / \$2.50

As it is well known [3], if $\nabla^T\psi = 0$ holds for a spinor field ψ, then at each point x of M we have that ψ_x is fixed by the holonomy algebra $\mathfrak{hol}_x^T$ at x of the spin connection ∇^T. Conversely, an algebraic spinor at a given point x of M, which is fixed by $\mathfrak{hol}_x^T$ extends, at least when the manifold is simply connected, to a ∇^T-parallel spinor field.

Therefore, one object of interest as far as ∇^T-parallel spinors are concerned is the holonomy algebra $\mathfrak{hol}^T$. Following [2] we will restrict to the special case of a flat manifold, where moreover T is assumed to have constant coefficients. In this situation, our goal in this paper is to obtain a full classification of the algebras $\mathfrak{hol}^T$ where T is a self-dual 4-form in 8-dimensions. Let us make the following definitions.

Definition 1.1. *Let $(V, \langle\cdot,\cdot\rangle)$ be an eight-dimensional Euclidean vector space and let Cl_8 be its Clifford algebra. Then:*

(i) the fix algebra of T in Cl_8 is the Lie-sub-algebra $\mathfrak{g}_T^*$ of Cl_8 generated by $\{X \lrcorner T : X \in V\}$.

(ii) the holonomy algebra of some T in Cl_8 is given as $\mathfrak{h}_T^* = [\mathfrak{g}_T^*, \mathfrak{g}_T^*]$.

In this situation $\mathfrak{hol}^T$ and $\mathfrak{h}_T^*$ coincide [1] due to the flatness of the metric and of the constancy of T, whence the geometric significance of the algebraic objects introduced in Definition 1.1. However, the holonomy algebra $\mathfrak{h}_T^*$ is still a rather non-trivial object, especially because its generators are fairly inaccessible. Consequently we will deal with the fix algebra $\mathfrak{g}_T^*$ rather than with $\mathfrak{h}_T^*$ itself. This is less restrictive than it might appear, in view of general structure results of [2]. In particular let us mention that we are not aware of any example of a fix algebra, which is neither perfect nor abelian and hence does not completely determine the holonomy algebra. Here we recall that a fix algebra $\mathfrak{g}_T^*$ is called *perfect* if it coincides with the holonomy algebra $\mathfrak{h}_T^*$.

Another object of relevance for our study is the space Z_T of spinors fixed by $\mathfrak{g}_T^*$, that is

$$Z_T = \{\psi \in \not{S} : \mathfrak{g}_T^*\psi = 0\}.$$

where $\not{S}$ is some irreducible $Cl(V)$-module and T is some form on V. This is the algebraic model, in the flat case, for the space of spinors which are parallel w.r.t the connection ∇^T. By [2, Prop. 3.8] the set Z_T is trivial in any dimension less than eight, while the cases when T has degree less than four have been studied ([1] for the case of 3-forms) or produce trivial results. This provides some further, algebraically inspired motivation, to treat the case of 4-forms in eight dimensions.

This is the precise working context in this paper which is structured as follows. Section 2 contains a few preliminaries on Clifford algebras and spinor modules and also a brief review of some results from [2] to be used later on. In Section 3 we give a complete classification of holonomy algebras generated by self-dual 4-forms in dimension 8. More precisely, we show:

Theorem 1.1. *Let V be an oriented Euclidean vector space of dimension 8 and let $T \neq 0$ be a self-dual four form on V. The fix algebra of the form T is perfect and its holonomy algebra is isomorphic to the Lie algebra $\mathfrak{so}(8, 8-\dim_{\mathbb{R}} Z_T)$ exception made of the cases when* (*i*) *T is proportional to a unipotent element, that is $T^2 = \lambda(1+\nu)$ in Cl_8, for some $\lambda > 0$*

or

(*ii*) $\dim_{\mathbb{R}} Z_T = 6$

when the holonomy algebras are isomorphic to $\mathfrak{so}(8,1)$ and $\mathfrak{so}(6,2)$ respectively.

The proof of Theorem 1.1, exception made of the case when $\dim_{\mathbb{R}} Z_T = 6$, uses the splitting of the space of two forms induced by a given self-dual four form. This is combined with the observation that raising the generating element to any odd Clifford power leaves the initial fix algebra unchanged. In Section 4 of the paper, we treat directly the special case appearing in (ii) of Theorem 1.1 using the one to one correspondence [8] between the existence of a such a form and that of an $SU(4)$-structure on our vector space.

2. Preliminaries

Let $(V^8, \langle \cdot, \cdot \rangle)$ be an eight-dimensional Euclidean vector space which is moreover assumed to be oriented by a volume form ν in $\Lambda^8(V)$. The Clifford algebra over V shall be denoted by Cl_8 with multiplication given by $(\varphi_1, \varphi_2) \mapsto \varphi_1\varphi_2$ for all φ_1, φ_2 in Cl_8. We shall also remind the reader of the expansion of the Clifford product in the exterior algebra $\Lambda^*(V)$

$$v\,\varphi = v \wedge \varphi - v \lrcorner \varphi, \qquad (-1)^k \varphi\, v = v \wedge \varphi + v \lrcorner \varphi, \tag{2}$$

where v is in V and φ is in $\Lambda^k(V) \subseteq Cl_8$. Note that here and in what follows we will systematically identify 1-forms and vectors by using the inner product on V.

Since in eight dimensions the volume element ν is an involution of Cl_8, in the sense that $\nu^2 = 1$, we have a splitting

$$Cl_8 = Cl_8^+ \oplus Cl_8^-,$$

where $\nu\varphi = \pm\varphi$ for $\varphi \in Cl_8^{\pm}$. The Clifford algebra Cl_8 is equipped with two canonical involutions $\alpha : Cl_8 \to Cl_8$ and $()^t : Cl_8 \to Cl_8$, the latter being referred to as the transpose, which are essentially described by

$$\alpha(\varphi) = (-1)^k \varphi, \quad \varphi^t = (-1)^{\frac{k}{2}(k-1)} \varphi, \quad \alpha(\varphi^t) = (-1)^{\frac{k}{2}(k+1)} \varphi, \tag{3}$$

whenever φ belongs to $\Lambda^k(V) \subset Cl_8$. The involution α is actually an automorphism of the Clifford algebra Cl_8 whereas the transpose is an anti-automorphism, that is

$$(\varphi_1\varphi_2)^t = \varphi_2^t \varphi_1^t,$$

for all φ_1, φ_2 in Cl_8. In view of subsequent computations it is convenient to record that

$$\alpha(\varphi)\nu = \nu\varphi,$$

for all φ in Cl_8. Moreover, there is a second direct sum splitting

$$Cl_8 = Cl_8^0 \oplus Cl_8^1,$$

where the summands above are given by the $\pm$-eigenspaces of α and thus consist in forms of even and odd degrees, respectively. For later use we also note that (2) may be rewritten as

$$v \lrcorner \varphi = -\tfrac{1}{2}(v\,\varphi - \alpha(\varphi)\,v), \qquad v \wedge \varphi = \tfrac{1}{2}(v\,\varphi + \alpha(\varphi)\,v). \tag{4}$$

for all v in V and φ in Cl_8 respectively.

To end this section we recall two more facts. The inner product on V induces an inner product $\langle\cdot,\cdot\rangle$ on Cl_8 such that

$$\langle \varphi_1\,\varphi_2, \varphi\rangle = \langle \varphi_2, \alpha(\varphi_1^t)\,\varphi\rangle = \langle \varphi_1, \varphi\,\alpha(\varphi_2^t)\rangle \tag{5}$$

holds for any $\varphi_1, \varphi_2, \varphi$ in Cl_8. Moreover the canonical decompositions of the Clifford algebra presented above are orthogonal ones w.r.t. the inner product Cl_8 has been equipped with. The operator $L : Cl_8 \to Cl_8$ defined by

$$L\varphi = \sum_{i=1}^{8} e_i \varphi e_i, \tag{6}$$

for all φ in Cl_8 and where $\{e_i, 1 \le i \le 8\}$ is some orthonormal basis, recovers the pure degree components of a form in $\Lambda^*(V) \subset Cl_8$, for

$$L = (-1)^k (2k - 8) 1_{\Lambda^k(V)}$$

on $\Lambda^k(V) \subset Cl_8, 0 \le k \le 8$.

2.1. Spinor products.

We consider now the real spinor representation

$$\mu : Cl_8 \to End_{\mathbb{R}}(\$),$$

where the irreducible and finite dimensional Clifford left module $\$$ is called the space of spinors. Most of the time we shall write $\mu_\varphi(\psi) = \varphi\psi$ for $\varphi \in Cl_8$ and $\psi \in \$$. The splitting of Cl_8 into self-dual and anti-self-dual components carries over to the space of spinors

$$\$ = \$^+ \oplus \$^-,$$

where $\nu\psi = \pm\psi$ for $\psi \in \$^\pm$. Note that this holds for any Clifford module, regardless of irreducibility. Later on in this paper we shall use frequently the following elementary observation, mainly when $W = Cl_8$ or $W = \$$.

Lemma 2.1. *Let W be any real Clifford module. The following stability conditions hold:*

$$\begin{aligned} \varphi W^\pm &\subseteq W^\pm, && \text{for all } \varphi \in Cl_8^0 \cap Cl_8^\pm, \\ \varphi W^\pm &= \{0\}, && \text{for all } \varphi \in Cl_8^0 \cap Cl_8^\mp \text{ or } Cl_8^1 \cap Cl_8^\pm, \\ \varphi W^\pm &\subseteq W^\mp, && \text{for all } \varphi \in Cl_8^1 \cap Cl_8^\mp. \end{aligned}$$

Now, following [7, Thm. 13.17 & Table 13.19] there exists a Euclidean inner product $\langle\cdot,\cdot\rangle$ on $\$$ such that

$$\langle \varphi\psi_1, \psi_2\rangle = \langle \psi_1, \alpha(\varphi^t)\psi_2\rangle, \tag{7}$$

whenever ψ_1, ψ_2 belong to $\$$ and φ is an element in Cl_8. Moreover, we have the following trace Lemma, to be used extensively in what follows. For the proof we refer the reader to [7, Thm. 9.65] or [2, Lemma 2.3].

Lemma 2.2. *Let W be any real finite dimensional Cl_8-module. Then:*

$$Tr_W(\mu_\varphi) = \dim_{\mathbb{R}} W \, \langle \varphi, 1 \rangle,$$

whenever φ belongs to Cl_8.

If the contrary is not specified, the Lemma above will be mostly used in the case when $W = \$$. Of particular relevance in what follows is that the Clifford multiplication $\mu : Cl_8 \to End_{\mathbb{R}}(\$)$ is actually an isomorphism (see e.g. [7, Thm. 11.3]). In fact, using also the inner product on $\$$ allows one to multiply spinors, in the sense of the Definition below.

Definition 2.1. *Let x and y be in $\$$. Then the spinor product $x \otimes y$ in Cl_8 is defined by*

$$(x \otimes y)\, \psi = \langle \psi, x \rangle \, y,$$

for all ψ in $\$$. Accordingly the symmetric and skew-symmetric spinor products of x and y are given by

$$\begin{aligned} x \odot y &= x \otimes y + y \otimes x \\ x \wedge y &= x \otimes y - y \otimes x. \end{aligned}$$

For simplicity we have defined the symmetrised spinor product in Definition 2.1 without any combinatorial factor. The isomorphism $\$ \otimes \$ \cong \Lambda^*(V)$ actually yields [12, Prop. 10.17]

$$\Lambda^2(\$) = \bigoplus_{k \equiv 1,2 \pmod 4} \Lambda^k(V), \qquad S^2(\$) = \bigoplus_{k \equiv 0,3 \pmod 4} \Lambda^k(V). \tag{8}$$

Here $\Lambda^2(\$)$ is defined to be the linear span of $\{x \wedge y : x, y \text{ in } \$\}$, and a completely similar interpretation is made regarding the second symmetric tensor power $S^2(\$)$. For self-dual spinors, we state the following

Lemma 2.3. *Let $(V^8, \langle \cdot, \cdot \rangle)$ be an oriented Euclidean vector space. Then*

$$\Lambda^2(\$^+) = (1 + \nu)\Lambda^2(V).$$

Proof. From Definition 2.1 it follows that $\Lambda^2(\$^+) = \Lambda^2(\$) \cap Cl_8^+$. But from (8) we have that $\Lambda^2(\$^+) = \Lambda^2(V) \oplus \Lambda^6(V)$ and the claim follows now by using that $\Lambda^6(V) = \nu \Lambda^2(V)$. ■

This is very peculiar to the case of dimension 8, which prevents the occurrence of higher degrees in $\Lambda^2(\$^+)$, as it appears from (8).

2.2. Model algebras in dimension 8.

We first recall that the Hodge star operator $* : \Lambda^k(V) \to \Lambda^{8-k}(V), 0 \leq k \leq 8$ can be alternatively viewed in the Clifford algebra as

$$*\varphi = \alpha(\varphi^t)\, \nu = \nu\, \varphi^t,$$

for all φ in Cl_8. Therefore $\Lambda^4(V)$ is stable under the Hodge star operator and moreover $*^2 = id$ on $\Lambda^4(V)$, allowing one to split

$$\Lambda^4(V) = \Lambda^4_+(V) \oplus \Lambda^4_-(V)$$

into the $\pm$-eigenspaces of $*$. Four forms in $\Lambda^4_\pm(V)$ are called self-dual respectively anti-self-dual, in analogy to the case of 2-forms in dimension 4.

Our aim in this paper is to obtain classification results for holonomy algebras $\mathfrak{h}^*_T$ generated by T in $\Lambda^4_+(V)$. To this extent we need to present a few preparatory results. Let us define

$$A = \{\varphi \in Cl_8 : \varphi^t = -\varphi\} \tag{9}$$

and recall [2, Lemma 3.1] that this is a Lie subalgebra of $(Cl_8, [\cdot,\cdot])$, where the commutator defined by

$$[\varphi_1, \varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1, \tag{10}$$

for all φ_1, φ_2 in Cl_8, gives Cl_8 the structure of a Lie algebra. A is preserved by the involution α and therefore splits as

$$A = A^0 \oplus A^1,$$

where $\alpha(\varphi) = \varphi$ for $\varphi \in A^0$ and $\alpha(\varphi) = -\varphi$ for $\varphi \in A^1$. Moreover, it can be easily seen that this splitting satisfies the relations

$$[A^0, A^0] \subseteq A^0, \quad [A^0, A^1] \subseteq A^1 \quad \text{and} \quad [A^1, A^1] \subseteq A^0. \tag{11}$$

This is actually saying that A is an orthogonal symmetric Lie algebra in the sense of [9, page 377]. The following is specialising Lemma 3.3 in [2] to the case of dimension 8.

Lemma 2.4. *The following hold:*

(i) A is isomorphic to $\mathfrak{so}(8,8)$;

(ii) the adjoint representation of A^0 on A^1 is irreducible.

Here the Lie algebra $\mathfrak{so}(8,8)$ arises as $\mathfrak{so}(\$, \widehat{\beta})$ where the split signature scalar product $\widehat{\beta}$ keeps $\$^\pm$ orthogonal and equals $\pm\langle\cdot,\cdot\rangle$ on $\$^\pm$. An important property of A^0 is to be stable under multiplication with the volume form ν. Therefore it can be split further as

$$A^0 = A^0_+ \oplus A^0_-,$$

where $A^0_\pm = A^0 \cap Cl_8^\pm$. It is easy to check that we have explicitly

$$A^0_\pm = (1 \pm \nu)\Lambda^2(V) \tag{12}$$

and also that $A^1 = \Lambda^3(V) \oplus \Lambda^7(V)$. It should be noted that (12) holds only in 8-dimensions. We also notice, for further use, that

$$\{\varphi \in Cl_8^0 \cap Cl_8^+ : \varphi^t = \varphi\} = \Lambda^4_+(V) \oplus \mathbb{R}(1+\nu). \tag{13}$$

Actually, A plays the rôle of a model algebra in the following sense.

Proposition 2.1. *[2, Prop. 3.1] For any T in $\Lambda^4_+(V)$, the following hold:*

(i) $\mathfrak{g}^*_T$ *is a Lie sub-algebra of A, which is preserved by α;*

(ii) *we have a splitting* $\mathfrak{g}^*_T = \mathfrak{g}^{*,0}_T \oplus \mathfrak{g}^{*,1}_T$ *where* $\mathfrak{g}^{*,k}_T = \mathfrak{g}^*_T \cap A^k, k = 0, 1$.

Let us recall now that for any form T in $\Lambda^*(V)$ one defines the set of spinors fixed by T by

$$Z_T = \{\psi \in \$: (X \lrcorner T)\psi = 0, \text{for all } X \in V\}. \tag{14}$$

For any T in $Cl^i_8, i = 0, 1$ the set Z_T splits along $\$ = \$^+ \oplus \$^-$ as

$$Z_T = Z^+_T \oplus Z^-_T,$$

where the obvious notations applies. Using Lemma 3.5 of [2] we also recall that

$$Z^-_T = \{0\} \quad \text{and} \quad Z_T = \{\psi \in \$^+ : T\psi = 0\}, \tag{15}$$

for any non-zero T in $\Lambda^4_+(V)$.

We end this section by presenting an embedding of the fix algebra of some self-dual form in dimension 8, taking into account the set of spinors it fixes. For some fixed T in $\Lambda^4_+(V)$ we consider the splitting

$$\$^+ = Z_T \oplus Z^\perp_T,$$

w.r.t. the positive definite scalar product $\langle \cdot, \cdot \rangle$. Let us define $\widehat{\beta}_T$ to be the restriction of $\widehat{\beta}$ to $\$^- \oplus Z^\perp_T$. From the definition of $\widehat{\beta}$ it follows that $\widehat{\beta}_T$ has signature $(8, 8 - \dim_{\mathbb{R}} Z_T)$ and moreover

Proposition 2.2. *For any T in $\Lambda^4_+(V)$ the Clifford multiplication*

$$\mu : \mathfrak{g}^*_T \to \mathfrak{so}(\$^- \oplus Z^\perp_T, \widehat{\beta}_T) \cong \mathfrak{so}(8, 8 - \dim_{\mathbb{R}} Z_T)$$

is a monomorphism of Lie algebras.

Proof. It is enough to show that $\mu(\mathfrak{g}^*_T) \subseteq \mathfrak{so}(\$^- \oplus Z^\perp_T, \widehat{\beta}_T)$. But from the definition of Z_T we have that $\mathfrak{g}^*_T Z_T = \{0\}$ and moreover from (7) we know that μ_φ, with φ in $\mathfrak{g}^*_T$, is skew-symmetric with respect to $\widehat{\beta}$ therefore with respect to $\widehat{\beta}_T$. Now because $\mu : Cl_8 \to End_{\mathbb{R}}(\$)$ is a faithful representation of algebras it follows by using (10) that $\mu_{|A} : A \to \Lambda^2(\$)$ is an injective morphism of Lie algebras. Consequently, the restriction of μ to $\mathfrak{g}^*_T \subseteq A$ is an injective Lie algebra morphism as well. ∎

3. The classification

3.1. Self-dual 4-forms.

Let us pick $T \neq 0$ in $\Lambda^4_+(V)$. Using (15) we have that $Z_T = Z_T^+$ and let us consider the symmetric and traceless operator $\mu_T : \$^+ \to \$^+$. Again from (15) one obtains that $Z_T = Ker(\mu_T)$. Let $\sigma_T = \{\lambda_q, 1 \leq q \leq p\}$ be the non-zero part of the spectrum of μ_T where we assume the eigenvalues $\lambda_q, 1 \leq q \leq p$ to be pairwise distinct and where we denote their multiplicities by $m_q, 1 \leq q \leq p$. Therefore we obtain a splitting

$$\$^+ = Z_T \oplus \$_1 \oplus \ldots \$_p, \tag{16}$$

where $\$_q$ are the eigenspaces of μ_T corresponding to the eigenvalues $\lambda_q, 1 \leq q \leq p$. Our aim here is to examine the splitting of $\Lambda^2(V)$ induced by (16) and to relate it directly to the form T. We need now to recall the following simple fact, which essentially exploits the squaring isomorphism in 8-dimensions as introduced in Definition 2.1.

Lemma 3.1. *Let x, y belong to $\$^+$. The following hold:*

(*i*) *$x \wedge y$ belongs to $Cl_8^0 \cap Cl_8^+$ and $(x \wedge y)^t = -x \wedge y$.*

(*ii*) *If moreover $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$, where T belongs to $\Lambda^4_+(V)$, then*

$$T(x \wedge y)T = \lambda_1\lambda_2\, x \wedge y, \quad \text{and} \quad T(x \wedge y) + (x \wedge y)T = (\lambda_1 + \lambda_2) x \wedge y.$$

(*iii*) *Under the assumptions in (ii), if $\lambda_1 = \lambda_2$ then $T(x \wedge y) = (x \wedge y)T = \lambda_1 x \wedge y$.*

(*iv*) *If x', y' is another pair of spinors in $\$^+$, then*

$$[x \wedge y, x' \wedge y'] = \langle x, y' \rangle x' \wedge y - \langle y, y' \rangle x' \wedge x - \langle x, x' \rangle y' \wedge y + \langle x', y \rangle y' \wedge x,$$

(*v*) *and also*

$$8\langle x \wedge y, x' \wedge y' \rangle = \langle y, y' \rangle \langle x, x' \rangle - \langle x, y' \rangle \langle y, x' \rangle.$$

Proof. (*i*) is elementary. We prove (*ii*) and (*iii*) at the same time. For any ψ in $\$$ we have

$$(x \wedge y)T\psi = \langle T\psi, x \rangle y - \langle T\psi, y \rangle x = \langle \psi, Tx \rangle y - \langle \psi, Ty \rangle x = \lambda_1 \langle \psi, x \rangle y - \lambda_2 \langle \psi, y \rangle x,$$

as $\langle T\psi, x \rangle = \langle \psi, Tx \rangle = \lambda_1 \langle \psi, x \rangle$ and similarly $\langle T\psi, y \rangle = \lambda_2 \langle \psi, y \rangle$. Moreover,

$$T(x \wedge y)\psi = \langle \psi, x \rangle Ty - \langle \psi, y \rangle Tx = \lambda_2 \langle \psi, x \rangle y - \lambda_1 \langle \psi, y \rangle x.$$

All claims in (*ii*) and (*iii*) follow now easily. The proof of (*iv*) is a straightforward direct computation involving only the definition of the exterior product of spinors. (*v*) A direct computation based on the definition of the wedge product of spinors shows that the trace of the Clifford multiplication with $(x \wedge y)(x' \wedge y')$ in Cl_8 is given by

$$-2\Big[\langle y, y' \rangle \langle x, x' \rangle - \langle x, y' \rangle \langle y, x' \rangle\Big].$$

The claim follows now by using Lemma 2.2. ■

For any $1 \leq i, j, k \leq p$ let us now define the spaces

$$\begin{aligned}
E_k &= \{\gamma \in \Lambda^2(V) : T\gamma T = 0, \quad T\gamma + \gamma T = \tfrac{\lambda_k}{2}(1+\nu)\gamma\} \\
F_{ij} &= \{\gamma \in \Lambda^2(V) : T\gamma T = \tfrac{\lambda_i \lambda_j}{2}(1+\nu)\gamma, \quad T\gamma + \gamma T = \tfrac{\lambda_i + \lambda_j}{2}(1+\nu)\gamma\} \qquad (17) \\
\iota_T^0 &= \{\gamma \in \Lambda^2(V) : \gamma T = T\gamma = 0\}.
\end{aligned}$$

Obviously we have $F_{ij} = F_{ji}$. Moreover, the spaces ι_T^0, E_k and F_{ij} with $1 \leq i, j, k \leq p$ are easily seen to stand in direct sum directly from their definition. For notational convenience, we set

$$E = \bigoplus_{k=1}^{p} E_k, \qquad F = \bigoplus_{1 \leq i \leq j \leq p} F_{ij}.$$

Another related object is

Definition 3.1. *The isotropy algebra ι_T of T in $\Lambda^4(V)$ is the subalgebra of $\mathfrak{so}(V)$ given by*

$$\{\gamma \in \mathfrak{so}(V) : [\gamma, T] = 0\}.$$

Here the Lie bracket is considered within the Lie algebra Cl_8.

For any two vector sub-spaces W_1, W_2 of $\$^+$ we denote by $W_1 \hat{\otimes} W_2$ the linear span of $\{w_1 \wedge w_2 : w_k \in W_k, k = 1, 2\}$. Note that $W \hat{\otimes} W = \Lambda^2(W)$ whenever $W \subseteq \$^+$. We can now explicitly relate the spaces defined in (17) to the spectral decomposition of μ_T and show they induce a splitting of $\Lambda^2(V)$.

Proposition 3.1. *The following hold :*

(*i*) *There is an orthogonal direct sum decomposition $\Lambda^2(V) = \iota_T^0 \oplus E \oplus F$.*

(*ii*) *We have the following string of isomorphisms:*

$$\iota_T^0 \cong \Lambda^2(Z_T), \quad E_k \cong Z_T \hat{\otimes} \$_k, \quad F_{kk} \cong \Lambda^2(\$_k), \quad F_{ij} \cong \$_i \hat{\otimes} \$_j, \ i \neq j.$$

(*iii*) $\iota_T \cong \iota_T^0 \oplus \bigoplus_{k=1}^{p} F_{kk}$.

Proof. We prove (*i*) and (*ii*) together. Letting now ι_T^0, E_k, F_{ij} be the images of $\Lambda^2(Z_T), Z_T \hat{\otimes} \$_k, \$_i \hat{\otimes} \$_j$ under the inverse of the linear isomorphism

$$\mu|_{(1+\nu)\Lambda^2(V)} : (1+\nu)\Lambda^2(V) \to \Lambda^2(\$^+)$$

proves our claims by making use of Lemma 3.1, (ii) and (iii).
(*iii*) Pick γ in ι_T. Then $T\gamma = \gamma T$ hence $T^2\gamma + \gamma T^2 = 2T\gamma T$. It is easy to see that the operator $\{T^2, \cdot\} - 2T \cdot T$ equals 0 on ι_T^0, $\frac{\lambda_k^2}{2}(1+\nu)1_{E_k}$ on E_k and $\frac{(\lambda_i - \lambda_j)^2}{2}(1+\nu)1_{E_{ij}}$ on F_{ij} thus $\iota_T \subseteq \iota_T^0 \oplus \bigoplus_{k=1}^{p} F_{kk}$. The reverse inclusion and therefore the equality follows from the construction of the spaces $F_{kk}, 1 \leq k \leq p$ and Lemma 3.1, (iii). ∎

In the Proposition above the fact that V is 8-dimensional, which implies that $A^0_{\pm} = (1 \pm \nu)\Lambda^2(V)$, see (12), has been used in a crucial way. The block structure of the isotropy algebra of a form T in $\Lambda^4_+(V)$ has been already obtained in [4] by a slightly different method and under the additional assumption that T is a calibration on V. In this case, the work in [4] gives a complete geometric description of the resulting orbits. In order to understand the structure of the holonomy algebra of T we need to have a look at the Lie algebraic features of the splitting above.

Corollary 3.1. *Let T in $\Lambda^4_+(V)$ be given. Then F is a Lie sub-algebra of $\Lambda^2(V)$ isomorphic to $\mathfrak{so}(Z_T^{\perp})$.*

Proof. From the construction of F the Clifford multiplication map gives an isometry $\mu : (1+\nu)F \to \Lambda^2(Z_T^{\perp})$ by Lemma 3.1, (v). Moreover, this is a Lie algebra isomorphism by (iv) of the same Lemma. ■

Lemma 3.2. *Let T be in $\Lambda^4_+(V)$. We have:*

(i) $[F_{ij}, F_{ik}] = F_{jk}$ *if i, j, k are mutually distinct,*

(ii) $[F_{ij}, F_{ij}] = F_{ii} \oplus F_{jj}$ *when $i \neq j$.*

Moreover, if $Z_T \neq \{0\}$ the following hold:

(iii) $[E_i, E_j] = F_{ij}$ *for $i \neq j$ and $[E_i, E_i] = F_{ii} \oplus \iota^0_T$.*

Proof. *(i)* Let x_i and x'_i belong to $\not{S}_i$, whereas $y_j \in \not{S}_j$ and $z_k \in \not{S}_k$. Then from (iv) of Lemma 3.1 we get

$$[x_i \wedge y_j, x'_i \wedge z_k] = \langle x_i, x'_i \rangle y_j \wedge z_k.$$

Given that F_{jk} is spanned by all $\{y_j \wedge z_k : y_j \in \not{S}_j,\ z_k \in \not{S}_k\}$, $1 \leq j, k \leq p$, (see Proposition 3.1, (ii)) it follows that $[F_{ij}, F_{ik}] \subseteq F_{jk}$. Equality follows easily from the above expression after choosing $x_i = x'_i$ and $|x_i| = 1$, because then

$$y_j \wedge z_k = [x_i \wedge y_j, x_i \wedge z_k],$$

for all $y_j \wedge z_k \in F_{jk}$.
(ii) That $[F_{ij}, F_{ij}] \subseteq F_{ii} \oplus F_{jj}, i \neq j$ follows as in (i) by using Lemma 3.1, (iv) and Proposition 3.1, (ii). To prove that equality holds, it is enough to observe that Lemma 3.1, (iv) gives

$$[x_i \wedge y_j, x_i \wedge y'_j] = y_j \wedge y'_j,$$

whenever x_i belongs to $\not{S}_i$ with $|x_i| = 1$ and y_j, y'_j are in $\not{S}_j$. Similarly,

$$[x_i \wedge y_j, x'_i \wedge y_j] = x_i \wedge x'_i,$$

whenever x_i, x'_i belong to $\not{S}_i$ and y_j in $\not{S}_j$ satisfies $|y_j| = 1$. ■

Point (iii) in Lemma 3.2 is peculiar to the case when $Z_T \neq \{0\}$, for when $Z_T = \{0\}$ we have $\iota^0_T = E = \{0\}$ by Proposition 3.1, (ii).

3.2. Structure of the commutators.

We shall give in this section a simplified expression, relying on the particular dimension, for the generating space of the even part of the fix algebra $\mathfrak{g}^*_T$, where T belongs to $\Lambda^4_+(V)$.

Lemma 3.3. *Let T belong to $Cl_8^0 \cap Cl_8^+$ satisfy $T^t = T$. We have*

$$-4[X \lrcorner T, Y \lrcorner T] = 2T\gamma T + \tfrac{1}{4}L(T^2\gamma + \gamma T^2) + 4|T|^2(1-\nu)\gamma,$$

for all X, Y in V, where γ in $\Lambda^2(V)$ is given as $\gamma = X \wedge Y$.

Proof. Let $a : \Lambda^2(V) \to Cl_8$ be defined by setting

$$a(\gamma) = \sum_{i=1}^{8} e_i T^2 (e_i \lrcorner \gamma),$$

for some orthonormal frame $\{e_i, 1 \leq i \leq 8\}$. From [2, Lemma 4.1] we get that

$$-4[X \lrcorner T, Y \lrcorner T] = 2T\gamma T + a(\gamma)$$

and we need only work out a simpler expression for the operator a. We compute

$$e_i T^2 (e_i \lrcorner \gamma) = \tfrac{1}{2} e_i (T^2\gamma) e_i - \tfrac{1}{2}(e_i T^2 e_i)\gamma,$$

leading to $a(\gamma) = \frac{1}{2}L(T^2\gamma) - \frac{1}{2}L(T^2)\gamma$. Using (13) it is easily checked that $[T^2, \gamma]$ belongs to $\Lambda^4_+(V) \oplus \mathbb{R}(1+\nu)$ and moreover, since

$$\begin{aligned} \langle [T^2, \gamma], 1+\nu \rangle &= \langle T^2\gamma - \gamma T^2, 1+\nu \rangle \\ &= \langle \gamma, T^2(1+\nu) - (1+\nu)T^2 \rangle = 0, \end{aligned}$$

we actually get that $[T^2, \gamma]$ is a 4-form. Since L vanishes on $\Lambda^4(V)$ it follows that $L(T^2\gamma) = \frac{1}{2}L(T^2\gamma + \gamma T^2)$. Similarly, one finds that $L(T^2) = -8|T|^2(1-\nu)$ and the claim follows. ∎

3.3. Computation of $\mathfrak{g}_T^{*,0}$ when $Z_T \neq \{0\}$.

For a given T in $\Lambda^4_+(V)$, we shall compute now the even part $\mathfrak{g}_T^{*,0}$ of its holonomy algebra, which has been introduced in Proposition 2.1, (ii). The main technical ingredient in this section is contained in the following observation.

Lemma 3.4. *Let T be in $\Lambda^4_+(V)$. Then $\mathfrak{g}^*_{T^{2k+1}} \subseteq \mathfrak{g}^*_T$ for all k in $\mathbb{N}$.*

Proof. Let $\{e_i, 1 \leq i \leq 8\}$ be an orthonormal basis in V and consider the partial Casimir operator $C_T : Cl_8 \to Cl_8$ given by

$$C_T = \sum_{i=1}^{8} [e_i \lrcorner T, [e_i \lrcorner T, \cdot]].$$

Obviously, C_T preserves the algebra $\mathfrak{g}_T^*$, that is $C_T(\mathfrak{g}_T^*) \subseteq \mathfrak{g}_T^*$. A straightforward computation actually shows that

$$C_T\varphi = \sum_{i=1}^{8} \left[(e_i \lrcorner T)^2 \varphi + \varphi (e_i \lrcorner T)^2 - 2(e_i \lrcorner T)\varphi(e_i \lrcorner T) \right],$$

for all φ in Cl_8. We shall now compute $C_T(X \lrcorner \varphi)$ where φ belongs to $Cl_8^0 \cap Cl_8^+$ is such that $\varphi^t = \varphi$ that is φ is in $\Lambda^4_+(V) \oplus \mathbb{R}(1+\nu)$ by (13). We compute using Lemma 2.1

$$\begin{aligned}
-8(e_i \lrcorner T)(X \lrcorner \varphi)(e_i \lrcorner T) &= (e_iT - Te_i)(X\varphi - \varphi X)(e_iT - Te_i) \\
&= (-Te_iX\varphi - e_iT\varphi X)(e_iT - Te_i) \\
&= -e_iT\varphi Xe_iT + Te_iX\varphi Te_i \\
&= e_iT\varphi(2\langle e_i, X\rangle + e_iX)T + T(-2\langle e_i, X\rangle - Xe_i)\varphi Te_i,
\end{aligned}$$

whenever X belongs to V. Hence after summation we get

$$4\sum_{i=1}^{8}(e_i \lrcorner T)(X \lrcorner \varphi)(e_i \lrcorner T) = -[X, T\varphi T] - \tfrac{1}{2}L(T\varphi)XT + \tfrac{1}{2}TXL(\varphi T).$$

Now

$$\begin{aligned}
4\sum_{i=1}^{8}(e_i \lrcorner T)^2 &= \sum_{i=1}^{8}(e_iT - Te_i)^2 \\
&= L(T)T - L(T^2) + 8T^2 + TL(T) = 8T^2 + 8|T|^2(1-\nu)
\end{aligned}$$

as $L(T) = 0$ and $L(T^2) = -8|T|^2(1-\nu)$. A short computation using the stability relations in Lemma 2.1 gives now

$$T^2(X \lrcorner \varphi) + (X \lrcorner \varphi)T^2 = -\tfrac{1}{2}(-T^2\varphi X + X\varphi T^2).$$

Hence in the end we obtain

$$\begin{aligned}
4C_T(X \lrcorner \varphi) &= -4(X\varphi T^2 - T^2\varphi X) + 2[X, T\varphi T] \\
&\quad + L(T\varphi)XT - TXL(\varphi T) + 16|T|^2(X \lrcorner \varphi),
\end{aligned}$$

for all X in V and where φ belongs to $\Lambda^4_+(V) \oplus \mathbb{R}(1+\nu)$. In particular, for $\varphi = T^k, k$ in $\mathbb{N}$ this yields

$$X \lrcorner T^{k+2} = C_T(X \lrcorner T^k) - 8\langle T^k, T\rangle X \lrcorner T - 4|T|^2(X \lrcorner T^k),$$

for all X in V, where we have used that

$$L(T^{k+1}) = -8\langle T^k, T\rangle(1-\nu). \tag{18}$$

By induction, given that C_T preserves $\mathfrak{g}_T^*$ and that the latter contains $\{X \lrcorner T : X \in V\}$ we arrive at $\{X \lrcorner T^{2k+1} : X \in V\} \subseteq \mathfrak{g}_T^*$, for all k in $\mathbb{N}$ and our claim follows. ∎

The following Lemma provides a spectral characterisation of self-dual 4-forms having 6 fixed spinors, which we shall need in our computations.

Lemma 3.5. *Let λ_i where $1 \le i \le p$ belong to σ_T. If*

$$8|T^{2k+1}|^2 = \lambda_i^{2(2k+1)}$$

holds for all k in $\mathbb{N}$ then $\dim_{\mathbb{R}} Z_T = 6$ and $\sigma_T = \{\lambda_i, -\lambda_i\}$ with multiplicities $(1,1)$, provided that $T \neq 0$.

Proof. By making use of Lemma 2.2 we have that

$$16|T^{2k+1}|^2 = \sum_{q=1}^{p} m_q \lambda_q^{2(2k+1)}. \tag{19}$$

The equation we have to solve becomes

$$\sum_{q=1}^{p} m_q \lambda_q^{2(2k+1)} = 2\lambda_i^{2(2k+1)}, \tag{20}$$

for all k in $\mathbb{N}$. We now divide by $\lambda_i^{2(2k+1)}$ and take the limit when $k \to \infty$. It follows that $|\lambda_q| \le |\lambda_i|$ for all $1 \le q \le p$ and also that $\sum_{|\lambda_q|=|\lambda_i|} m_q = 2$. It follows easily that $m_i = 1$, otherwise we would have $m_i = 2$ and further $\sigma_T = \{\lambda_i\}$ by making use of (20), which contradicts that μ_T is traceless. Therefore $-\lambda_i$ belongs to σ_T, with multiplicity 1 and our claim follows again from (20). ■

In the situation where the set of spinors fixed by some self-dual 4-form is not reduced to zero we shall conclude, with one exception, that:

Proposition 3.2. *Let T belong to $\Lambda^4_+(V)$ with $\dim_{\mathbb{R}} Z_T \neq 0, 6$. Then:*

$$\mathfrak{g}_T^{*,0} = (1+\nu)F \oplus (1-\nu)\Lambda^2(V).$$

Proof. We shall prove first that one has

$$(1+\nu)F \oplus (1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}. \tag{21}$$

Indeed, making use of Lemma 3.3 we have that $\mathfrak{g}_T^{*,0}$ contains the set

$$\{2T\gamma T + \tfrac{1}{4}L(T^2\gamma + \gamma T^2) + 4|T|^2(1-\nu)\gamma : \gamma \in \Lambda^2(V)\} \tag{22}$$

as this is just spanned by double commutators of elements in its generating set. For further use, let us note that (22) remains valid when $\dim_{\mathbb{R}} Z_T = 0, 6$.
From (22) we find that $(1-\nu)\iota_T^0$ is contained in $\mathfrak{g}_T^{*,0}$. Actually, by using Lemma 3.4 we have that $X \lrcorner T^{2k+1}$ belongs to $\mathfrak{g}_T^{*}$ and therefore, after taking double commutators of such elements and using again Lemma 3.3 we get that

$$2T^{2k+1}\gamma T^{2k+1} + \tfrac{1}{4}L(T^{2(2k+1)}\gamma + \gamma T^{2(2k+1)}) + 4|T^{2k+1}|^2(1-\nu)\gamma \tag{23}$$

belongs to $\mathfrak{g}_T^{*,0}$ for any γ in $\Lambda^2(V)$. Now if γ is in E_i, for some $1 \le i \le p$, we have $T\gamma T = 0$ and an easy computation by induction shows

$$T^{2(2k+1)}\gamma + \gamma T^{2(2k+1)} = \tfrac{1}{2}\lambda_i^{2(2k+1)}(1+\nu)\gamma,$$

for all k in $\mathbb{N}$. We are led eventually to having

$$(4|T^{2k+1}|^2 - \tfrac{1}{2}\lambda_i^{2(2k+1)})(1-\nu)E_i$$

contained in $\mathfrak{g}_T^{*,0}$ for all $1 \le i \le p$ and all k in $\mathbb{N}$.
Now since σ_T has not the form in Lemma 3.5, in other words $\dim_{\mathbb{R}} Z_T \neq 6$, for

each $1 \leq i \leq p$ the factor above will be non-vanishing for some k in $\mathbb{N}$ whence $(1-\nu)E_i \subseteq \mathfrak{g}_T^{*,0}$ whenever $1 \leq i \leq p$. Now taking commutators and using (iii) of Lemma 3.2 it follows that $(1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$. But $L(T^2\gamma + \gamma T^2)$ belongs to $(1-\nu)\Lambda^2(V)$ for all γ in $\Lambda^2(V)$ hence we get from (22) that $\{T\gamma T : \gamma \in \Lambda^2(V)\}$ is contained in $\mathfrak{g}_T^{*,0}$. Making use of the splitting in Proposition 3.1 this actually says that $(1+\nu)F \subseteq \mathfrak{g}_T^{*,0}$ and we have showed that

$$(1+\nu)F \oplus (1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0},$$

in other words (21) holds. Therefore it is enough to see that

$$\mathfrak{g}_T^{*,0} \subseteq (1+\nu)F \oplus (1-\nu)\Lambda^2(V)$$

and this will be achieved by showing that $\mathfrak{g}_T^{*,0}$ is orthogonal to $(1+\nu)(\iota_0^T \oplus E)$, since one knows that $\mathfrak{g}_T^{*,0} \subseteq A^0 = (1-\nu)\Lambda^2(V) \oplus (1+\nu)\Lambda^2(V)$, see Proposition 2.1, (ii) and (12).
Indeed, by the definition of Z_T we have $(X \lrcorner T)Z_T = 0$ for all X in V, therefore $\mathfrak{g}_T^* Z_T = \{0\}$. We now pick φ in $\mathfrak{g}_T^{*,0}$, x in Z_T and y in $\$^+$. From the definition of $x \wedge y$ and $\varphi x = 0$ follows that

$$\begin{aligned} \varphi(x \wedge y)\psi &= \langle x, \psi \rangle \varphi y \\ (x \wedge y)\varphi\psi &= -\langle \varphi\psi, y \rangle x = \langle \psi, \varphi y \rangle x, \end{aligned}$$

for all ψ in $\$^+$, where we used that $\varphi^t = \varphi$ by Proposition 2.1, (i). Since $\varphi(x \wedge y) + (x \wedge y)\varphi$ is a symmetric element in Cl_8^0 after taking the trace we get

$$Tr(\mu_{\varphi(x\wedge y)+(x\wedge y)\varphi}) = 2\langle \varphi y, x \rangle = -2\langle y, \varphi x \rangle = 0.$$

Now using Lemma 2.2 it follows that $\langle \varphi(x \wedge y) + (x \wedge y)\varphi, 1 \rangle = 0$ and since $\alpha(\varphi^t) = -\varphi$ we arrive at $\langle \varphi, x \wedge y \rangle = 0$. Because $\{x \wedge y : x \in Z_T, y \in \$\}$ spans $(1+\nu)(\iota_T^0 \oplus E)$ it follows that $\mathfrak{g}_T^{*,0}$ is orthogonal to $(1+\nu)(\iota_T^0 \oplus E)$, hence contained in $(1+\nu)F$ and the claim follows. ■

We leave out now the case when $\dim_{\mathbb{R}} Z_T = 6$, to be treated further on, and consider the situation when there are no non-zero fixed spinors.

3.4. The case when $Z_T = \{0\}$.

We will first treat two particular cases in the following Lemma. We shall show later on that it is possible to actually reduce the classification problem in the present case of interest to these two cases only.

Lemma 3.6. *Let T in $\Lambda^4_+(V)$ with $Z_T = \{0\}$ be such that its spectrum is of the form*

(i) $\sigma_T = \{\lambda_1, \lambda_2, \lambda_3\}$ *where* $\lambda_2 = -\lambda_1$ *and* $|\lambda_1| \neq |\lambda_3|$

or

(ii) $\sigma_T = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ *where* $\lambda_2 = -\lambda_1, \lambda_4 = -\lambda_3$ *and* $|\lambda_1| \neq |\lambda_3|$.

Then we must have $\mathfrak{g}_T^{*,0} = A^0$.

Proof. Using Lemma 3.4 we have that $[X \lrcorner T^{2k+1}, Y \lrcorner T^{2k+1}]$ belongs to $\mathfrak{g}_T^{*,0}$ for all X, Y in V and any natural number k. Now Lemma 3.3 implies that

$$2T^{2k+1}\gamma T^{2k+1} + \tfrac{1}{4}L(T^{2(2k+1)}\gamma + \gamma T^{2(2k+1)}) + 4|T^{2k+1}|^2(1-\nu)\gamma$$

belongs to $\mathfrak{g}_T^{*,0}$ for any γ in $\Lambda^2(V)$ and any k in $\mathbb{N}$. Let us pick now γ in F_{ij} and k in $\mathbb{N}$. An easy computation by induction based on (17) gives

$$\begin{aligned} T^{2k+1}\gamma T^{2k+1} &= \tfrac{1}{2}(\lambda_i\lambda_j)^{2k+1}(1+\nu)\gamma \\ T^{2(2k+1)}\gamma + \gamma T^{2(2k+1)} &= \tfrac{1}{2}(\lambda_i^{2(2k+1)} + \lambda_j^{2(2k+1)})(1+\nu)\gamma. \end{aligned}$$

Hence $G_{ij}^{(k)}$, defined by

$$G_{ij}^{(k)} = (\lambda_i\lambda_j)^{2k+1}(1+\nu)\gamma + \Big[4|T^{2k+1}|^2 - \tfrac{1}{2}(\lambda_i^{2(2k+1)} + \lambda_j^{2(2k+1)})\Big](1-\nu)\gamma, \qquad (24)$$

belongs to $\mathfrak{g}_T^{*,0}$ for all γ in F_{ij} and all k in $\mathbb{N}$.
We shall show that in both cases (i) and (ii) we have $(1\pm\nu)F \subseteq \mathfrak{g}_T^{*,0}$, from which we conclude by Proposition 3.1 and the fact that $Z_T = \{0\}$ that $(1\pm\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$ and hence $A^0 \subseteq \mathfrak{g}_T^{*,0}$. Consequently, by Proposition 2.1, we have that $\mathfrak{g}_T^{*,0} = A^0$.
(i) In this case, by use of (19), we get for $|T^{2k+1}|^2$

$$4|T^{2k+1}|^2 = \frac{1}{4}\sum_{q=1}^{3} m_q\lambda_q^{2(2k+1)} = \frac{m_1+m_2}{4}\lambda_1^{2(2k+1)} + \frac{m_3}{4}\lambda_3^{2(2k+1)}.$$

Using this we compute $G_{13}^{(k)}$ and get

$$G_{13}^{(k)} = (\lambda_1\lambda_3)^{2k+1}(1+\nu)\gamma + \frac{1}{2}\Big(\frac{m_1+m_2-2}{2}\lambda_1^{2(2k+1)} + \frac{m_3-2}{2}\lambda_3^{2(2k+1)}\Big)(1-\nu)\gamma.$$

(a) Let us assume that $|\lambda_1| < |\lambda_3|$, then

$$\lim_{k\to\infty}\frac{G_{13}^{(k)}}{\lambda_3^{2(2k+1)}} = \frac{m_3-2}{4}(1-\nu)\gamma.$$

That is, $(1-\nu)F_{13}$ and consequently, since $\lambda_1\lambda_3 \neq 0$, $(1+\nu)F_{13}$ are contained in $\mathfrak{g}_T^{*,0}$, unless $m_3 = 2$. If $m_3 = 2$ we have

$$\frac{G_{13}^{(k)}}{\lambda_1^{2k+1}} = \lambda_3^{2k+1}(1+\nu)\gamma + \frac{m_1+m_2-2}{4}\lambda_1^{2k+1}(1-\nu)\gamma,$$

where we take the limit

$$\lim_{k\to\infty}\frac{G_{13}^{(k)}}{\lambda_1^{2k+1}|\lambda_3|^{2k+1}} = (1+\nu)\gamma.$$

Hence $(1+\nu)F_{13} \subset \mathfrak{g}_T^{*,0}$ and so is $(1-\nu)F_{13}$, unless $m_1+m_2 = 2$. But this impossible since we must have $m_1+m_2+m_3 = 8$. Consequently $(1\pm\nu)F_{13}$ is, in

all cases, contained in $\mathfrak{g}_T^{*,0}$.
(b) Let us assume that $|\lambda_1| > |\lambda_3|$, then

$$\lim_{k\to\infty} \frac{G_{13}^{(k)}}{\lambda_1^{2(2k+1)}} = \frac{m_1+m_2-2}{4}(1-\nu)\gamma.$$

That is, $(1-\nu)F_{13}$ and, since $\lambda_1\lambda_3 \neq 0$, $(1+\nu)F_{13}$ are contained in $\mathfrak{g}_T^{*,0}$, unless $m_1+m_2=2$. In the case of $m_1+m_2=2$ we have

$$\frac{G_{13}^{(k)}}{\lambda_3^{2k+1}} = \lambda_1^{2k+1}(1+\nu)\gamma + \frac{m_3-2}{4}\lambda_3^{2k+1}(1-\nu)\gamma,$$

where we take the limit

$$\lim_{k\to\infty} \frac{G_{13}^{(k)}}{\lambda_3^{2k+1}|\lambda_1|^{2k+1}} = (1+\nu)\gamma,$$

by which $(1+\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$ and thus $(1-\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$, unless also $m_3 = 2$. The case of $m_1+m_2=m_3=2$ cannot occur by the same argument as in (a).
Since $\lambda_1 = -\lambda_2$ we can treat $G_{23}^{(k)}$ in complete analogy, it yields that $(1\pm\nu)F_{23}$ is contained in $\mathfrak{g}_T^{*,0}$. Now, by Lemma 3.2, (ii) we have

$$\begin{array}{rcl} (1\pm\nu)[F_{13},F_{13}] & = & (1\pm\nu)(F_{11}\oplus F_{33}) \\ (1\pm\nu)[F_{23},F_{23}] & = & (1\pm\nu)(F_{22}\oplus F_{33}), \end{array}$$

whereas by (i) of the same Lemma we have $(1\pm\nu)[F_{13},F_{23}] = (1\pm\nu)F_{12}$. Hence $(1\pm\nu)F \subseteq \mathfrak{g}_T^{*,0}$.
(ii) In this case, by use of (19), we have for $|T^{2k+1}|^2$

$$4|T^{2k+1}|^2 = \frac{1}{4}\sum_{q=1}^{4} m_q\lambda_q^{2(2k+1)} = \frac{m_1+m_2}{4}\lambda_1^{2(2k+1)} + \frac{m_3+m_4}{4}\lambda_3^{2(2k+1)}.$$

Let us again compute $G_{13}^{(k)}$. We have

$$G_{13}^{(k)} = (\lambda_1\lambda_3)^{2k+1}(1+\nu)\gamma + \frac{1}{2}\Big(\frac{m_1+m_2-2}{2}\lambda_1^{2(2k+1)} + \frac{m_3+m_4-2}{2}\lambda_3^{2(2k+1)}\Big)(1-\nu)\gamma.$$

Let us assume that $|\lambda_1| < |\lambda_3|$, then

$$\lim_{k\to\infty} \frac{G_{13}^{(k)}}{\lambda_3^{2(2k+1)}} = \frac{m_3+m_4-2}{4}(1-\nu)\gamma.$$

That is, $(1-\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$ and, since $\lambda_1\lambda_3 \neq 0$, $(1+\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$, unless $m_3+m_4=2$. If $m_3+m_4=2$, then we have

$$\frac{G_{13}^{(k)}}{\lambda_1^{2k+1}} = \lambda_3^{2k+1}(1+\nu)\gamma + \frac{m_1+m_2-2}{4}\lambda_1^{2k+1}(1-\nu)\gamma,$$

and we take the limit

$$\lim_{k\to\infty} \frac{G_{13}^{(k)}}{\lambda_1^{2k+1}|\lambda_3|^{2k+1}} = (1+\nu)\gamma.$$

That is, $(1+\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$ and $(1-\nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$, unless also $m_1 + m_2 = 2$. If $m_3 + m_4 = 2$ and $m_1 + m_2 = 2$, then we get again a contradiction with $m_1 + m_2 + m_3 + m_4 = 8$. Consequently, in all cases, $(1 \pm \nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$.
The case of $|\lambda_1| > |\lambda_3|$ can be treated in complete analogy, yielding again that $(1 \pm \nu)F_{13} \subseteq \mathfrak{g}_T^{*,0}$. Moreover it is evident that, since $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_3$, in the same manner we get that $(1 \pm \nu)F_{14}, (1 \pm \nu)F_{23}$ and $(1 \pm \nu)F_{24}$ are contained in $\mathfrak{g}_T^{*,0}$.
Now, by Lemma 3.2, (i) we get $(1\pm\nu)[F_{13}, F_{23}] = (1\pm\nu)F_{12}$ and $(1\pm\nu)[F_{14}, F_{13}] = (1 \pm \nu)F_{34}$, hence $(1\pm\nu)F_{12}$ and $(1\pm\nu)F_{34}$ are in $\mathfrak{g}_T^{*,0}$ as well. By (ii) of the same Lemma we get that $(1 \pm \nu)[F_{12}, F_{12}] = (1 \pm \nu)(F_{11} \oplus F_{22})$ and $(1 \pm \nu)[F_{34}, F_{34}] = (1 \pm \nu)(F_{33} \oplus F_{44})$, hence $(1 \pm \nu)F \subseteq \mathfrak{g}_T^{*,0}$ and the proof is finished. ∎

As a consequence of the arguments in the proof above we obtain the following simple maximality criterion, based on the classification in Proposition 3.2, completed in the previous section.

Corollary 3.2. *Let T belong to $\Lambda^4_+(V)$, such that $Z_T = \{0\}$. Then:*

- (i) *If $(1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$, we must have $\mathfrak{g}_T^* = A$.*
- (ii) *If there exists a φ in $\Lambda^4_+(V)$, such that $\mathfrak{g}_\varphi^* \subseteq \mathfrak{g}_T^*$ and $\dim_{\mathbb{R}} Z_\varphi \neq 0, 6$ hold, then $\mathfrak{g}_T^* = A$.*

Proof. (i) From (24) used for $k = 0$, and the assumption that $(1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$ it follows that $\lambda_i\lambda_j(1+\nu)\gamma$ belongs to $\mathfrak{g}_T^{*,0}$, whenever γ is in $F_{ij}, 1 \le i, j \le p$. Given that $Z_T = \{0\}$ we get $(1+\nu)\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$ hence $A^0 \subseteq \mathfrak{g}_T^{*,0}$ by means of (12). By Proposition 2.1, (ii), it follows that equality holds and since the adjoint representation of A^0 on A^1 is irreducible, as asserted in Lemma 2.4, (ii), we obtain that $\mathfrak{g}_T^{*,1} = A^1$, and the claim follows.
(ii) From Proposition 3.2 it follows that $(1-\nu)\Lambda^2(V) \subseteq \mathfrak{g}_\varphi^{*,0} \subseteq \mathfrak{g}_T^{*,0}$ where we have used the first assumption for the second inclusion. We conclude using (i). ∎

Another useful observation for what follows is contained in:

Lemma 3.7. *Let T belong to $\Lambda^4_+(V)$. If $\Lambda^2(V) \subseteq \mathfrak{g}_T^{*,0}$ then $\mathfrak{g}_T^*$ is maximal, that is $\mathfrak{g}_T^* = A$.*

Proof. Since $[\mathfrak{g}_T^{*,0}, \mathfrak{g}_T^{*,1}] \subseteq \mathfrak{g}_T^{*,1}$, which follows easily from (11), it follows that $\mathfrak{g}_T^{*,1} \subseteq A^1$ is an invariant subspace under the action of $\Lambda^2(V) \cong \mathfrak{so}(V)$. But the only proper invariant subspaces of $A^1 = \Lambda^3(V) \oplus \Lambda^7(V)$ are $\Lambda^3(V)$ and $\Lambda^7(V)$. Since $\mathfrak{g}_T^{*,1}$ contains 3-forms it follows that either $\mathfrak{g}_T^{*,1} = A^1$ or $\mathfrak{g}_T^{*,1} = \Lambda^3(V)$ so that $\Lambda^3(V) \subseteq \mathfrak{g}_T^{*,1}$. It follows that $[\Lambda^3(V), \Lambda^3(V)] \subseteq \mathfrak{g}_T^{*,0}$. Now, the Clifford product of the forms ω_1 and ω_2 of arbitrary degree is a linear combination of elements of the form

$$\sum_I (e_I \lrcorner \omega_1) \wedge (e_I \lrcorner \omega_2),$$

where $e_I = e_{i_1} \wedge ... \wedge e_{i_{|I|}}$ and $0 \leq |I| \leq deg(\omega_1) + deg(\omega_2)$, see e.g. [14, Thm. 9.2]. It is then straightforward to see that $[\Lambda^3(V), \Lambda^3(V)] = \Lambda^2(V) \oplus \Lambda^6(V)$, whence $A^0 \subseteq \mathfrak{g}_T^{*,0}$. In particular $Z_T = \{0\}$ since any spinor fixed by $\Lambda^2(V)$ must vanish (one may use that $e_i \wedge e_j, 1 \leq i \neq j \leq 8$ are in the group of invertible elements of Cl_8). Thus $\mathfrak{g}_T^{*,0} = A^0$ and therefore (i) in Corollary 3.2 yields $\mathfrak{g}_T^{*,1} = A^1$, and the claim follows. ■

We now need some additional information concerning our self-dual 4-form T with $Z_T = \{0\}$. We split $\mu_T : \$^+ \to \$^+$ as

$$\mu_T = \sum_{q=1}^{p} \lambda_q \Pi_q, \tag{25}$$

where Π_q is the orthogonal projection on the space $\$_q, 1 \leq q \leq p$. Using Definition 2.1 one obtains immediately that $\Pi_q = \mu_{T_q}$, where we define

$$T_q = \sum_{i=1}^{m_q} x_i \otimes x_i,$$

for some orthonormal basis $\{x_i, 1 \leq i \leq m_q\}$ in $\$_q, 1 \leq q \leq p$. By construction, T_q belongs to $\Lambda^4_+(V) \oplus \mathbb{R}(1+\nu)$, see also (13), and moreover the component on $\mathbb{R}(1+\nu)$ is determined from

$$16\langle T_q, 1\rangle = Tr\mu_{T_q} = Tr\Pi_q = m_q, \tag{26}$$

for all $1 \leq q \leq p$, by making use of Lemma 2.2. We also have

$$\sum_{q=1}^{p} T_q = \tfrac{1}{2}(1+\nu) \tag{27}$$

as a straightforward consequence of $\sum_{q=1}^{p} \Pi_q = 1_{\$^+}$. Since $\Pi_k \circ \Pi_q = \delta_{kq}\Pi_k$, for all $1 \leq k, q \leq p$, where δ is the Kronecker symbol, it follows by using the injectivity of $\mu : Cl_8^0 \cap Cl_8^+ \to End(\$^+)$ that

$$T_k T_q = 0, \; k \neq p \quad \text{and} \quad T_k^2 = T_k, \tag{28}$$

in Cl_8, for all $1 \leq k, q \leq p$.

Using that $\Pi_q = \mu_{T_q}, 1 \leq q \leq p$, in (25) and since $\mu : Cl_8^0 \cap Cl_8^+ \to End(\$^+)$ is injective we get that

$$T = \sum_{q=1}^{p} \lambda_q T_q,$$

and further

$$T^{2k+1} = \sum_{q=1}^{p} \lambda_q^{2k+1} T_q, \tag{29}$$

for all k in $\mathbb{N}$, after making use of (28).

We are now in position to prove our first structure result concerning holonomy algebras of self-dual 4-forms without fixed spinors. Our arguments mainly consist in making use of Lemma 3.4 and working on various linear combinations of the elements $\{T_i, 1 \leq i \leq d\}$ of Cl_8 to see that either the hypothesis in the maximality criterion in Corollary 3.2, (ii) are satisfied for the generating form T in $\Lambda^4_+(V)$, or its spectrum has the special form given in Lemma 3.6, (i) or (ii). By recalling the isomorphism $A \cong \mathfrak{so}(8,8)$ from Lemma 2.4 we can now make the following.

Theorem 3.1. *Let T belong to $\Lambda^4_+(V)$ such that $Z_T = \{0\}$. Then either*

(i) $\mathfrak{g}^*_T = A \cong \mathfrak{so}(8,8)$, *or*

(ii) $\mathfrak{g}^*_T \cong \mathfrak{so}(8,1)$,

where the latter case occurs when T is proportional to a unipotent element of Cl_8^+, in the sense that $T^2 = \lambda(1+\nu)$ for some $\lambda > 0$. In both cases the fix algebra is perfect, that is $\mathfrak{h}^*_T = \mathfrak{g}^*_T$.

Proof. The use of Lemma 3.4 combined with (29) leads to

$$\sum_{q=1}^{p} \lambda_q^{2k+1} X \lrcorner T_q \in \mathfrak{g}^*_T \tag{30}$$

for all k in $\mathbb{N}$ and all X in V.

Let now $\sigma^s_T = \{\lambda \in \sigma_T : -\lambda \in \sigma_T\}$ be the symmetric part of the spectrum of μ_T. We shall label the eigenvalues in σ^s_T by $\lambda_i, 1 \leq i \leq 2d$ where $\lambda_{j+d} = -\lambda_j, 1 \leq j \leq d$, and moreover $|\lambda_i| \neq |\lambda_j|$ for $1 \leq i, j \leq d$. The remaining part of the spectrum will be denoted by $\sigma^r_T = \sigma_T \backslash \sigma^s_T$. From (31) it follows that

$$\sum_{q=1}^{d} \lambda_q^{2k+1} X \lrcorner (T_q - T_{q+d}) + \sum_{q=2d+1}^{p} \lambda_q^{2k+1} X \lrcorner T_q \tag{31}$$

belongs to $\mathfrak{g}^*_T$, for all k in $\mathbb{N}$ and all X in V. Let M in $M_{p-d}(\mathbb{R})$ be the matrix given by

$$M = V(\lambda_1^2, ..., \lambda_d^2, \lambda_{2d+1}^2, ..., \lambda_p^2)\ diag(\lambda_1, ..., \lambda_d, \lambda_{2d+1}, ..., \lambda_p),$$

where $V(\lambda_1^2, ..., \lambda_d^2, \lambda_{2d+1}^2, ..., \lambda_p^2)$ is the Vandermonde-type matrix in the formerly listed entries. Hence

$$det(M) = \prod_{i<j} (\lambda_i^2 - \lambda_j^2) \prod_k \lambda_k,$$

where i, j, k take values from 1 to d and from $2d+1$ to p. It follows that M is invertible as a consequence of the fact that $\lambda_1^2, ..., \lambda_d^2, \lambda_{2d+1}^2, ..., \lambda_p^2$ are mutually distinct by construction and $0 \notin \sigma_T$.

Taking $k = 0, ..., p-d-1$ in (31) and formally multiplying the resulting $\mathfrak{g}_T^*$-valued vector with M^{-1} we find easily that

$$\begin{array}{rcll} \mathfrak{g}^*_{T_q - T_{q+d}} & \subseteq & \mathfrak{g}^*_T, & 1 \leq q \leq d, \\ \mathfrak{g}^*_{T_q} & \subseteq & \mathfrak{g}^*_T, & 2d+1 \leq q \leq p. \end{array} \tag{32}$$

We shall now split the proof into several cases. These essentially amount to counting the maximal number of eigenvalues in σ_T^s and σ_T^r which do not yield, a priori, the maximality of $\mathfrak{g}_T^*$. Each time when Corollary 3.2, (ii) is invoked we keep in mind that (32) implies that the fix algebra of any linear combination of the forms

$$T_q - T_{q+d}, T_s,$$

where $1 \leq q \leq d$ and $2d+1 \leq s \leq p$ is contained in $\mathfrak{g}_T^*$.

• **The case when $d \geq 2$:** In this case σ_T contains at least two pairs of symmetric eigenvalues, say $\lambda_1 = -\lambda_{d+1}$ and $\lambda_2 = -\lambda_{d+2}$ with $|\lambda_1| \neq |\lambda_2|$. Then $\widehat{T}$, defined by

$$\widehat{T} = (m_2 - m_{d+2})(T_1 - T_{d+1}) - (m_1 - m_{d+1})(T_2 - T_{d+2}),$$

belongs to $\Lambda^4_+(V)$ by (26). Several subcases enter now naturally the discussion.

(i) $\widehat{T} \neq 0$: From the definition of the forms $T_i, 1 \leq i \leq p$ we have that $Z_{\widehat{T}}$ is given by

$$\begin{array}{ll} (\$_1 \oplus \$_{1+d})^{\perp}, & \text{if } m_1 = m_{1+d} \\ (\$_2 \oplus \$_{2+d})^{\perp}, & \text{if } m_2 = m_{2+d} \\ (\$_1 \oplus \$_{1+d} \oplus \$_2 \oplus \$_{2+d})^{\perp}, & \text{if } (m_1 - m_{1+d})(m_2 - m_{2+d}) \neq 0. \end{array}$$

1. $(m_1 - m_{1+d})(m_2 - m_{2+d}) \neq 0$: Clearly $Z_{\widehat{T}} = \{0\}$ if and only if $d = 2$, in other words the spectrum of T has the form in (ii) of Lemma 3.6, which can then be used together with Corollary 3.2, (i) to conclude that $\mathfrak{g}_T^* = A$. Otherwise $Z_{\widehat{T}}$ cannot be 6-dimensional because $\$_1 \oplus \$_{1+d} \oplus \$_2 \oplus \$_{2+d}$ is at least 4-dimensional. It follows that $\dim_{\mathbb{R}} Z_{\widehat{T}} \neq 0, 6$ and given that $\mathfrak{g}^*_{\widehat{T}} \subseteq \mathfrak{g}^*_T$ by (32), we use Corollary 3.2, (ii) to conclude that $\mathfrak{g}_T^* = A$.

2. $m_1 = m_{1+d}$: Let us recall that here we must have $m_2 \neq m_{2+d}$ since $\widehat{T} \neq 0$. Since μ_T is tracefree we must have $\#\sigma_T \geq 5$. Suppose now that there exists another pair of symmetric eigenvalues, say $\lambda_3, \lambda_{3+d} = -\lambda_3$. The form

   $$\check{T} = (m_3 - m_{3+d})(T_2 - T_{2+d}) - (m_2 - m_{2+d})(T_3 - T_{3+d})$$

   in $\Lambda^4_+(V)$ is actually non-vanishing and enables us to obtain the desired conclusion, by making use of Case (1), unless $m_3 = m_{3+d}$. If this is the case, one constructs the form

   $$\widetilde{T} = T_1 - T_{1+d} + T_3 - T_{3+d}$$

   in $\Lambda^4_+(V)$, which satisfies

   $$Z_{\widetilde{T}} = (\$_1 \oplus \$_{1+d} \oplus \$_3 \oplus \$_{3+d})^{\perp}.$$

   This cannot be 6-dimensional and if it vanishes the spectrum of T is of the form given in (ii) of Lemma 3.6 leading to the maximality of $\mathfrak{g}_T^*$ by Corollary

3.2, (i). If $\dim_{\mathbb{R}} Z_{\widetilde{T}} \neq 0$ we conclude again by Corollary 3.2, (ii) that $\mathfrak{g}_T^*$ is maximal, when $m_3 = m_{3+d}$.
Suppose now that σ_T contains an eigenvalue, say λ_3, in σ_T^r. Then the self-dual 4-form

$$\overline{T} = m_3(T_2 - T_{2+d}) - (m_2 - m_{2+d})T_3$$

cannot vanish and has

$$Z_{\overline{T}} = (\$_2 \oplus \$_{2+d} \oplus \$_3)^{\perp}.$$

It is easily seen that this cannot be of dimension 6. If $Z_{\overline{T}} = \{0\}$ the spectrum of T is of the form in (i) of Lemma 3.6 and the maximality of $\mathfrak{g}_T^*$ follows from Corollary 3.2, (i). Otherwise the same conclusion is obtained from Corollary 3.2, (ii).

3. $m_2 = m_{2+d}$: This can be treated in complete analogy with case (2) above and it is therefore left to the reader.

(ii) $\widehat{T} = 0$: Since $\{T_i, 1 \leq i \leq p\}$ are linearly independent, as it follows from (28), we must have that $m_k - m_{d+k} = 0$, $k = 1, 2$. Then, again by (26), the forms

$$\widehat{T}_k = T_k - T_{k+d}, \quad k = 1, 2$$

belong to $\Lambda_+^4(V)$. Any linear combination $k\widehat{T}_1 + \widehat{T}_2, |k| \neq 1$ is easily seen to satisfy $\dim_{\mathbb{R}} Z_{k\widehat{T}_1+\widehat{T}_2} \neq 6$ and moreover $\mathfrak{g}^*_{k\widehat{T}_1+\widehat{T}_2} \subseteq \mathfrak{g}_T^*$, by (32), thus the maximality of the fix algebra of T follows from Corollary 3.2, (ii), provided that $\dim_{\mathbb{R}} Z_{k\widehat{T}_1+\widehat{T}_2} \neq 0$. On the other hand, if $\dim_{\mathbb{R}} Z_{k\widehat{T}_1+\widehat{T}_2} = 0$, the spectrum of $\mu_{k\widehat{T}_1+\widehat{T}_2}$ has again the form given in (ii) of Lemma 3.6 and the maximality of the fix algebra of T follows as above.

• **The case when $d = 1$:**
(i) $\#\sigma_T^r \geq 1$: To fix ideas, let us suppose that λ_3 belongs to σ_T^r and consider the form

$$\widetilde{T} = m_3(T_1 - T_2) - (m_1 - m_2)T_3$$

in $\Lambda_+^4(V)$. As before this is not zero and from the definition of $T_i, 1 \leq i \leq d$ we have

$$Z_{\widetilde{T}} = \begin{cases} (\$_1 \oplus \$_2 \oplus \$_3)^{\perp}, & \text{if } m_1 \neq m_2 \\ (\$_1 \oplus \$_2)^{\perp}, & \text{if } m_1 = m_2. \end{cases}$$

1. $m_1 \neq m_2$: A direct verification shows that $\dim_{\mathbb{R}} Z_{\widetilde{T}} \neq 6$. If $Z_{\widetilde{T}} = \{0\}$ we conclude by using Lemma 3.6 and Corollary 3.2, (i) for the spectrum of T must have the form given in case (i) of the above mentioned Lemma. It remains to consider the situation when $\dim_{\mathbb{R}} Z_{\widetilde{T}} \neq 0, 6$, where one uses Corollary 3.2, (ii) to get the maximality of $\mathfrak{g}_T^*$.

2. $m_1 = m_2$: Here we use that μ_T is tracefree to conclude that actually $\#\sigma_T^r \geq 2$. To fix ideas let us suppose that λ_4 belongs to σ_T^r and note that $T_1 - T_2$ is in $\Lambda_+^4(V)$. This time we construct a 4-form in $\Lambda_+^4(V)$ by setting

$$\overline{T} = T_1 - T_2 + m_4 T_3 - m_3 T_4.$$

We have that

$$Z_{\overline{T}} = (\$_1 \oplus \$_2 \oplus \$_3 \oplus \$_4)^{\perp},$$

since $m_3 m_4 \neq 0$. Since σ_T contains at least 4 eigenvalues $Z_{\overline{T}}$ is never 6-dimensional. Hence if $Z_{\overline{T}} \neq 0$ we use Corollary 3.2, (ii) to find that $\mathfrak{g}_T^*$ is maximal.

If $Z_{\overline{T}} = \{0\}$, in other words if $\$^+ = \$_1 \oplus \$_2 \oplus \$_3 \oplus \$_4$ we will use the self-dual 4-form

$$\check{T} = m_4 T_3 - m_3 T_4,$$

having $Z_{\check{T}} = (\$_3 \oplus \$_4)^{\perp}$. This cannot be of dimension 6 for that would imply $m_3 = m_4 = 1$ hence $\lambda_3 + \lambda_4 = 0$ since μ_T is trace free, which contradicts that λ_3 is not in σ_T^s. Neither can $Z_{\check{T}}$ vanish, so again (ii) in Corollary 3.2 yields that $\mathfrak{g}_T^* = A$.

(ii) $\sigma_T^r = \{0\}$: This corresponds to the case when T is proportional to a unipotent element. The computation of $\mathfrak{g}_T^*$ has been done in [2, Thm. 4.1].

• **The case when** $d = 0$**:** In this case, from (32) we know that $X \lrcorner T_i$ belong to $\mathfrak{g}_T^*$, for all X in V and all $1 \leq i \leq 8$. In particular

$$X \lrcorner \sum_{i=1}^{8} T_i = X \lrcorner \tfrac{1}{2}(1+\nu) = \tfrac{1}{2} X \lrcorner \nu$$

is an element of $\mathfrak{g}_T^*$ for all X in V, after using (27). But

$$[X \lrcorner \nu, Y \lrcorner \nu] = -2X \wedge Y,$$

for all X, Y in V (see for instance the proof of Proposition 3.5 in [2]). Therefore $\Lambda^2(V)$ is contained in $\mathfrak{g}_T^*$ and we conclude by Lemma 3.7.

We have finished to prove that $\mathfrak{g}_T^*$ is either isomorphic to $\mathfrak{so}(8,8)$ or to $\mathfrak{so}(8,1)$, which are semisimple, thus perfect (see [11, page 59]). Hence $\mathfrak{g}_T^*$ is perfect as well and the proof of the theorem is now complete. ∎

Theorem 1.1 in the introduction is now proved in the case of self-dual four forms without fixed spinors.

3.5. The full holonomy algebra when $Z_T \neq \{0\}$.

In order to have a complete description of holonomy algebras of self-dual 4-forms in 8-dimensions it remains to understand the odd part $\mathfrak{g}_T^{*,1}$ of $\mathfrak{g}_T^*$ for some T in $\Lambda^4_+(V)$ when $Z_T \neq \{0\}$. We recall that in this situation $\mathfrak{g}_T^{*,0}$ has been computed in Proposition 3.2, provided that also $\dim_{\mathbb{R}} Z_T \neq 6$. Let us now define

$$Q = \{\varphi \in A^1 : T\varphi + \varphi T = 0\}.$$

To study the space Q we shall make use of the symmetric tensor product of spinors, as defined in Definition 2.1.

Lemma 3.8. *Let T belong to $\Lambda^4_+(V)$. The following hold:*

(i) $Q = \{\varphi \in A^1 : T\varphi = \varphi T = 0\}$;

(ii) the Clifford multiplication provides an isomorphism $\$^- \otimes Z_T \to Q$ defined by $(x, y) \to x \odot y$ for all x in $\$^-$ and all y in Z_T;

(iii) the Clifford multiplication provides an isomorphism $\$^- \otimes Z_T^{\perp} \to Q^{\perp}$ defined by $(x, y) \to x \odot y$ for all x in $\$^-$ and all y in $Z_T^{\perp}$.

Proof. (i) If $T\varphi+\varphi T=0$ with φ in A^1, left multiplication with ν gives $T\varphi-\varphi T=0$, hence our claim, while using that $\nu T=T$ and $\nu\varphi+\varphi\nu=0$.
(ii) It is easy to see from (i) that for any φ in Q the map μ_φ is a symmetric endomorphism of $\$$ such that $\mu_\varphi \$^- \subseteq Z_T$. An elementary observation is that $x\odot y\in Cl_8^1$ for all x in $\$^+$ and for all y in $\$^-$. Moreover it is easy to check that $\alpha(x\odot y)^t=x\odot y$ for all x,y in $\$$ hence $x\odot y\in A^1$ whenever $(x,y)\in \$^+\times \$^-$. To see that μ is surjective we first observe that directly from the definition of the spinor product the following hold

$$\begin{aligned} T(x_1\otimes x_2) &= Tx_1\otimes x_2\\ (x_1\otimes x_2)T &= x_1\otimes Tx_2 \end{aligned}$$

for all x_1,x_2 in $\$$. After symmetrisation of the formulae above it is easy to conclude that

$$T(x\odot y)+(x\odot y)T=0,$$

for all x in $\$^-$ and for all y in Z_T, hence $x\odot y$ is in Q. But by construction of the spinor product $\{\mu_{x\odot y}: x\in \$^-, y\in Z_T\}$ spans $\$^-\otimes Z_T$ and the surjectivity of $\mu: Q\to \$^-\otimes Z_T$ follows. Moreover the restriction of ν to Q is injective since $\mu: Cl_8\to \$\otimes\$$ is injective.
(iii) Since $\mu: A^1\to \$^-\otimes\$^+$ is self-adjoint and the orthogonal complement of $\$^-\otimes Z_T$ in $\$^+\otimes\$^-$ is $\$^-\otimes Z_T^\perp$ the claim follows from (ii). ■

Proposition 3.3. *Let T belong to $\Lambda^4_+(V)$ with $Z_T\neq\{0\}$. We have that $\mathfrak{g}_T^{*,1}=Q^\perp$.*

Proof. A direct computation shows that

$$-[(1-\nu)\gamma, X\lrcorner T]=(\gamma\wedge X)T+T(\gamma\wedge X)+[X\lrcorner\gamma, T],$$

for all γ in $\Lambda^2(V)$ and all X in V. Given that $(1-\nu)\Lambda^2(V)\subseteq\mathfrak{g}_T^{*,0}$ it follows easily that $T\varphi+\varphi T$ belongs to $\mathfrak{g}_T^{*,1}$ for all φ in $\Lambda^3(V)$ and further that this actually holds for all φ in $A^1=\Lambda^3(V)\oplus\Lambda^7(V)$. This is because $T\Lambda^7(V)+\Lambda^7(V)T$ just gives the generating set of $\mathfrak{g}_T^*$ since $\Lambda^7(V)=\nu\Lambda^1(V)$.
Therefore $\mathfrak{g}_T^{*,1}$ contains the image of the symmetric operator $\{T,\cdot\}: A^1\to A^1$ hence $Q^\perp$. To show that $\mathfrak{g}_T^{*,1}\subseteq Q^\perp$ it is enough to see, by using Lemma 3.8, (ii) that $\langle\varphi, x\odot y\rangle=0$ for all x in $\$^-$ and for all y in Z_T. But an argument similar to that in the last part of the proof of Proposition 3.2 gives $\langle\varphi, x\odot y\rangle=\frac{1}{8}\langle\varphi y, x\rangle=0$ because $\varphi Z_T=0$, from the definition of the set of fixed spinors (14). The promised inclusion and hence the claim follow now easily. ■

Therefore our main result on holonomy algebras of self-dual 4-forms with fixed spinors from this section is

Theorem 3.2. *Let T be in $\Lambda^4_+(V)$ with $\dim_{\mathbb{R}} Z_T\neq 0,6$. Then the Clifford multiplication realises a Lie algebra isomorphism*

$$\mu:\mathfrak{g}_T^*\to\mathfrak{so}(8, 8-\dim_{\mathbb{R}} Z_T).$$

In particular, we must have $\mathfrak{h}_T^*=\mathfrak{g}_T^*$.

Proof. From Proposition 2.2 we know that the Clifford multiplication

$$\mu : \mathfrak{g}_T^* \to \mathfrak{so}(\not{S}^- \oplus Z_T^\perp, \widehat{\beta}_T) \cong \mathfrak{so}(8, 8-k)$$

is a Lie algebra monomorphism. Using this we shall prove the claim by counting dimensions. Let us set $k = \dim_{\mathbb{R}} Z_T$. From Proposition 3.2, (i) we get that the dimension of $\mathfrak{g}_T^{*,0}$ equals

$$\dim_{\mathbb{R}} F + 28 = \frac{(8-k)(7-k)}{2} + 28,$$

after using the isomorphism between F and $\mathfrak{so}(Z_T^\perp)$ in Corollary 3.1. Now since by Proposition 3.3 we have $\mathfrak{g}_T^{*,1} = Q^\perp$ and the latter is isomorphic to $\not{S}^- \otimes Z_T^\perp$ by Lemma 3.8, (iii), it follows that the dimension of $\mathfrak{g}_T^{*,1}$ is $8(8-k)$. Therefore the dimension of $\mathfrak{g}_T^*$ is

$$\frac{(8-k)(7-k)}{2} + 28 + 8(8-k) = \frac{(16-k)(15-k)}{2} = \dim_{\mathbb{R}} \mathfrak{so}(8, 8-k).$$

It follows that μ is surjective as well, therefore an isomorphism. The equality of $\mathfrak{g}_T^*$ and $\mathfrak{h}_T^*$ follows from the fact that the Lie algebras $\mathfrak{so}(8, 8-k), 0 \leq k \leq 8$ are semisimple, thus perfect (see [11, page 59]). ■

The proof of the Theorem 1.1 in the introduction is now complete (see also Theorem 3.1), except for the case when the space of fixed spinors is 6-dimensional.

4. The case when $\dim_{\mathbb{R}} Z_T = 6$

In this section we shall continue to work on an 8-dimensional Euclidean vector space $(V^8, \langle\cdot,\cdot\rangle)$ which is furthermore supposed to be oriented, with orientation form given by ν in $\Lambda^8(V)$. We will assume that T in $\Lambda^4_+(V)$ satisfies $\dim_{\mathbb{R}} Z_T = 6$, and our primary aim will be to compute the algebra $\mathfrak{g}_T^*$. As we have seen this situation cannot be covered only by the previous methods so we need more information about the structure of such forms. Let therefore $\sigma_T = \{\lambda_1, \lambda_2\}$ be the non-zero part of the spectrum of $\mu_T : \not{S}^+ \to \not{S}^+$ with multiplicities $(1,1)$ and let us also recall that $\lambda_1 + \lambda_2 = 0$. We equally recall that in this case the splitting of $\Lambda^2(V)$ from Proposition 3.1 becomes

$$\Lambda^2(V) = \iota_T^0 \oplus E_1 \oplus E_2 \oplus F_{12} \tag{33}$$

and in particular F is reduced to the 1-dimensional component F_{12}. In what follows we shall use the normalisation $\lambda_1 = 1$ as it is clear that rescaling the generating form leaves a holonomy algebra unchanged.

4.1. Spinor 2-planes.

We start by recalling the following

Definition 4.1. *Let $(V^8, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space. An almost Hermitian structure consists in a linear almost complex structure J which is orthogonal w.r.t. the scalar product $\langle\cdot,\cdot\rangle$. If moreover V is oriented, with orientation given by ν in $\Lambda^8(V)$, J is positive if $\omega^4 = \lambda\nu$ for some $\lambda > 0$ where $\omega = \langle J\cdot,\cdot\rangle$ is the so-called Kähler form of $(\langle\cdot,\cdot\rangle, J)$.*

In what follows we shall keep all previous notations and also recall the following well known fact, see [8] for instance.

Proposition 4.1. *Let $L \subset \$^+$ be any oriented 2-dimensional sub-space of positive spinors. Then L determines a unique positive almost Hermitian structure, say J, on V.*

For later use, and by sending again the reader to [8, page119], we mention that J is constructed such that $(1+\nu)\omega = x_1 \wedge x_2$ for any oriented orthogonal basis $\{x_1, x_2\}$ in L with the convention that $|x_1|^2 = |x_2|^2 = 2$, where $\omega = \langle J\cdot, \cdot\rangle$. Note that here one uses the isomorphism in Lemma 2.3. It is not difficult to see that the converse of Proposition 4.1 also holds, in the sense that any compatible almost Hermitian structure J defines a 2-dimensional sub-space L of $\$^+$ which is explicitly given as $L = Ker(\mu_\omega^2 + 16)$. For any compatible almost Hermitian structure J we denote by λ^4 the underlying real bundle of the canonical line bundle of J. Explicitly, $\lambda^4 = \{T \in \Lambda^4(V) : T(J\cdot, J\cdot, \cdot, \cdot) = -T\}$ and if moreover J is positive λ^4 is contained in $\Lambda^4_+(V)$ (see [13, page 112]). Note that if the contrary is not specified all forms are real valued in this setting. We also recall that in presence of an almost Hermitian structure $\Lambda^2(V) = \Lambda^2_0(V) \oplus \mathbb{R}\omega$, an orthogonal direct sum and that $\lambda^{1,1}$ denotes the space of J-invariant 2-forms on V.

Lemma 4.1. *Let $L \subset \$^+$ be two dimensional and oriented and let J be the complex structure determined by L. Then T in $\Lambda^2(V)$ satisfies $TL = 0$ iff T belongs to $\lambda_0^{1,1}(V)$.*

Proof. This is an easy exercise taking into account that from the construction of J it follows

$$JYx_1 = -Yx_2, \qquad JYx_2 = Yx_1, \tag{34}$$

for all Y in V, where $\{x_1, x_2\}$ is an oriented orthonormal basis in L. ■

This essentially leads to having $\iota_T^0 = \lambda_0^{1,1}$ fact to be used later on and which encodes the well-known special isomorphism $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ [12, Theorem 8.1]. Moving within the same circle of arguments we have:

Proposition 4.2. *Given any 2-dimensional sub-space $L \subset \$^+$, the map $(x, y) \to x \odot y$ induces an isomorphism $S_0^2(L) \to \lambda^4$.*

This is proved by considering an oriented, orthogonal basis $\{x_1, x_2\}$ in L as before. Then $\omega x_1 = 4x_2, \omega x_2 = -4x_1$ and the proof follows by a short computation on the generators of $S_0^2(L)$, when taking into account that $\lambda^4 \subseteq \Lambda^4_+(V)$ is characterised as

$$\{\varphi \in \Lambda^4_+(V) : ad_\omega^2 \varphi = -16\varphi\}.$$

Here $ad_\omega : Cl_8 \to Cl_8$ is given by $ad_\omega = [\omega, \cdot]$.

Proposition 4.3. *Any 4-form T in $\Lambda^4_+(V)$ with $\dim_{\mathbb{R}} Z_T = 6$ determines uniquely an $SU(4)$-structure. That is, there exists a unique compatible and positive almost Hermitian structure J on V such that T belongs to λ^4. The isotropy algebra of T is isomorphic to $\mathfrak{su}(4)$.*

Proof. Let $L = Z_T^{\perp}$ be the orthogonal complement of Z_T in $\$^+$. Since this is 2-dimensional we get a positive almost Hermitian structure J on V. Now μ_T is completely determined by its restriction to L which gives an element in $S_0^2(L)$ and the fact that T belongs to λ^4 follows from Proposition 4.2. The claim concerning the isotropy algebra follows from Proposition 3.1, (iii) by making use of the above mentioned special isomorphism $\mathfrak{su}(4) \cong \mathfrak{so}(6)$. ∎

This shows how to construct examples of self-dual 4-forms T such that Z_T is of dimension 6. Similarly, from the classification of self-dual 4-forms on $\mathbb{R}^8$ obtained in [4] one can easily give a geometric description of the cases when Z_T has smaller dimension, but for considerations of time and space we shall not present those here.

4.2. The holonomy algebra.

As a convenient intermediary object, we shall make use of the Lie subalgebra $\mathfrak{g}_T^{*,2}$ of $\mathfrak{g}_T^{*,0} \subseteq A^0$ generated by the subset

$$\{[X \lrcorner T, Y \lrcorner T] : X, Y \in V\}$$

of A^0. We point out that the even part of $\mathfrak{h}_T^*$ is *a priori* not equal to $\mathfrak{g}_T^{*,2}$.

Lemma 4.2. *We have*

$$\mathfrak{g}_T^{*,2} = (3+\nu)F \oplus (1-\nu)\iota_T^0.$$

Proof. Follows by a straightforward computation based on the fact that $\mathfrak{g}_T^{*,2}$ is generated by the set given in (22) and on the equations (17) defining the spaces E and F. ∎

For notational convenience let us set $Q_T^1 = \{X \lrcorner T : X \in V\}$ and also $Q_T^2 = \{X \lrcorner (T\gamma_{12}) : X \in V\}$, where γ_{12} is the generator of F_{12} with the convention that $|\gamma_{12}| = 2$. Given that $T\gamma_{12} + \gamma_{12}T = 0$ is easily seen that $T\gamma_{12}$ belongs to $\Lambda_+^4(V)$ and hence $Q_T^k, k = 1, 2$ are both contained in $\Lambda^3(V)$.

Lemma 4.3. *The following hold:*

(*i*) $[(1-\nu)\iota_T^0, Q_T^1] = Q_T^1$,

(*ii*) $[(3+\nu)F_{12}, Q_T^1] \subseteq Q_T^1 \oplus Q_T^2$,

(*iii*) $[Q_T^1, Q_T^2] = \mathfrak{g}_T^{*,2}$.

Proof. (*i*) If γ belongs to ι_T^0 and X is in V, an easy computation using essentially that $T\gamma = \gamma T = 0$ and the self-duality of T yields

$$[(1-\nu)\gamma, X \lrcorner T] = -2[X \lrcorner \gamma, T].$$

(*ii*) Recall that $T\gamma_{12} + \gamma_{12}T = 0$ and again using the self-duality of T we obtain after a short computation

$$[(3+\nu)\gamma_{12}, X \lrcorner T] = 3[X, T\gamma_{12}] - 2[X \lrcorner \gamma_{12}, T],$$

for all X in V.
(iii) Because we also have $T\gamma_{12}T = -\frac{1}{2}(1+\nu)\gamma_{12}$ it follows that $T^2\gamma_{12} = \gamma_{12}T^2 = \frac{1}{2}(1+\nu)\gamma_{12}$. Therefore, by using mainly the stability relations in Lemma 2.1 and that T is self-dual, we arrive after computing at some length at

$$-4[X \lrcorner T, Y \lrcorner (T\gamma_{12})] = (TXYT)\gamma_{12} + \gamma_{12}(TYXT) + \tfrac{1}{2}(1-\nu)(X\gamma_{12}Y + Y\gamma_{12}X),$$

whenever X, Y belong to V. Now using (33) it is easily seen that $[T\gamma T, \gamma_{12}] = 0$ for all γ in $\Lambda^2(V)$ hence our commutator becomes

$$-4[X \lrcorner T, Y \lrcorner (T\gamma_{12})] = -\langle X, Y\rangle(1+\nu)\gamma_{12} + \tfrac{1}{2}(1-\nu)(X\gamma_{12}Y + Y\gamma_{12}X),$$

for all X, Y in V. On the other hand, given that γ_{12} induces a compatible almost complex structure J on V such that $\gamma_{12} = \langle J\cdot, \cdot\rangle$ we actually have

$$\begin{aligned} X\gamma_{12}Y + Y\gamma_{12}X &= 2(X \wedge (Y \lrcorner \gamma_{12}) + Y \wedge (X \lrcorner \gamma_{12})) - 2\langle X, Y\rangle\gamma_{12} \\ &= 2(X \wedge JY + Y \wedge JX)_0 - \langle X, Y\rangle\gamma_{12}, \end{aligned}$$

where the subscript indicates orthogonal projection onto $\Lambda^2_0(V)$. Hence, our commutator reads finally

$$-4[X \lrcorner T, Y \lrcorner (T\gamma_{12})] = -\tfrac{1}{2}\langle X, Y\rangle(3+\nu)\gamma_{12} + (1-\nu)(X \wedge JY + Y \wedge JX)_0,$$

for all X, Y in V. Obviously, $(X \wedge JY + Y \wedge JX)_0$ belongs to $\lambda_0^{1,1} = \iota_T^0$ hence $[Q_T^1, Q_T^2] \subseteq \mathfrak{g}_T^{*,2}$ and the equality follows at once when using the linear isomorphism $S_0^2 \to \lambda_0^{1,1}, S \to SJ$. ∎

Theorem 4.1. *Let T belong to $\Lambda^4_+(V)$ satisfy $\dim_{\mathbb{R}} Z_T = 6$. Then $\mathfrak{g}_T^*$ is isomorphic to $\mathfrak{so}(6,2)$ and moreover $\mathfrak{h}_T^* = \mathfrak{g}_T^*$.*

Proof. It now easy to infer from the above that $\mathfrak{g}_T^* = \mathfrak{g}_T^{*,2} \oplus Q_T^1 \oplus Q_T^2$, therefore the claim on $\mathfrak{g}_T^*$ follows. The proof is completed when recalling that the Lie algebra $\mathfrak{so}(6,2)$ has trivial center. ∎

The proof of Theorem 1.1 is now complete, by putting together results in Theorems 3.1, 3.2 and 4.1.

We end this section by pointing out that in the case above the Clifford multiplication map $\mu : \mathfrak{g}_T^* \to Hom(Z_T^{\perp}, \$^-)$ is no longer surjective.

Acknowledgements: During the preparation of this paper both authors were partially supported by University of Auckland grants. They warmly acknowledge the Centro di Giorgi in Pisa and S. Salamon for giving them the opportunity to attend the meeting "Geometry and topology" where parts of this work were completed. During an early stage of this paper the research of P.-A. N. has been supported through the VW-foundation at the HU-Berlin, while the work of N. B. was funded by the DFG. Special thanks go to I. Agricola, S. Chiossi, A. R. Gover and Th. Friedrich for a number of useful discussions. We are grateful to the referee for pointing out some missing cases in an earlier version of this paper and for valuable suggestions on how to improve this work.

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Received June 27, 2007
and in final form September 14, 2007