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Three Term Recursion Relation for Spherical Functions Associated to the Complex Hyperbolic Plane

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Abstract. The symmetric space duality between the complex hyperbolic plane $H_2(\mathbb{C}) = \mathrm{SU}(2,1)/\mathrm{U}(2)$ and the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ also becomes apparent in the theory of matrix valued spherical functions associated to both spaces. This is stressed in this paper by proving a three term recursion relation for a family of matrix valued functions built up from the spherical functions associated to $H_2(\mathbb{C})$.
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1. Introduction

The analogy between the spherical geometries and the hyperbolic geometries is a special case of a general duality for symmetric spaces. This analogy reappears in the function theory on these spaces, as the one between the trigonometric functions and the hyperbolic functions, or between the Jacobi polynomials $P_n^{(\alpha,\beta)}$ and the Jacobi functions $F_u^{(\alpha,\beta)}$. These analogies become apparent once we recall the expressions of the trigonometric and hyperbolic functions in terms of the exponential function, and the ones of the Jacobi polynomials and the Jacobi functions in terms of Gauss' hypergeometric function. In particular the last analogy, for $\alpha = n$ and $\beta = 1$, is an instance of a more general one found while studying the irreducible spherical functions of any type, associated to the complex projective plane $P_2(\mathbb{C})$ or to the complex hyperbolic plane $H_2(\mathbb{C})$ (see [4] and [13]). These two are dual Hermitian symmetric spaces, and as such, the compact one $P_2(\mathbb{C})$ contains the noncompact dual $H_2(\mathbb{C})$ as an open submanifold (see Proposition 7.14, Ch.VII in [8]).

In [4] we obtain all irreducible spherical functions of any K-type associated to the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$. In [16] and [13] we carry out the same program for the complex hyperbolic plane $H_2(\mathbb{C}) = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$. These spherical functions are closely related with the

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matrix valued hypergeometric function, introduced in [20]. In [14], by tensoring certain representations of SU(3) and decomposing them into irreducible representations, we obtain a multiplication formula for spherical functions. From this formula we derive a three term recursion relation for certain "packages" of spherical functions. Restricting this to a one real variable (the variable that parameterizes a section of the K-orbits in $P_2(\mathbb{C})$), we obtain a three term recursion relation for a sequence of matrix valued orthogonal polynomials, closely related with the spherical functions. These results and the content of this paper, together with those on matrix valued orthogonal polynomials of several authors (see for example [1], [2], [6], [15], [12]), and the forthcoming paper [17] on the matrix spherical transform and its inversion formula on a locally compact group, should be considered as part of a large research project aimed at the analysis of matrix valued special functions.

In the present paper for $G = \mathrm{SU}(2,1)$ and $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ we construct, out of several spherical functions of (G,K) of a given K-type $\pi_{n,\ell}$, $n \in \mathbb{Z}$ and $\ell \in \mathbb{N}_0$, a family parameterized by $v \in \mathbb{C}$ of $(\ell+1) \times (\ell+1)$-matrix valued functions $\widetilde{H}(t;v)$, $t \geq 1$. One of the main purposes of this paper is to prove that, as functions of the spectral parameter v the functions $\widetilde{H}(t;v)$ satisfy a three term recursion relation of the form

$$t\widetilde{H}(t;v) = A_v\widetilde{H}(t;v-2) + B_v\widetilde{H}(t;v) + C_v\widetilde{H}(t;v+2), \tag{1}$$

where A_v, B_v, C_v are matrices independent of t. On the other hand the functions $\widetilde{H}(t;v)$ satisfy a differential equation of the form

$$D\widetilde{H}(t;v)^{\mathrm{t}} = \widetilde{H}(t;v)^{\mathrm{t}}\Lambda, \tag{2}$$

where D is a second order differential operator in the variable t whose coefficients depend on t and not on v, see [13] and [16]. Here Λ is a diagonal matrix with entries that depend on v but not on t and the superscript denote transpose. Thus the family $\widetilde{H}(t;v)$ is an instance of a solution of a matrix valued bispectral problem in the variables t, v. For a discussion of the bispectral problem see [3] and [9].

For $\ell = 0$ the function $\widetilde{H}(t;v)$ is scalar valued and it is given by

$$\widetilde{H}(t;v) = {}_2F_1\left(\begin{smallmatrix}-w,\, w+n+2\\ 2\end{smallmatrix}; 1-t\right),$$

where $w = -(n+v+2)/2$. Then, by making the change of variables $u = 1-t$, (1) comes down to the following three term recursion relation, in the spectral parameter w, for the Jacobi functions:

$$\begin{aligned} u\,{}_2F_1\left(\begin{smallmatrix}-w\,,\, w+n+2\\ 2\end{smallmatrix}; u\right) &= \tilde{a}_w\,{}_2F_1\left(\begin{smallmatrix}-w+1\,,\, w+n+1\\ 2\end{smallmatrix}; u\right) \\ &\quad + \tilde{b}_w\,{}_2F_1\left(\begin{smallmatrix}-w\,,\, w+n+2\\ 2\end{smallmatrix}; u\right) + \tilde{c}_w\,{}_2F_1\left(\begin{smallmatrix}-w-1\,,\, w+n+3\\ 2\end{smallmatrix}; u\right), \end{aligned}$$

where $\tilde{a}_w = -A_v$, $\tilde{b}_w = 1 - B_v$, $\tilde{c}_w = -C_v$.

Inspired on [14] we construct a family parameterized by $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ of matrix valued functions $\widetilde{\Phi}(g;\nu)$ on G of size $(\ell+1)^2 \times (\ell+1)$ built up of $\ell+1$ spherical functions of a given type (n,ℓ). The three term recursion relation that constitutes our main result, Theorem 5.1, is given by

$$\phi(g)\psi(g)\tilde{\Phi}(g;\nu) = \tilde{A}_\nu\tilde{\Phi}(g;\nu-\rho) + \tilde{B}_\nu\tilde{\Phi}(g;\nu) + \tilde{C}_\nu\tilde{\Phi}(g;\nu+\rho) \tag{3}$$

and this gives a highly nontrivial extension of (1). Here $\tilde{A}_\nu = A_\nu \otimes I$, $\tilde{B}_\nu = B_\nu \otimes I$, $\tilde{C}_\nu = C_\nu \otimes I$, where I denotes the $(\ell+1)\times(\ell+1)$ identity matrix and A_ν, B_ν, C_ν are the matrices appearing in (1) with $v = \nu(H_0)$.

It is important to stress that this relation is valid on G, and not just on a one dimensional submanifold of G. On the other hand from Proposition 2.1 it follows that

$$[D\widetilde{\Phi}(\cdot\,;\nu)](g) = \widetilde{\Phi}(g;\nu)[D\widetilde{\Phi}(\cdot\,;\nu)](e)$$

for any differential operator D on G left invariant under G and right invariant under K. Thus the family $\widetilde{\Phi}(g;\nu)$ provides an extension to G of a matrix valued version of the bispectral problem ([3], [9]) and to the best of our knowledge gives the first instance of a bispectral situation where one of the variables ranges over a set that is not one dimensional. Moreover the recursion relation (1) follows easily from (3) by restriction, see Proposition 6.2.

The first step in the proof of (3) consists in establishing a three term multiplication formula of (Theorem 4.4) for spherical functions, which is obtained from an explicit decomposition $Y^{\sigma,\nu}\otimes W = Y_1\oplus Y_2\oplus Y_3$ into irreducible submodules of the tensor product of the Harish-Chandra module $Y^{\sigma,\nu}$ of the principal series representation $U^{\sigma,\nu}$ and the standard module $W = \mathbb{C}^3$. Here we use Kostant's results on the tensor product of a finite dimensional and an infinite dimensional representation (see [11]). Then the closures $\overline{Y}_i$ of Y_i in $U^{\sigma,\nu}\otimes W$ are linearly independent $\mathrm{SU}(2,1)$-modules but $\overline{Y}_1\oplus\overline{Y}_2\oplus\overline{Y}_3$ is not closed. Because of this the proof of Theorem 4.4 is delicate. Concerning the equivalence of $\overline{Y}_i$ with certain principal series representations see Remark 3.19.

In this multiplication formula there appear spherical functions of types (n,ℓ) and $(n-1,\ell)$. To take care of this problem it is necessary to combine the multiplication formula with its dual, obtaining a multiplication formula involving seven spherical functions of type (n,ℓ). Then these spherical functions are appropriately package into three $(\ell+1)^2\times(\ell+1)$ matrix valued function $\widetilde{\Phi}(g;\nu)$, $\widetilde{\Phi}(g;\nu+\rho)$, $\widetilde{\Phi}(g;\nu-\rho)$, which yields (3).

In the last section, by restricting to the abelian Iwasawa subgroup A, we derive the three term recursion relation (1) for the family of $(\ell+1)\times(\ell+1)$ matrix valued functions $\widetilde{H}(t;v)$ closely related to the function $\widetilde{\Phi}(g;\nu)$, $v=\nu(H_0)$.

It is a pleasure to thank the referee for comments which led to the addition of Subsection 6.3.

2. Preliminaries

2.1. Spherical functions.

In this subsection we recall some facts on spherical functions which are useful to understand the rest of the paper.

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta\in\hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose the Haar measure dk on K normalized by $\int_K dk = 1$. We shall denote by V a finite dimensional vector space over $\mathbb{C}$ and by $\mathrm{End}(V)$ the space of all linear transformations of V into V.

A *spherical function* Φ on G *of type* $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that $\Phi(e) = I$ (I= identity transformation) and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk,$$

for all $x, y \in G$. See [19],[7].
If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ then $\pi : k \mapsto \Phi(k)$ is a representation of K such that any irreducible subrepresentation belongs to δ. The number of times that δ occurs in the representation π is called the *height* of Φ.

Spherical functions of type δ arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a complete, locally convex, Hausdorff topological vector space E, then

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k)\, dk$$

is a continuous projection of E onto $P(\delta)E = E(\delta)$; $E(\delta)$ consists of those vectors in E, the linear span of whose K-orbit is finite dimensional and splits into irreducible K-subrepresentations of type δ. Whenever $E(\delta)$ is finite dimensional, the function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by $\Phi(g)a = P(\delta)U(g)a$, $g \in G, a \in E(\delta)$ is a spherical function of type δ. If the representation $g \mapsto U(g)$ is topologically irreducible (i.e. E admits no non-trivial closed G-invariant subspace) then the associated spherical function Φ is also irreducible.

If a spherical function Φ is associated to a Banach representation of G then it is quasi-bounded, in the sense that there exists a semi-norm ρ on G and $M \in \mathbb{R}$ such that $\|\Phi(g)\| \leq M\rho(g)$ for all $g \in G$. Conversely, if Φ is an irreducible quasi-bounded spherical function on G, then it is associated to a topologically completely irreducible (TCI) Banach representation of G (Godement, see [19]). Thus if G is compact any ir reducible spherical function on G is associated to a Banach representation of G, which is finite dimensional by Peter-Weyl theorem.

When G is a connected Lie group then it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. From the differential point of view a spherical function of type δ can be characterized in the following way. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translation by elements in K. Let (V, π) be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$. Then we have

Proposition 2.1. *([19],[7]) A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*
i) Φ is analytic.
ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.
iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

Proposition 2.2. *([19]) Let $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ be two spherical functions on a connected Lie group G such that $\Phi(k) = \Psi(k)$ for all $k \in K$. Then $\Phi = \Psi$ if and only if $[D\Phi](e) = [D\Psi](e)$ for all $D \in D(G)^K$.*

If G is a noncompact connected semisimple Lie group with finite center, and K is a maximal compact subgroup of G, then, from the Subquotient Theorem of Harish-Chandra (see [21] Theorem 5.5.1.5) or from Casselman's Subrepresentation Theorem (see [10] p. 238), we know that any TCI Banach representation of G is infinitesimally equivalent to a subquotient, respectively to a subrepresentation, of a nonunitary principal series $U^{\sigma,\nu}$. Thus, if Φ is a quasi-bounded irreducible spherical function on G of type $\delta \in \hat{K}$, there exists (σ,ν) and a K-projection $Q(\delta)$, of $U^{\sigma,\nu}$ onto the δ-isotypic component of an irreducible subrepresentation of $U^{\sigma,\nu}$, such that Φ is equivalent to $Q(\delta)U^{\sigma,\nu}Q(\delta)$. This follows from Proposition 2.2. In particular when $D(G)^K$ is abelian Φ is equivalent to $\Phi_\delta^{\sigma,\nu} = P(\delta)U^{\sigma,\nu}P(\delta)$, because in such a case the multiplicity of δ in $U^{\sigma,\nu}$ is one. For our group $G = \mathrm{SU}(2,1)$ we know that all irreducible spherical functions on G are quasi-bounded since all appear associated to a principal series representation. (See [13] and [16]).

The group $G = \mathrm{SU}(2,1)$ consists of all 3×3 complex matrices of determinant one that preserve the Hermitian form $q(z) = z_1\overline{z}_1 + z_2\overline{z}_2 - z_3\overline{z}_3$. The group G acts naturally in $P_2(\mathbb{C})$. The G-orbit of the point $(0,0,1)$ is the set

$$B = \{(x,y,1) \in P_2(\mathbb{C}) : |x|^2 + |y|^2 < 1\},$$

and the corresponding isotropy subgroup is $K = \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$. Thus $H_2(\mathbb{C}) = G/K$ can be identified with B, the open ball of radius one centered at the origin in $\mathbb{C}^2$.

The set $\hat{K}$ can be identified with the set $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$. If $k = \left(\begin{smallmatrix} A & 0 \\ 0 & a \end{smallmatrix}\right)$, with $A \in \mathrm{U}(2)$ and $a = (\det A)^{-1}$, then

$$\pi(k) = \pi_{n,\ell}(A) = (\det A)^n\, A^\ell, \tag{4}$$

where A^ℓ denotes the ℓ-symmetric power of A, defines an irreducible representation of K in the class $(n,\ell) \in \mathbb{Z}\times\mathbb{Z}_{\geq 0}$.

The representation $\pi_{n,\ell}$ of $\mathrm{U}(2)$ extends to a unique holomorphic representation of $\mathrm{GL}(2,\mathbb{C})$ into $\mathrm{End}(V_\pi)$, which we still denote by $\pi_{n,\ell}$. For any $g \in SU(2,1)$, we denote by $A(g)$ the left upper 2×2 block of g. It is easy to see that $\det A(g) \neq 0$.

For any $\pi = \pi_{(n,\ell)}$ let $\Phi_\pi : G \longrightarrow \mathrm{End}(V_\pi)$ be defined by

$$\Phi_\pi(g) = \Phi_{n,\ell}(g) = \pi_{n,\ell}(A(g)).$$

This function Φ_π is always an irreducible spherical function on G of type π. Since $D(G)^K$ is commutative all irreducible spherical functions on G are of height one (see [7], [19]). Thus such functions of type π are functions on G with values in $\mathrm{End}(V_\pi)$. To determine them we define a function H by

$$H(g) = \Phi(g)\,\Phi_\pi(g)^{-1}, \tag{5}$$

where Φ is supposed to be a spherical function of type π. Then H satisfies
i) $H(e) = I$.
ii) $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$.
iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.

Property ii) says that H may be considered as a function on B, and moreover from iii) it follows that H is determined by the function $H : r \mapsto H(r,0)$ on the interval $[0,1)$. Let M be the closed subgroup of K of all diagonal matrices of the form $\Delta(e^{i\theta}, e^{-2i\theta}, e^{i\theta})$, $\theta \in \mathbb{R}$. Then M fixes all points $(r,0) \in B$. Therefore iii) also implies that $H(r) = \pi(m)H(r)\pi(m^{-1})$ for all $m \in M$. Since any V_π as an M-module is multiplicity free, it follows that there exists a basis of V_π such that $H(r)$ is simultaneously represented by a diagonal matrix for all $0 \le r < 1$. Thus we can identify $H(r) \in \operatorname{End}(V_\pi)$ with a vector $H(r) = (h_0(r), \dots, h_\ell(r)) \in \mathbb{C}^{\ell+1}$. In this case the algebra $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$. The algebras $D(G)^G$ and $D(K)^K$ are polynomial algebras (Harish-Chandra's theorem) in two algebraically independent generators. A particular choice of two algebraically independent generators Δ_2 and Δ_3 of $D(G)^G$ is given in Proposition 3.1 of [4] and rewritten in this paper in (6). The fact that a spherical function is a simultaneous eigenfunction of Δ_2 and Δ_3 implies that the function H is an eigenfunction of certain ordinary second order differential operators D and E given in [16] and in [13]. For an explicit expression of the functions H in terms of the matrix valued hypergeometric function introduced in [20] see Theorems 5.1 and 5.2 in [16].

2.2. The principal series of $\mathrm{SU}(2,1)$**.** The Lie algebra of G is $\mathfrak{g} = \{X \in \mathfrak{gl}(3,\mathbb{C}) : JXJ = -X^*,\ \operatorname{tr} X = 0\}$. Its complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C})$.

Let $\mathfrak{h}_{\mathbb{C}}$ be the compact Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ of all diagonal matrices. We denote by α, β, γ the positive roots given by $\alpha(x_1E_{11} + x_2E_{22} + x_3E_{33}) = x_1 - x_2$, $\beta(x_1E_{11} + x_2E_{22} + x_3E_{33}) = x_2 - x_3$ and $\gamma = \alpha + \beta$. The corresponding root space decomposition is given by

$$X_\alpha = E_{12},\ X_{-\alpha} = E_{21},\ X_\beta = E_{23},\ X_{-\beta} = E_{32},\ X_\gamma = E_{13}, X_{-\gamma} = E_{31},$$

$$H_\alpha = E_{11} - E_{22},\ H_\beta = E_{22} - E_{33},\ H_\gamma = E_{11} - E_{33}.$$

We observe that $Z = H_\alpha + 2H_\beta$ belongs to the center of $\mathfrak{k}_{\mathbb{C}}$. In [4] we have proved the following proposition:

Proposition 2.3. *$D(G)^G$ as a polynomial algebra is generated by*

$$\Delta_2 = -H_\alpha^2 - \tfrac{1}{3}Z^2 - 2H_\alpha - 2Z - 4X_{-\alpha}X_\alpha - 4X_{-\beta}X_\beta - 4X_{-\gamma}X_\gamma$$

and

$$\begin{aligned}\Delta_3 &= \tfrac{2}{9}H_\alpha^3 - \tfrac{2}{9}H_\beta^3 + \tfrac{1}{3}H_\alpha^2H_\beta - \tfrac{1}{3}H_\alpha H_\beta^2 + 2H_\alpha^2 + H_\alpha H_\beta + 4H_\alpha + 2H_\beta \\ &+ X_{-\alpha}X_\alpha H_\alpha + 2X_{-\alpha}X_\alpha H_\beta + 6X_{-\alpha}X_\alpha + 3X_{-\beta}X_\beta + 3X_{-\gamma}X_\gamma \\ &- X_{-\beta}X_\beta\,\tilde H_1 - X_{-\gamma}X_\gamma\,\tilde H_2 + 3X_{-\beta}X_\gamma X_{-\alpha} + 3X_{-\gamma}X_\beta X_\alpha.\end{aligned}$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated to the Cartan involution $\theta(X) = -X^*$. Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} k & 0 \\ 0 & y \end{pmatrix} : k \in \mathfrak{u}(2),\ y = -\operatorname{tr}(k) \right\} \text{ and } \mathfrak{p} = \left\{ \{ \begin{pmatrix} 0 & b \\ \overline{b}^t & 0 \end{pmatrix} : b \in \mathbb{C}^2 \right\}.$$

Let $H_0 = E_{13} + E_{31}$ and $T = H_\alpha - H_\beta$. Thus $\mathfrak{a} = \mathbb{R}H_0$ is a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{m} = \mathbb{R}iT_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\tilde{\mathfrak{h}} = \mathfrak{m} \oplus \mathfrak{a}$ is a split

Cartan subalgebra of $\mathfrak{g}$. The root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\tilde{\mathfrak{h}}_{\mathbb{C}}$ is given by

$$\tilde{\alpha}(H_0) = 1, \tilde{\alpha}(T) = 3,\ \tilde{\beta}(H_0) = 1, \tilde{\beta}(T) = -3,\ \tilde{\gamma}(H_0) = 2, \tilde{\gamma}(T) = 0,$$

and the corresponding root vectors are

$$X_{\tilde{\alpha}} = E_{12} + E_{32}, X_{-\tilde{\alpha}} = E_{21} + E_{23}, X_{\tilde{\beta}} = E_{21} - E_{23}, X_{-\tilde{\beta}} = E_{12} - E_{32},$$
$$X_{\tilde{\gamma}} = E_{13} - E_{31} - E_{11} + E_{33}, X_{-\tilde{\gamma}} = E_{31} - E_{13} - E_{11} + E_{33}.$$

In this new basis, the differential operators Δ_2 and Δ_3 are given by

$$\begin{aligned} \Delta_2 &= -H_0^2 - \tfrac{1}{3}T^2 + 4H_0 - 2X_{\tilde{\alpha}}X_{-\tilde{\alpha}} - 2X_{\tilde{\beta}}X_{-\tilde{\beta}} - X_{\tilde{\gamma}}X_{-\tilde{\gamma}}, \\ 4\Delta_3 &= -\tfrac{1}{9}T^3 + T^2 + 4T + 3H_0^2 + TH_0^2 - 4H_0T - 12H_0 - X_{\tilde{\alpha}}TX_{-\tilde{\alpha}} \\ &\quad - X_{\tilde{\beta}}TX_{-\tilde{\beta}} + X_{\tilde{\gamma}}TX_{-\tilde{\gamma}} + 3X_{\tilde{\alpha}}H_0X_{-\tilde{\alpha}} - 3X_{\tilde{\beta}}H_0X_{-\tilde{\beta}} \\ &\quad - 3X_{\tilde{\beta}}X_{\tilde{\alpha}}X_{-\tilde{\gamma}} - 3X_{\tilde{\gamma}}X_{-\tilde{\alpha}}X_{-\tilde{\beta}} + 12X_{\tilde{\beta}}X_{-\tilde{\beta}} + 6X_{\tilde{\gamma}}X_{-\tilde{\gamma}}. \end{aligned} \tag{6}$$

Let $\lambda \in \mathfrak{a}^*$ be the restricted root defined by $\lambda(H_0) = 1$ and let $\mathfrak{n} = \mathfrak{g}_\lambda + \mathfrak{g}_{2\lambda}$ be the sum of the corresponding restricted root subspaces of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition.

Let N be the analytic subgroup of G with Lie algebra $\mathfrak{n}$, and observe that M is the centralizer of A in K. Then MAN is a minimal parabolic subgroup of G, and

$$M = \left\{ m_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \right\}, \quad A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \right\}.$$

For $r \in \mathbb{Z}$ and $v \in \mathbb{C}$ we define $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\sigma(m_\theta) = e^{ir\theta} \quad \text{and} \quad \nu(tH_0) = vt. \tag{7}$$

Then $man \mapsto e^{\nu(\log a)}\sigma(m)$ is a one dimensional representation of MAN, and it is this representation that we induce to G to construct its generalized principal series representation. Thus we put $U^{\sigma,\nu} = Ind_{MAN}^G$.

There are different "pictures" or realizations of these representations which have different uses and advantages. We choose the compact picture. In this case a dense subspace of the representation space of $U^{r,v}$ is

$$\{F : K \longrightarrow \mathbb{C} \text{ continuous } : F(km) = \sigma(m)^{-1}F(k); k \in K, m \in M\},$$

with norm

$$||F||^2 = \int_K |F(k)|^2\, dk.$$

If g decomposes under $G = KAN$ as $g = \kappa(g)\exp(H(g))n$, then the action is

$$U^{r,v}(g)F(k) = e^{-(\nu+\rho)H(g^{-1}k)}F(\kappa(g^{-1}k)).$$

The actual Hilbert space and representation are then obtain by completion. We recall that $\rho \in \mathfrak{a}^*$ is the half-sum of the positive restricted roots counted with multiplicities. In this case it is given by $\rho(H_0) = 2$.

Let $Y^{r,v}$ be the Harish-Chandra module of all K-finite vectors in $U^{r,v}$. Then $U^{r,v}$ admits an infinitesimal character since any element in $D(G)^G$ reduces to an scalar operator on $Y^{r,v}$ (see Proposition 8.22 of [10]). Then using the induced picture and the expressions (6) it is easy to prove the following proposition.

Proposition 2.4. *The infinitesimal character $\chi_{r,v}$ of the principal series $U^{r,v}$ is given by*

$$\begin{aligned}\chi_{r,v}(\Delta_2) &= -v^2+4-\tfrac{1}{3}r^2\\ \chi_{r,v}(\Delta_3) &= \tfrac14\left(-\tfrac19 r^3+r^2+rv^2+3v^2-12\right).\end{aligned}$$

We observe that $U^{r,v}$ and $U^{r,-v}$ have the same infinitesimal character. This is an instance of the general invariance of the infinitesimal character of the principal series representations under the restricted Weyl group.

Now we want to describe the structure of $Y^{r,v}$ as a K-module. Since K is a compact group, by the Peter-Weyl theorem we know that we have the following unitary direct sum:

$$L^2(K) = \oplus_{\pi\in\hat K} V_\pi \otimes V_\pi^*,$$

where the identification of $V_\pi \otimes V_\pi^*$ as a subspace of $L^2(K)$ is given by $(v\otimes\lambda)\mapsto (v\otimes\lambda)(k) = (k\cdot\lambda)(v)$. The restrictions to $V_\pi\otimes V_\pi^*$ of the left and right regular representations of K are, respectively, $L|_{V_\pi\otimes V_\pi^*} = \pi\otimes 1$ and $R|_{V_\pi\otimes V_\pi^*} = 1\otimes\pi^*$.

If $\pi=\pi_{n,\ell}$ there exists a basis $\{v_j^{(n,\ell)}\}_{j=0}^{\ell}$ of V_π, unique up to a multiplicative constant, such that

$$\begin{aligned}\dot\pi(H_\alpha)v_j^{(n,\ell)} &= (\ell-2j)v_j^{(n,\ell)}, &\quad \dot\pi(Z)v_j^{(n,\ell)} &= (2n+\ell)v_j^{(n,\ell)},\\ \dot\pi(X_\alpha)v_j^{(n,\ell)} &= j\,v_{j-1}^{(n,\ell)}, & \dot\pi(X_{-\alpha})v_j^{(n,\ell)} &= (\ell-j)\,v_{j+1}^{(n,\ell)}.\end{aligned} \tag{8}$$

Let us consider a $U(2)$ invariant inner product on $V_{(n,\ell)}$. In Lemma 3.1 of [14] we proved that the basis $\{v_j^{(n,\ell)}\}_{j=0}^{\ell}$ is an orthogonal basis such that

$$\|v_j^{(n,\ell)}\|^2 = \binom{\ell}{j}^{-1}\|v_0^{(n,\ell)}\|^2. \tag{9}$$

Let $\{\lambda_j^{(n,\ell)}\}_{j=0}^{\ell}$ be the dual basis of $\{v_j^{(n,\ell)}\}_{j=0}^{\ell}$.

Proposition 2.5. *The Harish-Chandra module $Y^{r,v}$ as a K-module decomposes in the following way*

$$Y^{r,v} = \bigoplus_{\ell=0}^{\infty}\bigoplus_{j=0}^{\ell}\left(V_{(-r+\ell-3j,\ell)}\otimes \mathbb{C}\lambda_j^{(-r+\ell-3j,\ell)}\right).$$

Proof. We first note that

$$Y^{r,v} = \Big\{F\in\bigoplus_{\pi\in\hat K} V_\pi\otimes V_\pi^* : R(m_\theta)F = e^{-ir\theta}F, m_\theta\in M\Big\}.$$

From $T=(3H_\alpha - Z)/2$ it follows that $\dot\pi(T)v_j = (\ell-n-3j)v_j$, and $\dot\pi^*(T)\lambda_j = -(\ell-n-3j)\lambda_j$.

Let us consider the representation of M in $\bigoplus_{\pi\in\hat K} V_\pi\otimes V_\pi^*$ defined by $m_\theta\mapsto e^{ir\theta}R(m_\theta)$. Then

$$Y^{r,v} = \Big(\bigoplus_{\pi\in\hat K} V_\pi\otimes V_\pi^*\Big)^M = \bigoplus_{\pi\in\hat K} V_\pi\otimes \left(V_\pi^*\right)^M:$$

If $\pi=\pi_{n,\ell}$ we have that $\left(V_\pi^*\right)^M = \mathbb{C}\lambda_j$ if $r=\ell-n-3j$ and $\left(V_\pi^*\right)^M=0$ otherwise. This completes the proof of the proposition. ∎

3. On the structure of $Y^{r,v} \otimes W$

As we shall see in the next section, a multiplication formula for the spherical functions associated to our pair (G, K) arises from a direct sum decomposition of $Y^{r,v} \otimes W$ into $D(G)$-modules which admit infinitesimal characters. It is well known that even if $Y^{r,v}$ is irreducible $Y^{r,v} \otimes W$ does not need to have such a direct sum decomposition. Nevertheless it always has a finite composition series, but we were not able to derive a multiplication formula in this general case.

We shall start considering the $D(G)$-module $Y^{r,v} \otimes W$, taking into account Kostant's contribution [11] on the tensor product of a finite and an infinite dimensional representation. To quote what it is needed we introduce the following notation.

Let $\mathfrak{g}_{\mathbb{C}}$ be a finite dimensional complex semisimple Lie algebra, $U(\mathfrak{g}_{\mathbb{C}})$ be its universal enveloping algebra and $Z(\mathfrak{g}_{\mathbb{C}})$ be the center of $U(\mathfrak{g}_{\mathbb{C}})$. Let V be a Harish-Chandra module with infinitesimal character χ_ξ and let V_λ be an irreducible finite dimensional representation with highest weight λ. Let $\Delta_\lambda = \{\mu_1, \dots, \mu_k\}$ be the set of all the distinct weights of V_λ. Now consider the sequence of k characters $\chi_{\xi+\mu_1}, \dots, \chi_{\xi+\mu_k}$ and put

$$Y_i = \{y \in V \otimes V_\lambda : u\, y = \chi_{\xi+\mu_i}(u)\, y \ \text{ for all } \ u \in Z(\mathfrak{g}_{\mathbb{C}})\}.$$

Then the following is the content of Corollary 5.5 of [11].

Theorem 3.1. *If the characters $\chi_{\xi+\mu_i}$, $i = 1, \dots, k$ are distinct and Y_i is not zero, then Y_i is the maximal submodule of $V \otimes V_\lambda$ which admits the infinitesimal character $\chi_{\xi+\mu_i}$, and*

$$V \otimes V_\lambda = Y_1 \oplus \cdots \oplus Y_k.$$

Now we consider the standard irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ on $W = \mathbb{C}^3$ and let $\{e_1, e_2, e_3\}$ denote the canonical basis of $\mathbb{C}^3$. Then $e_1 + e_3, e_2, e_1 - e_3$ are weight vectors with respect to $\tilde{\mathfrak{h}}_{\mathbb{C}}$ of weights μ_1, μ_2, μ_3, respectively, given by

$$\begin{aligned} \mu_1(H_0) &= 1, \quad \mu_2(H_0) = 0, \quad \mu_3(H_0) = -1 \\ \mu_1(T) &= 1, \quad \mu_2(T) = -2, \quad \mu_3(T) = 1. \end{aligned}$$

We observe that $\lambda = \mu_1$ is the highest weight of W. In terms of the dual basis of $\{H_0, T\}$ we have $\mu_1 = (1, 1)$, $\mu_2 = (0, -2)$ and $\mu_3 = (-1, 1)$.

Lemma 3.2. *The infinitesimal character $\chi_\xi = \chi_{r,v}$ of the Harish-Chandra module $Y^{r,v}$ is given by $\xi = (-v - 2, r)$.*

Proof. We first recall the definition of the character χ_ξ of $Z(\mathfrak{g}_{\mathbb{C}})$ for $\xi \in \tilde{\mathfrak{h}}_{\mathbb{C}}^*$. We know that given $u \in Z(\mathfrak{g}_{\mathbb{C}})$ there exists a unique $f_u \in U(\tilde{\mathfrak{h}}_{\mathbb{C}})$ such that $u - f_u \in U(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\mathbb{C}}^+$. Then $\chi_\xi(u) = f_u(\xi)$. By using the isomorphism $D(G)^G \simeq Z(\mathfrak{g}_{\mathbb{C}})$, obtained by restricting the canonical isomorphism between $D(G)$ and $U(\mathfrak{g}_{\mathbb{C}})$, from Proposition 2.4 we can compute $\chi_\xi(\Delta_2)$ and $\chi_\xi(\Delta_3)$. But first we need to rewrite these operators in such a way that the positive root vectors appear on the right. To do this we use the transpose anti-automorphism of $U(\mathfrak{g})$ defined by: if $X \in \mathfrak{g}$ then $X' = -X$, thus

$$(X_1 \cdots X_r)' = (-1)^r X_r \cdots X_1, \qquad \text{for any } \ X_i \in \mathfrak{g}.$$

Then $\Delta_2' = \Delta_2$, since Δ_2 is the Casimir operator of $\mathfrak{g}_{\mathbb{C}}$, and furthermore we have that $\Delta_3' = -\Delta_3 - 6\Delta_2$. Thus from (6), we get $f_{\Delta_2} = -(H_0^2 + \frac{1}{3}T^2 + 4H_0)$, and $f_{\Delta_3} = -\frac{1}{9}T^3 + T^2 + 4T + 3H_0^2 + 4H_0T + TH_0^2 + 12H_0$. Now it is easy to check that

$$f_{\Delta_2}(-v-2, r) = \chi_{r,v}(\Delta_2) \quad \text{and} \quad f_{\Delta_3}(-v-2, r) = \chi_{r,v}(\Delta_3).$$

This completes the proof of the proposition. ∎

Theorem 3.3. *Let $r \in \mathbb{Z}$ and $v \in \mathbb{C}$. If $v(v+r)(v-r) \neq 0$ then*

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3,$$

where Y_1, Y_2, Y_3 are Harish-Chandra modules with infinitesimal characters $\chi_{r+1,v-1}$, $\chi_{r-2,v}$, $\chi_{r+1,v+1}$, respectively. Moreover for $i = 1, 2, 3$

$$Y_i = \{y \in Y^{r,v} \otimes W : \Delta_2\, y = c_i\, y\},$$

where $c_1 = \chi_{r+1,v-1}(\Delta_2)$, $c_2 = \chi_{r-2,v}(\Delta_2)$, $c_3 = \chi_{r+1,v+1}(\Delta_2)$.

Proof. By Theorem 3.1 we just need to understand when $\chi_{\xi+\mu_i} = \chi_{\xi+\mu_j}$ for $i \neq j$ and $\xi = (-v-2, r)$ (Lemma 3.2).

We know that in general $\chi_\lambda = \chi_{\lambda'}$ $(\lambda, \lambda' \in \mathfrak{h}_{\mathbb{C}}^*)$ if and only if there exists and element $\tilde{w}$ in the translated Weyl group $\tilde{W}$ such that $\tilde{w}(\lambda) = \lambda'$ (see Section 2.1 of [11]). We recall that for w in the Weyl group W we have

$$\tilde{w}(\lambda) = w(\lambda + \rho) - \rho.$$

If $\lambda = (\lambda_1, \lambda_2)$, in terms of the dual basis of $\{H_0, T\}$, then

$$\begin{aligned} \tilde{s}_{\tilde{\alpha}}(\lambda) &= \tfrac{1}{2}(\lambda_1 - \lambda_2, -3\lambda_1 - \lambda_2), \\ \tilde{s}_{\tilde{\beta}}(\lambda) &= \tfrac{1}{2}(\lambda_1 + \lambda_2 - 2, 3\lambda_1 - \lambda_2 + 6), \\ \tilde{s}_{\tilde{\gamma}}(\lambda) &= (-\lambda_1 - 4, \lambda_2). \end{aligned}$$

If $\xi = (-v-2, r)$ we have $\xi + \mu_1 = (-v-1, r+1)$, $\xi + \mu_2 = (-v-2, r-2)$ and $\xi + \mu_3 = (-v-3, r+1)$. Then it is easy to see that

$$\begin{aligned} \tilde{s}_{\tilde{\alpha}}(\xi + \mu_1) &= \xi + \mu_2, && \text{if and only if} \quad v = r, \\ \tilde{s}_{\tilde{\beta}}(\xi + \mu_2) &= \xi + \mu_3, && \text{if and only if} \quad v = -r, \\ \tilde{s}_{\tilde{\gamma}}(\xi + \mu_1) &= \xi + \mu_3, && \text{if and only if} \quad v = 0, \end{aligned}$$

and that no further identifications under $\tilde{W}$ occur in $\{\xi + \mu_j : j = 1, 2, 3\}$.

From Lemma 3.2 it follows that $\chi_{\xi+\mu_1} = \chi_{r+1,v-1}$, $\chi_{\xi+\mu_2} = \chi_{r-2,v}$, $\chi_{\xi+\mu_3} = \chi_{r+1,v+1}$, which completes the proof of the first assertion.

Now let

$$\tilde{Y}_i = \{y \in Y^{r,v} \otimes W : \Delta_2\, y = c_i\, y\}.$$

Since $\Delta_2 \in D(G)^G$ it follows that $\tilde{Y}_i$ is a $D(G)$-submodule of $Y^{r,v} \otimes W$ such that $Y_i \subset \tilde{Y}_i$. On the other hand it is easy to verify, under the hypothesis $v(v+r)(v-r) \neq 0$, that all c_1, c_2, c_3 are different. Therefore $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{Y}_3$ are direct summands of $Y^{r,v} \otimes W$. Thus

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3 \subset \tilde{Y}_1 \oplus \tilde{Y}_2 \oplus \tilde{Y}_3,$$

which implies that $Y_i = \tilde{Y}_i$ for $i = 1, 2, 3$, completing the proof of the theorem. ∎

Theorem 3.4. *Let $r \in \mathbb{Z}$ and $v \in \mathbb{C}$. If $v(v+r)(v-r) \neq 0$ then*

$$Y^{r,v} \otimes W^* = Z_1 \oplus Z_2 \oplus Z_3,$$

where Z_1, Z_2, Z_3 are Harish-Chandra modules with infinitesimal characters $\chi_{r-1,v-1}$, $\chi_{r+2,v}$, $\chi_{r-1,v+1}$, respectively. Moreover for $i = 1,2,3$

$$Z_i = \{y \in Y^{r,v} \otimes W : \Delta_2 \, y = d_i \, y\},$$

where $d_1 = \chi_{r-1,v-1}(\Delta_2)$, $d_2 = \chi_{r+2,v}(\Delta_2)$, $d_3 = \chi_{r-1,v+1}(\Delta_2)$.

Proof. The theorem follows from Theorem 3.3 by duality. First of all we recall that the dual G-module of $U^{r,v}$ is isomorphic to $U^{-r,-v}$. In fact $\langle f, h\rangle = \int_K f(k)h(k)\,dk$ defines a pairing between $U^{r,v}$ and $U^{-r,-v}$ which is G-invariant because

$$\begin{aligned}\langle U^{r,v}(g)f, U^{-r,-v}(g)h\rangle &= \int_K e^{-2\rho H(g^{-1}k)} f(\kappa(g^{-1}k))h(\kappa(g^{-1}k))\,dk\\ &= \int_K f(k)h(k)\,dk = \langle f, h\rangle.\end{aligned}$$

The second equality is a known integral identity (see [21] Section 5.5.1). Moreover this pairing is nonsingular because an f is in the representation space of $U^{r,v}$ if and only if $\bar f$ is in the representation space of $U^{-r,-v}$.

On the other hand if a Harish-Chandra module Y has infinitesimal character χ then its K-finite dual Y^* is a Harish-Chandra module Y with infinitesimal character χ^* given by $\chi^*(D) = \chi(D')$, $D \in D(G)^G$. Then using Proposition 2.4 it follows that $\chi^*_{r,v} = \chi_{-r,v}$.

Now from Theorem 3.3 we get $Y^{-r,-v} \otimes W^* = Y_1^* \oplus Y_2^* \oplus Y_3^*$. Then by taking $Z_1 = Y_3^*$, $Z_2 = Y_2^*$, $Z_3 = Y_1^*$ and changing signs the theorem follows. ∎

3.1. Explicit decomposition of the tensor product. The aim of this subsection is to obtain an explicit description of the $D(G)$ modules Y_1, Y_2, Y_3 appearing in Theorem 3.3.

From now on we shall choose a particular basis $\{a_s^{j,\ell} : 0 \le s \le \ell\}$ of weight vectors of $V_{(-r+\ell-3j,\ell)}$ such that (8) holds.

We realize the K-module $V_{(-r+\ell-3j,\ell)}$ as the ℓ-symmetric power of $\mathbb{C}^2$ (see (4)) and take $a_s^{j,\ell} = e_1^{\ell-s} e_2^s$. The weight of $a_s^{j,\ell}$ with respect to the diagonal Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$ is $(-r+2\ell-3j-s)x_1 + (-r+\ell-3j+s)x_2$.

We shall identify $a_s^{j,\ell} \in V_{(-r+\ell-3j,\ell)}$ with a function on K: If $\{\lambda_s^{j,\ell}\}$ denotes the dual basis of $\{a_s^{j,\ell}\}$ then we put

$$v_s^{j,\ell}(k) = \lambda_j^{j,\ell}(k^{-1} a_s^{j,\ell}). \tag{10}$$

In this way $V_{(-r+\ell-3j,\ell)} \simeq V_{(-r+\ell-3j,\ell)} \otimes \mathbb{C}\lambda_j^{j,\ell} \subset Y^{r,v}$ and

$$Y^{r,v} = \bigoplus_{0 \le j \le \ell} \langle v_0^{j,\ell}\rangle_K, \tag{11}$$

in accordance with Proposition 2.5. Explicitly, if

$$k = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ -be^{-i\theta} & ae^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \qquad k^{-1} = \begin{pmatrix} a & -\bar{b}e^{i\theta} & 0 \\ b & \bar{a}e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix},$$

then

$$v_s^{j,\ell}(k) = e^{i\theta(-r+\ell-3j)}\lambda_j^{j,\ell}\left((ae_1+be_2)^{\ell-s}(-\bar{b}e^{i\theta}e_1+\bar{a}e^{i\theta}e_2)^s\right). \tag{12}$$

Then we know that $\{v_s^{j,\ell}\}_s$ is an orthogonal basis of $V_{(-r+\ell-3j,\ell)} \subset Y^{r,v}$ such that

$$\begin{aligned} H_\alpha\, v_s^{j,\ell} &= (\ell-2s)v_s^{j,\ell}, \quad & Z\, v_s^{j,\ell} &= (-2r+3\ell-6j)v_s^{j,\ell}, \\ X_\alpha\, v_s^{j,\ell} &= s\, v_{s-1}^{j,\ell}, & X_{-\alpha}\, v_s^{j,\ell} &= (\ell-s)v_{s+1}^{j,\ell}. \end{aligned} \tag{13}$$

The following lemma is a consequence of the so called Pieri's formula, see [22] §77.

Lemma 3.5. *The following decomposition of K-modules holds*

$$Y^{r,v}\otimes W \simeq \bigoplus_{\ell=0}^{\infty}\bigoplus_{j=0}^{\ell} V_{(-r+\ell-3j,\ell+1)} \oplus V_{(-r+\ell-3j+1,\ell-1)} \oplus V_{(-r+\ell-3j-1,\ell)}.$$

Moreover for $0\le j\le \ell$ let

$$\begin{aligned} u_0^{j,\ell} &= v_1^{j,\ell}\otimes e_1 - v_0^{j,\ell}\otimes e_2 \in V_{(-r+\ell+1-3j,\ell-1)} \\ z_0^{j,\ell} &= v_0^{j,\ell}\otimes e_3 \in V_{(-r+\ell-3j-1,\ell)} \\ w_0^{j,\ell} &= v_0^{j,\ell}\otimes e_1 \in V_{(-r+\ell-3j,\ell+1)}. \end{aligned}$$

Then the elements $u_0^{j,\ell}, z_0^{j-1,\ell-1}, w_0^{j-1,\ell-2}$ are dominant vectors of weight

$$\mu^{j,\ell} = (-r+2\ell-3j)x_1 + (-r+\ell-3j+1)x_2.$$

Now we notice that the standard G-module W realizes in the principal series $U^{1,-3}$ in the following way: let $\lambda \neq 0$ be a dominant weight vector in W^* with respect to the Cartan subalgebra $\tilde{\mathfrak{h}}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, and for any w in W let $h_w(k) = \lambda(k^{-1}w)$ for $k\in K$.

Lemma 3.6. *The map $w\mapsto h_w$ defines an injective homomorphism of G-modules from W into $U^{1,-3}$.*

Proof. It is convenient to use the induced picture to realize $U^{1,-3}$. We extend our function h_w to G by $h_w(g) = \lambda(g^{-1}w)$ for $g\in G$. Then we only need to prove that

$$h_w(gm_\theta an) = e^{-(\nu+\rho)\log a}\sigma(m_\theta)^{-1}h_w(g),$$

for $m_\theta\in M, a\in A, n\in N$, where $\nu(H_0) = -3$, $\sigma(m_\theta) = e^{ir\theta}$.
If $X\in\mathfrak{g}$ then

$$\begin{aligned} [Xh_w](g) &= \left(\frac{d}{dt}\right)_{t=0}\lambda((g\exp tX)^{-1}w) \\ &= \left(\frac{d}{dt}\right)_{t=0}(\exp tX\,\lambda)(g^{-1}w) = [X\lambda](g^{-1}w). \end{aligned}$$

Since $X\lambda = 0$ for any $X \in \mathfrak{n}$ it follows that h_w is right invariant under N. In terms of the dual basis $\{\lambda_i\}_{i=1}^3$ of the canonical basis $\{e_i\}_{i=1}^3$ of W, λ is a nonzero multiple of $\lambda_1 - \lambda_3$. Then $H_0\lambda = \lambda$ and $T\lambda = -\lambda$. Thus $H_0 h_w = h_w$ and $iTh_w = -ih_w$. Therefore

$$h_w(g \exp tH_0) = e^t h_w(g) \quad \text{and} \quad h_w(g \exp(\theta iT)) = e^{-i\theta} h_w(g),$$

which are the required properties for A and M. The lemma is proved. ■

If M is abelian we have a G-morphism from $U^{\sigma,\nu} \otimes U^{\sigma',\nu'}$ into $U^{\sigma+\sigma',\nu+\nu'+\rho}$ defined by the multiplication of functions. In particular we consider the G-morphism defined by

$$P : U^{r,v} \otimes W \longrightarrow U^{r+1,v-1}, \qquad f \otimes w \longmapsto fh_w.$$

Proposition 3.7. *The G-morphism $P : U^{r,v} \otimes W \longrightarrow U^{r+1,v-1}$ is surjective and the Harish-Chandra module of* $\ker(P)$ *is given by*

$$\bigoplus_{\ell \geq 0} \bigoplus_{j=0}^{\ell} \langle u_0^{j,\ell} + z_0^{j-1,\ell-1},\, (\ell-1)w_0^{j-1,\ell-2} + (\ell-j)z_0^{j-1,\ell-1} \rangle,$$

where $\langle u, v\rangle$ denotes the K-module generated by $\{u, v\}$.

Proof. We recall that $h_w(k) = \lambda(k^{-1}w)$ and we may assume that $\lambda = \lambda_1 - \lambda_3 \in W^*$. If

$$k^{-1} = \begin{pmatrix} a & -\bar{b}e^{i\theta} & 0 \\ b & \bar{a}e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}$$

then

$$h_{e_1}(k) = a, \quad h_{e_2}(k) = -\bar{b}e^{i\theta}, \quad h_{e_3}(k) = -e^{-i\theta}.$$

On the other hand, by (12) we have

$$\begin{aligned} v_0^{j,\ell}(k) &= \tbinom{\ell}{j}(e^{i\theta})^{-r+\ell-3j} a^{\ell-j} b^j \\ v_1^{j,\ell}(k) &= (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j-1} b^{j-1} \left(|a|^2 \tbinom{\ell-1}{j-1} - |b|^2 \tbinom{\ell-1}{j} \right). \end{aligned} \tag{14}$$

Therefore

$$\begin{aligned} P(u_0^{j,\ell})(k) &= v_1^{j,\ell}(k) h_{e_1}(k) - v_0^{j,\ell}(k) h_{e_2}(k) \\ &= (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j} b^{j-1} \left(|a|^2 \tbinom{\ell-1}{j-1} - |b|^2 \tbinom{\ell-1}{j} + |b|^2 \tbinom{\ell}{j} \right) \\ &= (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j} b^{j-1} \left(|a|^2 \tbinom{\ell-1}{j-1} + |b|^2 \tbinom{\ell-1}{j-1} \right) \\ &= \tbinom{\ell-1}{j-1} (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j} b^{j-1}. \end{aligned}$$

In particular we note that $P(u_0^{j,\ell}) = v_0^{j-1,\ell-1} \in U^{r+1,v-1}$, proving that P is surjective. In a similar way we compute

$$P(z_0^{j-1,\ell-1})(k) = v_0^{j-1,\ell-1}(k) h_{e_3}(k) = -\tbinom{\ell-1}{j-1} (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j} b^{j-1}$$

and

$$P(w_0^{j-1,\ell-2})(k) = v_0^{j-1,\ell-2}(k) h_{e_1}(k) = \tbinom{\ell-2}{j-1} (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j} b^{j-1}.$$

To prove the statement about the kernel of P it is enough to find the K-dominant vectors in $\ker(P)$. Now it is easy to see that $u_0^{j,\ell}+z_0^{j-1,\ell-1}$ and $(\ell-1)w_0^{j-1,\ell-2}+(\ell-j)z_0^{j-1,\ell-1}$ are in $\ker(P)$. Moreover if $v\in\ker(P)$ is a K-dominant vector of weight $\mu^{j,\ell}$ it follows that it must be a linear combination of $(\ell-1)w_0^{j-1,\ell-2}+(\ell-j)z_0^{j-1,\ell-1}$ and $u_0^{j,\ell}+z_0^{j-1,\ell-1}$. The proposition is proved. ∎

Lemma 3.8. *The Iwasawa decomposition $g=\kappa(g)a(g)n(g)$ of an element $g=(g_{ij})\in \mathrm{SU}(2,1)$ is given by*

$$a(g)=\begin{pmatrix}\cosh s & 0 & \sinh s\\ 0 & 1 & 0\\ \sinh s & 0 & \cosh s\end{pmatrix},\qquad \kappa(g)=\begin{pmatrix}\overline{a} & \overline{b} & 0\\ -be^{-i\theta} & ae^{-i\theta} & 0\\ 0 & 0 & e^{i\theta}\end{pmatrix},$$

with

$$\overline{a}=\frac{g_{11}+g_{13}}{|g_{31}+g_{33}|},\qquad b=-\frac{(g_{21}+g_{23})(g_{31}+g_{33})}{|g_{31}+g_{33}|^2},$$
$$e^s=|g_{31}+g_{33}|,\qquad e^{i\theta}=\frac{g_{31}+g_{33}}{|g_{31}+g_{33}|}.$$

Proof. Evaluating both sides of $g=\kappa(g)a(g)n(g)$ at e_1+e_3 we get

$$\begin{pmatrix}g_{11}+g_{13}\\ g_{21}+g_{23}\\ g_{31}+g_{33}\end{pmatrix}=\kappa(g)a(g)\begin{pmatrix}1\\0\\1\end{pmatrix}=(\cosh s+\sinh s)\begin{pmatrix}\overline{a}\\ -be^{-i\theta}\\ e^{i\theta}\end{pmatrix},$$

because e_1+e_3 is left fixed by N. Then, from the third row it follows that $e^s=|g_{31}+g_{33}|$ and that $e^{i\theta}=(g_{31}+g_{33})/|g_{31}+g_{33}|$. Now from the first and the second rows we obtain the expressions for $\overline{a}$ and b. ∎

Proposition 3.9. *For any $0\le s\le\ell$ we have*

i) $X_\beta\, v_s^{j,\ell}=\frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)}v_{s+1}^{j,\ell+1}+\frac{(\ell-s)(r-v+2j)}{2(\ell+1)}v_s^{j-1,\ell-1}.$

ii) $X_{-\beta}\, v_s^{j,\ell}=-\frac{(j+1)(r+v+2j+2)}{2(\ell+1)}v_s^{j+1,\ell+1}+\frac{s(r+v-2\ell+2j)}{2(\ell+1)}v_{s-1}^{j,\ell-1}.$

Proof. We start by computing $X_\beta\, v_0^{j,\ell}$. Let $X_\beta=\frac12(Y_5-iY_6)$, where $Y_5,Y_6\in\mathfrak{g}$. We point out that from (12) we get

$$v_0^{j,\ell}(k)=\tbinom{\ell}{j}e^{i\theta(-r+\ell-3j)}a^{\ell-j}b^j,$$

where

$$k=\begin{pmatrix}\overline{a} & \overline{b} & 0\\ -be^{-i\theta} & ae^{-i\theta} & 0\\ 0 & 0 & e^{i\theta}\end{pmatrix}.$$

Now

$$(Y_5\, v_0^{j,\ell})(k)=\left(\frac{d}{dt}\right)_{t=0}\left(e^{-(\nu+\rho)(H(\exp(-tY_5)k))}v_0^{j,\ell}(\kappa(\exp(-tY_5)k))\right).$$

We have

$$\exp(-tY_5)k = \begin{pmatrix} \overline{a} & \overline{b} & 0 \\ -be^{-i\theta}\cosh t & ae^{-i\theta}\cosh t & -e^{i\theta}\sinh t \\ be^{-i\theta}\sinh t & -ae^{-i\theta}\sinh t & e^{i\theta}\cosh t \end{pmatrix}.$$

From Lemma 3.8 we obtain

$$H(\exp(-tY_5)k) = \log|\Delta(t)|H_0$$

and

$$\kappa(\exp(-tY_5)k) = \begin{pmatrix} \overline{\alpha} & \overline{\beta} & 0 \\ -\beta e^{-i\varphi} & \alpha e^{-i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= \frac{a}{\Delta(t)}, & \beta &= \frac{\Delta(t)(be^{-i\theta}\cosh t + e^{i\theta}\sinh t)}{|\Delta(t)|^2}, \\ e^{i\varphi} &= \frac{\Delta(t)}{|\Delta(t)|}, & \Delta(t) &= be^{-i\theta}\sinh t + e^{i\theta}\cosh t. \end{aligned}$$

Then

$$Y_5\, v_0^{j,\ell}(k) = \tbinom{\ell}{j}\left(\frac{d}{dt}\right)_{t=0}\left(\tfrac{\Delta(t)^{-r+\ell-2j}}{|\Delta(t)|^{-r+2\ell-2j+v+2}}a^{\ell-j}(be^{-i\theta}\cosh t + e^{i\theta}\sinh t)^j\right).$$

We observe that

$$\begin{aligned} \left(\frac{d}{dt}\right)_{t=0}&\Big((be^{-i\theta}\cosh t + e^{i\theta}\sinh t)^j\Big) = jb^{j-1}e^{i\theta(-j+2)}, \\ \left(\frac{d}{dt}\right)_{t=0}&\left(\tfrac{\Delta(t)^{-r+\ell-2j}}{|\Delta(t)|^{-r+2\ell-2j+v+2}}\right) = (-r+\ell-2j)be^{i\theta(-r+\ell-2j-2)} \\ &\qquad - (-r+v+2\ell-2j+2)\operatorname{Re}(\overline{b}e^{2i\theta})e^{i\theta(-r+\ell-2j)}. \end{aligned}$$

Therefore

$$\begin{aligned} Y_5\, v_0^{j,\ell}(k) = &\tbinom{\ell}{j}(-r+\ell-2j)e^{i\theta(-r+\ell-3j-2)}a^{\ell-j}b^{j+1} \\ &- \tbinom{\ell}{j}(-r+v+2\ell-2j+2)e^{i\theta(-r+\ell-3j)}a^{\ell-j}b^j\operatorname{Re}(\overline{b}e^{2i\theta}) \\ &+ \tbinom{\ell}{j}j\, e^{i\theta(-r+\ell-3j+2)}a^{\ell-j}b^{j-1}. \end{aligned}$$

Similarly for Y_6 we obtain

$$\begin{aligned} Y_6\, v_0^{j,\ell}(k) = &-i\tbinom{\ell}{j}(-r+\ell-2j)e^{i\theta(-r+\ell-3j-2)}a^{\ell-j}b^{j+1} \\ &+ \tbinom{\ell}{j}(-r+v+2\ell-2j+2)e^{i\theta(-r+\ell-3j)}a^{\ell-j}b^j\operatorname{Im}(\overline{b}e^{2i\theta}) \\ &+ i\tbinom{\ell}{j}j\, e^{i\theta(-r+\ell-3j+2)}a^{\ell-j}b^{j-1}. \end{aligned}$$

Therefore

$$X_\beta\, v_0^{j,\ell} = \tfrac{1}{2}(Y_5 - iY_6)v_0^{j,\ell} = \tbinom{\ell}{j}e^{i\theta(-r+\ell-3j+2)}a^{\ell-j}b^{j-1}\left(-\tfrac{(-r+v+2\ell-2j+2)}{2}|b|^2 + j\right).$$

On the other hand, by (14), we have that

$$\begin{aligned} v_0^{j-1,\ell-1}(k) &= \tbinom{\ell-1}{j-1} e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1}, \\ v_1^{j,\ell+1}(k) &= e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1} \left(\tbinom{\ell}{j-1} |a|^2 - \tbinom{\ell}{j} |b|^2 \right) \\ &= e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1} \left(-\tbinom{\ell+1}{j} |b|^2 + \tbinom{\ell}{j-1} \right). \end{aligned}$$

Then it is easy to verify that

$$X_\beta\, v_0^{j,\ell} = \tfrac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)} v_1^{j,\ell+1} + \tfrac{\ell\,(r-v+2j)}{2(\ell+1)} v_0^{j-1,\ell-1}.$$

Now by using that $X_{-\alpha}^s$ and X_β commute and the fact that

$$X_{-\alpha}^s\, v_0^{j,\ell} = \ell(\ell-1)\dots(\ell-s+1) v_s^{j,\ell}, \quad X_{-\alpha}^s\, v_1^{j,\ell} = (\ell-1)\dots(\ell-s) v_{s+1}^{j,\ell}$$

(see (13)) we prove that

$$X_\beta\, v_s^{j,\ell} = \tfrac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)} v_{s+1}^{j,\ell+1} + \tfrac{(\ell-s)\,(r-v+2j)}{2(\ell+1)} v_s^{j-1,\ell-1}.$$

This completes the proof of i). In the same way ii) follows. ∎

Corollary 3.10. *The following relations hold*

$$\begin{aligned} X_{-\beta}\, u_0^{j,\ell} &= -\tfrac{(j+1)(r+v+2j+2)}{2(\ell+1)} u_0^{j+1,\ell+1} - z_0^{j,\ell} + \tfrac{(r+v+2j-2\ell)}{2(\ell+1)} w_0^{j,\ell-1}, \\ X_{-\beta}\, z_0^{j-1,\ell-1} &= -\tfrac{j(r+v+2j)}{2\ell} z_0^{j,\ell}, \\ X_{-\beta}\, w_0^{j-1,\ell-2} &= -\tfrac{j(r+v+2j)}{2(\ell-1)} w_0^{j,\ell-1}. \end{aligned}$$

Proof. These are straightforward consequences of the definitions given in Lemma 3.5 and the previous proposition. ∎

Corollary 3.11. *We have*

$$\begin{aligned} X_\beta(X_{-\alpha}^{\ell-1}(u_0^{j,\ell})) &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}^{\ell}(u_0^{j,\ell+1}) \\ &\quad + \tfrac{(-r+v-2j)}{2\ell(\ell+1)} X_{-\alpha}^{\ell}(w_0^{j-1,\ell-1}), \\ X_\beta(X_{-\alpha}^{\ell-1}(z_0^{j-1,\ell-1})) &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell^2} X_{-\alpha}^{\ell}(z_0^{j-1,\ell}) + \tfrac{1}{\ell} X_{-\alpha}^{\ell}(w_0^{j-1,\ell-1}), \\ X_\beta(X_{-\alpha}^{\ell-1}(w_0^{j-1,\ell-2})) &= \tfrac{(\ell-j)(-r+v+2\ell-2j)}{2\ell(\ell-1)} X_{-\alpha}^{\ell}(w_0^{j-1,\ell-1}). \end{aligned}$$

Proof. First by induction on k we establish the following relations for $k = 1, \dots, \ell-1$

$$\begin{aligned} X_{-\alpha}^k(u_0^{j,\ell}) &= (\ell-1)\dots(\ell-k) \left(v_{k+1}^{j,\ell} \otimes e_1 - v_k^{j,\ell} \otimes e_2 \right), \\ X_{-\alpha}^k(z_0^{j-1,\ell-1}) &= (\ell-1)\dots(\ell-k) \left(v_k^{j-1,\ell-1} \otimes e_3 \right), \\ X_{-\alpha}^k(w_0^{j-1,\ell-2}) &= (\ell-2)\dots(\ell-k)\Big((\ell-k-1) v_k^{j-1,\ell-2} \otimes e_1 \\ &\qquad + k v_{k-1}^{j-1,\ell-2} \otimes e_2 \Big). \end{aligned}$$

Then the corollary is a direct consequence of Proposition 3.9. ∎

We recall that (Theorem 3.3) $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$ where

$$\begin{aligned}
Y_1 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r+1,v-1}(\Delta_2) f\},\\
Y_2 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r-2,v}(\Delta_2) f\},\\
Y_3 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r+1,v+1}(\Delta_2) f\}.
\end{aligned}$$

Proposition 3.12. *If $v(r+v)(r-v) \neq 0$, then the Harish-Chandra modules Y_1, Y_2, Y_3 are cyclic $D(G)$-modules, in fact they are generated by the minimal K-type dominant vectors given below*

i) $Y_1 = \langle (r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0} \rangle$,

ii) $Y_2 = \langle u_0^{0,1} \rangle$,

iii) $Y_3 = \langle u_0^{1,1} + z_0^{0,0} \rangle$.

Proof. The elements $u_0^{0,1}$, $u_0^{1,1}$ and $z_0^{0,0}$ are K-dominant vectors of minimal type in $U^{r,v} \otimes \mathbb{C}^3$, see Lemma 3.5. In fact $u_0^{1,1}$ and $z_0^{0,0}$ are of type $(-r-1, 0)$ and the type of $u_0^{0,1}$ is $(-r+2, 0)$. Moreover, through a careful calculation, using Propositions 2.3, 3.9 and the normalization (13) one can verify that,

$$\begin{aligned}
\Delta_2(u_0^{0,1}) &= (-v^2 + 4 - \tfrac{1}{3}(r-2)^2) u_0^{0,1},\\
\Delta_2(u_0^{1,1}) &= (-v^2 + 4 - \tfrac{1}{3}r^2 + \tfrac{4}{3}r + \tfrac{8}{3}) u_0^{1,1} + 2(r-v+2) z_0^{0,0},\\
\Delta_2(z_0^{0,0}) &= (-v^2 + 4 - \tfrac{1}{3}r^2 - \tfrac{8}{3}r - \tfrac{16}{3}) z_0^{0,0} - 2(r+v+2) u_0^{1,1}.
\end{aligned}$$

From the last two identities we obtain the following eigenvectors of Δ_2:

$$\begin{aligned}
&\Delta_2(u_0^{1,1} + z_0^{0,0}) = (-(v+1)^2 + 4 - \tfrac{1}{3}(r+1)^2)(u_0^{1,1} + z_0^{0,0}),\\
&\Delta_2((r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0})\\
&\qquad = (-(v-1)^2 + 4 - \tfrac{1}{3}(r+1)^2)((r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0}).
\end{aligned}$$

The first eigenvector $u_0^{0,1}$ is of weight $\mu^{0,1}$, the second and the third, $u_0^{1,1} + z_0^{0,0}$ and $(r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0}$, are of weight $\mu^{1,1}$. It is worth to observe that the eigenvalues are respectively: $\chi_{r-2,v}(\Delta_2)$, $\chi_{r+1,v-1}(\Delta_2)$ and $\chi_{r+1,v+1}(\Delta_2)$, see Proposition 2.4. Therefore

$$\langle (r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0} \rangle \subseteq Y_1, \quad \langle u_0^{0,1} \rangle \subseteq Y_2, \quad \langle u_0^{1,1} + z_0^{0,0} \rangle \subseteq Y_3.$$

The $D(G)$-module structure of $Y^{r,v} \otimes W$ can be visualized in the following diagram of all the highest weights of its K-submodules.

$$\begin{tikzcd}
 & \mu^{0,1} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{0,2} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{0,3} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \cdots \\
\mu^{1,1} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{1,2} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{1,3} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{1,4} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \cdots \\
\mu^{2,2} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{2,3} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{2,4} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \mu^{2,5} \arrow[r, "X_\beta"] \arrow[d, "X_{-\beta}"'] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Over each $\mu^{j,\ell}$, with $0\le j\le \ell$ we place the irreducible K-submodules of that highest weight contained, respectively, in Y_1, Y_2 and Y_3. In each place of the first row there is only one K-module of highest weight $\mu^{0,\ell}$ contained in Y_2, and on the first column there are two irreducible K-modules of highest weight $\mu^{\ell,\ell}$, one contained in Y_1 and the other in Y_3.

Let $f_0^{j,\ell}\in Y^{r,v}\otimes W$ be a K-dominant vector of weight $\mu^{j,\ell}$ and let $f_\ell^{j,\ell}$ be a corresponding lowest weight vector. Then $X_{-\beta}(f_0^{j,\ell})$ is a dominant vector of weight $\mu^{j+1,\ell+1}$, and $X_\beta(f_\ell^{j,\ell})$ is a lowest weight vector in a K-module of highest weight $\mu^{j,\ell+1}$. This follows from $[X_{-\beta},X_\alpha]=0$ and $[X_\beta,X_{-\alpha}]=0$.
In particular, by induction on $\ell\ge 1$ and on $k\ge 1$ it is not difficult to prove the following:

$$
\begin{aligned}
X_\beta^{\ell-1}(u_0^{0,1}) &= \tfrac{1}{(\ell-1)!}\left(\tfrac{-r+v+4}{2}\right)_{\ell-1} X_{-\alpha}^{\ell-1}(u_0^{0,\ell}),\\
X_{-\beta}^{k}(u_0^{0,\ell}) &= (-1)^k\tfrac{k!}{(\ell)_{k+1}}\left(\tfrac{r+v+2}{2}\right)_{k-1}\Big(\tfrac12(r+v+2k)\ell u_0^{k,\ell+k}\\
&\quad+\ell(\ell+k)z_0^{k-1,\ell+k-1}-\tfrac12(r+v-2\ell)(\ell+k-1)w_0^{k-1,\ell+k-2}\Big).
\end{aligned}
$$

From the first equality and the hypothesis we see that $u_0^{0,\ell}\in Y_2$ for all $\ell\ge 1$. Now using the second one we get that the vector

$$
\begin{aligned}
D_2^{j,\ell} &= (r+v+2j)(\ell-j)u_0^{j,\ell}+2\ell(\ell-j)z_0^{j-1,\ell-1}\\
&\quad-(r+v-2\ell+2j)(\ell-1)w_0^{j-1,\ell-2}
\end{aligned}\tag{15}
$$

is a dominant weight vector in Y_2 of highest weight $\mu^{j,\ell}$. We observe that $D_2^{j,\ell}\neq 0$ if and only if $j\neq\ell$.
In a similar way for $\ell\ge 1$ and $k\ge 0$ one establishes that

$$
\begin{aligned}
X_{-\beta}^{\ell-1}(u_0^{1,1}+z_0^{0,0}) &= (-1)^{\ell-1}\left(\tfrac{r+v+4}{2}\right)_{\ell-1}(u_0^{\ell,\ell}+z_0^{\ell-1,\ell-1}),\\
X_\beta^k(X_{-\alpha}^{\ell-1}(u_0^{\ell,\ell}+z_0^{\ell-1,\ell-1})) &= \tfrac{k!}{(\ell)_k(\ell)_{k+1}}\left(\tfrac{-r+v+2}{2}\right)_k X_{-\alpha}^{\ell+k-1}\Big(\ell u_0^{\ell,\ell+k}\\
&\quad+(\ell+k)z_0^{\ell-1,\ell+k-1}+(\ell+k-1)w_0^{\ell-1,\ell+k-2}\Big).
\end{aligned}
$$

From the first equality and the hypothesis we see that $u_0^{k+1,k+1}+z_0^{k,k}\in Y_3$ for all $k\ge 0$. Now using the second one we get that the vector

$$
D_3^{j,\ell}=ju_0^{j,\ell}+\ell z_0^{j-1,\ell-1}+(\ell-1)w_0^{j-1,\ell-2}\tag{16}
$$

is a dominant weight vector in Y_3 of highest weight $\mu^{j,\ell}$, moreover it is nonzero if $j\neq 0$.

Similarly for $\ell\ge 1$ and $k\ge 1$ one can prove that

$$
\begin{aligned}
X_{-\beta}^{\ell-1}\left((r+v+2)u_0^{1,1}+(r-v+2)z_0^{0,0}\right) &= (-1)^{\ell-1}\left(\tfrac{r+v+2}{2}\right)_{\ell-1}\\
&\quad\times\left((r+v+2\ell)u_0^{\ell,\ell}+(r-v+2\ell)z_0^{\ell-1,\ell-1}\right),\\
X_\beta^k\Big(X_{-\alpha}^{\ell-1}&\left((r+v+2\ell)u_0^{\ell,\ell}+(r-v+2\ell)z_0^{\ell-1,\ell-1}\right)\Big)\\
=\tfrac12\tfrac{k!}{(\ell)_k(\ell)_{k+1}}&\left(\tfrac{-r+v+2}{2}\right)_{k-1}X_{-\alpha}^{\ell+k-1}\Big(\ell(r+v+2\ell)(-r+v+2k)u_0^{\ell,\ell+k}\\
&+(\ell+k)(r-v+2\ell)(-r+v+2k)z_0^{\ell-1,\ell+k-1}\\
&-(\ell+k-1)(r-v+2\ell)(r+v-2k)w_0^{\ell-1,\ell+k-2}\Big).
\end{aligned}
$$

From the first equality and the hypothesis we see that $(r+v+2\ell)u_0^{\ell,\ell} + (r-v+2\ell)z_0^{\ell-1,\ell-1} \in Y_1$ for all $k \geq 0$. Now using the second one we get that the vector

$$\begin{aligned} D_1^{j,\ell} &= j(r+v+2j)(-r+v+2\ell-2j)u_0^{j,\ell} \\ &\quad + \ell(r-v+2j)(-r+v+2\ell-2j)z_0^{j-1,\ell-1} \\ &\quad - (\ell-1)(r-v+2j)(r+v-2\ell+2j)w_0^{j-1,\ell-2} \end{aligned} \tag{17}$$

is a dominant weight vector in Y_1 of highest weight $\mu^{j,\ell}$. It is easy to prove that $D_1^{j,\ell} \neq 0$ if $j \neq 0$ and $v(r+v)(r-v) \neq 0$.

Finally, since $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$ and as a K-module is the direct sum of all the K-modules generated by

$$\{D_1^{j,\ell}\}_{0<j\leq\ell} \cup \{D_2^{j,\ell}\}_{0\leq j<\ell} \cup \{D_3^{j,\ell}\}_{0<j\leq\ell}$$

the proposition follows. ■

In the following theorem we generalize a bit the previous proposition and we exhibit the K-module structure of Y_1, Y_2 and Y_3.

Theorem 3.13. *Let $v(v+r)(v-r) \neq 0$ and let $D_1^{j,\ell}$, $D_2^{j,\ell}$ and $D_3^{j,\ell}$ be defined respectively by* (17), (15) *and* (16). *Then they are K- dominant vectors of highest weight $\mu^{j,\ell}$ and*

$$Y_1 = \bigoplus_{0<j\leq\ell} \langle D_1^{j,\ell}\rangle_K, \quad Y_2 = \bigoplus_{0\leq j<\ell} \langle D_2^{j,\ell}\rangle_K, \quad Y_3 = \bigoplus_{0<j\leq\ell} \langle D_3^{j,\ell}\rangle_K.$$

Moreover

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3.$$

We used above $\langle D_i^{j,\ell}\rangle_K$ to denote the K-module generated by $D_i^{j,\ell}$.

Proof. That $D_i^{j,\ell}$ is K-dominant and of highest weight $\mu^{j,\ell}$ follows from (17), (15) and (16). Through a careful and long calculation, using Propositions 2.3, 3.9 and the normalization (13) one can verify that $D_i^{j,\ell} \in Y_i$ for $i = 1,2,3$. Finally, since $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$ and as a K-module is the direct sum of all the K-modules generated by

$$\{D_1^{j,\ell}\}_{0\leq j\leq\ell} \cup \{D_2^{j,\ell}\}_{0<j\leq\ell} \cup \{D_3^{j,\ell}\}_{0\leq j\leq\ell},$$

the theorem follows. ■

We consider in $U^{r,v} \otimes W$ the Hilbert structure given by tensoring the inner product of $L^2(K)$ with the standard inner product of $\mathbb{C}^3$. Then the closure $\overline{Y}_i$ of Y_i (i=1,2,3) in $U^{r,v} \otimes W$ is a G module, because G is connected. But it is worth to observe that $\overline{Y}_1$, $\overline{Y}_2$ and $\overline{Y}_3$ are not orthogonal subspaces.

Remark 3.14. If $v(r+v)(r-v) \neq 0$ then $\overline{Y}_1$, $\overline{Y}_2$, $\overline{Y}_3$ are linearly independent but $\overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3$ is not closed in $U^{r,v} \otimes W$.

In fact, if $\overline{Y}_1 \cap \overline{Y}_2$ were not zero then $\overline{Y}_1$ and $\overline{Y}_2$ would contain a common K-irreducible submodule which would imply that $Y_1 \cap Y_2 \neq \{0\}$, which is a contradiction. Similarly if $(\overline{Y}_1 \oplus \overline{Y}_2) \cap \overline{Y}_3 \neq \{0\}$ then $(Y_1 \oplus Y_2) \cap Y_3 \neq \{0\}$ which is also a contradiction. Thus $\overline{Y}_1$, $\overline{Y}_2$ and $\overline{Y}_3$ are linearly independent.

For the second assertion in the remark we recall that one may have closed linearly independent subspaces M_1 and M_2 in a Hilbert space such that $M_1 \oplus M_2$ is not closed. But the following is true (see [18], 4.8): If X is a Banach space and M_1 and M_2 are closed linearly independent subspaces of X then $M_1 \oplus M_2$ is closed if and only if there exists $d > 0$ such that $\|x_1 - x_2\| \geq d$ whenever $x_1 \in M_1, x_2 \in M_2$ and $\|x_1\| = \|x_2\| = 1$.

In our case if we let

$$y_\ell = \|D_2^{j,\ell}\|^{-1}\, D_2^{j,\ell} \quad \text{and} \quad z_\ell = \|D_3^{j,\ell}\|^{-1}\, D_3^{j,\ell},$$

one can verify, by using Lemma 3.22, that

$$\|y_\ell - z_\ell\|^2 = 2 - 2\left(\frac{8j(\ell-j)}{|r+v|^2 + 8j(\ell-j)}\right)^{1/2}.$$

If we fix $0 < j < \ell$ then $\lim_{\ell\to\infty} \|y_\ell - z_\ell\| = 0$. Therefore $\overline{Y}_2 \oplus \overline{Y}_3$ and $\overline{Y}_2 \oplus \overline{\overline{Y}_1 \oplus \overline{Y}_3}$ are not closed in $U^{r,v} \otimes W$. Therefore

$$U^{r,v} \otimes W \supsetneq \overline{Y}_2 \oplus \overline{\overline{Y}_1 \oplus \overline{Y}_3} \supset \overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3,$$

which implies that $\overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3$ is not closed, since $Y_1 \oplus Y_2 \oplus Y_3$ is dense in $U^{r,v} \otimes W$.

The following lemmas are consequences of Proposition 3.9.

Lemma 3.15. *We have*

$$\begin{aligned}
X_\beta(D_1^{j,\ell}) &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j)}{2\ell(\ell+1)} X_{-\alpha}(D_1^{j,\ell+1}) + \tfrac{(r-v+2j)}{2} D_1^{j-1,\ell-1},\\
X_\beta(D_2^{j,\ell}) &= \tfrac{(\ell-j)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}(D_2^{j,\ell+1}) + \tfrac{(r-v+2j-2)}{2} D_2^{j-1,\ell-1},\\
X_\beta(D_3^{j,\ell}) &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}(D_3^{j,\ell+1}) + \tfrac{(r-v+2j+2)}{2} D_3^{j-1,\ell-1}.
\end{aligned}$$

Lemma 3.16. *We have*

$$\begin{aligned}
&X_{-\beta}(X_{-\alpha}^{\ell-1}(D_1^{j,\ell}))\\
&\quad = -\tfrac{j(r+v+2j)}{2\ell(\ell+1)} X_{-\alpha}^{\ell-1}(D_1^{j+1,\ell+1}) + \tfrac{(\ell-1)(r+v-2\ell+2j)}{2} X_{-\alpha}^{\ell-2}(D_1^{j,\ell-1}),\\
&X_{-\beta}(X_{-\alpha}^{\ell-1}(D_2^{j,\ell}))\\
&\quad = -\tfrac{(j+1)(r+v+2j)}{2\ell(\ell+1)} X_{-\alpha}^{\ell-1}(D_2^{j+1,\ell+1}) + \tfrac{(\ell-1)(r+v-2\ell+2j)}{2} X_{-\alpha}^{\ell-2}(D_2^{j,\ell-1}),\\
&X_{-\beta}(X_{-\alpha}^{\ell-1}(D_3^{j,\ell}))\\
&\quad = -\tfrac{j(r+v+2j+2)}{2\ell(\ell+1)} X_{-\alpha}^{\ell-1}(D_3^{j+1,\ell+1}) + \tfrac{(\ell-1)(r+v-2\ell+2j+2)}{2} X_{-\alpha}^{\ell-2}(D_3^{j,\ell-1}).
\end{aligned}$$

Proposition 3.17. *If $v(v+r)(v-r) \neq 0$ then*

$$Y_1 \simeq Y^{r+1,v-1}, \quad Y_2 \simeq Y^{r-2,v}, \quad Y_3 \simeq Y^{r+1,v+1}$$

as $(\mathfrak{g}_{\mathbb{C}}, K)$-modules.

Proof. From (11) and Theorem 3.13 we have

$$Y^{r+1,v-1} = \bigoplus_{0\le j\le \ell} \langle v_0^{j,\ell}\rangle_K, \quad \text{and} \quad Y_1 = \bigoplus_{0<j\le \ell} \langle D_1^{j,\ell}\rangle_K,$$

where $v_0^{j,\ell}$ and $D_1^{j,\ell}$ are K-dominant vectors of highest weights $\mu^{j+1,\ell+1}$ and $\mu^{j,\ell}$, respectively. Thus an isomorphism of K-modules $\eta : Y^{r+1,v-1} \longrightarrow Y_1$ is characterized by

$$\eta(v_0^{j,\ell}) = c_{j,\ell}\, D_1^{j+1,\ell+1}, \quad c_{j,\ell}\neq 0, \quad 0\le j\le \ell.$$

Since $\mathfrak{g}_{\mathbb{C}}$ is generated by $\mathfrak{k}_{\mathbb{C}}$ and $X_{\pm\beta}$, to say that η is a $\mathfrak{g}_{\mathbb{C}}$-morphism is equivalent to require that η commutes with $X_{\pm\beta}$. This in turn is equivalent to

$$X_\beta(\eta(v_0^{j,\ell})) = \eta(X_\beta(v_0^{j,\ell})), \qquad X_{-\beta}(\eta(v_\ell^{j,\ell})) = \eta(X_{-\beta}(v_\ell^{j,\ell})),$$

for $0\le j\le \ell$, because $[X_\alpha, X_{-\beta}] = [X_{-\alpha}, X_\beta] = 0$.
From Lemma 3.15 we obtain

$$\begin{aligned} X_\beta(\eta(v_0^{j,\ell})) &= c_{j,\ell} X_\beta(D_1^{j+1,\ell+1}) \\ &= c_{j,\ell}\tfrac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)(\ell+2)} X_{-\alpha}(D_1^{j+1,\ell+2}) + c_{j,\ell}\tfrac{(r-v+2j+2)}{2} D_1^{j,\ell}. \end{aligned}$$

On the other hand from Proposition 3.9, changing r by $r+1$ and v by $v-1$, and using (13) we get

$$\begin{aligned} X_\beta\, v_0^{j,\ell} &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)} v_1^{j,\ell+1} + \tfrac{\ell(r-v+2j+2)}{2(\ell+1)} v_0^{j-1,\ell-1} \\ &= \tfrac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)^2} X_{-\alpha}(v_0^{j,\ell+1}) + \tfrac{\ell(r-v+2j+2)}{2(\ell+1)} v_0^{j-1,\ell-1}. \end{aligned}$$

Then

$$\eta(X_\beta(v_0^{j,\ell})) = c_{j,\ell+1}\tfrac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)^2} X_{-\alpha}(D_1^{j+1,\ell+2}) + c_{j-1,\ell-1}\tfrac{\ell(r-v+2j+2)}{2(\ell+1)} D_1^{j,\ell}.$$

Now $X_\beta(\eta(v_0^{j,\ell})) = \eta(X_\beta(v_0^{j,\ell}))$ if and only if

$$(\ell+1)c_{j,\ell} = (\ell+2)c_{j,\ell+1} = \ell c_{j-1,\ell-1}, \tag{18}$$

since $X_{-\alpha}(D_1^{j+1,\ell+2})$ and $D_1^{j,\ell}$ are linearly independent.
Similarly one gets that $X_{-\beta}(\eta(v_\ell^{j,\ell})) = \eta(X_{-\beta}(v_\ell^{j,\ell}))$ if and only if

$$(\ell+1)c_{j,\ell} = (\ell+2)c_{j+1,\ell+1} = \ell c_{j,\ell-1}. \tag{19}$$

From (18) and (19) we get

$$c_{j,\ell+1} = c_{j+1,\ell+1} \quad \text{and} \quad (\ell+1)c_{j,\ell} = (\ell+2)c_{j,\ell+1}.$$

Now it follows that $c_{j,\ell} = c_{0,0}/(\ell+1)$. This proves that

$$\eta_1 : Y^{r+1,v-1} \longrightarrow Y_1, \qquad \eta_1(v_0^{j,\ell}) = \tfrac{1}{\ell+1} D_1^{j+1,\ell+1}, \tag{20}$$

is the unique isomorphism of $(\mathfrak{g}_{\mathbb{C}}, K)$-module, up to a nonzero constant, from $Y^{r+1,v-1}$ onto Y_1.

In a similar way we can prove that, up to a non zero constant,

$$\eta_2 : Y^{r-2,v} \longrightarrow Y_2, \qquad \eta_2(v_0^{j,\ell}) = \tfrac{1}{\ell+1} D_2^{j,\ell+1} \tag{21}$$

and

$$\eta_3 : Y^{r+1,v+1} \longrightarrow Y_3, \qquad \eta_3(v_0^{j,\ell}) = \tfrac{1}{\ell+1} D_3^{j+1,\ell+1} \tag{22}$$

are, respectively the unique isomorphisms of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. ■

Remark 3.18. If $v(r+v)(r-v) \neq 0$ the $(\mathfrak{g}_\mathbb{C}, K)$ isomorphisms η_1 and η_2 defined in (20) and (21) are not continuous while η_3 defined in (22) is bicontinuous.

In fact, by using Lemmas 3.20, 3.22 below it follows that

$$\begin{aligned}\frac{\|\eta_1(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} &= \frac{j!\,(\ell-j)!}{(\ell+1)!}\,\|D_1^{j+1,\ell+1}\|^2\\ &= |r-v+2j+2|^2|r-v-2\ell+2j|^2\\ &\quad + |r-v+2j+2|^2|r+v-2\ell+2j|^2\\ &\quad + 8\,\mathrm{Re}(v)\left(|v|^2-(r+2j+2)^2+2(\ell+1)(r+2j+2)\right),\end{aligned}$$

and

$$\frac{\|\eta_2(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} = \frac{j!\,(\ell-j)!}{(\ell+1)!}\,\|D_2^{j,\ell+1}\|^2 = |r+v|^2 + 8j(\ell-j+1),$$

which implies that the left hand sides go to infinite when $\ell \to \infty$. But η_3 is bicontinuous since on one hand we have

$$\frac{\|\eta_3(X_{-\alpha}^s v_0^{j,\ell})\|^2}{\|X_{-\alpha}^s v_0^{j,\ell}\|^2} = \frac{\|X_{-\alpha}^s(\eta_3(v_0^{j,\ell}))\|^2}{\|X_{-\alpha}^s v_0^{j,\ell}\|^2} = \frac{\|\eta_3(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} = \frac{j!\,(\ell-j)!}{(\ell+1)!}\,\|D_3^{j+1,\ell+1}\|^2 = 2,$$

and on the other hand $\{X_{-\alpha}^s v_0^{j,\ell}\}$ and $\{\eta_3(X_{-\alpha}^s v_0^{j,\ell})\}$ are, respectively, orthogonal basis of $Y^{r+1,v+1}$ and Y_3. Therefore $\|\eta_3\| = \sqrt{2}$.

Remark 3.19. The pairs of G-modules $U^{r+1,v-1}$ and $\overline{Y}_1$ and $U^{r-2,v}$ and $\overline{Y}_2$ are infinitesimally equivalent, but not equivalent, while $U^{r+1,v+1}$ and $\overline{Y}_3$ are equivalent.
This is a consequence of the following facts proved in Proposition 3.17

$$\dim\ \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(Y^{r+1,v-1}, Y_1) = \dim\ \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(Y^{r-2,v}, Y_2) = 1.$$

We close this section with the following lemmas that were used in the previous remarks.

Lemma 3.20. *If $v_s^{j,\ell}$, $0 \le s \le \ell$, are the functions defined in* (12) *then we have*

$$\|v_s^{j,\ell}\|^2 = \frac{1}{\ell+1}\binom{\ell}{s}^{-1}\binom{\ell}{j}.$$

Proof. If V is an irreducible unitary K-module, then the following orthogonality relations are well known: $\int_K \langle k\cdot v, v'\rangle\overline{\langle k\cdot w, w'\rangle}\,dk = (\dim V)^{-1}\langle v, w\rangle\overline{\langle v', w'\rangle}$. for all $v, v', w, w' \in V$. From (10) we have

$$v_s^{j,\ell}(k) = \lambda_j^{j,\ell}(k^{-1}a_s^{j,\ell}) = \frac{\langle k^{-1}\cdot a_s^{j,\ell}, a_j^{j,\ell}\rangle}{\|a_j^{j,\ell}\|^4}.$$

Then, by using (9), we get $\|v_s^{j,\ell}\|^2 = \frac{\|a_s^{j,\ell}\|^2}{(\ell+1)\|a_j^{j,\ell}\|^2} = \frac{1}{\ell+1}\binom{\ell}{s}^{-1}\binom{\ell}{j}$. ■

The following lemma is a direct consequence of the definitions given in Lemma 3.5 and the lemma above.

Lemma 3.21. *We have*

$$\|u_0^{j,\ell}\|^2 = \tfrac{1}{\ell}\tbinom{\ell}{j}, \quad \|z_0^{j-1,\ell-1}\|^2 = \tfrac{1}{\ell}\tbinom{\ell-1}{j-1}, \quad \|w_0^{j-1,\ell-2}\|^2 = \tfrac{1}{\ell-1}\tbinom{\ell-2}{j-1}.$$

The next lemma follows from the definitions (17), (15) and (16) and Lemma 3.21.

Lemma 3.22. *We have*

$$\begin{aligned}
\|D_1^{j,\ell}\|^2 &= \ell\tbinom{\ell-1}{j-1}\Big(|r-v+2j|^2|r-v-2\ell+2j|^2 + |r-v+2j|^2|r+v-2\ell+2j|^2 \\
&\qquad + 8j\,\mathrm{Re}(v)\left(|v|^2-(r+2j)^2+2\ell(r+2j)\right)\Big),\\
\|D_2^{j,\ell}\|^2 &= (\ell-j)\tbinom{\ell}{j}\left(|r+v|^2+8j(\ell-j)\right),\\
\|D_3^{j,\ell}\|^2 &= 2\ell\tbinom{\ell-1}{j-1}.
\end{aligned}$$

4. Multiplication formulas

In this section we shall establish two three term multiplication formulas, one dual of the other, for matrix valued irreducible spherical functions, obtained from Theorem 3.13.

Proposition 4.1. *Let $v(r+v)(r-v)\neq 0$. Then*

$$\begin{aligned}
v_0^{j,\ell}\otimes e_3 = \tfrac{1}{2(\ell+1)v(r-v)}D_1^{j+1,\ell+1} &- \tfrac{2(j+1)}{(\ell+1)(r+v)(r-v)}D_2^{j+1,\ell+1}\\
&+ \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)}D_3^{j+1,\ell+1}.
\end{aligned}$$

Proof. Since $v_0^{j,\ell}\otimes e_3 = z_0^{j,\ell}$ is a K-dominant vector of weight $\mu^{j+1,\ell+1}$ it is a linear combination of $D_1^{j+1,\ell+1}$, $D_2^{j+1,\ell+1}$ and $D_3^{j+1,\ell+1}$. The elements $u_0^{j+1,\ell+1}, z_0^{j,\ell}$ and $w_0^{j,\ell-1}$ are linear independent, hence it is straightforward to verify, using (15),(16) and (17) that the identity in the statement follows. ■

We recall that (see (8)) given an irreducible K-module V of type (n,ℓ) there exists a basis $\{v_k\}_{k=0}^{\ell}$ of V, unique up to a multiplicative constant, such that

$$\begin{aligned}
H_\alpha\, v_k &= (\ell-2k)\,v_k, & Z\,v_k &= (2n+\ell)\,v_k\\
X_\alpha\, v_k &= k\,v_{k-1}, & X_{-\alpha}\,v_k &= (\ell-k)\,v_{k+1}.
\end{aligned} \tag{23}$$

Therefore the matrix of a linear map $T: V\longrightarrow V$ associated to any basis of V satisfying (23) is the same for all these bases.

Let V be a K-isotypic component of a representation of G in a Hilbert space U and let $P: U\longrightarrow V$ be the orthogonal projection. Let $\{v_i\}_{i=0}^{\ell}$ be a basis of V satisfying (23), and let $\{\lambda_i\}_{i=0}^{\ell}$ be its dual basis. Then the matrix coefficients of the spherical function Φ associated to (U,V) in the basis $\{v_i\}_{i=0}^{\ell}$ are given by

$$\Phi(g)_{ik} = \lambda_i(P(g\,v_k)). \tag{24}$$

Lemma 4.2. *Let $v(r+v)(r-v)\neq 0$ and let $\{v_i^{j,\ell}\}_{i=0}^{\ell}$ be a basis of the K-module $V_{(-r+\ell-3j,\ell)}\subset U^{r,v}$ such that (23) holds. Let*

$$v_i^{j,\ell}\otimes e_3 = w_i^{(1)} + w_i^{(2)} + w_i^{(3)} \in \overline{Y}_1\oplus\overline{Y}_2\oplus\overline{Y}_3.$$

For $p=1,2,3$, if $w_0^{(p)}\neq 0$ then $\{w_i^{(p)}\}_{i=0}^{\ell}$ is a basis of the K irreducible module of $\overline{Y}_p$ of type $(n,\ell)=(-r+\ell-3j-1,\ell)$, such that (23) holds.

Proof. We have that each $w_0^{(p)}$ is a K dominant vector of weight $\mu^{j+1,\ell+1}$. On the other hand, for $0 \leq i \leq \ell$ we have

$$\begin{aligned} X_{-\alpha}^i(w_0^{(1)}) + X_{-\alpha}^i(w_0^{(2)}) + X_{-\alpha}^i(w_0^{(3)}) &= X_{-\alpha}^i(v_0 \otimes e_3) = \tfrac{\ell!}{(\ell-i)!} v_i \otimes e_3 \\ &= \tfrac{\ell!}{(\ell-i)!}\left(w_i^{(1)} + w_i^{(2)} + w_i^{(2)}\right). \end{aligned}$$

Therefore $X_{-\alpha}^i(w_0^{(p)}) = \frac{\ell!}{(\ell-i)!}\, w_i^{(p)}$ for $p = 1, 2, 3$. This completes the proof of the lemma.

■

Remark 4.3. When $v(r+v)(r-v) \neq 0$, from Proposition 4.1, and the definitions (15), (16) and (17) it follows that

$$w_0^{(1)} = \tfrac{1}{2(\ell+1)v(r-v)} D_1^{j+1,\ell+1} \neq 0,$$

$$w_0^{(2)} = \tfrac{-2(j+1)}{(\ell+1)(r+v)(r-v)} D_2^{j+1,\ell+1} \neq 0, \qquad \text{if } j \neq \ell,$$

and if $(r+v+2j+2)(r+v-2\ell+2j) \neq 0$, we have

$$w_0^{(3)} = \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)} D_3^{j+1,\ell+1} \neq 0.$$

Theorem 4.4. *Let $v(r+v)(r-v) \neq 0$ and let $\Phi^{r,v}_{(-r+\ell-3j,\ell)}$ be the irreducible matrix valued spherical function associated to the G-module $U^{r,v}$ and to the K-submodule $V_{(-r+\ell-3j,\ell)}$. Let ϕ be the spherical function of type $(-1,0)$ associated to the G module W. Then*

$$\begin{aligned} \phi(g)\Phi^{r,v}_{(-r+\ell-3j,\ell)}(g) = a_1\, \Phi^{r+1,v-1}_{(-r+\ell-3j-1,\ell)}(g) + a_2\, \Phi^{r-2,v}_{(-r+\ell-3j-1,\ell)}(g) \\ + a_3\, \Phi^{r+1,v+1}_{(-r+\ell-3j-1,\ell)}(g), \end{aligned} \tag{25}$$

where

$$\begin{aligned} a_1 &= a_1(r,v,j,\ell) = \tfrac{(r-v+2j+2)(-r+v+2\ell-2j)}{2v(r-v)} \\ a_2 &= a_2(r,v,j,\ell) = -\tfrac{4(j+1)(\ell-j)}{(r+v)(r-v)} \\ a_3 &= a_3(r,v,j,\ell) = \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2v(r+v)} \end{aligned} \tag{26}$$

Proof. Let us consider the basis $\{v_i^{j,\ell}\}_{i=0}^{\ell}$ of $V_{(-r+\ell-3j,\ell)} \subset U^{r,v}$ introduced in (10) and let $\{\lambda_i\}_{i=0}^{\ell}$ be its dual basis. Also let $\{\mu_i\}_{i=1}^{3}$ be the dual basis of the canonical basis $\{e_i\}_{i=1}^{3}$ of $\mathbb{C}^3$.
Let $D_p = \langle D_p^{j+1,\ell+1}\rangle_K$ for $p = 1, 2, 3$. From Lemma 4.2 we have

$$v_i^{j,\ell} \otimes e_3 = w_i^{(1)} + w_i^{(2)} + w_i^{(3)} \in D_1 \oplus D_2 \oplus D_3,$$

where $\{w_i^{(p)}\}_{i=0}^{\ell}$ is a basis of D_p satisfying (23). As it was pointed out in Remark 4.3, $w_0^{(p)}$ may be zero. In such a case $D_p = 0$ and the corresponding coefficient $a_p = 0$, and everything is all right.

Let $\{\chi_i^{(p)}\}_{i=0}^{\ell}$ be the dual basis of $\{w_i^{(p)}\}_{i=0}^{\ell}$. We consider $\chi_i^{(p)} \in (D_1 \oplus D_2 \oplus D_3)^*$ by using the canonical isomorphism

$$(D_1 \oplus D_2 \oplus D_3)^* \simeq D_1^* \oplus D_2^* \oplus D_3^*.$$

We also consider $\lambda_i \otimes \mu_3 \in (D_1 \oplus D_2 \oplus D_3)^*$ by setting $\lambda_i \otimes \mu_3$ equal to zero on the orthogonal complement of $V_{(-r+\ell-3j,\ell)} \otimes \mathbb{C}e_3$ in $D_1 \oplus D_2 \oplus D_3$. Now let

$$\lambda_i \otimes \mu_3 = \nu_i^{(1)} + \nu_i^{(2)} + \nu_i^{(3)} \in D_1^* \oplus D_2^* \oplus D_3^*. \tag{27}$$

Since $\lambda_i \otimes \mu_3$ is of weight $-(\ell-2i)$ with respect to H_α it follows that $\nu_i^{(p)} = a_i^{(p)} \chi_i^{(p)}$ where

$$a_i^{(p)} = \nu_i^{(p)}(w_i^{(p)}) = (\lambda_i \otimes \mu_3)(w_i^{(p)}) \quad \text{ for } p = 1, 2, 3. \tag{28}$$

Now we observe that $a_i^{(p)}$ does not depend on i, that is

$$(\lambda_i \otimes \mu_3)(w_i^{(p)}) = (\lambda_0 \otimes \mu_3)(w_0^{(p)}). \tag{29}$$

Because $\{v_i^{j,\ell}\}_{i=0}^{\ell}$ satisfies (23) we have $v_i^{j,\ell} = \frac{(\ell-i)!}{\ell!} X_{-\alpha}^i(v_0^{j,\ell})$, which implies that $X_{-\alpha}^i(\lambda_i) = \frac{(-1)^i \ell!}{(\ell-i)!}\lambda_0$. Also from Lemma 4.2 we get that $w_i^{(p)} = \frac{(\ell-i)!}{\ell!} X_{-\alpha}^i(w_0^{(p)})$. Then

$$\begin{aligned}
(\lambda_i \otimes \mu_3)(w_i^{(p)}) &= \tfrac{(\ell-i)!}{\ell!}(\lambda_i \otimes \mu_3)(X_{-\alpha}^i w_0^{(p)}) \\
&= \tfrac{(-1)^i(\ell-i)!}{\ell!}\big(X_{-\alpha}^i(\lambda_i) \otimes \mu_3\big)(w_0^{(p)}) = (\lambda_0 \otimes \mu_3)(w_0^{(p)}).
\end{aligned}$$

Thus from (27) we obtain

$$\lambda_i \otimes \mu_3 = a_1 \chi_i^{(1)} + a_2 \chi_i^{(2)} + a_3 \chi_i^{(3)} \tag{30}$$

where $a_p = (\lambda_0 \otimes \mu_3)(w_0^{(p)})$.
From Proposition 4.1 we have $v_0^{j,\ell} \otimes e_3 = w_0^{(1)} + w_0^{(2)} + w_0^{(3)}$, with

$$\begin{aligned}
w_0^{(1)} &= \tfrac{1}{2\,(\ell+1)\,v(r-v)} D_1^{j+1,\ell+1}, \qquad w_0^{(2)} = -\tfrac{2(j+1)}{(\ell+1)\,(r+v)(r-v)} D_2^{j+1,\ell+1} \\
w_0^{(3)} &= \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2\,(\ell+1)\,v(r+v)} D_3^{j+1,\ell+1}.
\end{aligned}$$

From (15), (16), (17) and recalling that $u_0^{j+1,\ell+1} = v_1^{j+1,\ell+1} \otimes e_1 - v_0^{j+1,\ell+1} \otimes e_2$, $z_0^{j,\ell} = v_0^{j,\ell} \otimes e_3$, $w_0^{j,\ell-1} = v_0^{j,\ell-1} \otimes e_1$, we get

$$\begin{aligned}
a_1 = (\lambda_0 \otimes \mu_3)(w_0^{(1)}) &= \tfrac{1}{2\,(\ell+1)\,v(r-v)}(\lambda_0 \otimes \mu_3)(D_1^{j+1,\ell+1}) \\
&= \tfrac{(r-v+2j+2)(-r+v+2\ell-2j)}{2v(r-v)} \\
a_2 = (\lambda_0 \otimes \mu_3)(w_0^{(2)}) &= -\tfrac{2(j+1)}{(\ell+1)\,(r+v)(r-v)}(\lambda_0 \otimes \mu_3)(D_2^{j+1,\ell+1}) \\
&= -\tfrac{4(j+1)(\ell-j)}{(r+v)(r-v)} \\
a_3 = (\lambda_0 \otimes \mu_3)(w_0^{(3)}) &= \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2\,(\ell+1)\,v(r+v)}(\lambda_0 \otimes \mu_3)(D_3^{j+1,\ell+1}) \\
&= \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2v(r+v)}.
\end{aligned} \tag{31}$$

Let Q be the orthogonal projection of $U^{r,v}$ onto $V_{(-r+\ell-3j,\ell)}$, the isotypic component of type $(-r+\ell-3j,\ell))$, and let Q_3 be the orthogonal projection of W onto $\mathbb{C}e_3$. Also let P be the orthogonal projection of $U^{r,v}\otimes W$ onto $D_1\oplus D_2\oplus D_3$, the K-isotypic component of type $(-r+\ell-3j-1,\ell)$.
We claim that

$$(\lambda_i\otimes\mu_3)\circ(Q\otimes Q_3)=(\lambda_i\otimes\mu_3)\circ P. \tag{32}$$

Since $\mathrm{Im}(Q\otimes Q_3)=V_{(-r+\ell-3j,\ell)}\otimes\mathbb{C}e_3\subset D_1\oplus D_2\oplus D_3=\mathrm{Im}(P)$ it follows that $\ker(Q\otimes Q_3)\supset\ker P$. Therefore to prove (32) it is enough to see that

$$((\lambda_i\otimes\mu_3)\circ(Q\otimes Q_3))\,|_{D_1\oplus D_2\oplus D_3}=(\lambda_i\otimes\mu_3)|_{D_1\oplus D_2\oplus D_3}. \tag{33}$$

We notice that $D_1\oplus D_2\oplus D_3=\langle u_0^{j+1,\ell+1}\rangle_K\oplus\langle z_0^{j,\ell}\rangle_K\oplus\langle w_0^{j,\ell-1}\rangle_K$ (see (15), (16) and (17)).
From Lemma 3.5 it is clear that

$$\ker\big((Q\otimes Q_3)\,|_{D_1\oplus D_2\oplus D_3}\big)=\langle u_0^{j+1,\ell+1}\rangle_K\oplus\langle w_0^{j,\ell-1}\rangle_K=\ker\big((\lambda_i\otimes\mu_3)|_{D_1\oplus D_2\oplus D_3}\big).$$

Moreover, since $\langle z_0^{j,\ell}\rangle_K=\mathrm{Im}(Q\otimes Q_3)$ the identity (33) follows, hence (32) is proved.

Now we are ready to prove the theorem. For $g\in G$ we apply the left hand side of (32) to $g(v_k^{j,\ell}\otimes e_3)$ and using (24) we obtain

$$\begin{aligned}(\lambda_i\otimes\mu_3)\Big((Q\otimes Q_3)(g\,(v_k^{j,\ell}\otimes e_3))\Big)&=(\lambda_i\otimes\mu_3)\Big(Q(g\,v_k^{j,\ell})\otimes Q_3(g\,e_3)\Big)\\&=\lambda_i(Q(g\,v_k^{j,\ell}))\,\mu_3(Q_3(g\,e_3))\\&=\phi(g)\,(\Phi(g))_{ik}\,,\end{aligned}$$

where Φ is the spherical function of type $(-r+\ell-3j,\ell)$ associated to the G-module $U^{r,v}$.
Let P_i be the orthogonal projection of $\overline{Y}_i$ onto D_i. Then $P|_{\overline{Y}_1\oplus\overline{Y}_2\oplus\overline{Y}_3}=P_1\oplus P_2\oplus P_3$. In fact $P_i(D_i^\perp\cap\overline{Y}_i)=0$ because P is the projection of $U^{r,v}\otimes W$ onto the isotypic component $D_1\oplus D_2\oplus D_3$ and $D_i^\perp\cap\overline{Y}_i$ does not contain such K-type. Thus $P|_{\overline{Y}_i}=P_i$.
Now we apply the right hand side of (32) to $g(v_k^{j,\ell}\otimes e_3)$ and using Lemma 4.2 and (30) we obtain

$$\begin{aligned}(\lambda_i\otimes\mu_3)P(g(v_k^{j,\ell}\otimes e_3))&=(\lambda_i\otimes\mu_3)P(g\,w_k^{(1)}+g\,w_k^{(2)}+g\,w_k^{(3)})\\&=(\lambda_i\otimes\mu_3)(P_1(g\,w_k^{(1)})+P_2(g\,w_k^{(2)})+P_3(g\,w_k^{(3)}))\\&=a_1\chi_i^{(1)}(P_1(g\,w_k^{(1)}))+a_2\chi_i^{(2)}(P_2(g\,w_k^{(2)}))+a_3\chi_i^{(3)}(P_3(g\,w_k^{(3)}))\\&=a_1\,(\Phi_1(g))_{ik}+a_2\,(\Phi_2(g))_{ik}+a_3\,(\Phi_3(g))_{ik}\end{aligned}$$

where Φ_p is the spherical function of type $(-r+\ell-3j-1,\ell)$ associated to the G-module $\overline{Y}_p$.
Since $\overline{Y}_1$ and $U^{r+1,v-1}$, $\overline{Y}_2$ and $U^{r-2,v}$, $\overline{Y}_3$ and $U^{r+1,v+1}$ are infinitesimally equivalent (see Proposition 3.17) and taking into account the remark below Proposition 2.2 the theorem follows. ∎

If $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ is an irreducible spherical function of type $\delta \in \hat{K}$ then the function $\Phi^* : G \longrightarrow \mathrm{End}(V_\pi^*)$ defined by $\Phi^*(g) = \Phi(g^{-1})^{\mathrm{t}}$ is a spherical function of type δ^*, where δ^* denotes the equivalence class of the contragradient representation π^* (see [4]). If π is of type (n, ℓ) then it is easy to verify that π^* is of type $(-n-\ell, \ell)$.

We recall that $(U^{r,v})^* \simeq U^{-r,-v}$, see the proof of Theorem 3.4. Therefore if $\Phi^{r,v}_{(-r+\ell-3j,\ell)}$ denotes the irreducible spherical function associated to the G-module $U^{r,v}$ of type $(-r+\ell-3j, \ell)$, then

$$\left(\Phi^{r,v}_{(-r+\ell-3j,\ell)}\right)^* = \Phi^{-r,-v}_{(r+\ell-3(\ell-j),\ell)}. \tag{34}$$

Theorem 4.5. *Let $v(r+v)(r-v) \neq 0$ and let $\Phi^{r,v}_{(-r+\ell-3j,\ell)}$ be the irreducible matrix valued spherical function associated to the G-module $U^{r,v}$ and to the K-submodule $V_{(-r+\ell-3j,\ell)}$. Let ψ be the spherical function of type $(1,0)$ associated to the G module W^*. Then*

$$\begin{aligned}\psi(g)\Phi^{r,v}_{(-r+\ell-3j,\ell)}(g) = b_1\,\Phi^{r-1,v+1}_{(-r+\ell-3j+1,\ell)}(g) &+ b_2\,\Phi^{r+2,v}_{(-r+\ell-3j+1,\ell)}(g)\\ &+ b_3\,\Phi^{r-1,v-1}_{(-r+\ell-3j+1,\ell)}(g),\end{aligned} \tag{35}$$

where

$$\begin{aligned} b_1 &= b_1(r,v,j,\ell) = \tfrac{(r-v+2j)(-r+v+2\ell-2j+2)}{2v(r-v)}\\ b_2 &= b_2(r,v,j,\ell) = -\tfrac{4j(\ell-j+1)}{(r+v)(r-v)}\\ b_3 &= b_3(r,v,j,\ell) = \tfrac{(r+v+2j)(r+v-2\ell+2j-2)}{2v(r+v)}\end{aligned} \tag{36}$$

Proof. We start from the following identity established in Theorem 4.4

$$\begin{aligned}\phi(g)\Phi^{r,v}_{(-r+\ell-3j,\ell)}(g) = a_1\,\Phi^{r+1,v-1}_{(-r+\ell-3j-1,\ell)}(g) &+ a_2\,\Phi^{r-2,v}_{(-r+\ell-3j-1,\ell)}(g)\\ &+ a_3\,\Phi^{r+1,v+1}_{(-r+\ell-3j-1,\ell)}(g),\end{aligned} \tag{37}$$

If we take $*$ on both sides of (37), and we use (34) we obtain

$$\begin{aligned}\phi(g)^*\Phi^{-r,-v}_{(r+\ell-3(\ell-j),\ell)}(g) = a_1\,\Phi^{-r-1,-v+1}_{(r+\ell-3(\ell-j)+1,\ell)}(g) &+ a_2\,\Phi^{-r+2,-v}_{(r+\ell-3(\ell-j)+1,\ell)}(g)\\ &+ a_3\,\Phi^{-r-1,-v-1}_{(r+\ell-3(\ell-j)+1,\ell)}(g).\end{aligned}$$

We note that $\psi(g) = \phi^*(g)$, because both spherical functions are associated to the same G module and are of the same K- type. Now if we change r by $-r$, v by $-v$, and j by $\ell - j$, then we obtain that

$$\begin{aligned}\psi(g)\Phi^{r,v}_{(-r+\ell-3j,\ell)}(g) = b_1\,\Phi^{r-1,v+1}_{(-r+\ell-3j+1,\ell)}(g) &+ b_2\,\Phi^{r+2,v}_{(-r+\ell-3j+1,\ell)}(g)\\ &+ b_3\,\Phi^{r-1,v-1}_{(-r+\ell-3j+1,\ell)}(g),\end{aligned}$$

with

$$b_i = b_i(r,v,j,\ell) = a_i(-r,-v,\ell-j,\ell). \tag{38}$$

Now the explicit expressions for $b_j(r,v,j,\ell)$ follow from Theorem 4.4 and this completes the proof of the theorem.

■

5. Three term recursion relation

Let $(v+i)(r+v+2i)(r-v+2i)\neq 0$ for $i=-1,0,1$. If we apply successively Theorems 4.4 and 4.5 and we set $n=-r+\ell-3j$ we get

$$\begin{aligned}
\psi(g)\varphi(g)\Phi^{r,v}_{(n,\ell)}(g) &= \psi(g)\left(a_1\Phi^{r+1,v-1}_{(n-1,\ell)}(g)+\,a_2\Phi^{r-2,v}_{(n-1,\ell)}(g)+a_3\,\Phi^{r+1,v+1}_{(n-1,\ell)}(g)\right)\\
&= a_1\Big(b_1(r+1,v-1,j,\ell)\,\Phi^{r,v}_{(n,\ell)}(g)+b_2(r+1,v-1,j,\ell)\,\Phi^{r+3,v-1}_{(n,\ell)}(g)\\
&\qquad +b_3(r+1,v-1,j,\ell)\,\Phi^{r,v-2}_{(n,\ell)}(g)\Big)\\
&\quad +a_2\Big(b_1(r-2,v,j+1,\ell)\,\Phi^{r-3,v+1}_{(n,\ell)}(g)+b_2(r-2,v,j+1,\ell)\,\Phi^{r,v}_{(n,\ell)}(g)\\
&\qquad +b_3(r-2,v,j+1,\ell)\,\Phi^{r-3,v-1}_{(n,\ell)}(g)\Big)\\
&\quad +a_3\Big(b_1(r+1,v+1,j,\ell)\,\Phi^{r,v+2}_{(n,\ell)}(g)+b_2(r+1,v+1,j,\ell)\,\Phi^{r+3,v+1}_{(n,\ell)}(g)\\
&\qquad +b_3(r+1,v+1,j,\ell)\,\Phi^{r,v}_{(n,\ell)}(g)\Big).
\end{aligned}\tag{39}$$

Now we want to *package* these spherical functions in larger matrices to obtain in such a way a three term recursion relation for these matrices.

We fix the type (n,ℓ) and we take $r=\ell-n$ and $v\in\mathbb{C}$. We define the $(\ell+1)^2\times(\ell+1)$ matrix valued function $\tilde\Phi(g;v)$ of $\ell+1$ spherical functions of type (n,ℓ) as follows

$$\widetilde\Phi(g;v)=\widetilde\Phi_{(n,\ell)}(g;v)=\left(\Phi^{r,v}_{(n,\ell)}(g),\dots,\Phi^{r-3j,v+j}_{(n,\ell)}(g),\dots,\Phi^{r-3\ell,v+\ell}_{(n,\ell)}(g)\right)^{\mathrm{t}}.$$

If we write (39) replacing r by $r-3j=\ell-n-3j$ and v by $v-j$ with $0\le j\le\ell$ we obtain

$$\begin{aligned}
\psi(g)\varphi(g)\Phi^{r-3j,v-j}_{(n,\ell)}(g) &= B_{j,j}\,\Phi^{r-3j,v-j}_{(n,\ell)}(g)+A_{j,j-1}\,\Phi^{r-3j+3,v-j-1}_{(n,\ell)}(g)\\
&\quad +A_{j,j}\,\Phi^{r-3j,v-j-2}_{(n,\ell)}(g)+C_{j,j+1}\,\Phi^{r-3j-3,v-j+1}_{(n,\ell)}(g)\\
&\quad +B_{j,j+1}\,\Phi^{r-3j-3,v-j-1}_{(n,\ell)}(g)+C_{j,j}\,\Phi^{r-3j,v-j+2}_{(n,\ell)}(g)\\
&\quad +B_{j,j-1}\,\Phi^{r-3j+3,v-j+1}_{(n,\ell)}(g),
\end{aligned}\tag{40}$$

where

$$\begin{aligned}
A_{j,j-1} &= a_1(r-3j,v-j,j,\ell)b_2(r-3j+1,v-j-1,j,\ell),\\
A_{j,j} &= a_1(r-3j,v-j,j,\ell)b_3(r-3j+1,v-j-1,j,\ell),\\
B_{j,j-1} &= a_3(r-3j,v-j,j,\ell)b_2(r-3j+1,v-j+1,j,\ell),\\
B_{j,j} &= a_1(r-3j,v-j,j,\ell)b_1(r-3j+1,v-j-1,j,\ell),\\
&\quad +a_2(r-3j,v-j,j,\ell)b_2(r-3j-2,v-j,j+1,\ell),\\
&\quad +a_3(r-3j,v-j,j,\ell)b_3(r-3j+1,v-j+1,j,\ell),\\
B_{j,j+1} &= a_2(r-3j,v-j,j,\ell)b_3(r-3j-2,v-j,j+1,\ell),\\
C_{j,j} &= a_3(r-3j,v-j,j,\ell)b_1(r-3j+1,v-j+1,j,\ell),\\
C_{j,j+1} &= a_2(r-3j,v-j,j,\ell)b_1(r-3j-2,v-j,j+1,\ell).
\end{aligned}\tag{41}$$

Taking into account the expressions of the coefficients a_i and b_i given in (26) and (36) we get

$$
\begin{aligned}
A_{j,j-1} &= \frac{2j(\ell-j+1)(r-v+2)(r-v-2\ell)}{(v-j)(r+v-4j)(r-v-2j)(r-v-2j+2)},\\
A_{j,j} &= -\frac{(r+v-2j)(r+v-2\ell-2j-2)(r-v+2)(r-v-2\ell)}{4(v-j)(v-j-1)(r+v-4j)(r-v-2j)},\\
B_{j,j-1} &= -\frac{2j(\ell-j+1)(r+v-2j+2)(r+v-2\ell-2j)}{(v-j)(r+v-4j)(r+v-4j+2)(r-v-2j)},\\
B_{j,j} &= \frac{(r-v+2)^2(r-v-2\ell)^2}{4(v-j)(v-j-1)(r-v-2j)(r-v-2j+2)}\\
&\quad+\frac{16(j+1)^2(\ell-j)^2}{(r+v-4j)(r+v-4j-2)(r-v-2j)(r-v-2j-2)}\\
&\quad+\frac{(r+v-2j+2)^2(r+v-2\ell-2j)^2}{4(v-j)(v-j+1)(r+v-4j)(r+v-4j+2)},\\
B_{j,j+1} &= -\frac{2(j+1)(\ell-j)(r+v-2j)(r+v-2\ell-2j-2)}{(v-j)(r+v-4j)(r+v-4j-2)(r-v-2j)},\\
C_{j,j} &= -\frac{(r-v)(r+v-2j+2)(r+v-2\ell-2j)(r-v-2\ell-2)}{4(v-j)(v-j+1)(r+v-4j)(r-v-2j)},\\
C_{j,j+1} &= \frac{2(j+1)(\ell-j)(r-v)(r-v-2\ell-2)}{4(v-j)(r+v-4j)(r-v-2j)(r-v-2j-2)}.
\end{aligned}
$$

Now we consider the following $(\ell+1)\times(\ell+1)$ matrices

$$
\begin{aligned}
A_v &= \sum_{k=1}^{\ell} A_{k,k-1}E_{k,k-1} + \sum_{k=0}^{\ell} A_{k,k}E_{k,k},\\
B_v &= \sum_{k=1}^{\ell} B_{k,k-1}E_{k,k-1} + \sum_{k=0}^{\ell} B_{k,k}E_{k,k} + \sum_{k=0}^{\ell-1} B_{k,k+1}E_{k,k+1}, \qquad (42)\\
C_v &= \sum_{k=0}^{\ell} C_{k,k}E_{k,k} + \sum_{k=0}^{\ell-1} C_{k,k+1}E_{k,k+1}.
\end{aligned}
$$

We recall that given two square matrices M and P the tensor product matrix $M\otimes P$ is the matrix obtained by blowing up each entry M_{ij} of M to the matrix $M_{ij}P$. Let

$$\widetilde{A}_v = A_v\otimes I, \qquad \widetilde{B}_v = B_v\otimes I, \qquad \widetilde{C}_v = C_v\otimes I,$$

where I denotes the $(\ell+1)\times(\ell+1)$ identity matrix.

Theorem 5.1. *For each K-type (n,ℓ) if $(v-k)(\ell-n-v-2k)\neq 0$ for $-1\le k\le \ell+1$ and $\ell-n+v-2k\neq 0$ for $0\le k\le 2\ell+1$, then the matrix valued function*

$$\widetilde{\Phi}(g;v) = \widetilde{\Phi}_{(n,\ell)}(g;v) = \Big(\Phi^{r,v}_{(n,\ell)}(g),\dots,\Phi^{r-3j,v-j}_{(n,\ell)}(g),\dots,\Phi^{r-3\ell,v-\ell}_{(n,\ell)}(g)\Big)^t,$$

with $r=\ell-n$, satisfies the following three term recursion relation

$$\phi(g)\psi(g)\widetilde{\Phi}(g;v) = \widetilde{A}_v\widetilde{\Phi}(g;v-2) + \widetilde{B}_v\widetilde{\Phi}(g;v) + \widetilde{C}_v\widetilde{\Phi}(g;v+2), \qquad (43)$$

for all $g\in G$.

Remark 5.2. If instead of the parameter $v \in \mathbb{C}$ we use $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, see (7), then (43) can be written as

$$\phi(g)\psi(g)\widetilde{\Phi}(g;\nu) = \widetilde{A}_\nu\widetilde{\Phi}(g;\nu-\rho) + \widetilde{B}_\nu\widetilde{\Phi}(g;\nu) + \widetilde{C}_\nu\widetilde{\Phi}(g;\nu+\rho).$$

Proof. An $(\ell+1)^2 \times (\ell+1)$ matrix V will be seen as an $(\ell+1)$-column vector $V = (V_0, \ldots, V_\ell)^{\mathrm{t}}$ of $(\ell+1)\times(\ell+1)$ matrices V_k.
If M is an $(\ell+1)\times(\ell+1)$ matrix and $\widetilde{M} = M \otimes I$, then

$$(\widetilde{M}\,\widetilde{\Phi}(g;v))_k = \sum_{j=0}^{\ell} M_{kj}\Phi_{(n,\ell)}^{r-3j,v-j}(g).$$

Thus

$$\begin{aligned}
(\widetilde{A}_v\widetilde{\Phi}(g;v-2))_k &= A_{k,k-1}\Phi_{(n,\ell)}^{r-3k+3,v-k-1}(g) + A_{k,k}\Phi_{(n,\ell)}^{r-3k,v-k-2}(g),\\
(\widetilde{B}_v\widetilde{\Phi}(g;v))_k &= B_{k,k-1}\Phi_{(n,\ell)}^{r-3k+3,v-k+1}(g) + B_{k,k}\Phi_{(n,\ell)}^{r-3k,v-k}(g)\\
&\quad + B_{k,k+1}\Phi_{(n,\ell)}^{r-3k-3,v-k-1}(g),\\
(\widetilde{C}_v\widetilde{\Phi}(g;v+2))_k &= C_{k,k}\Phi_{(n,\ell)}^{r-3k,v-k+2}(g) + C_{k,k+1}\Phi_{(n,\ell)}^{r-3k-3,v-k+1}(g).
\end{aligned}$$

Therefore, from the identities (40) with $j = k$ and $0 \le k \le \ell$, we obtain

$$\phi(g)\psi(g)\widetilde{\Phi}(g;v)_k = (\widetilde{A}_v\widetilde{\Phi}(g;v-2))_k + (\widetilde{B}_v\widetilde{\Phi}(g;v))_k + (\widetilde{C}_v\widetilde{\Phi}(g;v+2))_k,$$

which proves the theorem. ∎

6. Reduction to one variable

It is of interest to see how we can write the above theorem when we restrict the spherical functions to the abelian subgroup A of G of all matrices of the form

$$a_s = \begin{pmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{pmatrix},$$

for any $s \in \mathbb{R}$. Recall that the centralizer of A in K is the subgroup M of all elements of the form $m_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}$, for any $\theta \in \mathbb{R}$.

If Φ is a spherical function on G of type $\pi = \pi_{(n,\ell)} \in \hat{K}$ then $\Phi(a_s)$ commutes with $\pi(m)$ for all $m \in M$. On the other hand we observed that in a basis $\{v_i^{(n,\ell)}\}_{i=0}^{\ell}$ satisfying (8) we have $m_\theta\, v_k^{(n,\ell)} = e^{i\theta(\ell-2k-n)}v_k$, $k = 0, \ldots, \ell$. Therefore we have that $\Phi(a_s)$ diagonalizes in such a basis for each $s \in \mathbb{R}$. We denote by $\Phi_k(a_s)$ the k-th diagonal entry of the matrix $\Phi(a_s)$.

In the open subset $\{a_s \in A : s > 0\}$ of A we introduce the coordinate $t = \cosh^2(s)$ and define the vector valued function

$$F(t) = (\Phi_0(a_s), \ldots, \Phi_\ell(a_s))$$

associated to the spherical function Φ. If $\Phi(g) = \Phi_{(n,\ell)}^{r,v}(g)$ then we shall also denote $F(t) = F_{(n,\ell)}^{r,v}(t)$, $t \ge 1$. In a similar way, corresponding to the function

$\widetilde{\Phi}(g;v)$ we consider the $(\ell+1)\times(\ell+1)$ matrix valued function $\widetilde{F}(t;v)$ whose j-th row is given by the vector $F_{(n,\ell)}^{r-3j,v-j}(t)$, with $r=\ell-n$, for $0\le j\le\ell$. More explicitly

$$\widetilde{F}(t;v)=(F_{jk}(t;v))\quad \text{with}\quad F_{jk}(t;v)=\left(\Phi_{(n,\ell)}^{-n+\ell-3j,v-j}(a_s)\right)_k.$$

Proposition 6.1. *For each K-type (n,ℓ) if both $(v-k)(\ell-n-v-2k)\neq 0$ for $-1\le k\le \ell+1$ and $\ell-n+v-2k\neq 0$ for $0\le k\le 2\ell+1$, then the matrix valued function $\widetilde{F}(t;v)$ satisfies the following three term recursion relation*

$$t\widetilde{F}(t;v)=A_v\widetilde{F}(t;v-2)+B_v\widetilde{F}(t;v)+C_v\widetilde{F}(t;v+2), \tag{44}$$

for all $t\ge 1$.

Proof. We recall that $\phi(g)$ is the spherical function of type $(-1,0)$ associated to the G-module $W=\mathbb{C}^3$ and that $\psi(g)$ is the spherical function of type $(1,0)$ associated to W^*. Then a direct computation gives

$$\phi(a(s))=\cosh s=t^{1/2}\quad\text{and}\quad \psi(a(s))=\phi^*(a(s))=t^{1/2}.$$

Therefore from the identity (40), for $g=a_s$, we obtain the following vector identity

$$\begin{aligned}
tF_{n,\ell}^{r-3j,v+j}(t)&=A_{j,j-1}F_{n,\ell}^{r-3j+3,v-j-1}(t)+A_{j,j}F_{n,\ell}^{r-3j,v-j-2}(t)\\
&+B_{j,j-1}F_{n,\ell}^{r-3j+3,v-j+1}(t)+B_{j,j}F_{n,\ell}^{r-3j,v-j}(t)+B_{j,j+1}F_{n,\ell}^{r-3j-3,v-j-1}(t)\\
&+C_{j,j}F_{n,\ell}^{r-3j,v-j+2}(t)+C_{j,j+1}F_{n,\ell}^{r-3j-3,v-j+1}(t).
\end{aligned}$$

This is nothing else that the equality of the j-th rows of the identity (44). ∎

In Section 2 we consider the function $H(g)=\Phi(g)\Phi_\pi(g)^{-1}$ (see (5)), associated to a spherical function $\Phi(g)$ of type $\pi=\pi_{n,\ell}\in\hat{K}$, and its restriction $H(t)=H(a(s))$ where $t=\cosh^2(s)$. We view the diagonal matrix $H(t)$ as a column vector. Then it is easy to verify that the functions H and F are related by the identity

$$F(t)=t^{n/2}H(t)^{\mathrm{t}}\begin{pmatrix} t^{1/2} & 0\\ 0 & 1\end{pmatrix}^{\ell},$$

where the exponent ℓ denotes the ℓ-th symmetric power of the matrix. Explicitly $\begin{pmatrix} t^{1/2} & 0\\ 0 & 1\end{pmatrix}^{\ell}$ is a diagonal matrix whose j-th entry is $t^{(\ell-j)/2}$, with $0\le j\le\ell$.

If $\Phi(g)=\Phi_{(n,\ell)}^{\ell-n-3j,v}(g)$ we denote $H(t)=H(t;v,j)$. Corresponding to the function $\widetilde{F}(t;v)$, we also consider the $(\ell+1)\times(\ell+1)$ matrix valued function $\widetilde{H}(t;v)$ whose j-th row is the vector $H(t;v-j,j)$, for $0\le j\le\ell$. Then

$$\widetilde{F}(t;v)=t^{n/2}\widetilde{H}(t;v)\begin{pmatrix} t^{1/2} & 0\\ 0 & 1\end{pmatrix}^{\ell}.$$

Then from (44) we obtain

Proposition 6.2. *For each K-type (n,ℓ) if both $(v-k)(\ell-n-v-2k)\neq 0$ for $-1\le k\le \ell+1$ and $\ell-n+v-2k\neq 0$ for $0\le k\le 2\ell+1$, then the matrix valued function $\widetilde{H}(t;v)$ satisfies the following three term recursion relation*

$$t\widetilde{H}(t;v)=A_v\widetilde{H}(t;v-2)+B_v\widetilde{H}(t;v)+C_v\widetilde{H}(t;v+2), \tag{45}$$

where the matrices A_v, B_v and C_v are defined in (42).

Remark 6.3. If we fix $(n,\ell)\in\hat{K}$, then the map $(v,j)\mapsto\Phi_{(n,\ell)}^{-n+\ell-3j,v}$ is a surjective map from $\{(v,j)\,, v\in\mathbb{C}, 0\le j\le\ell\}$ onto the set of all equivalence classes of matrix valued spherical functions of type (n,ℓ). In order to relate (45) with the three term recursion relation established at the end of Section 5 in [14], we introduce a new parameter $w=w(v,j)$ by

$$w=\tfrac{-(v+n+\ell+2+j)}{2},$$

and we define the function $h(t;w,j)=H(t;v,j)$. The matrix function $\widetilde{H}(t;v)$ whose j-th row is $H(t;v-j,j)$ corresponds to the function $\tilde{h}(t;w)$ whose j-th row is $h(t;w(v-j,j),j)$. Observe that $w(v-j,j)=w(v,0)$.

Given $v\in\mathbb{C}$ we set $w=\frac{-(v+n+\ell+2)}{2}$, then the function $\widetilde{H}(t;v)$ associated to the matrix function

$$\widetilde{\Phi}(g;v)=\Big(\Phi_{(n,\ell)}^{r,v}(g),\dots,\Phi_{(n,\ell)}^{r-3j,v-j}(g),\dots,\Phi_{(n,\ell)}^{r-3\ell,v-\ell}(g)\Big)^{\mathrm{t}},$$

corresponds, for the parameter w, to the matrix function

$$\tilde{h}(t;w)=\left(h(t;w,0),\dots,h(t;w,j),\dots,h(t;w,\ell)\right)^{\mathrm{t}}.$$

Then we have that the function $\tilde{h}(t;w)$ satisfies the following three term recursion relation

$$t\,\tilde{h}(t;w)=A'_w\,\tilde{h}(t;w-1)+B'_w\,\tilde{h}(t;w)+C'_w\,\tilde{h}(t;w+1), \tag{46}$$

with

$$A'_w=C_v,\quad B'_w=B_v\quad\text{and}\quad C'_w=A_v,$$

because the functions $\tilde{H}(t;v-2)$ and $\tilde{H}(t;v+2)$ correspond with $\tilde{h}(t,w+1)$ and $\tilde{h}(t,w-1)$, respectively.

With this choice of the parameter w we get exactly the same three term recursion relation that the one obtained in [14], when w is an integer such that $w\ge 0$ and $w+n\ge 0$. This can be explained in the following way: the functions $h(t;w)$ associated to the spherical functions of $(\mathrm{SU}(3),\mathrm{U}(2))$, are polynomial eigenfunctions of D and E and $h(1;w)=(1,\dots,1)^t$. Therefore $H(t;v)=h(t;w)$ is the function associated to a spherical function of $(\mathrm{SU}(2,1),\mathrm{U}(2))$.

6.1. The case $\ell=0$. In this subsection we shall display the results obtained in Proposition 6.2 when $\ell=0$ and n arbitrary. In these cases the spherical functions are complex valued, and for $n=0$ they are the zonal or classical spherical functions.

As we mentioned in the introduction of this paper, the irreducible spherical function $\Phi_{(n,0)}^{-n,v}$ of type $(n,0)$ corresponds precisely to the complex valued functions $h(t)=h(t;v)$ which are: eigenfunctions of D, analytic in the interval $[1,\infty)$ and $h(1)=1$. In this case the differential operator E is a scalar multiple of D. Then $h(t)$ satisfies

$$t(1-t)h''(t)+(n+1-(n+3)t)h'(t)-\lambda h(t)=0,$$

with $\lambda=\frac{1}{4}(n+2+v)(n+2-v)$. This is a hypergeometric equation with

$$a=\tfrac{n+2+v}{2},\quad b=\tfrac{n+2-v}{2},\quad c=n+1.$$

Therefore the analytic solution on $[1,\infty)$ with $h(1)=1$ is

$$h(t)=t^{-(n+2+v)/2}\,{}_2F_1\left(\begin{smallmatrix}\frac12(n+2+v)\,,\,\frac12(-n+2+v)\\ 2\end{smallmatrix};1-\frac{1}{t}\right). \tag{47}$$

The matrices A_v, B_v and C_v in this case are the following numbers

$$a_v=\tfrac{(v+n-2)(v-n-2)}{4v(v-1)},\quad b_v=\tfrac{n^2+v^2-2}{2(v-1)(v+1)},\quad c_v=\tfrac{(v+n+2)(v-n+2)}{4v(v+1)}.$$

In the parameter $w=\frac{-(v+n+2)}{2}$ introduced before, we have

$$h(t)=h(t;w)=t^{w}{}_2F_1\left(\begin{smallmatrix}-w,-w-n\\ 2\end{smallmatrix};1-\frac{1}{t}\right)\qquad t>1,$$

and

$$\begin{aligned}a'_w&=\tfrac{w(w+n)}{(2w+n+2)(2w+n+1)}\,,\quad c'_w=\tfrac{(w+2)(w+n+2)}{(2w+n+2)(2w+n+3)},\\ b'_w&=\tfrac{(w+1)(w+2)}{(2w+n+2)(2w+n+3)}+\tfrac{(w+n)(w+n+1)}{(2w+n+2)(2w+n+1)}=\tfrac{2w(w+n+2)+(n+1)^2}{(2w+n+1)(2w+n+3)}.\end{aligned}$$

By using the Pfaff's identity we have that

$$h(t)=t^{w}{}_2F_1\left(\begin{smallmatrix}-w,-w-n\\ 2\end{smallmatrix};1-\frac{1}{t}\right)={}_2F_1\left(\begin{smallmatrix}-w,w+n+2\\ 2\end{smallmatrix};1-t\right).$$

We observe that $h(t)$ is a polynomial precisely when w is an integer and $w\geq 0$ or $w\leq -n-2$. Then Proposition 6.2 for $\ell=0$ gives

$$\begin{aligned}t\,{}_2F_1\left(\begin{smallmatrix}-w\,,\,w+n+2\\ 2\end{smallmatrix};1-t\right)&=a'_w\,{}_2F_1\left(\begin{smallmatrix}-w+1\,,\,w+n+1\\ 2\end{smallmatrix};1-t\right)\\&\quad+b'_w\,{}_2F_1\left(\begin{smallmatrix}-w\,,\,w+n+2\\ 2\end{smallmatrix};1-t\right)+c'_w\,{}_2F_1\left(\begin{smallmatrix}-w-1\,,\,w+n+3\\ 2\end{smallmatrix};1-t\right).\end{aligned}$$

By making the change of variables $u=1-t$ we obtain the three term recursion relation, in the spectral parameter w, for the Jacobi functions

$$\begin{aligned}u\,{}_2F_1\left(\begin{smallmatrix}-w\,,\,w+n+2\\ 2\end{smallmatrix};u\right)&=\tilde a_w\,{}_2F_1\left(\begin{smallmatrix}-w+1\,,\,w+n+1\\ 2\end{smallmatrix};u\right)\\&\quad+\tilde b_w\,{}_2F_1\left(\begin{smallmatrix}-w\,,\,w+n+2\\ 2\end{smallmatrix};u\right)+\tilde c_w\,{}_2F_1\left(\begin{smallmatrix}-w-1\,,\,w+n+3\\ 2\end{smallmatrix};u\right),\end{aligned}$$

where $\tilde a_w=-a'_w$, $\tilde b_w=1-b'_w$, $\tilde c_w=-c'_w$.

6.2. The case $\ell=1$. Any irreducible spherical function of type $(n,1)$ is equivalent to $\Phi^{1-n-3j,v}_{(n,1)}$ for some $v\in\mathbb{C}$ and $j=0,1$. The corresponding functions $H(t;v,j)=(H_0(t;v,j),H_1(t;v,j))^t$ are eigenfunctions of D and E with respective eigenvalues $\lambda_j(v)$ and $\mu_j(v)$. In [13] we prove that

$$\begin{aligned}\lambda_j(v)&=\tfrac14(n+3-j+v)(n+3-j-v)+j(2-j)\\ \mu_j(v)&=\lambda_j(n+3j-1)-3j(2-j)(n+j+1),\end{aligned}$$

and that

$$H_0(t;v,0)=t^{-(n+3+v)/2}\,{}_3F_2\left(\begin{smallmatrix}\frac{(n+3+v)}{2}\,,\,\frac{(-n+1+v)}{2}\,,\,\frac{2(n-v)}{n+1-v}\\ 3\,,\,\frac{n-1-v}{n+1-v}\end{smallmatrix};1-\frac{1}{t}\right),$$

$$H_1(t;v,0) = t^{-(n+3+v)/2}\,{}_2F_1\left(\begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+3+v)}{2} \\ 3 \end{matrix}; 1-\frac{1}{t}\right),$$

$$H_0(t;v,1) = t^{-(n+4+v)/2}\,{}_2F_1\left(\begin{matrix} \frac{(n+4+v)}{2}, \frac{(-n+2+v)}{2} \\ 3 \end{matrix}; 1-\frac{1}{t}\right),$$

$$H_1(t;v,1) = t^{-(n+2+v)/2}\,{}_3F_2\left(\begin{matrix} \frac{(n+2+v)}{2}, \frac{(-n+2+v)}{2}, \frac{2(n+v+1)}{n+v} \\ 3 \quad , \quad \frac{n+2+v}{n+v} \end{matrix}; 1-\frac{1}{t}\right).$$

The first row of the matrix function $\tilde{H}(t;v)$ is $H(t;v,0)$ and the second one is $H(t;v-1,1)$. Then

$$\tilde{H}(t;v) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

with

$$F_{11} = t^{-(n+3+v)/2}\,{}_3F_2\left(\begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+1+v)}{2}, \frac{2(n-v)}{n+1-v} \\ 3 \quad , \quad \frac{n-1-v}{n+1-v} \end{matrix}; 1-\frac{1}{t}\right),$$

$$F_{12} = t^{-(n+3+v)/2}\,{}_2F_1\left(\begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+3+v)}{2} \\ 3 \end{matrix}; 1-\frac{1}{t}\right),$$

$$F_{21} = t^{-(n+3+v)/2}\,{}_2F_1\left(\begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+1+v)}{2} \\ 3 \end{matrix}; 1-\frac{1}{t}\right),$$

$$F_{22} = t^{-(n+1+v)/2}\,{}_3F_2\left(\begin{matrix} \frac{(n+1+v)}{2}, \frac{(-n+1+v)}{2}, \frac{2(n+v)}{n-1+v} \\ 3 \quad , \quad \frac{n+1+v}{n-1+v} \end{matrix}; 1-\frac{1}{t}\right).$$

The matrices A_v, B_v and C_v are

$$A_v = \begin{pmatrix} \frac{(v+n+1)(v+n-3)(v-n-3)}{4v(v-1)(v+n-1)} & 0 \\ \frac{2(v+n-3)}{(v-1)(v+n-1)(v-n-3)} & \frac{(v+n-3)(v+n-1)(v-n-5)}{4(v-1)(v-2)(v-n-3)} \end{pmatrix},$$

$$B_v = \begin{pmatrix} \frac{v^4-n^4-4n^3+16n-6v^2+21}{2(v+1)(v-1)(v+n-1)(v-n-1)} & \frac{2(v-n-3)}{v(v+n-1)(v-n-1)} \\ \frac{2(v-n-1)}{(v-1)(v+n+1)(v-n-1)} & \frac{v^4-4v^3+8v+3+6n^2-n^4-8n}{2v(v-2)(v+n-1)(v-n-1)} \end{pmatrix},$$

$$C_v = \begin{pmatrix} \frac{(v+n+3)(v-n-3)(v-n-3)}{4v(v+1)(v-n+1)} & \frac{2(v+n+3)}{v(v+n+1)(v-n-1)} \\ 0 & \frac{(v+n-1)(v+n+3)(v-n-1)}{4v(v-1)(v+n+1)} \end{pmatrix}.$$

6.3. The general case. If we use infinite matrices the three term recursion relation (46) can be written, for any $w \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$, in the following way:

$$t\begin{vmatrix} \cdot \\ \cdot \\ \cdot \\ \tilde{h}_{w-1} \\ \tilde{h}_w \\ \tilde{h}_{w+1} \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} = \begin{Vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & A'_{w-1} & B'_{w-1} & C'_{w-1} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & A'_w & B'_w & C'_w & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & A'_{w+1} & B'_{w+1} & C'_{w+1} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}\begin{vmatrix} \cdot \\ \cdot \\ \cdot \\ \tilde{h}_{w-1} \\ \tilde{h}_w \\ \tilde{h}_{w+1} \\ \cdot \\ \cdot \\ \cdot \end{vmatrix}. \qquad (48)$$

Also it is of interest to point out that if $w=\max\{0,-n\}$ then (46) implies that

$$t\begin{bmatrix}\tilde h_w\\ \tilde h_{w+1}\\ \tilde h_{w+2}\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}=\begin{bmatrix}B'_w & C'_w & 0 & \cdot & \cdot & \cdot\\ A'_{w+1} & B'_{w+1} & C'_{w+1} & 0 & \cdot & \cdot\\ 0 & A'_{w+2} & B'_{w+2} & C'_{w+2} & 0 & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}\tilde h_w\\ \tilde h_{w+1}\\ \tilde h_{w+2}\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}. \tag{49}$$

This is precisely the three term recursion relation obtained in the case of the complex projective plane, see [14].

The coefficient matrices appearing in (48) and (49) have the following interesting property: the sum of all the matrix elements in any row is equal to one. See the next proposition. Moreover all the entries of the coefficient matrix in (49) are nonnegative real numbers. This may have important applications in the modeling of some stochastic phenomena.

Proposition 6.4. *If $2w=-v-n-\ell-2$ and $(v-k)(\ell-n-v-2k)\neq 0$ for $-1\le k\le \ell+1$ and $\ell-n+v-2k\neq 0$ for $0\le k\le 2\ell+1$, then*

$$\sum_j (A'_w)_{i,j}+\sum_j (B'_w)_{i,j}+\sum_j (C'_w)_{i,j}=1. \tag{50}$$

Proof. We refer the reader to (46). Then (50) is equivalent to,

$$\sum_j (A_v)_{i,j}+\sum_j (B_v)_{i,j}+\sum_j (C_v)_{i,j}=1. \tag{51}$$

Taking into account the definitions of A_v, B_v and C_v given in (42) the above equation becomes

$$(A_v)_{j,j-1}+(A_v)_{j,j}+(B_v)_{j,j-1}+(B_v)_{j,j}+(B_v)_{j,j+1}+(C_v)_{j,j}+(C_v)_{j,j+1}=1. \tag{52}$$

Using (41) and writing $a_1=a_1(r-3j,v-j,j,\ell)$, $a_2=a_2(r-3j,v-j,j,\ell)$ and $a_3=a_3(r-3j,v-j,j,\ell)$ we have to check that

$$\begin{aligned}&a_1\big(b_2(r-3j+1,v-j-1,j,\ell)+b_3(r-3j+1,v-j-1,j,\ell)\\&\qquad+b_1(r-3j+1,v-j-1,j,\ell)\big)\\&+a_2\big(b_2(r-3j-2,v-j,j+1,\ell)+b_3(r-3j-2,v-j,j+1,\ell)\\&\qquad+b_1(r-3j-2,v-j,j+1,\ell)\big)\\&+a_3\big(b_2(r-3j+1,v-j+1,j,\ell)+b_3(r-3j+1,v-j+1,j,\ell)\\&\qquad+b_1(r-3j+1,v-j+1,j,\ell)\big)=1,\end{aligned} \tag{53}$$

where $r=\ell-n$. The proof will be completed once we established the following lemma. ∎

Lemma 6.5. *If $v\in\mathbb{C}$ and $v(r-v)(r+v)\neq 0$ then*

i) $a_1(r,v,j,\ell)+a_2(r,v,j,\ell)+a_3(r,v,j,\ell)=1$,

ii) $b_1(r,v,j,\ell)+b_2(r,v,j,\ell)+b_3(r,v,j,\ell)=1$.

Proof. i) From (26) we get

$$a_1 = \tfrac{(r-v+2j+2)(-r+v+2\ell-2j)}{2v(r-v)}, \; a_2 = -\tfrac{4(j+1)(\ell-j)}{(r+v)(r-v)}, \; a_3 = \tfrac{(r+v+2j+2)(r+v-2\ell+2j)}{2v(r+v)}.$$

Now it is a simple matter to check (i).
ii) It is a straightforward consequence of (i) since

$$b_i(r,v,j,\ell) = a_i(-r.-v.\ell-j,\ell), \quad i=1,2,3,$$

see (38). This completes the proof of the lemma. ∎

In [16] an explicit expression of the irreducible spherical functions associated to the complex hyperbolic plane is given in terms of the matrix valued hypergeometric functions introduced in [20].

For the reader's benefit we recall the main facts. If Φ is an irreducible spherical function of type (n,ℓ) then $\Delta_2\Phi = \tilde\lambda\Phi$ and $\Delta_3\Phi = \tilde\mu\Phi$ and moreover, up to equivalence, Φ is determined by the pair $(\tilde\lambda,\tilde\mu)$. Then for a particular irreducible spherical function Φ_π of type (n,ℓ), which is nonsingular everywhere, the function $H(g)=\Phi(g)\Phi_\pi(g)^{-1}$ is introduced. Now $H(a_s)$ is a diagonal matrix identified with the vector $(H_0(a_s),\dots,H_\ell(a_s))^t\in\mathbb{C}^{\ell+1}$. If we make the change of variables $u=1-\cosh^2(s)$ then $\bar H(u)=H(a_s)$ is analytic in the interval $(-\infty,0]$ and it is an eigenfunction of the following differential operators given in (3.1) and (3.2) of [16]

$$\begin{aligned}\bar D\bar H &= u(1-u)\bar H'' + (2-uA_1)\bar H' + \frac{1}{u}(B_0-B_1+uB_1)\bar H,\\ \bar E\bar H &= u(1-u)M\bar H'' + (C_1-C_0-uC_1)\bar H' + \frac{1}{u}(D_0+D_1-uD_1)\bar H,\end{aligned} \tag{54}$$

for $u\in(-\infty,0)$. The corresponding eigenvalues are, respectively,

$$\begin{aligned}\lambda &= \tfrac14\tilde\lambda + \tfrac13(\ell^2+n\ell+n^2)+\ell+n,\\ \mu &= \tilde\mu + 3\lambda - \tfrac19(\ell-n)(2\ell^2+5\ell n+2n^2)-(\ell+2)(2\ell+n).\end{aligned} \tag{55}$$

Then the matrix valued polynomial function ψ of degree ℓ defined by $\psi(u)=XT(u)$, where X is the Pascal matrix given by $X_{i,j}=\binom{i}{j}$ and $T(u)$ is the diagonal matrix such that $T(u)_{i,i}=u^i$, has the interesting property that the function $F(u)=\psi(u)^{-1}\bar H(u)$ satisfies

$$u(1-u)F'' + (C-uU)F' - (\lambda+V)F = 0, \tag{56}$$

$$(1-u)(Q_0+uQ_1)F'' + (P_0+uP_1)F' - (\mu-R)F = 0, \tag{57}$$

where the coefficient matrices are given in Lemmas 3.4 and 4.1 of [16].
Since the eigenvalues of C are not $0,-1,-2,\dots$ and F is a solution of (56) analytic at $u=0$ it follows that

$$F(u) = {}_2H_1(U;V+\lambda;C;u)F(0), \tag{58}$$

where ${}_2H_1$ is the matrix hypergeometric function introduced in [20]. Moreover F satisfies (57) if and only if $F(0)$ is the unique μ-eigenvector of the matrix $M(\lambda)$, introduced in (4.1) of [16], normalized by $F(0) = (1, x_1, \dots, x_\ell)^t$.

Any irreducible spherical function of type (n,ℓ) is equivalent to $\Phi_{(n,\ell)}^{\ell-n-3j,v}$ for some $v \in \mathbb{C}$, and $j \in \mathbb{Z}$, $0 \le j \le \ell$. The corresponding function $\bar H(u; v, j)$ is an eigenfunction of $\bar D$ and $\bar E$ with eigenvalues, respectively,

$$\begin{aligned}\lambda_j(v) &= \tfrac{1}{4}(\ell+n+2-j+v)(\ell+n+2-j-v) + j(\ell+1-j),\\ \mu_j(v) &= \lambda_j(v)(n+3j-\ell) - 3j(\ell+1-j)(n+j+1),\end{aligned} \tag{59}$$

as can be deduced from (55) replacing $r = \ell - n - 3j$, $\tilde\lambda = -v^2 + 4 - \frac{1}{3}r^2$ and $\tilde\mu = \frac{1}{4}\left(-\frac{1}{9}r^3 + r^2 + rv^2 + 3v^2 - 12\right)$, see Proposition 2.4. Thus

$$\bar H(u) = \psi(u)\, {}_2H_1(U; V + \lambda_j(v); C; u) F_j(v),$$

where $F_j(v)$ is the unique $\mu_j(v)$-eigenvector of the matrix $M(\lambda_j(v))$ normalized by $F_j(v) = (1, x_1, \dots, x_\ell)^t$.

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