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New Applications of Graded Lie Algebras to Lie Algebras, Generalized Lie Algebras, and Cohomology

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Abstract. We give new applications of graded Lie algebras to: identities of standard polynomials, deformation theory of quadratic Lie algebras, cyclic cohomology of quadratic Lie algebras, $2k$-Lie algebras, generalized Poisson brackets and so on.
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1. Introduction

Graded Lie algebras (**gla**) are commonly used in many areas of Mathematics and Physics. One of the reasons is that they offer a very convenient framework for the development of theories such as Cohomology Theory, Deformation Theory, among others, very often avoiding heavy computations. The aim of this paper is to give some new applications of classical well-known **gla** related to Deformation Theory.

Let us start with some notations: $\mathfrak{g}$ will be a complex vector space and $\bigwedge \mathfrak{g}$ the Grassmann algebra of $\mathfrak{g}$, that is, the algebra of skew multilinear forms on $\mathfrak{g}$, with the wedge product. When $\mathfrak{g}$ is finite-dimensional, we have $\bigwedge \mathfrak{g} = \mathrm{Ext}(\mathfrak{g}^*)$, where $\mathrm{Ext}(\mathfrak{g}^*)$ denotes the exterior algebra of the dual space $\mathfrak{g}^*$. However, when $\mathfrak{g}$ is not finite-dimensional, the strict inclusion $\mathrm{Ext}(\mathfrak{g}^*) \subset \bigwedge \mathfrak{g}$ holds. A *quadratic* vector space is a vector space endowed with a nondegenerate symmetric bilinear form. In the case of a *quadratic Lie algebra* this bilinear form has to be invariant. A theory of finite-dimensional quadratic Lie algebras based on the notion of double extension, was developed in [8] in the solvable case, following Kac's arguments [13], and in [18] in the general case by Medina and Revoy. In this paper, we shall present another interpretation based on the concept of super Poisson bracket.

The **gla** we shall use here are:

(1) Gerstenhaber's graded Lie algebras $\mathscr{M}(\mathfrak{g})$, related to associative algebra structures on $\mathfrak{g}$ (see Section 2).

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(2) Gerstenhaber-Nijenhuis's graded Lie algebras $\mathscr{M}_a(\mathfrak{g})$, related to Lie algebra structures on $\mathfrak{g}$ (see Section 2).

(3) The graded Lie algebra $\mathscr{D}(\mathfrak{g})$ of derivations of the Grassmann algebra $\bigwedge \mathfrak{g}$ (often called $W(n)$ when $n = \dim(\mathfrak{g})$, see Section 3).

(4) For finite-dimensional $\mathfrak{g}$, the graded Lie algebra $\mathscr{W}(\mathfrak{g})$ of skew symmetric polynomial multivectors on $\mathfrak{g}^*$ with the Schouten bracket (see Section 4).

(5) Given a quadratic finite-dimensional space $\mathfrak{g}$, the super Poisson graded Lie algebra structure on the Grassmann algebra $\bigwedge \mathfrak{g}$ (see Section 5)and the superalgebra $\mathscr{H}(\mathfrak{g})$ of Hamiltonian derivations of $\bigwedge \mathfrak{g}$.

For (1) and (2), we refer to [11], for (3) to [20], for (4) to Koszul's presentation [17] (though [11] could be convenient as well). For (5), though it is a known algebra, we have no references, probably because of the lack of applications up to now (we shall show, e.g. in Sections 6 to 9, that there are some natural and interesting ones). Since we want to fix our conventions and notations, we give an introduction to all the above **gla**, recalling the main properties that will be used all along this paper.

Section 2 is a review of $\mathscr{M}(\mathfrak{g})$ and $\mathscr{M}_a(\mathfrak{g})$. We conclude the Section with a notion of Generalized Lie Algebras structures, that we call $2k$*-Lie algebras*, namely the elements F in $\mathscr{M}_a^{2k}(\mathfrak{g})$ that satisfy $[F,F]_a = 0$. Such structures are introduced in [9] and many other papers (e.g. [3]), under various names.

In Section 3 we recall how to go from $\mathscr{M}_a(\mathfrak{g})$ to $\mathscr{D}(\mathfrak{g})$, an operation that can be translated as going from a structure to its cohomology, as we shall now explain. The argument is given by Proposition 3.1: there exists a one to one **gla** homomorphism from $\mathscr{M}_a(\mathfrak{g})[1]$ to $\mathscr{D}(\mathfrak{g})$, which turns out to be an isomorphism when $\mathfrak{g}$ is finite-dimensional. So given a $2k$-Lie algebra structure on $\mathfrak{g}$, there is an associated derivation D of $\bigwedge \mathfrak{g}$, and the (generalized) Jacobi identity $[F,F]_a = 0$ is equivalent to $D^2 = 0$, so that D defines a cohomology complex (3.3). This is well-known for Lie algebras since the corresponding complex is the Chevalley complex of trivial cohomology. The existence of a cohomology complex for a $2k$-Lie algebra was pointed out (without the **gla** interpretation), e.g. in [3]. We then recall the definition and properties of the Schouten bracket for a finite-dimensional $\mathfrak{g}$. As in [3], we define a *Generalized Poisson Bracket* (GPB) as an element W of $\mathscr{W}^{2k}(\mathfrak{g})$ satisfying $[W,W]_S = 0$ (Definition 3.4), the obvious generalization of the classical definition of a Poisson bracket. We show that there exists a one to one **gla** homomorphism from $\mathscr{D}(\mathfrak{g})$ into $\mathscr{W}(\mathfrak{g})[1]$ (Proposition 3.5), so that any $2k$-Lie algebra structure on $\mathfrak{g}$ has an associated GPB, generalizing the classical Lie-Kostant-Kirillov bracket associated to a Lie algebra.

We apply the results of Sections 2 and 3 to standard polynomials in Section 4. Standard polynomials $\mathscr{A}_k$ $(k \geq 0)$ on an associative algebra $\mathfrak{g}$, appear in the PI algebras theory (see [12]) and also in cohomology theory (for instance, the cohomology of $\mathfrak{gl}(n)$ is $\mathrm{Ext}[a_1, a_3, \ldots, a_{2n-1}]$ where $a_k = \mathrm{Tr}(\mathscr{A}_k)$, and the cohomology of $\mathfrak{gl}(\infty)$ is $\mathrm{Ext}[a_1, a_3, \ldots]$ [10]). There are two different structures on the space $\mathscr{A} = \mathrm{span}\{\mathscr{A}_k \mid k \geq 0\}$, both with interesting consequences. The first one comes from the Gerstenhaber bracket of $\mathscr{M}_a(\mathfrak{g})$: we compute explicitly $[\mathscr{A}_k, \mathscr{A}_{k'}]_a$, and it results that $\mathscr{A}$ is a subalgebra of the **gla** $\mathscr{M}_a(\mathfrak{g})$ (Proposition 4.2). Since $[\mathscr{A}_{2k}, \mathscr{A}_{2k}]_a = 0$, any even standard polynomial define a $2k$-Lie algebra structure on $\mathfrak{g}$ (Proposition 4.3). Moreover, $\mathscr{A}_{2k}$ is a coboundary (an invariant one) of the adjoint cohomology of the Lie algebra $\mathscr{A}_2$ defined by the associative algebra $\mathfrak{g}$. The second product, denoted by $\times$, is the cup-product on $\mathscr{M}_a(\mathfrak{g})$. We find that $\mathscr{A}$ is an Abelian algebra for $\times$, and in fact a very simple one, since $\mathscr{A}_k = (\mathscr{A}_1)^{\times k}$, $\forall k$ (Corollary 4.5). For instance, for $\mathfrak{g} = \mathfrak{gl}(n)$, $\mathscr{A}$ with its $\times$-product is isomorphic to $\mathbb{C}[x]/x^{2n}$, since $\mathscr{A}_k = 0$, $\forall k \geq 2n$ (the Amitsur-Levitzki theorem [1, 14]). From the identities in Corollary 4.5, we deduce some classical identities of standard polynomials (e.g. $\mathscr{A}_{2k} = (\mathscr{A}_2)^{\times k}$, $\forall k$, usually proved by hand). When $\mathfrak{g}$ has a trace, we prove that $\mathrm{Tr}\left([F,G]_\times\right) = 0$, for all F, $G \in \mathscr{M}_a(\mathfrak{g})$ (Corollary 4.8), and then (keeping the notation $a_k = \mathrm{Tr}\left(\mathscr{A}_k\right)$), that $a_{2k} = 0$, $\forall k > 0$, and that a_{2k+1} is an invariant Lie algebra cocycle (Proposition 4.10). To conclude Section 4, we compute the cohomology of the Lie algebra $\mathfrak{g}$ of finite-rank operators in an infinite-dimensional space. Obviously, $\mathfrak{gl}(\infty) \subset \mathfrak{g}$, but this inclusion is strict. Our result is $H^\star(\mathfrak{g}) = \mathrm{Ext}[a_1, a_3, \ldots]$ (Proposition 4.14), so the above inclusion induces an isomorphism in cohomology.

The first part of Section 5 is devoted to the construction of the super Poisson bracket defined on $\bigwedge \mathfrak{g}$, when $\mathfrak{g}$ is a finite-dimensional quadratic vector space. We follow a deformation argument as in [15]: the Clifford algebra $\mathrm{Cliff}(\mathfrak{g}^*)$ can be seen as a quantization of the algebra $\bigwedge \mathfrak{g}$ of skew polynomials, similarly to the Moyal quantization of polynomials by the Weyl algebra. In 5.1, we introduce some formulas for the construction of the Clifford algebra that are convenient since they easily provide a transparent formula for the deformed product (Proposition 5.1), with leading term the super Poisson bracket, explicitly computed in Proposition 5.4. The relation with the superalgebra $\widetilde{H}(n)$ [20] is given by Definition 5.3, and a Moyal type formula is obtained in Proposition 5.5 (an equivalent formula without the super Poisson bracket can be found in [15]). In the second part of Section 5, we use the **gla** $\bigwedge \mathfrak{g}$ and the super Poisson bracket to study quadratic Lie algebras. We obtain that quadratic Lie algebra structures on $\mathfrak{g}$ with bilinear form B are in one to one correspondence with elements I in $\bigwedge^3 \mathfrak{g}$ satisfying $\{I,I\} = 0$; more precisely, $I(X,Y,Z) = B([X,Y],Z)$, $\forall X$, Y, $Z \in \mathfrak{g}$, and the differential ∂ of $\bigwedge \mathfrak{g}$ is $\partial = -\frac{1}{2}\,\mathrm{ad}_{\mathrm{P}}(I)$ (Propositions 5.11, 5.12). We prove that any quadratic deformation of a quadratic Lie algebra is equivalent to a deformation with unchanged invariant bilinear form (Proposition 5.10), and finally, we propose a **gla** framework well-adapted to the deformation theory of quadratic Lie algebras (Remark 5.13).

We use the results of Section 5 in Section 6 to give a complete description of finite-dimensional elementary quadratic Lie algebras, i.e. those with decomposable associated element I in $\bigwedge^3 \mathfrak{g}$ (Definition 6.3). We first give a simple characterization in Proposition 6.4: if $\mathfrak{g}$ is a non Abelian quadratic Lie algebra, then $\dim\left([\mathfrak{g},\mathfrak{g}]\right) \geq 3$, and $\mathfrak{g}$ is elementary if and only if $\dim([\mathfrak{g},\mathfrak{g}]) = 3$. We then show that any non Abelian quadratic Lie algebra reduces, up to a central factor, to a quadratic Lie algebra with totally isotropic center (Proposition 6.7); the property of being elementary is preserved under the reduction. This reduces the problem of finding all non Abelian elementary quadratic Lie algebras to algebras of dimension 3 to 6 (Corollary 6.8), that we completely describe in 6.2 and Proposition 6.10. Some remarks: as we show in Proposition 6.5, if $\mathfrak{g}$ is an elementary quadratic Lie algebra, all coadjoint orbits have dimension at most 2. Now, a classification of Lie algebras whose coadjoint orbits are of dimension at most 2, is given in Arnal et al. [2], and the proof, using Lie algebra theory, is not at all trivial. With some effort, the elementary quadratic algebras can be identified in their classification. Our proof is completely different, based on basic properties of quadratic forms.

In Section 7 we study cyclic cohomology of quadratic Lie algebras. Given a quadratic vector space $\mathfrak{g}$, we use $\mathfrak{g}$-valued cochains (rather than $\mathfrak{g}^*$-valued, by analogy to the associative case [7]) to define cyclic cochains (Definition 7.1) (both notions are equivalent when $\mathfrak{g}$ is finite-dimensional). Thanks to this definition, we can use the Gerstenhaber bracket of $\mathscr{M}_a(\mathfrak{g})$ and we show that cyclic cochains are well-behaved with respect to this bracket: the space $\mathscr{C}_c(\mathfrak{g})$ of cyclic cochains is a subalgebra of the **gla** $\mathscr{M}_a(\mathfrak{g})$ (Proposition 7.2) and if $\mathfrak{g}$ is a Lie algebra, $\mathscr{C}_c(\mathfrak{g})$ is a subcomplex of the adjoint cohomology complex $\mathscr{M}_a(\mathfrak{g})$ (Proposition 7.4); we define the cyclic cohomology $H_c^*(\mathfrak{g})$ as the cohomology of this subcomplex (Definition 7.5). There is a natural one to one map from $\mathscr{C}_c(\mathfrak{g})$ into $\bigwedge_Q \mathfrak{g} = \bigwedge \mathfrak{g}/\mathbb{C}$ (Proposition 7.2) which induces a map from $H_c^*(\mathfrak{g})$ into $H_Q^*(\mathfrak{g}) = H^*(\mathfrak{g})/\mathbb{C}$. When $\mathfrak{g}$ is finite-dimensional, $\bigwedge_Q \mathfrak{g}$ is a **gla** for the (quotient) Poisson bracket, isomorphic to $\mathscr{H}(\mathfrak{g})$, and there is an induced **gla** structure on $H_Q^*(\mathfrak{g})$. We show that there is a **gla** isomorphism from $\mathscr{C}_c(\mathfrak{g})$ onto $\bigwedge_Q \mathfrak{g}$ (Proposition 7.7), and from $H_c^*(\mathfrak{g})$ onto $H_Q^*(\mathfrak{g})$ (Proposition 7.15). We also introduce a wedge product on $\mathscr{C}_c(\mathfrak{g})$, and on $H_c^*(\mathfrak{g})$ (Definition 7.12 and Proposition 7.13) which proves to be useful to describe $H_c^*(\mathfrak{g})$ (Proposition 7.15). When $\mathfrak{g}$ is not finite-dimensional, the isomorphism between $H_c^*(\mathfrak{g})$ and $H_Q^*(\mathfrak{g})$ is no longer true: we give an example where the natural map is neither one to one, nor onto (Propositions 7.18, 7.19). So the cyclic cohomology $H_c^*(\mathfrak{g})$ can have its own life, independently of the reduced cohomology $H_Q^*(\mathfrak{g})$.

Section 8 starts with the study of invariant cyclic cochains in the case of a finite-dimensional quadratic Lie algebra. We first prove that any invariant cyclic cochain is a cocycle (Proposition 8.2). When $\mathfrak{g}$ is reductive, we demonstrate that the inclusion of invariant cyclic cochains into cocycles induces an isomorphism in cohomology (Proposition 8.2), so that $H_c^*(\mathfrak{g}) \simeq \mathscr{C}_c(\mathfrak{g})^{\mathfrak{g}}$. Assuming that $\mathfrak{g}$ is a semisimple Lie algebra, we prove in Proposition 8.3:

If $I, I' \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$, then $\{I, I'\} = 0$

As a corollary, when $\mathfrak{g}$ is semisimple, the Gerstenhaber bracket induces the null bracket on $H_c^*(\mathfrak{g})$. Applying the preceding results, we give a complete description of the super Poisson bracket in $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$, and of the **gla** $H_c^*(\mathfrak{g})$, when $\mathfrak{g} = \mathfrak{gl}(n)$ (Example 8.8).

We develop in Section 9 the theory of quadratic $2k$-Lie algebra structures defined from cyclic cochains on a semi-simple Lie algebra $\mathfrak{g}$: we show that any invariant even cyclic cochain F defines a quadratic $2k$-Lie algebra (Proposition 9.4) and that $(\bigwedge \mathfrak{g})^{\mathfrak{g}} = H^\star(\mathfrak{g})$ is contained in $H^\star(F)$. We also give an interpretation of some interesting examples given in [3] of $2k$-Lie algebras in terms of the techniques developed in the present paper, pointing out where these examples come from. Finally, we give some examples of quadratic $2k$-Lie algebra structures on $\mathfrak{gl}(n)$ (9.3).

2. $\mathscr{M}(\mathfrak{g})$, $\mathscr{M}_a(\mathfrak{g})$ and $2k$-Lie algebra structures

This Section is essentially a review, except 2.3. For more details, see [11] and [19]. Let $\mathfrak{g}$ be a complex vector space. We denote by $\mathscr{M}(\mathfrak{g})$ the space of multilinear mappings from $\mathfrak{g}$ to $\mathfrak{g}$. The space $\mathscr{M}(\mathfrak{g})$ is graded as follows:

$$\mathscr{M}(\mathfrak{g}) = \sum_{k \geq 0} \mathscr{M}^k(\mathfrak{g})$$

where $\mathscr{M}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathscr{M}^k(\mathfrak{g}) = \{F : \mathfrak{g}^k \to \mathfrak{g} \mid F \ k\text{-linear}\}$, for $k \geq 1$.

2.1. . The theory of associative algebra structures on $\mathfrak{g}$ is conveniently described in a graded Lie algebra framework as follows: first, consider $\mathscr{M}(\mathfrak{g})$ with shifted grading $\mathscr{M}^k[1] = \mathscr{M}^{k+1}(\mathfrak{g})$ and denote it $\mathscr{M}[1]$. Then define a graded Lie bracket on $\mathscr{M}[1]$ as follows: for all $F \in \mathscr{M}^p[1]$, $G \in \mathscr{M}^q[1]$, then $[F,G] \in \mathscr{M}^{p+q}[1]$ and

$$
\begin{aligned}
&[F,G](X_1,\dots,X_{p+q+1}) := \\
&\quad (-1)^{pq}\sum_{j=1}^{p+1}(-1)^{q(j-1)} \quad F(X_1,\dots,X_{j-1},G(X_j,\dots,X_{j+q}),X_{j+q+1},\dots,X_{p+q+1}) \\
&\qquad -\sum_{j=1}^{q+1}(-1)^{p(j-1)} \quad G(X_1,\dots,X_{j-1},F(X_j,\dots,X_{j+p}),X_{j+p+1},\dots,X_{p+q+1}).
\end{aligned}
$$

for $X_1,\dots,X_{p+q+1} \in \mathfrak{g}$.

When $X \in \mathscr{M}^{-1}[1] = \mathfrak{g}$, then $[X,G]$ is defined by:

$$
[X,G](X_1,\dots,X_q) = -\sum_{j=1}^{q+1}(-1)^{j-1}\, G(X_1,\dots,X_{j-1},X,X_j,\dots,X_q).
$$

Notice that when F and G are in $\mathscr{M}^0[1] = \mathrm{End}(\mathfrak{g})$, then $[F,G]$ is the usual bracket of the two linear maps F and G.

Any $F \in \mathscr{M}^1[1]$ defines a product on $\mathfrak{g}$ by:

$$
X \cdot Y = F(X,Y), \forall\, X,Y \in \mathfrak{g}.
$$

This product is associative if and only if $[F,F] = 0$. In this case, the derivation $\mathrm{ad}(F)$ of the graded Lie algebra $\mathscr{M}[1]$ satisfies $(\mathrm{ad}(F))^2 = 0$, so it defines a complex on $\mathscr{M}(\mathfrak{g})$ which turns out to be the Hochschild cohomology complex of the associative algebra defined by F [11].

2.2. . In the remaining of the paper, we use $\mathfrak{S}_{p,q}$ to denote the set of all (p,q)-*unshuffles*, that is, elements σ in the permutation group $\mathfrak{S}_{p+q}$ satisfying $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

The theory of Lie algebra structures on $\mathfrak{g}$ is also conveniently described in a graded Lie algebra framework. First, let $\mathscr{M}_a = \mathscr{M}_a(\mathfrak{g})$ be the space of skew symmetric elements in $\mathscr{M}(\mathfrak{g})$. One has $\mathscr{M}_a = \sum_{k\geq 0} \mathscr{M}_a^k$ with $\mathscr{M}_a^0 = \mathfrak{g}$ and $\mathscr{M}_a^1 = \mathrm{End}(\mathfrak{g})$. Then consider $\mathscr{M}_a$ with shifted grading denoted by $\mathscr{M}_a[1]$, and define a graded Lie bracket as follows: for all $F \in \mathscr{M}_a^p[1]$, $G \in \mathscr{M}_a^q[1]$, then $[F,G]_a \in \mathscr{M}_a^{p+q}[1]$ and

$$
\begin{aligned}
&[F,G]_a\,(X_1,\dots,X_{p+q+1}) := \\
&\qquad (-1)^{pq}\sum_{\sigma\in\mathfrak{S}_{q+1,p}} \varepsilon(\sigma)\, F(G(X_{\sigma(1)},\dots,X_{\sigma(q+1)}),X_{\sigma(q+2)},\dots,X_{\sigma(p+q+1)}) \\
&\qquad\quad -\sum_{\sigma\in\mathfrak{S}_{p+1,q}} \varepsilon(\sigma)\, G(F(X_{\sigma(1)},\dots,X_{\sigma(p+1)}),X_{\sigma(p+2)},\dots,X_{\sigma(p+q+1)})
\end{aligned}
$$

for $X_1,\dots,X_{p+q+1} \in \mathfrak{g}$.

When $X \in \mathscr{M}_a^{-1}[1] = \mathfrak{g}$, then

$$
[X,G]_a\,(X_1,\dots,X_q) = -G(X,X_1,\dots,X_q)\ \ (= -\iota_X(G)(X_1,\dots,X_q)).
$$

Moreover, when F, $G \in \mathscr{M}_a^0[1] = \mathrm{End}(\mathfrak{g})$, then $[F,G]_a$ is the usual bracket of the linear maps F and G.

Now, any $F \in \mathscr{M}_a^1[1]$ defines a bracket on $\mathfrak{g}$ by

$$[X,Y] = F(X,Y), \forall\, X,Y \in \mathfrak{g}.$$

The Jacobi identity is satisfied if and only if $[F,F]_a = 0$. In this case, the derivation $\mathrm{ad}(F)$ of the graded Lie algebra $\mathscr{M}_a[1]$ satisfies $(\mathrm{ad}(F))^2 = 0$, so it defines a complex on $\mathscr{M}_a$ which turns out to be the Chevalley cohomology complex with coefficients in the adjoint representation, of the Lie algebra structure defined by F.

At this point, let us quickly explain the relations between the two brackets defined in 2.1 and 2.2. First, define the *skew symmetrization map* $\mathrm{A}\colon \mathscr{M}(\mathfrak{g}) \to \mathscr{M}_a(\mathfrak{g})$:

$$\mathrm{A}(F)(X_1,\dots,X_k) = \sum_{\sigma\in\mathfrak{S}_k} \varepsilon(\sigma)\, F(X_{\sigma(1)},\dots,X_{\sigma(k)})$$

with $F \in \mathscr{M}^k(\mathfrak{g})$ and $X_1,\dots,X_k \in \mathfrak{g}$. One has:

Proposition 2.1. *For all F, $G \in \mathscr{M}(\mathfrak{g})$, $\mathrm{A}([F,G]) = [\mathrm{A}(F),\mathrm{A}(G)]_a$.*

Obviously, when $F \in \mathscr{M}^1[1]$ is an associative product on $\mathfrak{g}$, then $\mathrm{A}(F)$ is a Lie algebra structure on $\mathfrak{g}$. However one should notice that from Proposition 2.1, Lie algebra structures of type $\mathrm{A}(F)$ can be obtained from a product F on $\mathfrak{g}$ satisfying other conditions than associativity, for instance:

Proposition 2.2. *Let $F \in \mathscr{M}^1[1]$ such that there exists $\tau \in \mathfrak{S}_3$ satisfying $\tau.[F,F] = -\varepsilon(\tau)\,[F,F]$. Then $\mathrm{A}(F)$ defines a Lie algebra structure on $\mathfrak{g}$.*

2.3. . Let us introduce a concept of generalized Lie algebra structures on $\mathfrak{g}$:

Definition 2.3. An element $F \in \mathscr{M}_a^{2k-1}[1]$ is a $2k$*-Lie algebra structure on* $\mathfrak{g}$ if

$$[F,F]_a = 0.$$

We shall often use a bracket notation: for $X_1,\dots,X_{2k} \in \mathfrak{g}$,

$$[X_1,\dots,X_{2k}] = F(X_1,\dots,X_{2k}).$$

The identity $[F,F]_a = 0$ can be seen as a generalized Jacobi identity (see [9, 3]).

Given a $2k$-Lie algebra structure F on $\mathfrak{g}$, $\mathrm{ad}(F)$ is an odd derivation of $\mathscr{M}_a[1]$ and satisfies $(\mathrm{ad}(F))^2 = 0$, so there is an associated cohomology defined by $\ker(\mathrm{ad}(F))/\mathrm{Im}(\mathrm{ad}(F))$, which can be interpreted as a generalization of the Chevalley complex of 2.2.

3. $\mathscr{D}(\mathfrak{g})$, $\mathscr{W}(\mathfrak{g})$, cohomology of $2k$-Lie algebras and GPB

In this Section, with exception made to 3.3 and 3.4, we recall classical material needed in the paper.

3.1. . We denote by $\mathscr{D} = \mathscr{D}(\mathfrak{g})$ the space of (graded) derivations of $\bigwedge \mathfrak{g}$. The space $\mathscr{D}$ is graded by $\mathscr{D} = \sum_{k=-1}^{n} \mathscr{D}^k$ with $D \in \mathscr{D}^d$ if $D(\bigwedge^p \mathfrak{g}) \subset \bigwedge^{p+d} \mathfrak{g}$, for all p, and has a graded Lie algebra structure with a bracket defined by:

$$[D,D'] = D \circ D' - (-1)^{dd'} D' \circ D, \forall\, D \in \mathscr{D}^d, D' \in \mathscr{D}^{d'}. \tag{1}$$

We denote by ι_X, $X \in \mathfrak{g}$, the elements of $\mathscr{D}^{-1}$ defined by

$$\iota_X(\Omega)(Y_1,\dots,Y_k) := \Omega(X,Y_1,\dots,Y_k), \forall\, \Omega \in \bigwedge\nolimits^{k+1}\mathfrak{g}, X, Y_1,\dots,Y_k \in \mathfrak{g}\ (k \geq 0),$$

and $\iota_X(1) = 0$. When $\mathfrak{g}$ is finite-dimensional, given a basis $\{X_1,\dots,X_n\}$ and its dual basis $\{\omega_1,\dots,\omega_n\}$, any element $D \in \mathscr{D}$ can be written in a unique way:

$$D = \sum_{r=1}^{n} D_r \wedge \iota_{X_r}$$

where $D_r = D(\omega_r)$. Moreover, $\mathscr{D}$ is a simple Lie superalgebra (often denoted by $W(n)$, see [20]) and there exists a vector space isomorphism $\mathbf{D}\colon \mathscr{M}_a[1] \to \mathscr{D}$ defined by $\mathbf{D}(\Omega \otimes X) = -\Omega \wedge \iota_X$, $\forall\, \Omega \in \bigwedge \mathfrak{g}$, $X \in \mathfrak{g}$ which turns out to be a **gla** isomorphism.

Since we do not want to restrict ourselves to the finite-dimensional case, we give a proof of the following result:

Proposition 3.1. *There exists a one to one* **gla** *homomorphism* $\mathbf{D}\colon \mathscr{M}_a[1] \to \mathscr{D}$ *such that*

$$\mathbf{D}(\Omega \otimes X) = -\Omega \wedge \iota_X, \forall\, \Omega \in \bigwedge \mathfrak{g}, X \in \mathfrak{g}.$$

When $\mathfrak{g}$ *is finite-dimensional,* $\mathbf{D}$ *is an isomorphism.*

Proof. Given a basis $\{X_r \mid r \in \mathscr{R}\}$ of $\mathfrak{g}$, and the forms ω_r, $r \in \mathscr{R}$, defined by $\omega_r(X_s) = \delta_{rs}$, $\forall r,s$, for $F \in \mathscr{M}_a^k$, let $\mathbf{D}(F) = -\sum_{r\in\mathscr{R}} {}^tF(\omega_r) \wedge \iota_{X_r}$. It is easy to see that though its indexes set is infinite, this sum, when applied to an element $\Omega \in \bigwedge^w \mathfrak{g}$, gives:

$$\mathbf{D}(F)(\Omega)(Y_1,\dots,Y_{k+w-1}) = \\ -\sum_{\sigma \in \mathfrak{S}_{k,w-1}} \varepsilon(\sigma)\Omega(F(Y_{\sigma(1)},\dots,Y_{\sigma(k)}), Y_{\sigma(k+1)},\dots,Y_{\sigma(k+w-1)}),$$

for all $Y_1,\dots,Y_{k+w-1} \in \mathfrak{g}$. It results that our definition of $\mathbf{D}$ does not depend on the basis of $\mathfrak{g}$, and that $\mathbf{D}(A \otimes X) = -A \wedge \iota_X$, $A \in \bigwedge^k \mathfrak{g}$, $X \in \mathfrak{g}$. Keeping in mind the remark about the sum defining $\mathbf{D}$, we compute for $G \in \mathscr{M}_a^{k'}$:

$$[\mathbf{D}(F),\mathbf{D}(G)] = \\ \sum_{r,s} \left({}^tF(\omega_r) \wedge \iota_{X_r}({}^tG(\omega_s)) - (-1)^{(k+1)(k'+1)}\,{}^tG(\omega_r) \wedge \iota_{X_r}({}^tF(\omega_s))\right) \wedge \iota_{X_s}.$$

By a direct computation:

$$\sum_r \left({}^tF(\omega_r) \wedge {}^tG(\omega_s)) - (-1)^{(k+1)(k'+1)}\,{}^tG(\omega_r) \wedge \iota_{X_r}({}^tF(\omega_s))\right)(Y_1,\dots,Y_{k+k'-1})$$
$$= \omega_s\Big(\sum_{\sigma\in\mathfrak{S}_{k,k'-1}} \varepsilon(\sigma) G(F(Y_{\sigma(1)},\dots,Y_{\sigma(k)}), Y_{\sigma(k+1)},\dots,Y_{\sigma(k+k'-1)}) -$$
$$(-1)^{(k+1)(k'+1)} \sum_{\sigma\in\mathfrak{S}_{k',k-1}} \varepsilon(\sigma) F(G(Y_{\sigma(1)},\dots,Y_{\sigma(k')}), Y_{\sigma(k'+1)},\dots,Y_{\sigma(k+k'-1)})\Big)$$
$$= -{}^t[F,G](\omega_s)(Y_1,\dots,Y_{k+k'-1}),$$

Hence:

$$[\mathbf{D}(F),\mathbf{D}(G)] = -\sum_s {}^t[F,G](\omega_s)\wedge \imath_{X_s} = \mathbf{D}([F,G])$$

■

In the sequel, given $F\in\mathscr{M}_a(\mathfrak{g})$, we denote by $\mathbf{D}_F$ the associated derivation of $\bigwedge\mathfrak{g}$. If $\mathfrak{g}$ is finite-dimensional, for $D\in\mathscr{D}$, we denote by $\mathbf{F}_D$ the associated element in $\mathscr{M}_a(\mathfrak{g})$. Here are some examples:

Example 3.2. If $T\in\mathrm{End}(\mathfrak{g}) = \mathscr{M}_a^0[1]$, then

$$\mathbf{D}_T(\Omega)(Y_1,\dots,Y_p) = -\sum_{i=1}^{p}\Omega(Y_1,\dots,Y_{i-1},T(Y_i),Y_{i+1},\dots,Y_p)$$

for all $\Omega\in\bigwedge^p\mathfrak{g}$, $Y_1,\dots,Y_p\in\mathfrak{g}$.

Example 3.3. If $F\in\mathscr{M}_a^1[1]$, then

$$\mathbf{D}_F(\Omega)(Y_1,\ \dots\ ,Y_{p+1}) = \sum_{i<j}(-1)^{i+j}\,\Omega(F(Y_i,Y_j),Y_1,\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots,Y_{p+1}) \qquad (2)$$

for all $\Omega\in\bigwedge^p\mathfrak{g}$, $Y_1,\dots,Y_{p+1}\in\mathfrak{g}$.

3.2. . Let F be a Lie algebra structure on $\mathfrak{g}$, then $F\in\mathscr{M}_a^1[1]$ and $[F,F]_a=0$. Let $\partial = \mathbf{D}_F$, then $[\partial,\partial]=0$ gives $\partial^2=0$ and formula (2) shows that the associated complex, in the Grassmann algebra $\bigwedge\mathfrak{g}$, is exactly the Chevalley cohomology complex of trivial cohomology of $\mathfrak{g}$. One defines θ_X:

$$\theta_X = [\imath_X,\partial] = \mathbf{D}_{\mathrm{ad}(X)}.$$

If $\{X_r \mid r\in\mathscr{R}\}$ is a basis of $\mathfrak{g}$, consider the forms ω_r, $r\in\mathscr{R}$, defined by $\omega_r(X_s)=\delta_{rs}$, $\forall r,s$. The map θ defines a Lie algebra representation of $\mathfrak{g}$ in $\bigwedge\mathfrak{g}$ and one has:

$$\partial = \frac{1}{2}\sum_{r\in\mathscr{R}}\omega_r\wedge\theta_{X_r}. \qquad (3)$$

Let us precise that this formula is well-known when $\mathfrak{g}$ is finite-dimensional (see [16]), and that a proof in the infinite-dimensional case can be found in the proof of Lemma 4.9 of the present paper. In any case, a very important consequence of formula (3) is that any invariant in $(\bigwedge\mathfrak{g})^{\mathfrak{g}}$ is a cocycle.

3.3. . Let us now check how 3.2 can be extended to $2k$-Lie algebra structures on $\mathfrak{g}$. Let $F\in\mathscr{M}_a^{2k-1}[1]$. Assume that $[F,F]_a=0$, and let

$$[Y_1,\dots,Y_{2k}] := F(Y_1,\dots,Y_{2k}) \qquad (4)$$

for $Y_1,\dots,Y_{2k}\in\mathfrak{g}$. Denote by $D=\mathbf{D}_F$ the associated derivation of $\bigwedge\mathfrak{g}$. Using Proposition 3.1, one concludes $D^2=0$, so one can define an associated cohomology $H^\star(F)=\ker(D)/\mathrm{Im}(D)$. One has

$$D\omega(Y_1,\dots,Y_{2k}) = -\omega([Y_1,\dots,Y_{2k}])$$

for $\omega \in \mathfrak{g}^*$, $Y_1, \ldots, Y_{2k} \in \mathfrak{g}$. We shall come back to cohomology of $2k$-Lie algebras in Section 9.

In the remaining of this Section, we will assume that $\mathfrak{g}$ is a finite-dimensional vector space with $\dim(\mathfrak{g}) = n$. We now recall some properties of the Schouten bracket. For more details, we refer to [17].

Let $\mathscr{W} = \mathscr{W}(\mathfrak{g}) = \mathscr{P} \otimes \bigwedge \mathfrak{g}$, graded by $\mathscr{W}^p = \mathscr{P} \otimes \bigwedge^p \mathfrak{g}$, where $\mathscr{P}$ is the symmetric algebra of $\mathfrak{g}^*$. Elements of $\mathscr{W}$ act as skew symmetric multivectors on $\mathscr{P}$ as follows: for $\Omega \in \bigwedge^p \mathfrak{g}$, $P \in \mathscr{P}$, $f_1, \ldots, f_p \in \mathscr{P}$,

$$(P \otimes \Omega)(f_1, \ldots, f_p)_\varphi = P(\varphi)\, \Omega((d f_1)_\varphi, \ldots, (d f_p)_\varphi).$$

For instance, if $\{X_1, \ldots, X_p\}$ is a basis of $\mathfrak{g}$ and $\{\omega_1, \ldots, \omega_p\}$ its dual basis, one has for all $i = 1, \ldots, p$:

$$\omega_i(f) = \frac{\partial f}{\partial X_i}, \forall f \in \mathscr{P}.$$

There is a natural $\wedge$-product on $\mathscr{W}$, defined by: for all $P, P' \in \mathscr{P}$, $\Omega, \Omega' \in \bigwedge \mathfrak{g}$:

$$(P \otimes \Omega) \wedge (P' \otimes \Omega') = PP' \otimes (\Omega \wedge \Omega').$$

Each $f \in \mathscr{P}$ defines a derivation ι_f of degree -1 of $\mathscr{W}$ by:

$$\begin{aligned} \iota_f(P \otimes \Omega)(f_1, \ldots, f_{p-1})_\varphi &= P(\varphi)\, \iota_{(d f)_\varphi}(\Omega)((d f_1)_\varphi, \ldots, (d f_{p-1})_\varphi) \\ &= P(\varphi)\, \Omega((d f)_\varphi, (d f_1)_\varphi, \ldots, (d f_{p-1})_\varphi). \end{aligned}$$

For instance, if $V \in \mathscr{W}^1$, one has $\iota_f(V) = V(f)$. There is a graded Lie bracket on $\mathscr{W}[1]$ called the *Schouten bracket*, and defined by: for all $W \in \mathscr{W}^p[1]$, $W' \in \mathscr{W}^q[1]$, then $[W, W']_S \in \mathscr{W}^{p+q}[1]$ and

$$\begin{aligned} &[W, W']_S\, (f_1, \ldots, f_{p+q+1}) = \\ &\quad (-1)^{pq} \sum_{\sigma \in \mathfrak{S}_{q+1,p}} \varepsilon(\sigma)\, W(W'(f_{\sigma(1)}, \ldots, f_{\sigma(q+1)}), f_{\sigma(q+2)}, \ldots, f_{\sigma(p+q+1)}) \\ &\quad - \sum_{\sigma \in \mathfrak{S}_{p+1,q}} \varepsilon(\sigma)\, W'(W(f_{\sigma(1)}, \ldots, f_{\sigma(p+1)}), f_{\sigma(p+2)}, \ldots, f_{\sigma(p+q+1)}) \end{aligned}$$

for $f_1, \ldots, f_{p+q+1} \in \mathscr{P}$.

Then for all $P, P' \in \mathscr{P}$, $\Omega \in \bigwedge^{p+1} \mathfrak{g}$, $\Omega' \in \bigwedge^{q+1} \mathfrak{g}$:

$$[P \otimes \Omega, P' \otimes \Omega']_S = (-1)^{pq} P \otimes (\Omega' \wedge \iota_{P'}(\Omega)) - P' \otimes (\Omega \wedge \iota_P(\Omega')).$$

As a particular case, one has $[\Omega, \Omega']_S = 0$, for all Ω, $\Omega' \in \bigwedge \mathfrak{g}$.

Let $W \in \mathscr{W}^1[1]$, then W defines a Poisson bracket on $\mathscr{P}$ by $\{P, P'\} = W(P, P')$ if and only if $[W, W]_S = 0$. More generally, as proposed in [3], one can define *Generalized Poisson Brackets* (GPB) as follows:

Definition 3.4. An element $W \in \mathscr{W}^{2k-1}[1]$ is a GPB if $[W, W]_S = 0$.

(see [3] where these structures are introduced and applications are proposed).

3.4. . Let us now show that $2k$-Lie algebras have associated GPB, exactly as Lie algebras have associated Poisson brackets. This will be a consequence of the following construction: define a map $V\colon \mathscr{D} = \mathscr{D}(\mathfrak{g}) \to \mathscr{W}$ by $V_D = V(D) := -X \otimes \Omega$ for $D = \Omega \wedge \iota_X$ with $\Omega \in \bigwedge \mathfrak{g}$, $X \in \mathfrak{g}$. Then, it is easy to check that:

Proposition 3.5. *One has $V_{[D,D']} = [V_D, V_{D'}]_S$, for D, $D' \in \mathscr{D}$. Moreover V is a one to one graded Lie algebras homomorphism from $\mathscr{D}$ into $\mathscr{W}[1]$.*

For example, given a $2k$-Lie algebra structure F on $\mathfrak{g}$, denoted by $[Y_1, \dots, Y_{2k}] = F(Y_1, \dots, Y_{2k})$, $\forall\, Y_1, \dots, Y_{2k} \in \mathfrak{g}$, let D be the associated derivation (see Proposition 3.1) in $\mathscr{D}$. Then one has:

$$V_D(f_1, \dots, f_{2k})_\varphi = \langle \varphi | [(df_1)_\varphi, \dots, (df_{2k})_\varphi] \rangle,$$

and since $[F,F]_a = 0$ (by Proposition 3.1), one has $[D,D] = 0$. Using Proposition 3.5 above, $[V_D, V_D]_S = 0$, so V_D defines a GPB on $\mathscr{P}$.

Finally, using 3.1 and Proposition 3.5, one deduces an inclusion of the simple Lie superalgebra $W(n)$ into the graded Lie algebra $\mathscr{W}[1]$, endowed with the Schouten bracket which provides a natural realization of $W(n)$.

4. Application to identities of standard polynomials, and cohomology

In this Section, $\mathfrak{g}$ denotes an associative algebra, with product m. We also use the notation: $X.Y = m(X,Y)$, $\forall\, X, Y \in \mathfrak{g}$. We assume that m has a unit $\mathbf{1}_m$, but this is not really necessary.

We first define the iterated m_k $(k \geq 0)$ of m as:

$$m_0 = \mathbf{1}_m,\ m_1 = \mathrm{Id}_{\mathfrak{g}},\ m_2 = m, \dots,\ m_k(Y_1, \dots, Y_k) = Y_1. \dots .Y_k,\ \forall\, Y_1, \dots, Y_k \in \mathfrak{g}, \dots$$

It is easy to check that:

Proposition 4.1. *For all k, $k' \geq 0$, one has:*

$$\begin{aligned}
[m_{2k}, m_{2k'}] &= 0,\\
[m_{2k}, m_{2k'+1}] &= (2k-1)\, m_{2k+2k'},\\
[m_{2k+1}, m_{2k'+1}] &= 2(k-k')\, m_{2k+2k'+1}.
\end{aligned}$$

Hence the space generated by $\{m_k, k \geq 0\}$ is a subalgebra of the **gla** $\mathscr{M}(\mathfrak{g})$ of Section 2.

Now define the *standard polynomials* $\mathscr{A}_k$ $(k \geq 0)$ on $\mathfrak{g}$ as:

$$\mathscr{A}_k := \mathrm{A}(m_k)$$

Using Propositions 2.1 and 4.1, one immediately obtains:

Proposition 4.2. *For all k, $k' \geq 0$, one has:*

$$\begin{aligned}
[\mathscr{A}_{2k}, \mathscr{A}_{2k'}]_a &= 0,\\
[\mathscr{A}_{2k}, \mathscr{A}_{2k'+1}]_a &= (2k-1)\, \mathscr{A}_{2k+2k'},\\
[\mathscr{A}_{2k+1}, \mathscr{A}_{2k'+1}]_a &= 2(k-k')\, \mathscr{A}_{2k+2k'+1}.
\end{aligned}$$

Let $\mathscr{A}$ be the subspace generated by $\{\mathscr{A}_k, k \geq 0\}$. Hence $\mathscr{A}$ is a subalgebra of the graded Lie algebra $\mathscr{M}_a(\mathfrak{g})$ of Section 2. The standard polynomial $\mathscr{A}_2$ is the Lie algebra structure on $\mathfrak{g}$ associated to m. Since $[\mathscr{A}_{2k}, \mathscr{A}_{2k}]_a = 0$, $\forall k$, we conclude:

Proposition 4.3. *The standard polynomials $\mathscr{A}_{2k}$, $k \geq 1$ define $2k$-Lie algebra structures on $\mathfrak{g}$.*

Remark that $\mathscr{A}_k$ is a $\mathfrak{g}$-invariant map from $\mathfrak{g}^k$ to $\mathfrak{g}$ for the Lie algebra structure. Moreover the standard polynomial $\mathscr{A}_{2k}$ is a coboundary of the adjoint representation of the Lie algebra $\mathfrak{g}$ since $[\mathscr{A}_2, \mathscr{A}_{2k-1}] = \mathscr{A}_{2k}$.

Let us now define an associative product on $\mathscr{A}$. First consider the cup-product $\circ$ on $\mathscr{M}(\mathfrak{g})$:

$$(F \circ G)(Y_1, \dots, Y_{p+q}) = F(Y_1, \dots, Y_p).G(Y_{p+1}, \dots, Y_{p+q}),$$

for all $F \in \mathscr{M}^p(\mathfrak{g}), G \in \mathscr{M}^q(\mathfrak{g})$, $Y_1, \dots, Y_{p+q} \in \mathfrak{g}$.

Then define an associative product $\times$ on $\mathscr{M}_a(\mathfrak{g})$ by:

$$(F \times G)(Y_1, \dots, Y_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{p,q}} \varepsilon(\sigma)\, F(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}) \,.\, G(Y_{\sigma(p+1)}, \dots, Y_{\sigma(p+q)}),$$

for all $F \in \mathscr{M}_a^p(\mathfrak{g}), G \in \mathscr{M}_a^q(\mathfrak{g})$, $Y_1, \dots, Y_{p+q} \in \mathfrak{g}$. By a straightforward computation, one has:

Proposition 4.4. *For all F, $G \in \mathscr{M}_a(\mathfrak{g})$, $\mathrm{A}(F \circ G) = \mathrm{A}(F) \times \mathrm{A}(G)$.*

It is obvious that $m_k = \underbrace{m_1 \circ \cdots \circ m_1}_{k \text{ times}}$, so:

Corollary 4.5. *$\mathscr{A}_k = \underbrace{\mathscr{A}_1 \times \cdots \times \mathscr{A}_1}_{k \text{ times}}$, for all $k \geq 1$ and $\mathscr{A}_k \times \mathscr{A}_\ell = \mathscr{A}_\ell \times \mathscr{A}_k = \mathscr{A}_{k+\ell}$, for all $k, \ell \geq 0$.*

As a consequence, $\mathscr{A}$ is a commutative algebra for the $\times$-product.

Any element $Z \in \mathfrak{g}$ defines a super derivation ι_Z of degree -1 of the $\times$-product of $\mathscr{M}_a(\mathfrak{g})$ by: for all $F \in \mathscr{M}_a^p(\mathfrak{g})$, $Y_1, \dots, Y_{p-1} \in \mathfrak{g}$,

$$\iota_Z(F)(Y_1, \dots, Y_{p-1}) := F(Z, Y_1, \dots, Y_{p-1}),$$

Denote by $\mathrm{Z}(\mathfrak{g})$ the center of the algebra $\mathfrak{g}$. If $Z \in \mathrm{Z}(\mathfrak{g})$, one has $\iota_Z(\mathscr{A}_2) = 0$. Hence using Corollary 4.5 and the derivation property of ι_Z, we deduce:

Proposition 4.6. *Assume that $Z \in \mathrm{Z}(\mathfrak{g})$. Then for all k,*

$$\iota_Z(\mathscr{A}_{2k}) = 0 \text{ and } \iota_Z(\mathscr{A}_{2k+1}) = Z \,.\, \mathscr{A}_{2k}.$$

This Proposition expresses classical identities of standard polynomials, generally written in the case $Z = \mathbf{1}_m$.

Let us now assume that $\mathfrak{g}$ is equipped with a trace, that is, a linear form $\mathrm{Tr} : \mathfrak{g} \to \mathbb{C}$ satisfying:

$$\mathrm{Tr}(X.Y) = \mathrm{Tr}(Y.X), \ \forall\, X, Y \in \mathfrak{g}.$$

Let $\bigwedge \mathfrak{g}$ be the Grassmann algebra of $\mathfrak{g}$. We extend the trace Tr to a map $\mathrm{Tr} : \mathscr{M}_a(\mathfrak{g}) \to \bigwedge \mathfrak{g}$ defined by:

$$\mathrm{Tr}(F)(Y_1, \dots, Y_p) = \mathrm{Tr}(F(Y_1, \dots, Y_p)),$$

for all $F \in \mathscr{M}_a^p(\mathfrak{g})$, $Y_1, \dots, Y_p \in \mathfrak{g}$.

Proposition 4.7. *One has* $\mathrm{Tr}\,(F\times G) = (-1)^{pq}\ \mathrm{Tr}\,(G\times F)$, *for all* $F\in\mathscr{M}_a^p(\mathfrak{g})$, $G\in\mathscr{M}_a^q(\mathfrak{g})$.

Proof. Let $F\in\mathscr{M}_a^p(\mathfrak{g})$, $G\in\mathscr{M}_a^q(\mathfrak{g})$, $Y_1,\ldots,Y_{p+q}\in\mathfrak{g}$:

$$\begin{aligned}
\mathrm{Tr}(F\times G)(Y_1,\ldots,Y_{p+q}) &= \\
&= \sum_{\sigma\in\mathfrak{S}_{p,q}} \varepsilon(\sigma)\quad \mathrm{Tr}(F(Y_{\sigma(1)},\ldots,Y_{\sigma(p)})\ .\ G(Y_{\sigma(p+1)},\ldots,Y_{\sigma(p+q)}))\\
&= \sum_{\sigma\in\mathfrak{S}_{p,q}} \varepsilon(\sigma)\quad \mathrm{Tr}(G(Y_{\sigma(p+1)},\ldots,Y_{\sigma(p+q)})\ .\ F(Y_{\sigma(1)},\ldots,Y_{\sigma(p)}))
\end{aligned}$$

Given $\sigma\in\mathfrak{S}_{p,q}$, define $\tau\in\mathfrak{S}_{q,p}$ as $\tau(1)=\sigma(p+1),\ \ldots,\ \tau(q)=\sigma(p+q)$ and $\tau(q+1)=\sigma(1),\ \ldots,\ \tau(q+p)=\sigma(p)$. Then one has $\varepsilon(\tau)=(-1)^{pq}\varepsilon(\sigma)$, so:

$$\begin{aligned}
&\mathrm{Tr}(F\times G)(Y_1,\ldots,Y_{p+q}) = \\
=\ &(-1)^{pq}\sum_{\tau\in\mathfrak{S}_{q,p}} \varepsilon(\tau)\ \mathrm{Tr}(G(Y_{\tau(1)},\ldots,Y_{\tau(q)})\ .\ F(Y_{\tau(q+1)},\ldots,Y_{\tau(p+q)}))\\
=\ &(-1)^{pq}\mathrm{Tr}(G\times F)(Y_1,\ldots,Y_{p+q}).
\end{aligned}$$

■

Hence our extension of the trace has, in fact, the properties of a $\bigwedge\mathfrak{g}$-valued super trace on the graded algebra $(\mathscr{M}_a(\mathfrak{g}),\times)$. Denoting the super bracket associated to the $\times$-product on $\mathscr{M}_a(\mathfrak{g})$ by:

$$[F,G]_\times = F\times G - (-1)^{pq} G\times F, \forall\ F\in\mathscr{M}_a^p(\mathfrak{g}), G\in\mathscr{M}_a^q(\mathfrak{g}),$$

one obtains

Corollary 4.8. $\mathrm{Tr}\,([F,G]_\times) = 0$.

Lemma 4.9. *Let* $\mathfrak{h}$ *be a Lie algebra. Then any invariant cochain in* $(\bigwedge\mathfrak{h})^{\mathfrak{h}}$ *is a cocycle.*

Proof. If $\mathfrak{h}$ is finite-dimensional, the result is well-known ([16]) and is a direct consequence of the formula $\partial = \frac{1}{2}\sum_{i=1}^n \omega_i\wedge\theta_{X_i}$ where ∂ is the differential, $\{X_1,\ldots,X_n\}$ a basis of $\mathfrak{h}$ and $\{\omega_1,\ldots,\omega_n\}$ its dual basis.

For the sake of completeness, we give a proof in the general case, let $\{X_i \mid i\in I\}$ be a basis of $\mathfrak{h}$, and $\{\omega_i \mid i\in I\}$ be the forms defined by $\omega_i(X_j)=\delta_{ij}$, $\forall\ i,j$. We claim that the formula $\partial = \frac{1}{2}\sum_{i\in I}\omega_i\wedge\theta_{X_i}$ is still valid. To prove this, let $D=\frac{1}{2}\sum_{i\in I}\omega_i\wedge\theta_{X_i}$. Though its indexes set is infinite, this sum exists since for $\Omega\in\bigwedge^p\mathfrak{h}$ and $Y_1,\ldots,Y_{p+1}\in\mathfrak{h}$, one has:

$$\frac{1}{2}\sum_{i\in I}\omega_i\wedge\theta_{X_i}(\Omega)(Y_1,\ldots,Y_{p+1}) = \frac{1}{2}\sum_{j=1}^{p+1}(-1)^{j+1}\sum_{i\in I}\omega_i(Y_j)\theta_{X_i}(\Omega)(Y_1,\ldots,\widehat{Y_j},\ldots,Y_{p+1})$$

Then

$$D(\Omega)(Y_1,\ldots,Y_{p+1}) = -\frac{1}{2}\sum_{j=1}^{p+1}(-1)^{j+1}\left(\sum_{k=1}^{j-1}\Omega(Y_1,\ldots,[Y_j,Y_k],\ldots,\widehat{Y_j},\ldots,Y_{p+1}) +\right.$$

$$\sum_{k=j+1}^{p+1} \Omega(Y_1,\ldots,\widehat{Y_j},\ldots,[Y_j,Y_k],\ldots,Y_{p+1})\Bigg)$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{p+1}(-1)^j \Bigg(\sum_{k<j}(-1)^{k+1}\Omega([Y_j,Y_k],Y_1,\ldots,\widehat{Y_k},\ldots,\widehat{Y_j},\ldots,Y_{p+1})+\\
&\qquad \sum_{j<k}(-1)^k\Omega([Y_j,Y_k],Y_1,\ldots,\widehat{Y_j},\ldots,\widehat{Y_k},\ldots,Y_{p+1})\Bigg)\\
&= \sum_{j<k}(-1)^{j+k}\Omega([Y_j,Y_k],Y_1,\ldots,\widehat{Y_j},\ldots,\widehat{Y_k},\ldots,Y_{p+1}) = \partial(\Omega)(Y_1,\ldots,Y_{p+1})
\end{aligned}$$

■

Proposition 4.10. *One has* $\mathrm{Tr}\,(\mathscr{A}_{2k}) = 0$ *(*$k \geq 1$*) and* $\mathrm{Tr}\,(\mathscr{A}_{2k+1})$ *(*$k \geq 0$*) is an invariant cocycle for the (trivial) cohomology of the Lie algebra* $\mathfrak{g}$.

Proof. For the first claim, use $[\mathscr{A}_1, \mathscr{A}_{2k-1}]_\times = 2\,\mathscr{A}_{2k}$ and apply Corollary 4.8. For the second, we remark that $\mathscr{A}_{2k+1}$ is a $\mathfrak{g}$-invariant map from $\mathfrak{g}^{2k+1}$ into $\mathfrak{g}$, so $\mathrm{Tr}\,(\mathscr{A}_{2k+1}) \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ and it is a cocycle by Lemma 4.9. ■

Now recall the well-known formula (e.g. [14]):

Proposition 4.11. *For all* $Y_1,\ldots,Y_{2k+1} \in \mathfrak{g}$,

$$\mathrm{Tr}(\mathscr{A}_{2k+1}(Y_1,\ldots,Y_{2k+1})) = (2k+1)\mathrm{Tr}(\mathscr{A}_{2k}(Y_1,\ldots,Y_{2k}).Y_{2k+1}).$$

This formula will be reinterpreted in Section 7 in terms of cyclic cohomology of the Lie algebra $\mathfrak{g}$: $\mathscr{A}_{2k}$ is a cocycle of the adjoint action (actually a coboundary since $[\mathscr{A}_2, \mathscr{A}_{2k-1}] = \mathscr{A}_{2k}$), and Proposition 4.11 tells that it is a cyclic cocycle, as will be defined in Section 7.

Example 4.12. Assume that $\mathfrak{g} = \mathfrak{gl}(n)$. Then $H^\star(\mathfrak{g})$ can be completely described in terms of standard polynomials (see e.g. [14] or [10]):

$$H^\star(\mathfrak{g}) = \mathrm{Ext}[\mathrm{Tr}(\mathscr{A}_1), \mathrm{Tr}(\mathscr{A}_3), \ldots, \mathrm{Tr}(\mathscr{A}_{2n-1})].$$

Moreover, by the Amitsur-Levitzki theorem ([1, 14]):

$$\mathscr{A}_k = 0, \text{ if } k \geq 2n.$$

So $\dim(\mathscr{A}) = 2n$. For the $\times$-product, $\mathscr{A} \simeq \mathbb{C}[X]/X^{2n}$. For the graded bracket of 2.2, the structure of $\mathscr{A}$ is given by Proposition 4.2, $\mathscr{A}_2$ is the Lie algebra structure on $\mathfrak{g}$ and the standard polynomials $\mathscr{A}_4, \ldots, \mathscr{A}_{2n-2}$ define $2k$-Lie algebra structures on $\mathfrak{g}$ by Proposition 4.3.

Example 4.13. More generally, let V be an infinite-dimensional vector space. Let $\mathfrak{g}$ be the space of finite-rank linear maps. So $\mathfrak{g}$ is an ideal of the associative algebra $\mathrm{End}(V)$. There is a vector spaces isomorphism $\mathfrak{g} \simeq V^* \otimes V$ defined by $(\omega \otimes v)(v') = \omega(v')\, v$, for all $v,\ v' \in V,\ \omega \in V^*$. So we can define the trace $\mathrm{Tr}\,(X)$ when $X \in \mathfrak{g}$ by $\mathrm{Tr}\,(\omega \otimes v) := \omega(v)$, for all $\omega \in V^*,\ v \in V$. It is easy to check that $\mathrm{Tr}\,([X,Y]) = 0$, for all $X,\ Y \in \mathfrak{g}$, so the preceding results apply. Moreover, the symmetric bilinear form B defined on $\mathfrak{g}$ by $B(X,Y) = \mathrm{Tr}\,(XY)$ is nondegenerate and invariant, therefore $\mathfrak{g}$ is a quadratic Lie algebra. Since $\mathfrak{gl}(n) \subset \mathfrak{g},\ \forall\, n$, by Example 4.12, we can conclude that

$$\mathrm{Ext}[\mathrm{Tr}(\mathscr{A}_1), \mathrm{Tr}(\mathscr{A}_3), \ldots, \mathrm{Tr}(\mathscr{A}_{2n-1}), \ldots] \subset H^\star(\mathfrak{g})$$

Proposition 4.14. *Let* $a_{2n+1} = \mathrm{Tr}\,(\mathscr{A}_{2n+1})$. *Then*

$$H^\star(\mathfrak{g}) = \mathrm{Ext}[a_1, a_3, \ldots, a_{2n+1}, \ldots].$$

Proof. Recall that for any Lie algebra $\mathfrak{h}$, there is an isomorphism $H^k(\mathfrak{h}) \simeq H_k(\mathfrak{h})^*$, induced by the restriction $\Omega \in Z^k(\mathfrak{h}) \mapsto \Omega|_{Z_k(\mathfrak{h})}$ where $H_k(\mathfrak{h})$ is the homology of $\mathfrak{h}$ defined as $H_k(\mathfrak{h}) = Z_k(\mathfrak{h})/B_k(\mathfrak{h})$ (with $Z_k(\mathfrak{h})$ the cycles and $B_k(\mathfrak{h})$ the boundaries).

Let us define $\mathscr{S} = \{S = (W,W') \mid W,W'$ complementary subspaces of V with $\dim(W) < \infty\}$ and for $S = (W,W') \in \mathscr{S}$, $\mathfrak{g}_S = \{X \in \mathfrak{g} \mid X(V) \subset W,\ X(W') = \{0\}\}$. Then $\mathfrak{g}_S$ is a subalgebra of the (associative or Lie) algebra $\mathfrak{g}$ and one has $\mathfrak{g}_S \simeq \mathfrak{gl}(\dim(W))$. It is easy to check that given $X_1,,\ \ldots,\ X_r \in \mathfrak{g}$, there exists $S \in \mathscr{S}$ such that $X_i \in \mathfrak{g}_S$, $\forall i = 1, \ldots, r$. It results that, if $c \in \mathrm{Ext}^k(\mathfrak{g})$, there exists S such that $c \in \mathrm{Ext}^k(\mathfrak{g}_S)$, so that $\mathrm{Ext}^k(\mathfrak{g}) = \cup_{S \in \mathscr{S}} \mathrm{Ext}^k(\mathfrak{g}_S)$.

Set $\mathscr{E} = \mathrm{Ext}[a_1, a_3, \ldots, a_{2n+1}, \ldots] \subset H^\star(\mathfrak{g})$ and $\mathscr{E}^k = \mathscr{E} \cap H^k(\mathfrak{g})$. Then $\dim(\mathscr{E}^k) = \sharp\, I_k$ with $I_k = \{(i_j) \in \{0,1\}^{\mathbb{N}} \mid \sum_{j \in \mathbb{N}} (2j+1)\ i_j = k\}$. We fix a basis $\{\Omega_i \mid i \in I_k\}$ of $\mathscr{E}^k$.

Given $c \in Z_k(\mathfrak{g})$, denote by $\overline{c}$ its class in $H_k(\mathfrak{g})$. Let us assume that $\Omega_i(\overline{c}) = 0$, $\forall i \in I_k$. Take $S \in \mathscr{S}$ such that $c \in \mathrm{Ext}^k(\mathfrak{g}_S)$, then by (4.12), $\{\Omega_i \mid i \in I_k\}$ generates $H^k(\mathfrak{g}_S) = H_k(\mathfrak{g}_S)^*$ and since $c \in Z_k(\mathfrak{g}_S)$, it results that $c \in B_k(\mathfrak{g}_S) \subset B_k(\mathfrak{g})$, therefore $\overline{c} = 0$. So, $\{\Omega_i \mid i \in I_k\}$ is free in $H_k(\mathfrak{g})^*$ and $\cap_{i \in I_k} \ker(\Omega_i) = \{0\}$. It results that $\dim(H_k(\mathfrak{g})) = \sharp\ I_k$. Since $H^k(\mathfrak{g}) = H_k(\mathfrak{g})^*$, one has $\dim(H^k(\mathfrak{g})) = \sharp\ I_k$ and since $\mathscr{E}^k \subset H^k(\mathfrak{g})$, one obtains $\mathscr{E}^k = H^k(\mathfrak{g})$. ∎

Remark 4.15. From $H^1(\mathfrak{g}) = \mathbb{C}\ \mathrm{Tr}$, we deduce that $[\mathfrak{g},\mathfrak{g}] = \ker(\mathrm{Tr})$. From $H^2(\mathfrak{g}) = \{0\}$, we deduce that $\mathfrak{g}$ has no (non trivial) central extension.

5. Super Poisson brackets and quadratic Lie algebras

The canonical Poisson bracket on $\mathbb{C}^{2n}$ appears as the leading term of a quantization of the algebra of polynomial functions by the Weyl algebra: the *Moyal product*. We will develop a similar formalism, replacing polynomials (i.e. commuting variables) by skew multilinear forms (i.e. skew commuting variables) and the Weyl algebra by the Clifford algebra. The leading term of the deformation will be the *super Poisson bracket*.

5.1. . Let us give a definition of the Clifford algebra that is well-adapted to the realization of this algebra as a deformation of the exterior algebra. Denote by $\mathscr{C}_t$, $t \in \mathbb{C}$, the associative algebra with basis $\{e_I, I \in \mathbb{Z}_2^n\}$ and product defined by

$$e_I \star e_J = (-1)^{\Omega(I,J)}\, t^{|IJ|}\, e_{\underset{(2)}{I+J}}$$

where Ω is the bilinear form associated to the matrix $(a_{ij})_{i,j=1}^n$ with $a_{ij} = 1$ if $i > j$ and 0 otherwise.

Take $I_i = (j_k) \in \mathbb{Z}_2^n$, with $j_i = 1$ and 0 otherwise. Set $e_i = e_{I_i}$, $i = 1, \dots, n$ and $V = \operatorname{span}\{e_1, \dots, e_n\}$. When $t = 0$, one obtains $\mathscr{C}_0 = \operatorname{Ext}(V)$. When $t \neq 0$, $\mathscr{C}_t$ is the *Clifford algebra*. The following relations hold:

$$e_i^2 = t, \ \forall\, i, \quad e_i \star e_j + e_j \star e_i = 0, \ i \neq j,$$

$$e_{i_1} \star e_{i_2} \star \cdots \star e_{i_p} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}, \text{ if } i_1 < i_2 < \cdots < i_p$$

So that $\mathscr{C}_t$ is the quotient algebra of the tensor algebra $T(V)$ by the relations:

$$v \otimes v = t.\, B(v,v).\, 1, v \in V,$$

where B is the bilinear form $B(e_i, e_j) = \delta_{ij}$, for all i, j, and we recover the usual definition of the Clifford algebra.

But we are mainly interested in realizing $\mathscr{C}_t$ as a deformation of $\operatorname{Ext}(V)$. Using:

$$t^k = 0^k + t\ \delta_{k,1} + t^2\ \delta_{k,2} + \dots\ .$$

this deformation becomes transparent:

Proposition 5.1. *One has*

$$e_I \star e_J = e_I \wedge e_J + \sum_{k=1}^{n} t^k\, D_k(e_I, e_J)$$

where $D_k(e_I, e_J) = \delta_{|IJ|,k}\, (-1)^{\Omega(I,J)}\, e_{\underset{(2)}{I+J}}$.

Symmetry properties of the coefficients are resumed in:

Proposition 5.2. *For all* $\Omega \in \operatorname{Ext}^w(V)$, $\Omega' \in \operatorname{Ext}^{w'}(V)$,

$$D_j(\Omega, \Omega') = (-1)^j (-1)^{ww'} D_j(\Omega', \Omega).$$

We insist on the fact that $\mathscr{C}_t$ is not a $\mathbb{Z}$-graded, but only a $\mathbb{Z}_2$-graded algebra. The associated Lie superalgebra has bracket:

$$[\Omega, \Omega']_\star = 2 \sum_{p \geq 0} t^{2p+1}\, D_{2p+1}(\Omega, \Omega').$$

Definition 5.3. We define the *super Poisson bracket* on $\operatorname{Ext}(V)$ by:

$$\{\Omega, \Omega'\} = 2\, D_1(\Omega, \Omega'), \ \forall\, \Omega, \Omega' \in \operatorname{Ext}(V).$$

Since $[.,.]_\star$ satisfies the super Jacobi identity, so does $\{.,.\}$. Moreover, since $\mathrm{ad}_\star(\Omega)$ is derivation of the $\mathscr{C}_t$-product, $\mathrm{ad}_\mathrm{P}(\Omega) := \{\Omega, .\}$ is a derivation of the $\wedge$-product (actually of degree $(w-2)$ if $\Omega \in \mathrm{Ext}^w(V)$).

Finally, by a straightforward computation, one gets:

$$\{v_1 \wedge \cdots \wedge v_p, w_1 \wedge \dots \wedge w_q\} = 2\,(-1)^{p+1} \times \sum_{\substack{i=1,\dots,p \\ j=1,\dots,q}} (-1)^{i+j}\, B(v_i, w_j)\ v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_q, \tag{5}$$

for all $v_1, \dots, v_p,\ w_1, \dots, w_q \in V$.

Comparing with the formulas given in [20], we conclude that the Lie superalgebra $\mathrm{Ext}(V)/\mathbb{C}$ is isomorphic to the simple Lie superalgebra $\widetilde{H}(n)$. Notice that $\mathrm{Ext}(V)/\mathbb{C} \simeq \mathrm{ad}_\mathrm{P}(\mathrm{Ext}(V)) \subset \mathrm{Der}(\mathrm{Ext}(V)) = \mathscr{D}(V^*)$, so we obtain the classical inclusion $\widetilde{H}(n) \subset W(n)$ ([20]).

5.2. . Let us modify slightly the formalism in 5.1 in order to apply it to Lie algebras deformation theory. We begin with a n-dimensional vector space $\mathfrak{g}$ and we set $V = \mathfrak{g}^*$. We assume that $\mathfrak{g}$ is a quadratic space with bilinear form B. Denote by $\{X_1, \dots, X_n\}$ an orthonormal basis of $\mathfrak{g}$ and by $\{\omega_1, \dots, \omega_n\}$ the dual basis; we define B on $\mathfrak{g}^*$ by $B(\omega_i, \omega_j) = \delta_{ij}$. Applying the construction in 5.1 with $e_i = \omega_i$, $i = 1, \dots, n$, we get a super Poisson bracket on $\bigwedge \mathfrak{g}$ and it is easy to check that:

Proposition 5.4. *For all $\Omega \in \bigwedge^w \mathfrak{g}$, $\Omega' \in \bigwedge \mathfrak{g}$, one has*

$$\{\Omega, \Omega'\} = 2\,(-1)^{w+1} \sum_{j=1}^{n} \iota_{X_j}(\Omega) \wedge \iota_{X_j}(\Omega').$$

This formula is valid in any orthonormal basis of $\mathfrak{g}$ and it is enough for our purpose in Section 5, but a general formula can be found in Lemma 6.9. There is a Moyal type formula which gives the Clifford product in terms of the super Poisson bracket: let $m_\wedge$ be the product from $\bigwedge \mathfrak{g} \otimes \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g}$, and define $\mathscr{F} : \bigwedge \mathfrak{g} \otimes \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g} \otimes \bigwedge \mathfrak{g}$ by

$$\mathscr{F}(\Omega \otimes \Omega') = (-1)^w \sum_{j=1}^{n} \iota_{X_j}(\Omega) \otimes \iota_{X_j}(\Omega')$$

for all $\Omega \in \bigwedge^w \mathfrak{g}$, $\Omega' \in \bigwedge \mathfrak{g}$. Then:

Proposition 5.5.

$$\Omega \star \Omega' = m_\wedge \circ \exp(-t\mathscr{F})(\Omega \otimes \Omega').$$

Proof. As in the beginning of 5.2, let $e_i = \omega_i$ and let $\partial_i = \iota_{X_i}$, $i = 1, \dots, n$. As in 5.1, for $I = (i_1, \dots, i_n) \in \mathbb{Z}_2^n$, let $e_I = e_1^{i_1} \wedge \cdots \wedge e_n^{i_n}$ and $\partial_I = \partial_1^{i_1} \circ \cdots \circ \partial_n^{i_n}$. For $J = (j_1, \dots, j_n) \in \mathbb{Z}_2^n$, let $J^I = j_1^{i_1} \dots j_n^{i_n}$. One has $\partial_I(e_J) = (-1)^{\Omega(I,J)} J^I e_{I \underset{(2)}{+} J}$. Since all $\partial_i \otimes \partial_i$ commute, and $\partial_i^2 = 0$, one has:

$$\left(\sum_i \partial_i \otimes \partial_i\right)^k = k! \sum_{|I|=k} \partial_I \otimes \partial_I$$

For $k>0$, one has:

$$\begin{aligned} & m_\wedge \circ \mathscr{F}^k(e_R\otimes e_S) \\ = \quad & (-1)^{|R|}(-1)^{|R|-1}\dots(-1)^{|R|-(k-1)} m_\wedge \circ \left(\sum_i \partial_i\otimes\partial_i\right)^k (e_R\otimes e_S) \\ = \quad & (-1)^{k|R|}(-1)^{\frac{k(k-1)}{2}} k! \sum_{|I|=k} (-1)^{\Omega(I,R)}(-1)^{\Omega(I,S)} R^I S^I \, e_{\underset{(2)}{I+R}} \wedge e_{\underset{(2)}{I+S}} \end{aligned}$$

This vanishes, except if $k=|RS|$, and in that case, the only remaining term in the sum is when $I=RS$. We compute this term:

$$m_\wedge \circ \mathscr{F}^k(e_R\otimes e_S) = (-1)^{k|R|}(-1)^{\frac{k(k-1)}{2}} k!(-1)^{\Omega(RS,R)+\Omega(RS,S)}(-1)^{\Omega(RS+R,RS+S)} e_{\underset{(2)}{R+S}}$$

But one has $\Omega(A,B)+\Omega(B,A)=|A||B|-|AB|$, so:

$$\Omega(RS,R)+\Omega(RS,S)=k|R|-k, \quad \text{and} \quad \Omega(RS,RS)=\frac{k(k-1)}{2}.$$

So finally, we have proved that

$$m_\wedge \circ \mathscr{F}^k(e_R\otimes e_S) = (-1)^k k!(-1)^{\Omega(R,S)}\delta_{|RS|,k} e_{\underset{(2)}{R+S}}$$

On the other hand, by Proposition 5.1, one has

$$e_R\star e_S = e_R\wedge e_S + (-1)^{\Omega(R,S)}\sum_{k=1}^n \delta_{|RS|,k} e_{\underset{(2)}{R+S}}$$

so the result follows. ∎

Remark 5.6. An equivalent formula is given in [15].

5.3. . A derivation $D\in\mathscr{D}$ is *Hamiltonian* if it belongs to $\mathrm{ad}_{\mathrm{P}}(\bigwedge\mathfrak{g})$. Actually, the space of Hamiltonian derivations is a subalgebra of $\mathscr{D}$, that we denote by $\mathscr{H}(\mathfrak{g})$, which is isomorphic to $\bigwedge_Q\mathfrak{g}=\bigwedge\mathfrak{g}/\mathbb{C}$ and therefore, by (5), isomorphic to the simple Lie superalgebra $\widetilde{H}(n)$. Here is a simple characterization of Hamiltonian derivations:

Proposition 5.7. *A derivation $D=\sum_r D_r\wedge\iota_{X_r}$ is Hamiltonian if and only if $\iota_{X_r}(D_s)+\iota_{X_s}(D_r)=0$, $\forall\, r,s$.*

Proof. When the condition is satisfied, one has $D=\mathrm{ad}_{\mathrm{P}}(\Omega)$ where $\Omega=\frac{1}{2w}\sum_r D_r\wedge\omega_r$ and $w=\deg(D)+2$. ∎

Remark 5.8. A Hamiltonian derivation is a derivation of the $\wedge$-product and also of the super Poisson bracket.

In fact, one has:

Proposition 5.9. *Let $D \in \mathscr{D}$. Then D is Hamiltonian if and only if D is a derivation of the super Poisson bracket.*

Proof. Let $D = \sum_r D_r \wedge \iota_{X_r}$ with $D \in \bigwedge^d \mathfrak{g}$, then $D_r = D(\omega_r)$. Since $\{\omega_r, \omega_s\} \in \mathbb{C}$, assuming that D is a derivation of the super Poisson bracket, one has:

$$0 = D(\{\omega_r, \omega_s\}) = 2\,(-1)^{d+1}\left(\iota_{X_r}(D_s) + \iota_{X_s}(D_r)\right)$$

and the result follows by Proposition 5.7. ■

5.4. . We now want to apply super Poisson brackets to the theory of quadratic Lie algebras, in a deformation framework that we will set up. Given a quadratic Lie algebra $(\mathfrak{g}_0, B_0)$ with bilinear form B_0 and product $[.,.]$, a *deformation* $(\mathfrak{g}_t, B_t)$ of $(\mathfrak{g}_0, B_0)$ is:

(1) a deformation $\mathfrak{g}_t$ of $\mathfrak{g}$ in the usual sense, so:

$$[X,Y]_t = [X,Y] + tC_1(X,Y) + \dots, \forall\, X,Y \in \mathfrak{g},$$

(2) a formal bilinear form $B_t = B_0 + tB_1 + \dots$ such that

$$B_t([X,Y]_t, Z) = -B_t(Y, [X,Z]_t), \forall\, X,Y,Z \in \mathfrak{g},$$

Two deformations $(\mathfrak{g}_t, B_t)$ and $(\mathfrak{g}'_t, B'_t)$ with respective brackets $[.,.]_t$ and $[.,.]'_t$ are *equivalent* if there exists $T_t = \mathrm{Id} + tT_1 + \dots$ such that:

$$[X,Y]'_t = T_t^{-1}([T_t(X), T_t(Y)]) \quad \text{and} \quad B'_t(X,Y) = B_t(T_t(X), T_t(Y)), \forall\, X,Y \in \mathfrak{g}.$$

Proposition 5.10. *Any deformation $(\mathfrak{g}_t, B_t)$ of $(\mathfrak{g}_0, B_0)$ is equivalent to a deformation with unchanged bilinear form.*

Proof. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$ with respect to B_0. By a Gram-Schmidt type strategy, one can construct $\{e_1(t), \dots, e_n(t)\}$ such that:

$$e_\ell(t) = \lambda_1(t)e_1(t) + \dots + \lambda_{\ell-1}(t)e_{\ell-1}(t) + e_\ell, \forall \ell \le n,$$

with $\lambda_j(t) \in t\ \mathbb{C}[[t]]$, and $B_t(e_\ell(t), e_m(t)) = 0$, for all $\ell, m \le n$. Since $[B_t(e_\ell(t), e_\ell(t))]_{t=0} = B_0(e_\ell, e_\ell) = 1$, $\forall \ell \le n$, $B_t(e_\ell(t), e_\ell(t))$ is invertible, and

$$e'_\ell(t) = \frac{1}{\left(B_t(e_\ell(t), e_\ell(t))\right)^{\frac{1}{2}}}\, e_\ell(t)$$

does satisfy $B_t(e'_\ell(t), e'_m(t)) = \delta_{\ell m}$, $\forall \ell, m$.

Now if we define T_t by $T_t(e_\ell) = e'_\ell(t)$, $\forall \ell \le n$, and a new deformation

$$[X,Y]_t = T_t([T_t(X), T_t(Y)]_t),\ \forall X,Y \in \mathfrak{g},$$

with bilinear form $B'_t(X,Y) = B_t(T_t(X), T_t(Y)) = B_0(X,Y)$, $\forall X,\ Y \in \mathfrak{g}$, we obtain a deformation that is equivalent to the initial one. ■

So if one wants to study quadratic Lie algebras in terms of deformation theory, one can restrict to quadratic Lie algebras with a specified bilinear form, and that is what we shall do next.

5.5. . The construction of the super Poisson bracket made at the beginning of this section can now be applied as follows: given a finite-dimensional quadratic Lie algebra $\mathfrak{g}$ with bilinear form B, let ∂ be the corresponding derivation of $\bigwedge \mathfrak{g}$ (i.e. the differential of the trivial cohomology complex of $\mathfrak{g}$, see 3.2), we define:

$$I(X,Y,Z) := B([X,Y],Z), \ \forall\, X,Y,Z \in \mathfrak{g}$$

Then one has:

Proposition 5.11.

(1) $I \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$.

(2) $\partial = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)$.

(3) $\{I,I\} = 0$.

Proof. The assertion (1) is obvious. To show (2), let $\{X_1,\ldots,X_n\}$ be an orthonormal basis of $\mathfrak{g}$ and $\{\omega_1,\ldots,\omega_n\}$ the dual basis. Then for all Y, $Z \in \mathfrak{g}$:

$$-\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)(\omega_i)(Y,Z) = -\left(\sum_j \iota_{X_j}(I)\wedge \iota_{X_j}(\omega_i)\right)(Y,Z) = -B([X_i,Y],Z) =$$

$$= -B(X_i,[Y,Z]) = -\omega_i([Y,Z]) = \partial\omega_i(Y,Z)$$

Hence, $\partial = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)$.

Finally $\mathrm{ad}_{\mathrm{P}}(\{I,I\}) = [\mathrm{ad}_{\mathrm{P}}(I),\mathrm{ad}_{\mathrm{P}}(I)] = 4[\partial,\partial] = 8\partial^2 = 0$. So $\{I,I\} = 0$ and that proves (3). ∎

Note that ∂, ι_X and $\theta_X = [\iota_X,\partial]$, $\forall\, X \in \mathfrak{g}$ are all Hamiltonian derivations.

5.6. . Conversely, assume that $\mathfrak{g}$ is a finite-dimensional quadratic vector space. Fix $I \in \bigwedge^3 \mathfrak{g}$ and define $\partial = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)$. Then the formula

$$\mathrm{ad}_{\mathrm{P}}\left(\{\Omega,\Omega'\}\right) = [\mathrm{ad}_{\mathrm{P}}(\Omega),\mathrm{ad}_{\mathrm{P}}(\Omega')], \forall\, \Omega,\Omega' \in \bigwedge \mathfrak{g}$$

leads to

$$[\partial,\partial] = 0 \text{ if and only if } \{I,I\} = 0. \tag{6}$$

Let $F = \mathbf{F}_\partial$ be the structure on $\mathfrak{g}$ associated to ∂ (see 3.1 and 3.2), then from (6), it follows:

Proposition 5.12. *F is a Lie algebra structure if and only if $\{I,I\} = 0$. In that case, with the notation $[X,Y] = F(X,Y)$, one has:*

$$I(X,Y,Z) = B([X,Y],Z), \ \forall\, X,Y,Z \in \mathfrak{g},$$

the form B is invariant and $\mathfrak{g}$ is a quadratic Lie algebra.

Proof. We have to prove that if F is a Lie algebra structure, then $I(X,Y,Z) = B([X,Y],Z)$, $\forall\, X,Y,Z \in \mathfrak{g}$.

Let $\{X_1,\ldots,X_n\}$ be an orthonormal basis, then $\partial = -\sum_k \iota_{X_k}(I)\wedge\iota_{X_k}$, so $F = \sum_k \iota_{X_k}(I)\otimes X_k$, and therefore $B([X_i,X_j],X_k) = \iota_{X_k}(I)(X_i,X_j) = I(X_i,X_j,X_k)$, for all i, j, k. ∎

Remark 5.13. Using 5.4, 5.5 and 5.6, it appears that $\bigwedge\mathfrak{g}[2]$ with super Poisson bracket is a **gla** associated to deformation theory of finite-dimensional quadratic Lie algebras: by 5.4, one can assume that B does not change, then quadratic Lie algebra structures with the same B are in one to one correspondence with elements $I \in \bigwedge^3\mathfrak{g}$ such that $\{I,I\} = 0$ (5.5, 5.6). An equivalent description can be given in terms of Hamiltonian derivations, i.e. of the **gla** $\mathscr{H}(\mathfrak{g}) = \mathrm{ad}_{\mathrm{P}}(\bigwedge\mathfrak{g}) \simeq \bigwedge\mathfrak{g}/\mathbb{C} = \bigwedge_Q\mathfrak{g}$.

Let us note that in this picture, one has to redefine equivalence: a priori, one might think that equivalence should be defined as Lie algebras isomorphism keeping B fixed. But this is too restrictive, since $[.,.]$ and $\lambda(t)[.,.]$, with $\lambda(t) = 1+t(\ldots)$ will not be equivalent in that sense as they should be. So one has rather to work with the notion of a conformal equivalence, i.e. an equivalence defined by a Lie algebras isomorphism $T(t) = \mathrm{Id}+t(\ldots)$ satisfying $B(T_t(X),T_t(Y)) = \mu(t)B(X,Y)$, with $\mu(t) = 1+t(\ldots)$. This will change the corresponding **gla** : one can consider the subalgebra $\mathbb{C}R\oplus\mathscr{H}(\mathfrak{g})$ of $\mathscr{D}(\mathfrak{g})$ (where $R = \sum_i \omega_i\wedge\iota_{X_i}$ is the *super radial vector field*), rather than $\mathscr{H}(\mathfrak{g})$. Hence, there are some adaptations to carry out, which will not be developed here since they are somewhat standard. Let us only indicate that in this framework, if $(\mathfrak{g}_0,B_0)$ is the initial quadratic Lie algebra with associated $I_0 \in \bigwedge^3(\mathfrak{g})$, then the obstruction to triviality of a quadratic deformation will lie in $H^3(\mathfrak{g})/\mathbb{C}\, I_0$. For instance, if $\mathfrak{g}_0$ is semisimple, it is shown in [16] that $H^3(\mathfrak{g}_0)$ and the space of symmetric invariant bilinear forms on $\mathfrak{g}_0$ are isomorphic, the isomorphism being $B\mapsto I_B$ where $I_B(X,Y,Z) = B([X,Y],Z)$, $\forall\, X$, Y, $Z\in\mathfrak{g}$. It results that when $\mathfrak{g}_0$ is simple, it is rigid in quadratic deformation theory.

6. Elementary quadratic Lie algebras

Let us recall two results:

Proposition 6.1. *Let V be a finite-dimensional vector space and I a k-form in $\bigwedge^k V$. Denote by V_I the orthogonal subspace in V^* of the subspace $\{X\in V \mid \iota_X(I) = 0\}$. Then* $\dim(V_I)\geq k$ *and if I is nonzero, I is decomposable if and only if* $\dim(V_I) = k$. *In this case, if $\{\omega_1,\ldots,\omega_k\}$ is a basis of V_I, one has $I = \alpha\ \omega_1\wedge\cdots\wedge\omega_k$, for some $\alpha\in\mathbb{C}$ ([4]).*

Proposition 6.2. *Let V be a finite-dimensional quadratic vector space with a nondegenerate symmetric bilinear form B. For a subspace W of V, denote by $W^\perp$ its orthogonal subspace in V with respect to B and $W^{\perp^*}$ its orthogonal in V^*. Let ϕ be the isomorphism from V onto V^* induced by B. Then $\phi|_{W^\perp}$ is an isomorphism from $W^\perp$ onto $W^{\perp^*}$, so* $\dim(W^\perp) = \dim(V)-\dim(W)$. *One has $V = W\oplus W^\perp$ if and only if $W\cap W^\perp = \{0\}$ and in this case the restriction of B to W or $W^\perp$ is nondegenerate.*

In the rest of this Section, $\mathfrak{g}$ will denote a finite-dimensional quadratic Lie algebra with bilinear form B. Denote by $\mathrm{Z}(\mathfrak{g})$ the center of $\mathfrak{g}$ and by $I_{\mathfrak{g}}$ the element of $\bigwedge^3\mathfrak{g}$ defined by $I_{\mathfrak{g}}(X,Y,Z) = B([X,Y],Z)$, $\forall\, X$, Y, $Z\in\mathfrak{g}$.

Definition 6.3. We say that $\mathfrak{g}$ is an *elementary* quadratic Lie algebra if $I_{\mathfrak{g}}$ is decomposable.

Remark that the obvious identity $\mathrm{Z}(\mathfrak{g})^{\perp} = [\mathfrak{g},\mathfrak{g}]$ holds here. As a consequence,

Proposition 6.4. *Let $\mathfrak{g}$ be a non Abelian quadratic Lie algebra. Then* $\dim([\mathfrak{g},\mathfrak{g}]) \geq 3$. *Moreover, $\mathfrak{g}$ is elementary if and only if the equality holds.*

Proof. Since $\mathrm{Z}(\mathfrak{g})^{\perp} = [\mathfrak{g},\mathfrak{g}]$ and $V_{I_{\mathfrak{g}}} = \mathrm{Z}(\mathfrak{g})^{\perp^*}$, the result follows directly from Proposition 6.1. ■

Corollary 6.5. *Let $\mathfrak{g}$ be an elementary quadratic Lie algebra. Then all coadjoint orbits have dimension at most 2.*

Proof. Let $\omega \in \mathfrak{g}^*$ and $X_\omega \in \mathfrak{g}$ such that $\omega = \phi(X_\omega)$. Then $\mathrm{ad}(\mathfrak{g})(\omega) = \phi([\mathfrak{g}, X_\omega]) \subset \phi([\mathfrak{g},\mathfrak{g}])$, so $\dim(\mathrm{ad}(\mathfrak{g})(\omega)) \leq 3$. Since all coadjoint orbits have even dimension, the result follows. ■

Remark 6.6. Suppose that $\mathfrak{g}$ is a finite-dimensional quadratic vector space and let I be a decomposable 3-form in $\bigwedge^3 \mathfrak{g}$. Then it is easy to check that $\{I, I\} = 0$ for the super Poisson bracket. So by Proposition 5.12 there is an elementary quadratic Lie algebra structure on $\mathfrak{g}$ such that $I(X,Y,Z) = B([X,Y],Z)$, $\forall\, X,\, Y,\, Z \in \mathfrak{g}$.

In the sequel, we classify all non Abelian elementary quadratic Lie algebras. This will be done in two steps: first, in 6.1, we show a result on quadratic Lie algebras that reduces the classification problem to small dimensions, namely between 3 and 6. Then in 6.2, we proceed by classifying these small dimensional elementary quadratic algebras. Explicit commutators in a canonical basis with respect to B are computed as well.

6.1. . Here is the reduction result on quadratic Lie algebras:

Proposition 6.7. *Let $\mathfrak{g}$ be a non Abelian quadratic Lie algebra with bilinear form B. Then there exist a central ideal $\mathfrak{z}$ and an ideal $\mathfrak{l} \neq \{0\}$ such that:*

(1) $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{l}$, and $\mathfrak{l}$ and $\mathfrak{z}$ are orthogonal with respect to B.

(2) $\mathfrak{z}$ and $\mathfrak{l}$ are quadratic (with bilinear forms induced by the restriction of B) and $\mathfrak{l}$ is non Abelian. Moreover, $\mathfrak{l}$ is elementary if and only if $\mathfrak{g}$ is elementary.

(3) the center $\mathrm{Z}(\mathfrak{l})$ is totally isotropic and

$$\dim(\mathrm{Z}(\mathfrak{l})) \leq \frac{1}{2}\dim(\mathfrak{l}) \leq \dim([\mathfrak{l},\mathfrak{l}]).$$

Proof. Let $\mathfrak{z}_0 = \mathrm{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$. Fix any subspace $\mathfrak{z}$ such that $\mathrm{Z}(\mathfrak{g}) = \mathfrak{z}_0 \oplus \mathfrak{z}$. Since $\mathrm{Z}(\mathfrak{g})^{\perp} = [\mathfrak{g},\mathfrak{g}]$, one has $B(\mathfrak{z}_0,\mathfrak{z}) = \{0\}$ and $\mathfrak{z} \cap \mathfrak{z}^{\perp} = \{0\}$. It results from Proposition 6.2 that $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{l}$ where $\mathfrak{l} = \mathfrak{z}^{\perp}$.

Since $B([\mathfrak{g},\mathfrak{g}],\mathfrak{z}) = \{0\}$, one has $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{l}$, so $\mathfrak{l} \neq \{0\}$. It is easy to check that $\mathrm{Z}(\mathfrak{l}) = \mathfrak{z}_0$, so $\mathrm{Z}(\mathfrak{l})$ is totally isotropic; moreover the restriction of B to $\mathfrak{z}$ and $\mathfrak{l}$

is nondegenerate, so $\mathfrak{l}$ is quadratic and clearly non Abelian since $\mathfrak{z}$ is central in $\mathfrak{g}$. If $\mathfrak{g}$ is elementary, $\mathfrak{l}$ is a fortiori elementary. If $\mathfrak{l}$ is elementary, let $I_{\mathfrak{l}} = \omega_1 \wedge \omega_2 \wedge \omega_3$, ω_1, ω_2, $\omega_3 \in \mathfrak{l}^*$, extend the ω_i to $\mathfrak{g}$ by $\omega_i|_{\mathfrak{z}} = 0$. Since $I_{\mathfrak{g}}(X,Y,Z) = 0$, $\forall X \in \mathfrak{z}$, Y, $Z \in \mathfrak{g}$, one concludes $I_{\mathfrak{g}} = \omega_1 \wedge \omega_2 \wedge \omega_3$, hence $\mathfrak{g}$ is elementary. Finally $\mathrm{Z}(\mathfrak{l}) \subset [\mathfrak{l},\mathfrak{l}] = \mathrm{Z}(\mathfrak{l})^\perp$ implies $\dim(\mathfrak{l}) - \dim([\mathfrak{l},\mathfrak{l}]) \leq \dim([\mathfrak{l},\mathfrak{l}])$ and the last inequality follows. ∎

Corollary 6.8. *Let $\mathfrak{l}$ be a nonzero elementary quadratic Lie algebra such that $\mathrm{Z}(\mathfrak{l})$ is totally isotropic. Then one has*

$$3 \leq \dim(\mathfrak{l}) \leq 6.$$

Proof. Use Propositions 6.7(3) and 6.4. ∎

6.2. . We shall now finish the classification of non Abelian elementary quadratic Lie algebras. This classification is reduced, by Proposition 6.7 and Corollary 6.8, to the case of nonzero elementary quadratic $\mathfrak{l}$ with a totally isotropic center $\mathrm{Z}(\mathfrak{l})$. Applying Proposition 6.8 one has $3 \leq \dim(\mathfrak{l}) \leq 6$. Note that if $\dim(\mathfrak{l}) = 3$, one has $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}]$ (Proposition 6.4), so $\mathfrak{l} \simeq \mathfrak{sl}(2)$ and B is the Killing form up to a scalar. So we have to consider $\dim(\mathfrak{l}) \geq 4$ (therefore $\dim(\mathrm{Z}(\mathfrak{l})) \geq 1$).

We need the following Lemma:

Lemma 6.9. *Let V be a quadratic vector space with bilinear form B. Define B on V^* by $B(\omega,\omega') := B(\phi^{-1}(\omega),\phi^{-1}(\omega'))$, $\forall\ \omega,\ \omega' \in V^*$ (ϕ as in Proposition 6.2). Let $\{\omega_1,\dots,\omega_n\}$ be a basis of V^*, $\{X_1,\dots,X_n\}$ its dual basis and $\{Y_1,\dots,Y_n\}$ the basis of $\mathfrak{g}$ defined by $Y_i = \phi^{-1}(\omega_i)$. Then the super Poisson bracket on $\bigwedge\mathfrak{g}$ is given by*

$$\{\Omega,\Omega'\} = 2\,(-1)^{w+1}\sum_{i,j} B(Y_i,Y_j)\iota_{X_i}(\Omega)\wedge\iota_{X_j}(\Omega'),\ \Omega \in \bigwedge{}^w\mathfrak{g}, \Omega' \in \bigwedge\mathfrak{g}.$$

Proof. Using Proposition 5.4, one has

$$\{\Omega,\Omega'\} = 2\,(-1)^{w+1}\sum_{i,j}\alpha_{ij}\iota_{X_i}(\Omega)\wedge\iota_{X_j}(\Omega'),$$

$\Omega \in \bigwedge^w\mathfrak{g}$, $\Omega' \in \bigwedge\mathfrak{g}$ and $\alpha_{ij} = \frac{1}{2}\{\omega_i,\omega_j\}$. But from 5.1, one has $\{\omega_i,\omega_j\} = 2B(\omega_i,\omega_j) = 2B(Y_i,Y_j)$. ∎

Proposition 6.10. *Let $\mathfrak{l}$ be an elementary quadratic Lie algebra with nonzero totally isotropic center $\mathrm{Z}(\mathfrak{l})$. Then:*

(1) If $\dim(\mathfrak{l}) = 6$, there exists a basis $\{Z_1,Z_2,Z_3,X_1,X_2,X_3\}$ of $\mathfrak{l}$ such that:

- *(i) $\{Z_1,Z_2,Z_3\}$ is a basis of $\mathrm{Z}(\mathfrak{l})$.*
- *(ii) $B(Z_i,Z_j) = B(X_i,X_j) = 0$, $B(Z_i,X_j) = \delta_{ij}$, $\forall\ i,j$.*
- *(iii) $[X_1,X_2] = Z_3$, $[X_2,X_3] = Z_1$, $[X_3,X_1] = Z_2$ and the other brackets vanish.*

(2) If $\dim(\mathfrak{l}) = 5$, there exists a basis $\{Z_1,Z_2,X_1,X_2,T\}$ of $\mathfrak{l}$ such that:

(i) $\{Z_1, Z_2\}$ *is a basis of* $\mathrm{Z}(\mathfrak{l})$.

(ii) $B(Z_i, Z_j) = B(X_i, X_j) = 0$, $B(Z_i, X_j) = \delta_{ij}$, $\forall\, i, j$, $B(T, Z_i) = B(T, X_i) = 0$, $B(T, T) = 1$.

(iii) $[X_1, T] = -Z_2$, $[X_2, T] = Z_1$, $[X_1, X_2] = T$ *and the other brackets vanish.*

(3) *If* $\dim(\mathfrak{l}) = 4$, *then* $\dim(\mathrm{Z}(\mathfrak{l})) = 1$ *and there exist totally isotropic subspaces* $\mathfrak{i}$ *with basis* $\{Z, P\}$ *and* $\mathfrak{i}'$ *with basis* $\{X, Q\}$ *such that* $\mathrm{Z}(\mathfrak{l}) \subset \mathfrak{i} \subset [\mathfrak{l}, \mathfrak{l}]$, $\mathfrak{l} = \mathfrak{i} \oplus \mathfrak{i}'$ *and :*

(i) $\mathrm{Z}(\mathfrak{l}) = \mathbb{C}\, Z$, $B(Z, X) = B(P, Q) = 1$, $B(Z, Q) = B(X, P) = 0$.

(ii) $[X, P] = P$, $[X, Q] = -Q$, $[P, Q] = Z$ *and the other brackets vanish.*

Proof.

(1) Assuming that $\dim(\mathfrak{l}) = 6$, one has $\dim(\mathrm{Z}(\mathfrak{l})) = 3$, so $\mathrm{Z}(\mathfrak{l}) = [\mathfrak{l}, \mathfrak{l}] = \mathrm{Z}(\mathfrak{l})^\perp$. Using [4], there is a totally isotropic subspace $\mathfrak{l}'$ such that $\mathfrak{l} = \mathrm{Z}(\mathfrak{l}) \oplus \mathfrak{l}'$. With the notation of Proposition 6.2, since $\phi|_{\mathfrak{l}'}$ is an isomorphism from $\mathfrak{l}'$ onto $\mathrm{Z}(\mathfrak{l})^*$, we can find a basis $\{Z_1, Z_2, Z_3\}$ of $\mathrm{Z}(\mathfrak{l})$ and a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{l}'$ such that $B(Z_i, X_j) = \delta_{ij}$. Then

$$\mathrm{Z}(\mathfrak{l})^{\perp^*} = \mathrm{span}\{X_1^*, X_2^*, X_3^*\} = \mathrm{span}\{\phi(Z_1), \phi(Z_2), \phi(Z_3)\}.$$

Let $I_\mathfrak{l} = B([X, Y], Z)$, $\forall\, X, Y, Z \in \mathfrak{l}$. Since $V_{I_\mathfrak{l}} = \mathrm{Z}(\mathfrak{l})^{\perp^*}$, it results from Proposition 6.1 that $I_\mathfrak{l} = \alpha\, X_1^* \wedge X_2^* \wedge X_3^*$, $\alpha \in \mathbb{C}$. Replacing X_1 by $\frac{1}{\alpha} X_1$ and Z_1 by αZ_1, we can assume that $\alpha = 1$. Using Proposition 5.11 and Lemma 6.9, $\partial = -\frac{1}{2}\mathrm{ad}_\mathrm{P}(I) = -\sum_{i=1}^3 \iota_{X_i}(X_1^* \wedge X_2^* \wedge X_3^*) \wedge \iota_{Z_i}$, so by 3.2 and 3.1, $[X, Y] = \sum_{i=1}^3 \iota_{X_i}(X_1^* \wedge X_2^* \wedge X_3^*)(X, Y)\, Z_i$, $\forall\, X,\, Y \in \mathfrak{l}$ and the commutation rules follow.

(2) Assuming $\dim(\mathfrak{l}) = 5$, one has $\dim(\mathrm{Z}(\mathfrak{l})) = 2$. Using [4], there is a totally isotropic subspace $\mathfrak{l}'$ and a one-dimensional subspace $\mathfrak{l}''$ such that $\mathfrak{l} = \mathrm{Z}(\mathfrak{l}) \oplus \mathfrak{l}' \oplus \mathfrak{l}''$ and $B(\mathrm{Z}(\mathfrak{l}) \oplus \mathfrak{l}', \mathfrak{l}'') = \{0\}$.Then one can find a basis $\{Z_1, Z_2\}$ of $\mathrm{Z}(\mathfrak{l})$, a basis $\{X_1, X_2\}$ of $\mathfrak{l}$ and a basis $\{T\}$ of $\mathfrak{l}''$ such that: $B(Z_i, X_j) = \delta_{ij}$, $\forall\, i,\ j$ and $B(T, T) = 1$. Therefore

$$\mathrm{Z}(\mathfrak{l})^{\perp^*} = \mathrm{span}\{X_1^*, X_2^*, T^*\} = \mathrm{span}\{\phi(Z_1), \phi(Z_2), \phi(T)\}.$$

So $I_\mathfrak{l} = \alpha\, X_1^* \wedge X_2^* \wedge T^*$, $\alpha \in \mathbb{C}$. Replacing X_1 by $\frac{1}{\alpha} X_1$ and Z_1 by αZ_1, we can assume that $\alpha = 1$. By Proposition 5.11 and Lemma 6.9, one obtains $\partial = -\frac{1}{2}\mathrm{ad}_\mathrm{P}(I) = -\sum_{i=1}^2 \iota_{X_i}(X_1^* \wedge X_2^* \wedge T^*) \wedge \iota_{Z_i}\ - \iota_T(X_1^* \wedge X_2^* \wedge T^*) \wedge \iota_T$, so by 3.2 and 3.1, $[X, Y] = \sum_{i=1}^2 \iota_{X_i}(X_1^* \wedge X_2^* \wedge T^*)(X, Y)\, Z_i + \iota_T(X_1^* \wedge X_2^* \wedge T^*)\, T$, $\forall\, X,\, Y \in \mathfrak{l}$ and the commutation rules follow.

(3) Assuming $\dim(\mathfrak{l}) = 4$, one has $\dim(\mathrm{Z}(\mathfrak{l})) = 1$. Using [4], there is a totally isotropic 2-dimensional subspace $\mathfrak{i}$ such that $\mathrm{Z}(\mathfrak{l}) \subset \mathfrak{i}$. Since $\mathrm{Z}(\mathfrak{l})^\perp = [\mathfrak{l}, \mathfrak{l}]$, one has $\mathfrak{i} \subset [\mathfrak{l}, \mathfrak{l}]$. Using [4] once more, there exists a totally isotropic $\mathfrak{i}'$ such that $\mathfrak{l} = \mathfrak{i} \oplus \mathfrak{i}'$. Let us write $\mathfrak{i} = \mathrm{span}\{Z, P\}$, $\mathfrak{i}' = \mathrm{span}\{X, Q\}$ with $\mathrm{Z}(\mathfrak{l}) = \mathbb{C}\, Z$ and $B(Z, X) = B(P, Q) = 1$, $B(Z, Q) = B(X, P) = 0$. Therefore

$$\mathrm{Z}(\mathfrak{l})^{\perp^*} = \mathrm{span}\{P^*, Q^*, X^*\} = \mathrm{span}\{\phi(Q), \phi(P), \phi(Z)\}.$$

So $I_{\mathfrak{l}} = \alpha\ P^* \wedge Q^* \wedge X^*,\ \alpha \in \mathbb{C}$. Replacing P by $\frac{1}{\alpha}P$ and Q by αQ, we can assume that $\alpha = 1$. Using Proposition 5.11, Lemma 6.9, 3.2 and 3.1 as above, one finds $[A,B] = [\iota_P(P^* \wedge Q^* \wedge X^*)\ Q + \iota_Q(P^* \wedge Q^* \wedge X^*)\ P + \iota_X(P^* \wedge Q^* \wedge X^*)\ Z](A,B),\ \forall\, A,\ B \in \mathfrak{l}$ and the commutation rules follow.

As a final remark, the brackets in (1), (2) and (3) do satisfy Jacobi identity thanks to Remark 6.6. ■

Remark 6.11. In the Proposition above, cases (1) and (2) are nilpotent Lie algebras and case (3) is a solvable, non-nilpotent Lie algebra, with derived algebra the Heisenberg algebra.

7. Cyclic cochains and cohomology of quadratic Lie algebras

7.1. . First we fix some notation: $\mathfrak{g}$ will be a n-dimensional quadratic vector space with bilinear form B and $\bigwedge_+ \mathfrak{g}$ will denote the associative algebra without unit $\sum_{k\geq 1} \bigwedge^k \mathfrak{g}$. If $\mathfrak{g}$ is a quadratic Lie algebra, we denote by F_0 its bracket (i.e. $F_0(X,Y) = [X,Y]$, X, $Y \in \mathfrak{g}$), by $\partial = \mathbf{D}_{F_0}$ (see 3.2) the differential of $\bigwedge \mathfrak{g}$, by $H^*(\mathfrak{g})$ the corresponding cohomology, and by $H^*_+(\mathfrak{g})$ the restricted cohomology, i.e. $H^*_+(\mathfrak{g}) = \sum_{k\geq 1} H^k(\mathfrak{g})$ which is an algebra without unit (for the induced wedge product).

When $\mathfrak{g}$ is a n-dimensional quadratic vector space, $\bigwedge \mathfrak{g}$ is a **gla** for the super Poisson bracket with grading $\bigwedge \mathfrak{g}[2]$. Denote by $\bigwedge_Q \mathfrak{g}$ the quotient **gla** $\bigwedge_Q \mathfrak{g} = \bigwedge \mathfrak{g}/\mathbb{C}$, and by $[.,.]_Q$ its bracket. The map $\mathrm{ad}_{\mathrm{P}}\colon \bigwedge \mathfrak{g} \to \mathscr{D}(\mathfrak{g})$ is a **gla** homomorphism, and we define the **gla** $\mathscr{H}(\mathfrak{g})$ of Hamiltonian derivations to be the image $\mathscr{H}(\mathfrak{g}) = \mathrm{ad}_{\mathrm{P}}(\bigwedge \mathfrak{g})$, as in 5.3. There is an obvious **gla** isomorphism from $\bigwedge_Q \mathfrak{g}$ onto $\mathscr{H}(\mathfrak{g})$, and since $\bigwedge_Q \mathfrak{g} \simeq \widetilde{H}(n)$ (see 5.3), the **gla** $\bigwedge_Q \mathfrak{g}$, $\mathscr{H}(\mathfrak{g})$ and $\widetilde{H}(n)$ are isomorphic. Moreover, if $\mathfrak{g}$ is a quadratic Lie algebra, since ∂ is Hamiltonian (see Proposition 5.11), the super Poisson bracket induces a **gla** structure on $H^*(\mathfrak{g})$ and also on $H^*_Q(\mathfrak{g}) = H^*(\mathfrak{g})/\mathbb{C}$.

Given $C \in \mathscr{M}^k_a(\mathfrak{g})$ (see 2.2.), we define $\widehat{C}$ by:

$$\text{if } k = 0,\ C \in \mathfrak{g},\ \widehat{C}(Y) := B(C,Y), \forall\, Y \in \mathfrak{g},$$
$$\text{if } k > 0,\ \widehat{C}(Y_1,\ldots,Y_{k+1}) := B(C(Y_1,\ldots,Y_k),Y_{k+1}),\ \forall\, Y_1,\ldots,Y_{k+1} \in \mathfrak{g}.$$

Definition 7.1. C is a *cyclic cochain* if

$$\widehat{C}(Y_1,\ldots,Y_{k+1}) = (-1)^k \widehat{C}(Y_{k+1},Y_1,\ldots,Y_k), \forall\, Y_1,\ldots,Y_{k+1} \in \mathfrak{g}.$$

We denote by $\mathscr{C}_c(\mathfrak{g})$ the space of cyclic cochains.

Proposition 7.2.

(1) *C is a cyclic cochain if and only if $\widehat{C} \in \bigwedge_+ \mathfrak{g}$. The map Θ from $\mathscr{C}_c(\mathfrak{g})$ into $\bigwedge_+ \mathfrak{g}$ defined by $\Theta(C) = \widehat{C}$, is one to one.*

(2) *When $\mathfrak{g}$ is finite-dimensional, the map $\Theta\colon \mathscr{C}_c(\mathfrak{g}) \to \bigwedge_+ \mathfrak{g}$ is an isomorphism.*

(3) *$\mathscr{C}_c(\mathfrak{g})$ is a subalgebra of the* **gla** *$\mathscr{M}_a(\mathfrak{g})$.*

Proof.

(1) Let τ be the cycle $\tau = (1\ 2\ \dots\ k+1) \in \mathfrak{S}_{k+1}$. Given $\sigma \in \mathfrak{S}_{k+1}$, let $\ell = \sigma^{-1}(k+1)$, then $\sigma' = \sigma \circ \tau^{\ell} \in \mathfrak{S}_k$. If C is cyclic, one has $\tau^{-1}.\widehat{C} = \varepsilon(\tau)\ \widehat{C}$. So $\sigma.\widehat{C} = (\sigma' \circ \tau^{-\ell}).\widehat{C} = \varepsilon(\sigma)\widehat{C}$ and therefore $\widehat{C} \in \bigwedge_+ \mathfrak{g}$. Since B is nondegenerate, Θ is clearly one to one.

(2) Given $\Omega \in \bigwedge^{k+1} \mathfrak{g}$, define $D \in \mathscr{M}_a^k(\mathfrak{g})$ by

$$\Omega(Y_1,\dots,Y_k,Y) = B(D(Y_1,\dots,Y_k),Y), \forall Y_1,\dots,Y_k,Y \in \mathfrak{g}.$$

Then $\Omega = \widehat{D}$.

(3) Let $F \in \mathscr{M}_a^p(\mathfrak{g})$, and $G \in \mathscr{M}_a^q(\mathfrak{g})$, from (1) we have to prove that:

$$B([F,G]_a(Y_1,\dots,Y_{p+q-1}),Y_{p+q}) = B([F,G]_a(Y_1,\dots,Y_{p+q-2},Y_{p+q}),Y_{p+q-1}),$$

for all $Y_1,\dots,Y_{p+q} \in \mathfrak{g}$. Using the formulas in 2.1, we can write the left hand side as a sum of four terms, $B([F,G]_a(Y_1,\dots,Y_{p+q-1}),Y_{p+q}) = \alpha+\beta+\gamma+\delta$ where:

$$\alpha = (-1)^{(p-1)(q-1)} \sum_{\substack{\sigma\in\mathfrak{S}_{q,p-1}\\ \sigma(p+q-1)=p+q}} (\dots) \quad \text{and} \quad \beta = (-1)^{(p-1)(q-1)} \sum_{\substack{\sigma\in\mathfrak{S}_{q,p-1}\\ \sigma(q)=p+q-1}} (\dots)$$

$$\gamma = -\sum_{\substack{\sigma\in\mathfrak{S}_{p,q-1}\\ \sigma(p+q-1)=p+q-1}} (\dots) \quad \text{and} \quad \delta = -\sum_{\substack{\sigma\in\mathfrak{S}_{p,q-1}\\ \sigma(p)=p+q-1}} (\dots)$$

In α, we can commute, up to a sign, Y_{p+q-1} and Y_{p+q}. In δ, we commute, up to a sign, $F(Y_{\sigma(1)},\dots,Y_{\sigma(p-1)},Y_{p+q-1})$ and Y_{p+q} to obtain:

$$\delta = \sum_{\substack{\sigma\in\mathfrak{S}_{p,q-1}\\ \sigma(p)=p+q-1}} \varepsilon(\sigma) \; B(G(Y_{p+q},Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q-1)}),F(Y_{\sigma(1)},\dots,Y_{\sigma(p-1)},Y_{p+q-1}))$$

Now commute, up to a sign, $G(Y_{p+q},Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q-1)})$ and Y_{p+q-1} to obtain:

$$\begin{aligned}\delta &= -\sum_{\substack{\sigma\in\mathfrak{S}_{p,q-1}\\ \sigma(p)=p+q-1}} \varepsilon(\sigma) \; B(F(Y_{\sigma(1)},\dots,Y_{\sigma(p-1)},G(Y_{p+q},Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q-1)})),Y_{p+q-1})\\ &= -(-1)^{p-1}(-1)^{q-1} \sum_{\substack{\sigma\in\mathfrak{S}_{p,q-1}\\ \sigma(p)=p+q-1}} \varepsilon(\sigma) \; B(F(G(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q-1)},Y_{p+q}),Y_{\sigma(1)},\dots,Y_{\sigma(p-1)}),Y_{p+q-1})\end{aligned}$$

Let $Z_i = Y_i$, $i = 1, \dots, p+q-2$ and $Z_{p+q-1} = Y_{p+q}$, then:

$$\begin{aligned} F(G(Y_{\sigma(p+1)}, & \dots ,Y_{\sigma(p+q-1)}, Y_{p+q}), Y_{\sigma(1)}, \dots, Y_{\sigma(p-1)}) \\ &= F(G(Z_{\sigma(p+1)}, \dots, Z_{\sigma(p+q-1)}, Z_{p+q-1}), Z_{\sigma(1)}, \dots, Z_{\sigma(p-1)}) \\ &= F(G(Z_{\tau(1)}, \dots, Z_{\tau(q)}), Z_{\tau(q+1)}, \dots, Z_{\tau(p+q-1)}) \end{aligned}$$

where $\tau(1) = \sigma(p+1)$, $\dots$, $\tau(q-1) = \sigma(p+q-1)$, $\tau(q) = p+q-1 = \sigma(p)$, $\tau(q+1) = \sigma(1)$, $\dots$, $\tau(q+p-1) = \sigma(p-1)$. Comparing the inversions of τ with the inversions of σ, it is easy to check that

$$\varepsilon(\tau) = (-1)^{p-1}(-1)^{q-1}(-1)^{(p-1)(q-1)}\varepsilon(\sigma).$$

Finally

$$\begin{aligned} \delta &= -(-1)^{(p-1)(q-1)} \sum_{\substack{\tau \in \mathfrak{S}_{q,p-1} \\ \tau(q) = p+q-1}} \varepsilon(\tau) \\ & \qquad B(F(G(Z_{\tau(1)}, \dots, Z_{\tau(q)}), Z_{\tau(q+1)}, \dots, Z_{\tau(p+q-1)}), Y_{p+q-1}) \end{aligned}$$

Then

$$\begin{aligned} \alpha + \delta &= -(-1)^{(p-1)(q-1)} \sum_{\tau \in \mathfrak{S}_{q,p-1}} \varepsilon(\tau) \\ & \qquad B(F(G(Z_{\tau(1)}, \dots, Z_{\tau(q)}), Z_{\tau(q+1)}, \dots, Z_{\tau(p+q-1)}), Y_{p+q-1}) \end{aligned}$$

Using similar arguments to compute $\beta + \gamma$, one obtains the required identity.

∎

Remark 7.3. When $\mathfrak{g}$ is finite-dimensional, there is a direct proof of (7.2)(3) (avoiding computations) that we shall give in the proof of Proposition 7.9, in Remark 7.10.

We assume now that $\mathfrak{g}$ is a quadratic Lie algebra.

Proposition 7.4. *$(\mathscr{C}_c(\mathfrak{g}), d)$ is a subcomplex of the adjoint cohomology complex $(\mathscr{M}_a(\mathfrak{g}), d)$ of $\mathfrak{g}$.*

Proof. It is enough to check that $d(\mathscr{C}_c(\mathfrak{g})) \subset \mathscr{C}_c(\mathfrak{g})$, but this is obvious from Proposition 7.2(3) because $d = \mathrm{ad}(F_0)$ and $F_0 \in \mathscr{C}_c(\mathfrak{g})$ since $\mathfrak{g}$ is quadratic. ∎

Definition 7.5. The cohomology of the complex $(\mathscr{C}_c(\mathfrak{g}), d)$ is called the *cyclic cohomology* of $\mathfrak{g}$, and denoted by $H_c^*(\mathfrak{g})$.

Remark 7.6. Since $d = \mathrm{ad}(F_0)$, the Gerstenhaber bracket induces a **gla** structure on $H_c^*(\mathfrak{g})$.

Proposition 7.7. *The map* $\Theta\colon \mathscr{C}_c(\mathfrak{g}) \to \bigwedge_+ \mathfrak{g}$ *is a homomorphism of complexes. Moreover,* Θ *induces a map* $\Theta^*\colon H^*_c(\mathfrak{g}) \to H^*_+(\mathfrak{g})$, *which is an isomorphism when* $\mathfrak{g}$ *is finite-dimensional.*

Proof. By an easy computation, one has $\Theta \circ d = \partial \circ \Theta$ and the two first claims follow. For the third claim, use Proposition 7.2. ∎

Example 7.8. Assume that $\mathfrak{g}$ is the Lie algebra associated to an associative algebra with a trace such that the bilinear form $\mathrm{Tr}\,(XY) := XY$, $\forall X,\ Y \in \mathfrak{g}$ is nondegenerate (e.g. $\mathfrak{g}$ is the Lie algebra of finite-rank operators on a given vector space, see Examples 4.12 and 4.13). Consider the standard polynomials $\mathscr{A}_k$, for $k \geq 0$ if $\mathfrak{g}$ had a unit, or for $k > 0$, if $\mathfrak{g}$ has no unit. Since $[\mathscr{A}_2, \mathscr{A}_{2k}]_a = 0$ by Proposition 4.2, each $\mathscr{A}_{2k}$ is a cocycle, then by Proposition 4.11, it is a cyclic cocycle, and one has $\Theta(\mathscr{A}_{2k}) = \frac{1}{2k+1}\,\mathrm{Tr}\,(\mathscr{A}_{2k+1})$.

7.2. . We assume now that $\mathfrak{g}$ is a n-dimensional quadratic vector space. Using the super Poisson bracket, we shall now go further into the structure of $\mathscr{C}_c(\mathfrak{g})$. We need to renormalize the map Θ, defining $\Phi := -\frac{1}{2}\Theta$. We denote by μ the canonical map from $\bigwedge \mathfrak{g}$ onto $\bigwedge_Q \mathfrak{g}$, and by Ψ the map $\Psi = \mu \circ \Phi$ from $\mathscr{C}_c(\mathfrak{g})$ into $\bigwedge_Q \mathfrak{g}$.

Proposition 7.9.

(1) If $C \in \mathscr{C}_c(\mathfrak{g})$, *one has* $\mathbf{D}(C) = \mathrm{ad}_{\mathtt{P}}(\Phi(C))$.

(2) The restriction map $\mathtt{H} = \mathbf{D}|_{\mathscr{C}_c(\mathfrak{g})}$ *is a* **gla** *isomorphism from* $\mathscr{C}_c(\mathfrak{g})[1]$ *onto* $\mathscr{H}(\mathfrak{g})$.

(3) Ψ *is a* **gla** *isomorphism from* $\mathscr{C}_c(\mathfrak{g})[1]$ *onto* $\bigwedge_Q \mathfrak{g}[2]$.

Proof. Fix an orthonormal basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ and $\{\omega_1, \dots, \omega_n\}$ the dual basis. Given $C \in \mathscr{C}_c(\mathfrak{g})$, $Y_1, \dots, Y_p \in \mathfrak{g}$,

$$\begin{aligned}
\mathrm{ad}_{\mathtt{P}}(\Phi(C))(\omega_k)(Y_1, \quad \dots \quad ,Y_p)&\\
&= 2(-1)^p \left(\sum_{r=1}^{n} \iota_{X_r}(\Phi(C)) \wedge \iota_{X_r}(\omega_r)\right)(Y_1, \dots, Y_p)\\
&= (-1)^{p+1} B(C(X_k, Y_1, \dots, Y_{p-1}), Y_p)\\
&= B(C(Y_1, \dots, Y_{p-1}, X_k), Y_p)\\
&= -B(C(Y_1, \dots, Y_{p-1}, Y_p), X_k)\\
&= -\omega_k(C(Y_1, \dots, Y_p)) = -\mathbf{D}(C)(\omega_k)(Y_1, \dots, Y_p)
\end{aligned}$$

by a formula given in 3.3, and this proves (1). From (1), we deduce that $\mathbf{D}$ maps $\mathscr{C}_c(\mathfrak{g})$ into $\mathscr{H}(\mathfrak{g})$.

To prove (2), we remark that $\mathrm{ad}_{\mathtt{P}} \circ \Phi$ is onto by Proposition 7.2 (2), so $\mathtt{H}$ is onto, one to one and a **gla** homomorphism by Proposition 3.1 and this proves (2).

To prove (3), we use the **gla** isomorphism $\nu\colon \bigwedge_Q \mathfrak{g} \to \mathscr{H}(\mathfrak{g})$ defined from $\mathrm{ad}_{\mathtt{P}}\colon \bigwedge \mathfrak{g} \to \mathscr{H}(\mathfrak{g})$, so one has $\nu(\mu(\Omega)) = \mathrm{ad}_{\mathtt{P}}(\Omega)$, $\Omega \in \bigwedge \mathfrak{g}$, and then $\nu(\Psi(C)) = \mathrm{ad}_{\mathtt{P}}(\Phi(C)) = \mathtt{H}(C)$, $\forall\, C \in \mathscr{C}_c(\mathfrak{g})$, so $\Psi = \nu^{-1} \circ \mathtt{H}$. ∎

Remark 7.10. Let us give a direct proof of 7.2 (2): given C, $C' \in \mathscr{C}_c(\mathfrak{g})$, from the preceding results, we can assume that $C = \mathbf{F}(\mathrm{ad}_{\mathrm{P}}(\Omega))$, $C' = \mathbf{F}(\mathrm{ad}_{\mathrm{P}}(\Omega'))$, with Ω, $\Omega' \in \bigwedge_+ \mathfrak{g}$. Then:

$$[C,C']_a = [\mathbf{F}(\mathrm{ad}_{\mathrm{P}}(\Omega)),\mathbf{F}(\mathrm{ad}_{\mathrm{P}}(\Omega'))]_a = \mathbf{F}\left([\mathrm{ad}_{\mathrm{P}}(\Omega),\mathrm{ad}_{\mathrm{P}}(\Omega')]\right) = \mathbf{F}\left(\mathrm{ad}_{\mathrm{P}}(\{\Omega,\Omega'\})\right)$$

Corollary 7.11. *The* **gla** $\mathscr{C}_c(\mathfrak{g})$ *is isomorphic to* $\mathscr{H}(\mathfrak{g})$, *and to* $\widetilde{H}(n)$.

Using Φ, we can pull back the $\wedge$-product of $\bigwedge_+ \mathfrak{g}$ on $\mathscr{C}_c(\mathfrak{g})$ defining:

Definition 7.12.

$$C \wedge C' := \Phi^{-1}\left(\Phi(C) \wedge \Phi(C')\right), \forall\, C, C' \in \mathscr{C}_c(\mathfrak{g}).$$

Hence $\mathscr{C}_c(\mathfrak{g})$ becomes an associative algebra (without unit), graded by $\mathscr{C}_c(\mathfrak{g})[-1]$. To describe the $\wedge$-product of $\mathscr{C}_c(\mathfrak{g})$, we define a natural $\bigwedge \mathfrak{g}$-module structure on $\mathscr{M}_a(\mathfrak{g})$ by:

$$\Omega \cdot (\alpha \otimes X) := (\Omega \wedge \alpha) \otimes X, \forall\, \Omega, \alpha \in \bigwedge \mathfrak{g},\ X \in \mathfrak{g}$$

Proposition 7.13. *If* $C \in \mathscr{C}_c^k(\mathfrak{g})$, $C' \in \mathscr{C}_c^{k'}(\mathfrak{g})$, *then* $C \wedge C' \in \mathscr{C}_c^{k+k'+1}(\mathfrak{g})$, *and one has:*

$$C \wedge C' = \Phi(C) \cdot C' + (-1)^{(k+1)(k'+1)} \Phi(C') \cdot C.$$

Proof. Let $C'' = \Phi(C) \cdot C' + (-1)^{(k+1)(k'+1)} \Phi(C') \cdot C$. Then

$$\Phi(C'')(Y_1, \ldots, Y_{k+k'+2}) = \sum_{\sigma \in \mathfrak{S}_{k+1,k'}} \varepsilon(\sigma)\Phi(C)(Y_{\sigma(1)}, \ldots, Y_{\sigma(k+1)})\Phi(C')((Y_{\sigma(k+2)}, \ldots, Y_{\sigma(k+k'+1)}, Y_{k+k'+2}) + (-1)^{(k+1)(k'+1)} \sum_{\sigma \in \mathfrak{S}_{k'+1,k}} \varepsilon(\sigma)\Phi(C')(Y_{\sigma(1)}, \ldots, Y_{\sigma(k'+1)})\Phi(C)((Y_{\sigma(k'+2)}, \ldots, Y_{\sigma(k+k'+1)}, Y_{k+k'+2}).$$

In the first term of the right hand side, for each σ define τ by $\tau(i) = \sigma(i)$, $i \leq k + k' + 1$, and $\tau(k + k' + 2) = k + k' + 2$. In the second term, for each σ define τ by $\tau(1) = \sigma(k' + 2)$, $\tau(k) = \sigma(k + k' + 1)$, $\tau(k + 1) = k + k' + 2$, $\tau(k + 2) = \sigma(1)$, $\tau(k + 3) = \sigma(2)$, $\ldots$, $\tau(k + k' + 2) = \sigma(k' + 1)$, then $\varepsilon(\tau) = (-1)^{(k+1)(k'+1)} \varepsilon(\sigma)$, and one has:

$$\Phi(C'')(Y_1, \ldots, Y_{k+k'+2}) = \sum_{\substack{\tau \in \mathfrak{S}_{k+1,k'+1} \\ \tau(k+k'+2)=k+k'+2}} \varepsilon(\tau)\Phi(C)(Y_{\tau(1)}, \ldots, Y_{\tau(k+1)})\Phi(C')((Y_{\tau(k+2)}, \ldots, Y_{\tau(k+k'+2)}) + \sum_{\substack{\tau \in \mathfrak{S}_{k+1,k'+1} \\ \tau(k+1)=k+k'+2}} \varepsilon(\tau)\Phi(C)(Y_{\tau(1)}, \ldots, Y_{\tau(k+1)})\Phi(C')((Y_{\tau(k+2)}, \ldots, Y_{\tau(k+k'+2)}) = \Phi(C) \wedge \Phi(C')(Y_1, \ldots, Y_{k+k'+2}).$$

■

One has to be careful that $\mathrm{ad}(C)$ ($C \in \mathscr{C}_c(\mathfrak{g})$) is generally not a derivation of the $\wedge$-product of $\mathscr{C}_c(\mathfrak{g})$, so the following result is of interest:

Proposition 7.14. *If $C \in \mathscr{C}_c^k(\mathfrak{g})$, $C' \in \mathscr{C}_c^{k'}(\mathfrak{g})$, $C'' \in \mathscr{C}_c(\mathfrak{g})$, with $k \geq 1$, then:*

$$\mathrm{ad}(C)(C' \wedge C'') = \mathrm{ad}(C)(C') \wedge C'' + (-1)^{(k+1)(k'+1)} C' \wedge \mathrm{ad}(C)(C'').$$

This means that when $C \in \mathscr{C}_c^k(\mathfrak{g})[1]$ with $k \geq 1$, then $\mathrm{ad}(C)$ is a derivation of degree k of the graded algebra $\mathscr{C}_c(\mathfrak{g})[-1]$ with the $\wedge$-product.

Proof. One has

$$\begin{aligned} \mu\left(\Phi([C,C']_a)\right) &= \Psi([C,C']) = [\Psi(C),\Psi(C')]_Q = [\mu\left(\Phi(C)\right), \mu\left(\Phi(C')\right)]_Q \\ &= \mu\left(\{\Phi(C),\Phi(C')\}\right) \end{aligned}$$

Since $\mathrm{ad}_{\mathrm{P}}(\Phi(C))\left(\bigwedge \mathfrak{g}\right) \subset \bigwedge_+ \mathfrak{g}$, it follows that $\Phi(\mathrm{ad}(C)(C')) = \mathrm{ad}_{\mathrm{P}}(\Phi(C))\left(\Phi(C')\right)$, and the result is obtained using the fact that $\mathrm{ad}_{\mathrm{P}}(\Phi(C))$ is a derivation of degree $k-1$ of $\bigwedge \mathfrak{g}$, and the definition of the $\wedge$-product of $\mathscr{C}_c(\mathfrak{g})$. ■

Using the Proposition above, and $d = \mathrm{ad}(F_0)$ with $F_0 \in \mathscr{C}_c^2(\mathfrak{g})$, it results that the $\wedge$-product of $\mathscr{C}_c(\mathfrak{g})$ induces a $\wedge$-product on $H_c^*(\mathfrak{g})$ and $\phi^* = -\frac{1}{2}\theta^*$ is clearly an isomorphism of graded algebras from $H_c^*(\mathfrak{g})$ onto $H_+^*(\mathfrak{g})$. From the definition of the **gla** bracket on $H_c^*(\mathfrak{g})$, denoting by μ^* the canonical map from $H^*(\mathfrak{g})$ onto $H_Q^*(\mathfrak{g}) = H^*(\mathfrak{g})/\mathbb{C}$, the map $\Psi^* = \mu^* \circ \Phi^*$ is a **gla** isomorphism from $H_c^*(\mathfrak{g})$ onto $H_Q^*(\mathfrak{g})$. We summarize in:

Proposition 7.15. *As a graded associative algebra, $H_c^*(\mathfrak{g})$ is isomorphic to $H_+^*(\mathfrak{g})$ and as a* **gla** *, $H_c^*(\mathfrak{g})$ is isomorphic to $H_Q^*(\mathfrak{g})$.*

Example 7.16. Let $\mathfrak{g} = \mathfrak{gl}(n)$. Then $H_+^*(\mathfrak{g}) = \mathrm{Ext}_+[a_1, a_3, \ldots, a_{2n-1}]$, where $a_k = \mathrm{Tr}\left(\mathscr{A}_k\right)$, $k \geq 0$ (e.g. [10]). One has $\Theta(\mathscr{A}_{2k}) = \frac{1}{2k+1}\,\mathrm{Tr}\left(\mathscr{A}_{2k+1}\right)$ (Example 7.8), so by Proposition 7.15, $H_c^*(\mathfrak{g}) = \mathrm{Ext}_+[\mathscr{A}_0, \mathscr{A}_2, \ldots, \mathscr{A}_{2n-2}]$. The **gla** bracket will be computed in 9.3.

Remark 7.17. When $\mathfrak{g}$ is not finite-dimensional, the map Θ^* of Proposition 7.7 is no longer an isomorphism, as shown with the following example: let V be an infinite-dimensional vector space, and $\mathfrak{g}$ be the quadratic Lie algebra of finite-rank operators of V, as defined in Example 4.13. Recall that the invariant bilinear form is $B(X,Y) = \mathrm{Tr}\,(XY)$, X, $Y \in \mathfrak{g}$. Notice that $B(X,Y)$ is well-defined when $X \in \mathfrak{g}$ and $Y \in \mathrm{End}(V)$. Moreover, the formula $B([X,Y],Z) = -B(Y,[X,Z])$ is valid if at least one argument is in $\mathfrak{g}$. By Remark 4.15, $H_c^0(\mathfrak{g}) = Z(\mathfrak{g}) = \{0\}$ and $H^1(\mathfrak{g}) = \mathbb{C}\,\mathrm{Tr}$, so:

Proposition 7.18. *The map $\Theta^* \colon H_c^0(\mathfrak{g}) \to H^1(\mathfrak{g})$ is not onto.*

Moreover,

Proposition 7.19. *The map $\Theta^* \colon H_c^1(\mathfrak{g}) \to H^2(\mathfrak{g})$ is not one to one.*

Proof. Fix $U \in \mathrm{End}(V)$ such that $U \notin \mathfrak{g} \oplus \mathbb{C}\,\mathrm{Id}_V$ and consider the skew symmetric derivation D of $\mathfrak{g}$ defined by $D = \mathrm{ad}(U)|_{\mathfrak{g}}$. The derivation D is a cyclic cocycle but $D = \mathrm{ad}(Y)$ with $Y \in \mathfrak{g}$ cannot be true because if $U' \in \mathrm{End}(V)$ commutes with $\mathfrak{g}$, then U' must be a multiple of Id_V. So D is not a cyclic coboundary. On the other hand, $\widehat{D}(X,Y) = B(D(X),Y) = \partial\omega(X,Y)$ where $\omega \in \bigwedge^1 \mathfrak{g}$ is defined by $\omega(X) = -B(U,X)$, $X \in \mathfrak{g}$. Hence $\widehat{D}$ is a coboundary, and if we denote by $\overline{D}$ the class of D in $H_c^*(\mathfrak{g})$, we get $\Theta^*(\overline{D}) = 0$, and $\overline{D} \neq 0$. ■

8. The case of reductive and semisimple Lie algebras

Let $\mathfrak{g}$ be a n-dimensional quadratic Lie algebra with bilinear form B. We recall the natural $\mathfrak{g}$-module structures on $\bigwedge \mathfrak{g}$ and $\mathscr{M}_a(\mathfrak{g})$ defined by:

$$\theta_X(\Omega)(Y_1,\ldots,Y_p) = -\sum_i \Omega(Y_1,\ldots,[X,Y_i],\ldots,Y_p),\ \forall\, X,Y_1,\ldots,Y_p \in \mathfrak{g}, \Omega \in \bigwedge^p \mathfrak{g}.$$

$$L_X(\Omega\otimes Y) = \theta_X(\Omega)\otimes Y + \Omega\otimes[X,Y],\ \forall X,Y \in \mathfrak{g}, \Omega \in \bigwedge \mathfrak{g}.$$

Using the notation in 7.2, it is easy to check that

$$\Phi\circ L_X = \theta_X\circ\Phi,\ \forall\, X\in\mathfrak{g}.$$

So we have:

Proposition 8.1. *$\mathscr{C}_c(\mathfrak{g})$ is a $\mathfrak{g}$-submodule of the $\mathfrak{g}$-module $\mathscr{M}_a(\mathfrak{g})$ and the isomorphism Φ (of 7.2) is a $\mathfrak{g}$-module isomorphism from $\mathscr{C}_c(\mathfrak{g})$ onto $\bigwedge_+ \mathfrak{g}$.*

It is well-known that any element of $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is a cocycle, and if $\mathfrak{g}$ is reductive, that $H^\star(\mathfrak{g}) = (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ [16]. Using Propositions 7.7, 7.15 and 8.1, we deduce:

Proposition 8.2. *Any invariant cyclic cochain is a cocycle. If $\mathfrak{g}$ is reductive, any cyclic cohomology class contains one, and only one invariant cyclic cocycle (for instance, the only invariant cyclic coboundary is 0).*

Hence, when $\mathfrak{g}$ is reductive, we can identify $H_c^*(\mathfrak{g})$ and $\mathscr{C}_c(\mathfrak{g})^{\mathfrak{g}}$. This identification is valid for the corresponding $\wedge$-products (actually isomorphic to the $\wedge$-product of $(\bigwedge_+ \mathfrak{g})^{\mathfrak{g}} \simeq H_+^\star(\mathfrak{g})$) and for the corresponding graded Lie bracket induced by the Gerstenhaber bracket (actually isomorphic to $\mathscr{H}(\mathfrak{g})^{\mathfrak{g}}$ and $\left(\bigwedge_Q \mathfrak{g}\right)^{\mathfrak{g}}$).

In the remaining of this Section, we assume that $\mathfrak{g}$ is a semisimple Lie algebra with invariant bilinear form B (not necessarily the Killing form).

Proposition 8.3. *If I and $I' \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$, then $\{I,I'\} = 0$.*

As a consequence of this Proposition and of Proposition 7.15, one has:

Corollary 8.4. *The Gerstenhaber bracket induces the null bracket on*

$$H_c^*(\mathfrak{g}) \simeq \mathscr{C}_c(\mathfrak{g})^{\mathfrak{g}}.$$

To prove Proposition 8.3, we need several lemmas: first, let $\mathfrak{h}$ be a Lie algebra and $I \in \left(\bigwedge^{p+1} \mathfrak{h}\right)^{\mathfrak{h}}$. Define a map $\Omega\colon \mathfrak{h} \to \bigwedge^p \mathfrak{h}$ by $\Omega(X) = \iota_X(I), \forall\, X \in \mathfrak{h}$. Then since I is invariant, one has:

Lemma 8.5. *Ω is a morphism of $\mathfrak{h}$-modules from $(\mathfrak{h},\mathrm{ad})$ into $(\bigwedge^p \mathfrak{h},\theta)$.*

Proof. For all X, Y and $Z \in \mathfrak{g}$, we have:

$$\theta_X(\Omega(Y)) = \theta_X(\iota_Y(I)) = [\theta_X,\iota_Y](I) + \iota_Y(\theta_X(I)) = \iota_{[X,Y]}(I) = \Omega([X,Y]).$$

■

As a second argument for the proof of Proposition 8.3:

Lemma 8.6. *Assuming that* $\mathfrak{h}$ *is a perfect Lie algebra (i.e.* $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$*), there exists a map* $\alpha\colon \mathfrak{h} \to \bigwedge^{p-1}\mathfrak{h}$ *such that* $\Omega = \partial \circ \alpha$ *(*∂ *is the differential of the trivial cohomology of* $\mathfrak{h}$*). Moreover, if* $\mathfrak{h}$ *is semisimple, there exist an* $\mathfrak{h}$*-homomorphism* α *such that* $\Omega = \partial \circ \alpha$*.*

Proof. If $X \in \mathfrak{h}$, we can find Z_i, $T_i \in \mathfrak{h}$ such that $X = \sum_i [Z_i, T_i]$. Then, $\Omega(X) = \sum_i \theta_{Z_i}(\Omega(T_i))$ by Lemma 8.5. But $\partial(\Omega(T_i)) = \partial(\iota_{T_i}(I)) = \theta_{T_i}(I) - \iota_{T_i}(\partial(I)) = 0$ since I is an invariant. But θ_{Z_i} maps $Z^p(\mathfrak{h})$ into $B^p(\mathfrak{h})$, so $\Omega(X) \in B^p(\mathfrak{h})$. To construct α, fix a section σ of the map $\partial\colon \bigwedge^{p-1}\mathfrak{h} \to B^p(\mathfrak{h})$, i.e. $\sigma\colon B^p(\mathfrak{h}) \to \bigwedge^{p-1}\mathfrak{h}$ such that $\partial \circ \sigma = \mathrm{Id}_{B^p(\mathfrak{h})}$ and then set $\alpha = \sigma \circ \Omega$. When $\mathfrak{g}$ is semisimple, one can fix a section σ which is a $\mathfrak{g}$-homomorphism. ■

Proof. *(of Proposition 8.3)*

Fix an orthonormal basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ with respect to B. Given I, $I' \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$, let $\Omega_r = \iota_{X_r}(I)$, $\Omega'_r = \iota_{X_r}(I')$, α, α' the $\mathfrak{g}$-homomorphisms given by Lemma 8.6 and finally $\alpha_r = \alpha(X_r)$, $\alpha'_r = \alpha'(X_r)$ so that $\Omega_r = \partial\ \alpha_r$ and $\Omega'_r = \partial\ \alpha'_r$. With these notations, in order to finish the proof, we need to show that $\sum_r \Omega_r \wedge \Omega'_r = 0$. But:

$$\sum_r \Omega_r \wedge \Omega'_r = \sum_r \partial\alpha_r \wedge \partial\alpha'_r = \partial(\sum_r \alpha_r \wedge \partial\alpha'_r) = 0$$

since $\sum_r \alpha_r \wedge \partial\alpha'_r \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$. ■

Remark 8.7. Proposition 8.3 can be directly deduced from a deep result of Kostant [15] about the structure of $\mathrm{Cliff}(\mathfrak{g}^*)^{\mathfrak{g}}$ seen as a deformation of $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$: by the Hopf-Koszul-Samelson theorem, $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is an exterior algebra $\mathrm{Ext}[a_1, \ldots, a_r]$ with $\mathrm{rank}(\mathfrak{g}) = r$ and $a_1, \ldots, a_r$ primitive (odd) invariants. Kostant shows that $\mathrm{Cliff}(\mathfrak{g}^*)^{\mathfrak{g}}$ is a Clifford algebra constructed on $a_1, \ldots, a_r$. Since the deformation from $\bigwedge \mathfrak{g}$ to $\mathrm{Cliff}(\mathfrak{g}^*)$ has leading term the Poisson bracket, it results that $\{a_i, a_j\} = 0$, $\forall\, i, j$, and then Proposition 8.3 follows.

Example 8.8. Using the results in Section 5 and Corollary 8.4, we will describe $H_c^*(\mathfrak{s})$ and $H_c^*(\mathfrak{g})$ when $\mathfrak{s} = \mathfrak{sl}(n)$ and $\mathfrak{g} = \mathfrak{gl}(n)$, both equipped with the bilinear form $B(X,Y) = \mathrm{Tr}\,(XY)$, $\forall\, X, Y$. Let $\mathbf{1}_{\mathfrak{g}}$ be the identity matrix.

One has $\bigwedge \mathfrak{s} = \{\Omega \in \bigwedge \mathfrak{g} \mid \iota_{\mathbf{1}_{\mathfrak{g}}}(\Omega) = 0\}$ and $\mathscr{M}_a(\mathfrak{s}) = \{F \in \mathscr{M}_a(\mathfrak{g}) \mid \iota_{\mathbf{1}_{\mathfrak{g}}}(F) = 0 \text{ and } F(\mathfrak{g}^p) \subset \mathfrak{s} (F \in \mathscr{M}_a^p(\mathfrak{g}))\}$. By Propositions 4.6 and 4.10, $\mathscr{A}_{2k} \in \mathscr{M}_a(\mathfrak{s})$. Moreover, let $a_k = \mathrm{Tr}\,(\mathscr{A}_k)$ $(k \geq 0)$, then by Proposition 4.10, $a_{2k+1} \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$, $\forall\, k \geq 0$, and by Proposition 4.6, $a_{2k+1} \in (\bigwedge \mathfrak{s})^{\mathfrak{s}}$, $\forall\, k \geq 0$.

(1) It is well-known that $H^*(\mathfrak{g}) = (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is the exterior algebra generated by the invariant cocycles a_1, a_3, …, a_{2n-1}, i.e. $(\bigwedge \mathfrak{g})^{\mathfrak{g}} = \mathrm{Ext}[a_1, a_3, \ldots, a_{2n-1}]$ and that $H^*(\mathfrak{s}) = (\bigwedge \mathfrak{s})^{\mathfrak{s}}$ is the exterior algebra generated by the invariant cocycles a_3, a_5, …, a_{2n-1}, i.e. $(\bigwedge \mathfrak{s})^{\mathfrak{s}} = \mathrm{Ext}[a_3, a_5, \ldots, a_{2n-1}]$ (see [15, 14, 10]).

(2) We need to compute the super Poisson bracket on $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$. Note that $\{\Omega, \Omega'\} = 0$, $\forall\, \Omega, \Omega' \in (\bigwedge \mathfrak{s})^{\mathfrak{s}}$ by Proposition 8.3. Using $\mathfrak{s}^{\perp} = \mathbb{C}\,\mathbf{1}_{\mathfrak{g}}$, an adapted orthonormal basis, and the formula in Proposition 5.4, one finds that $\{a_1, a_1\} = 2n$. Then,

since any element in $(\bigwedge\mathfrak{g})^{\mathfrak{g}}$ decomposes as $\Omega+\Omega'\wedge a_1$, with $\Omega,\ \Omega'\in(\bigwedge\mathfrak{s})^{\mathfrak{s}}$, we have only to compute the following brackets:

$$\{\Omega,\Omega'\wedge a_1\}=0,\forall\ \Omega,\Omega'\in\mathrm{Ext}^{w'}[a_3,\dots,a_{2n-1}],$$
$$\{\Omega\wedge a_1,\Omega'\wedge a_1\}=2n(-1)^{w'}\ \Omega\wedge\Omega',$$
$$\forall\ \Omega\in\mathrm{Ext}[a_3,\dots,a_{2n-1}],\Omega'\in\mathrm{Ext}^{w'}[a_3,\dots,a_{2n-1}].$$

(3) Use the isomorphism Φ^* of Proposition 7.15 to find $H_c^*(\mathfrak{s})=\mathscr{C}_c(\mathfrak{s})^{\mathfrak{s}}$ and $H_c^*(\mathfrak{g})=\mathscr{C}_c(\mathfrak{g})^{\mathfrak{g}}$. One has $[\mathscr{A}_2,\mathscr{A}_{2k}]=0$ by Proposition 4.2, so $\mathscr{A}_{2k}$ is a cocycle, obviously $\mathfrak{g}$-invariant. By Proposition 4.11, it is a cyclic cocycle, and $\Phi(\mathscr{A}_{2k})=-\frac{1}{2(2k+1)}a_{2k+1}$. It results that

$$H_c^*(\mathfrak{s})=\mathrm{Ext}_+[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}]\text{ and }H_c^*(\mathfrak{g})=\mathrm{Ext}_+[\mathscr{A}_0,\mathscr{A}_2,\dots,\mathscr{A}_{2n-2}].$$

(4) Now we compute the Gerstenhaber bracket. For $H_c^*(\mathfrak{s})$, by Corollary 8.4, the Gerstenhaber bracket vanishes. For $H_c^*(\mathfrak{g})$, we use the isomorphism Ψ^* (see Proposition 7.15) combined with 8.8 (3) and the commutation rules in $H^*(\mathfrak{g})$ computed in 8.8 (2) from which the commutation rules in $H_Q^*(\mathfrak{g})=H^*(\mathfrak{g})/\mathbb{C}$ are deduced. Finally the result is the following:

$$[F,F']_a=0,\forall\ F,F'\in\mathrm{Ext}_+[\mathscr{A}_2,\dots,\mathscr{A}_{2n-2}],$$
$$[\mathscr{A}_0,F]_a=0,\forall\ F\in\mathrm{Ext}_+[\mathscr{A}_0,\mathscr{A}_2,\dots,\mathscr{A}_{2n-2}],$$
$$[F,F'\wedge\mathscr{A}_0]_a=0,\forall\ F,F'\in\mathrm{Ext}_+[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}],$$
$$[\mathscr{A}_0,F'\wedge\mathscr{A}_0]_a=\frac{n}{2}(-1)^{f'}\ F',\forall\ F'\in\mathrm{Ext}_+^{f'}[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}],$$
$$[F\wedge\mathscr{A}_0,F'\wedge\mathscr{A}_0]_a=\frac{n}{2}(-1)^{f'}\ F\wedge F',$$
$$\forall\ F\in\mathrm{Ext}_+[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}],F'\in\mathrm{Ext}_+^{f'}[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}],$$

Remark that for the last result, one uses: if $F'\in\mathrm{Ext}_+^{f'}[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}]\cap\mathscr{C}_c^{p'}(\mathfrak{g})$, then $f'=p'+1$ mod 2 and $\Phi(F')\in\bigwedge^{p'+1}\mathfrak{g}$.

9. Quadratic $2k$-Lie algebras and cyclic cochains

9.1.. Let $\mathfrak{g}$ be a finite-dimensional quadratic vector space with bilinear form B. Given $D\in\mathscr{D}^{2k-1}$, $k\geq 1$ denote by $F=\mathbf{F}_D$ the associated (even) structure on $\mathfrak{g}$ (see Sections 2 and 3), 3.), that we also denote by a bracket notation:

$$[Y_1,\dots,Y_{2k}]=F(Y_1,\dots,Y_{2k}),\forall\ Y_1,\dots,Y_{2k}\in\mathfrak{g}.$$

Definition 9.1. The bilinear form B is F*-invariant* (or F is a *quadratic structure with bilinear form* B) if $B([Y_1,\dots,Y_{2k-1},Y],Z)=-B(Y,[Y_1,\dots,Y_{2k-1},Z])$, $\forall\ Y_1,\dots,Y_{2k-1},\ Y,\ Z\in\mathfrak{g}$.

We introduce the linear maps $\mathrm{ad}_{Y_1,\dots,Y_{2k-1}}:\mathfrak{g}\to\mathfrak{g}$ by:

$$\mathrm{ad}_{Y_1,\dots,Y_{2k-1}}(Y)=[Y_1,\dots,Y_{2k-1},Y],\ \forall\ Y_1,\dots,Y_{2k-1},Y\in\mathfrak{g}.$$

It is obvious that

Proposition 9.2. *The bilinear form B is F-invariant if and only if* $\mathrm{ad}_{Y_1,\dots,Y_{2k-1}} \in \mathfrak{o}(B)$, $\forall\, Y_1,\dots,Y_{2k-1} \in \mathfrak{g}$.

The next Proposition results directly from Propositions 7.2 and 7.9.

Proposition 9.3.

(1) F is quadratic if and only if it is a cyclic cochain.

(2) F is quadratic if and only if there exists $I \in \bigwedge^{2k+1}\mathfrak{g}$ such that $D = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)$ and in that case, one has $I(Y_1,\dots,Y_{2k+1}) = B([Y_1,\dots,Y_{2k}],Y_{2k+1})$, $\forall\, Y_1,\dots,\ Y_{2k+1} \in \mathfrak{g}$.

9.2. . Keeping the notation of Proposition 9.3, a quadratic F will define a $2k$-Lie algebra structure on $\mathfrak{g}$ (namely a quadratic $2k$-Lie algebra) if and only if:

$$[F,F]_a = 0 \quad \text{or} \quad [D,D] = 0 \quad \text{or} \quad \{I,I\} = 0. \tag{7}$$

Examples of quadratic $2k$-Lie algebras can be directly deduced from Proposition 8.3: let us assume in the remaining of 9.2, that $\mathfrak{g}$ is a semisimple Lie algebra with bilinear form B (not necessarily the Killing form). Then one has:

Proposition 9.4. *Any invariant even cyclic cochain in $\mathscr{M}_a(\mathfrak{g})$ defines a quadratic $2k$-Lie algebra on $\mathfrak{g}$.*

These examples were introduced for the first time in [3], in the case of primitive elements in $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ (we shall come back to the construction in [3] later in this Section).

Let F be an invariant even cyclic cochain, denote by:

$$[Y_1,\dots,Y_{2k}] = F(Y_1,\dots,Y_{2k}), \forall\, Y_1,\dots,Y_{2k} \in \mathfrak{g}$$

the associated quadratic $2k$-bracket on $\mathfrak{g}$. Let us introduce, as in 9.1:

$$I([Y_1,\dots,Y_{2k+1}) = B([Y_1,\dots,Y_{2k}],Y_{2k+1}),\ \forall\, Y_1,\dots,Y_{2k+1} \in \mathfrak{g},$$

and the associated derivation $D = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I)$ of $\bigwedge\mathfrak{g}$. Since $[D,D] = 2D^2 = 0$, we can define the associated cohomology on $\bigwedge\mathfrak{g}$ by

$$H^\star(F) = Z(D)/B(D)$$

where $Z(D) = \ker(D)$ and $B(D) = \mathrm{Im}(D)$.

The following Lemma has to be compared with Formula 3 of 3.2:

Proposition 9.5. *Let $\{X_1,\dots,X_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to B. Then there exist $\beta_1,\dots,\beta_n \in \bigwedge^{2k-1}\mathfrak{g}$ such that:*

$$D = \frac{1}{2}\sum_r \beta_r \wedge \theta_{X_r}.$$

Proof. Let $\{\omega_1,\dots,\omega_n\}$ be the dual basis of $\{X_1,\dots,X_n\}$. One has $\theta_{X_r}(\omega_s)(Y) = B([X_r,X_s],Y)$ for all $Y\in\mathfrak{g}$. So $\theta_{X_r}(\omega_s) = -\theta_{X_s}(\omega_r)$ for all r, s. Define $\Omega(X) = \iota_X(I)$, $X\in\mathfrak{g}$. By Lemma 8.6, there exists a $\mathfrak{g}$-homomorphism $\alpha\colon \mathfrak{g}\to\bigwedge^{2k-1}\mathfrak{g}$ such that $\Omega = \partial\circ\alpha$. Define $\alpha_r = \alpha(X_r)$, then $\theta_{X_r}(\alpha_s) = \alpha([X_r,X_s])$, so one has $\theta_{X_r}(\alpha_s) = -\theta_{X_s}(\alpha_r)$. Define $\Omega_r = \Omega(X_r) = \partial\alpha_r$, the one has:

$$D = -\frac{1}{2}\mathrm{ad}_{\mathrm{P}}(I) = -\sum_r \Omega_r\wedge\iota_{X_r}.$$

So $D(\omega_r) = -\partial\alpha_r$. Then using $\partial = \frac{1}{2}\sum_s \omega_s\wedge\theta_{X_s}$ ([16]), one has:

$$\partial\alpha_r = -\frac{1}{2}\sum_s \omega_s\wedge\theta_{X_r}(\alpha_s) = -\frac{1}{2}\left(\theta_{X_r}(\sum_s\omega_s\wedge\alpha_s) - \sum_s\theta_{X_r}(\omega_s)\wedge\alpha_s\right).$$

But $\sum_s\omega_s\wedge\alpha_s$ is $\mathfrak{g}$-invariant, so :

$$\partial\alpha_r = \frac{1}{2}\sum_s\theta_{X_r}(\omega_s)\wedge\alpha_s = -\frac{1}{2}\sum_s\alpha_s\wedge\theta_{X_r}(\omega_s) = \frac{1}{2}\sum_s\alpha_s\wedge\theta_{X_s}(\omega_r).$$

Therefore, since D and $\sum_s\alpha_s\wedge\theta_{X_s}$ are derivations of $\bigwedge\mathfrak{g}$, one has $D = -\frac{1}{2}\sum_s\alpha_s\wedge\theta_{X_s}$, and if we set $\beta_s = -\alpha_s$, the Proposition is proved. ∎

From Proposition 9.5, we deduce:

Proposition 9.6. *One has* $(\bigwedge\mathfrak{g})^{\mathfrak{g}}\subset Z(D)$.

From the fact that $I\in(\bigwedge\mathfrak{g})^{\mathfrak{g}}$, D is a $\mathfrak{g}$-homomorphism of the $\mathfrak{g}$-module $\bigwedge\mathfrak{g}$, which is semisimple. By standard arguments ([16]), one deduces:

Proposition 9.7. *One has* $(\bigwedge\mathfrak{g})^{\mathfrak{g}}\subset H^\star(F)$.

When F is the Lie algebra structure of $\mathfrak{g}$, it is well-known that $H^\star(F) = (\bigwedge\mathfrak{g})^{\mathfrak{g}}$ ([16]).

Let us now place the constructions in [3] in our context. We assume that $\mathfrak{g}$ is a semisimple Lie algebra of rank r and fix a non degenerate symmetric bilinear form B (not necessarily Killing) on $\mathfrak{g}$. Let $\mathrm{S}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}^*)$. Using Chevalley's theorem, there exist homogeneous invariants $Q_1,\dots,Q_r$ with $q_i = \deg(Q_i)$ such that $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[Q_1,\dots,Q_r]$. Let $t\colon \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}\to(\bigwedge\mathfrak{g})^{\mathfrak{g}}$ be the Cartan-Chevalley transgression operator ([5], [6]). By the Hopf-Koszul-Samelson theorem ([5], [6], [15]), one has $(\bigwedge\mathfrak{g})^{\mathfrak{g}} = \mathrm{Ext}[t(Q_1),\dots,t(Q_r)]$ and $\deg(t(Q_i)) = 2q_i - 1$. By (7) and Proposition 8.3, any odd element I in $(\bigwedge\mathfrak{g})^{\mathfrak{g}}$ defines a quadratic $2k$-Lie algebra structure on $\mathfrak{g}$ (and corresponding generalized Poisson bracket on $\mathfrak{g}^*$). As a particular case, this works for $t(Q_i)$, $i=1,\dots,r$ which define a $(2q_i-2)$-Lie algebra structure on $\mathfrak{g}$ and a GPB on $\mathfrak{g}^*$, and these are exactly the examples given in [3], though in these papers there are no citations, neither to Chevalley [6], nor to Cartan [5]. Let us insist that not only primitive invariants (as sometimes claimed in [3]), but actually all odd invariants do define $2k$-Lie algebra structures on $\mathfrak{g}$ (Propositions 9.3 and 9.4).

9.3. . Using the notation and the results of Example 8.8, let us consider the case of $\mathfrak{g} = \mathfrak{gl}(n)$, with bilinear form $B(X,Y) = \mathrm{Tr}\,(XY)$, $\forall X,Y\in\mathfrak{g}$. Consider $C = F + F'\wedge\mathscr{A}_0$ with F, $F'\in\mathrm{Ext}_+[\mathscr{A}_2,\mathscr{A}_4,\dots,\mathscr{A}_{2n-2}]$. In order to have C an even element of $\mathscr{M}_a(\mathfrak{g})$, we have to assume that $F \in \mathrm{Ext}_+^{\mathrm{odd}}[\mathscr{A}_2, \mathscr{A}_4, \ldots, \mathscr{A}_{2n-2}]$ and $F' \in \mathrm{Ext}_+^{\mathrm{even}}[\mathscr{A}_2, \ldots, \mathscr{A}_{2n-2}]$ (see the last remark in Example 8.8 (4)). Moreover, we have to assume that F and $F' \wedge \mathscr{A}_0$ have the same degree in $\mathscr{M}_a(\mathfrak{g})$, say $2k$. Then, from commutation rules in 8.8 (4), C defines a $2k$-Lie algebra structure on $\mathfrak{g}$ if and only if $F' \wedge F' = 0$. This last condition is obviously satisfied if F' is decomposable. For instance, if $n \geq 3$, $\alpha\mathscr{A}_8 + \beta\mathscr{A}_0 \wedge \mathscr{A}_2 \wedge \mathscr{A}_4$, α, $\beta \in \mathbb{C}$, defines a 8-Lie algebra structure on $\mathfrak{g}$; if $n \geq 4$, $\alpha\mathscr{A}_{14} + \beta\mathscr{A}_0 \wedge \mathscr{A}_4 \wedge \mathscr{A}_8$, α, $\beta \in \mathbb{C}$, defines a 14-Lie algebra structure on $\mathfrak{g}$.

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