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Almost Horospherical Subalgebras of Semisimple Lie Algebras

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Abstract. It is well known that the maximal subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are certain semisimple subalgebras and the maximal parabolic subalgebras. This paper provides a classification of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ such that every subalgebra strictly containing $\mathfrak{h}$ is horospherical, i.e., contains a maximal unipotent subalgebra of $\mathfrak{g}$.
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Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In his doctoral thesis [5] V.V. Morozov showed that any maximal subalgebra of $\mathfrak{g}$ is either semisimple or a maximal parabolic subalgebra (Morozov's proof can also be found in [4]). Later in his paper [2] E.B. Dynkin classified all semisimple maximal subalgebras.

Let us call an algebraic subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ *horospherical* if it contains a maximal unipotent subalgebra of $\mathfrak{g}$. These subalgebras can be described as follows: there is a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $[\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{p}_u \subset \mathfrak{h} \subset \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$ is a Levi decomposition of $\mathfrak{p}$.

The inclusion relation on the horospherical subalgebras is very simple. It looks natural to classify all non-horospherical algebraic subalgebras that are maximal in this class. This is the main result of this paper.

Definition 0.1. A maximal non-horospherical subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *almost horospherical*. The pair $(\mathfrak{g}, \mathfrak{h})$ in this case is also said to be *almost horospherical*.

Let us recall the concept of parabolic induction of subalgebras introduced in [8].

Let $\mathfrak{g}$ be a reductive algebraic Lie algebra, $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, and ϕ be a homomorphism from $\mathfrak{p}$ onto some reductive algebraic Lie algebra $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{h}}$ be a subalgebra of $\tilde{\mathfrak{g}}$. Set $\mathfrak{h} = \phi^{-1}(\tilde{\mathfrak{h}})$. The pair $(\mathfrak{g}, \mathfrak{h})$ is said to be obtained from the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ *by parabolic induction* via the parabolic subalgebra $\mathfrak{p}$. We will denote this by $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \underset{\mathfrak{p}}{\rightarrow} (\mathfrak{g}, \mathfrak{h})$.

We will call a pair $(\mathfrak{g}, \mathfrak{h})$ *cuspidal* if it cannot be obtained by a parabolic induction with $\tilde{\mathfrak{g}} \neq \mathfrak{g}$.

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Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and ϕ be a homomorphism from $\mathfrak{p}$ onto some reductive algebraic Lie algebra $\tilde{\mathfrak{g}}$. Then the subalgebra $\mathfrak{p}$ will be called *admissible* for $\tilde{\mathfrak{g}}$. We will call a parabolic subalgebra $\mathfrak{p}_1 \supset \mathfrak{p}$ of $\mathfrak{g}$, an *admissible enlargement* of $\mathfrak{p}$ for $\tilde{\mathfrak{g}}$ if there exists a homomorphism ϕ_1 from $\mathfrak{p}_1$ onto $\tilde{\mathfrak{g}}$ such that $\phi_1(a) = \phi(a)$ for all $a \in \mathfrak{p}$. If $\mathfrak{p}$ has no admissible enlargements then it will be called *maximal admissible* for $\tilde{\mathfrak{g}}$. In this case the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is maximal among reductive subalgebras of $\mathfrak{g}$ containing $\tilde{\mathfrak{g}}$ as an ideal.

Theorem 0.2. *Almost horospherical pairs are exactly the pairs obtained from the cuspidal almost horospherical pairs by parabolic induction via maximal admissible parabolic subalgebras.*

Therefore, the question is reduced to classifying cuspidal almost horospherical pairs. Let us introduce some notions and notation necessary to formulate the classification.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, Δ be the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, $\Delta^+ \subset \Delta$ be the set of positive roots with respect to $\mathfrak{b}$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the system of simple roots, $e_\alpha \in \mathfrak{g}$ be a root vector corresponding to the root $\alpha \in \Delta$, h_α be the element of $\mathfrak{t}$ such that h_α is orthogonal to $\operatorname{Ker}\alpha$ (with respect to the Cartan scalar product) and $\alpha(h_\alpha) = 2$.

Now we can give a description of admissible and maximal admissible parabolic subalgebras in terms of roots. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra containing $\mathfrak{b}$ and $\mathfrak{l}$ be its Levi subalgebra containing $\mathfrak{t}$. If there is a homomorphism $\phi : \mathfrak{p} \to \tilde{\mathfrak{g}}$ for an algebraic reductive $\tilde{\mathfrak{g}}$ then we may assume $\tilde{\mathfrak{g}}$ to be an ideal of $\mathfrak{l}$. If $\tilde{\mathfrak{g}}$ is semisimple then it is defined by the set Π' of simple roots α such that $e_\alpha \in \tilde{\mathfrak{g}}$ (or $e_{-\alpha} \in \tilde{\mathfrak{g}}$). Similarly, $\mathfrak{p}$ is defined by the set Π'' of simple roots α such that $e_{-\alpha} \in \mathfrak{p}$. The subdiagram corresponding to Π' is a connected component of the subdiagram corresponding to Π''. And in case when $\mathfrak{p}$ is maximal admissible for $\tilde{\mathfrak{g}}$, Π'' consists of Π' and all simple roots orthogonal to all elements of Π'.

For each set $\Pi_1 \subset \Pi$ let us define a grading of $\mathfrak{g}$ in the following way. Let α be a root and $\alpha = \sum n_j \alpha_j$ be its decomposition into a linear combination of simple roots. Let us define the Π_1-height of α as the sum of coefficients n_j such that $\alpha_j \in \Pi_1$, and denote it by $\operatorname{ht}_{\Pi_1}(\alpha)$. Set

$$\mathfrak{g}_i = \sum_{\alpha \mid \operatorname{ht}_{\Pi_1}(\alpha) = i} \mathbb{C} e_\alpha, \ i \neq 0,$$

$$\mathfrak{g}_0 = \mathfrak{t} + \sum_{\alpha \mid \operatorname{ht}_{\Pi_1}(\alpha) = 0} \mathbb{C} e_\alpha.$$

It is clear that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Below we will prove that $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i \geq 1$ (Lemma 1.2). Set $\mathfrak{g}_{\geq i} = \sum_{j \geq i} \mathfrak{g}_j$ and $\mathfrak{g}_{\leq i} = \sum_{j \leq i} \mathfrak{g}_j$. Then $\mathfrak{p} = \mathfrak{p}(\Pi_1) := \mathfrak{g}_{\geq 0}$ is a parabolic subalgebra. We will say that it is defined by the set Π_1. Set $\operatorname{depth}(\Pi_1) = \max\{i | \mathfrak{g}_i \neq \{0\}\}$. This number is the nilpotent degree of the unipotent radical $\mathfrak{p}_u = \mathfrak{g}_{\geq 1}$ of $\mathfrak{p}$.

Theorem 0.3. *Let $\mathfrak{h}$ be an algebraic subalgebra of a semisimple Lie algebra $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is cuspidal almost horospherical exactly in the following cases:*

1. $\mathfrak{g}$ *is a simple Lie algebra,* $\mathfrak{h}$ *is a semisimple maximal subalgebra (as it was said above, the list of such pairs can be found in [2]);*

2. $\mathfrak{g}$ *is a direct sum of two copies of a simple Lie algebra,* $\mathfrak{h}$ *is the diagonal subalgebra;*

3. $\mathfrak{g}$ *is a simple Lie algebra,* $\mathfrak{h}$ *is a Levi subalgebra of a maximal parabolic subalgebra* $\mathfrak{p}(\{\alpha\}) \subset \mathfrak{g}$ *such that* $\mathrm{depth}(\{\alpha\}) = 1$*;*

4. $\mathfrak{g}$ *is a simple Lie algebra,*

$$\mathfrak{h} = \mathbb{C}h_0 + \{\sum_{\alpha\in\Pi} c_\alpha e_\alpha | \sum_{\alpha\in\Pi} c_\alpha = 0\} + \mathfrak{g}_{\geq 2} \subset \mathfrak{g},$$

where $h_0 \in \mathfrak{t}$ *is the element such that* $\alpha(h_0) = 1$ *for all* $\alpha \in \Pi$ *and the grading is defined by the set of all simple roots;*

5. $\mathfrak{g}$ *is a simple Lie algebra of type* D_4*,*

$$\mathfrak{h} = \mathbb{C}h_{\alpha_2} + \mathbb{C}(h_{\alpha_1} + h_{\alpha_3} + h_{\alpha_4}) + \mathbb{C}e_{\alpha_2} + \mathbb{C}e_{-\alpha_2} + \mathbb{C}(e_{\alpha_1} - e_{\alpha_3}) +$$

$$\mathbb{C}(e_{\alpha_1} - e_{\alpha_4}) + \mathbb{C}(e_{\alpha_1+\alpha_2} - e_{\alpha_3+\alpha_2}) + \mathbb{C}(e_{\alpha_1+\alpha_2} - e_{\alpha_4+\alpha_2}) + \mathfrak{g}_{\geq 2}$$

where the grading is defined by the set $\{\alpha_1, \alpha_3, \alpha_4\}$*.*

6. $\mathfrak{g}$ *is an exceptional simple Lie algebra and*

$$\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_{\geq 2},$$

where the grading is defined by the set $\{\alpha\}$ *for a simple root* α *such that* $\mathrm{depth}(\{\alpha\}) \geq 3$*.*

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Notation:

$\mathfrak{g}$	a semisimple Lie algebra;
$\mathfrak{t}$	a Cartan subalgebra of $\mathfrak{g}$;
$\mathfrak{b}$	a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$;
Δ	the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$;
$\Delta^+ \subset \Delta$	the system of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{b}$;
$\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta^+$	the system of simple roots of $\mathfrak{g}$ (see [6]);
$\{\pi_1, \ldots, \pi_n\}$	the set of fundamental weights of $\mathfrak{g}$;
$\mathfrak{p}_u$	the unipotent radical of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$;

We will also use the notation ε_i for weights as accepted in [6].

If $\alpha \in \mathfrak{t}(\mathbb{R})^*$, then h_α is an element of $\mathfrak{t}$ such that h_α is orthogonal to $\mathrm{Ker}\,\alpha$ (with respect to the Cartan scalar product) and $\alpha(h_\alpha) = 2$.

If $\alpha \in \Delta$, then e_α will denote the root vector, corresponding to α (defined up to a scalar factor). We will assume that $[e_\alpha, e_{-\alpha}] = h_\alpha$ (thus, the set $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ will be a $\mathfrak{sl}_2$-triple).

The sign $\oplus$ will be used for semidirect sums of Lie algebras.

1. Proofs of the theorems

Let us formulate Theorem 1.3 from [8] that will be used in the proof:

Theorem 1.1. *Let $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \overrightarrow{_{\mathfrak{p}}} (\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{h}_1 \supset \mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h}_1)$ can be obtained from the pair $(\mathfrak{g}, \mathfrak{h})$ in the following way:*

- *first of all, the subalgebra $\tilde{\mathfrak{h}}$ is is enlarged to a subalgebra $\tilde{\mathfrak{h}}' \supset \tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$, so we replace the $(\mathfrak{g}, \mathfrak{h})$ with $(\mathfrak{g}, \mathfrak{h}')$, where $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}') \overrightarrow{_{\mathfrak{p}}} (\mathfrak{g}, \mathfrak{h}')$*
- *we pass to a deeper parabolic induction: if $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}')$ is not cuspidal by itself, then $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}) \overrightarrow{_{\tilde{\mathfrak{p}}}} (\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}')$ for some cuspidal pair $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$, therefore $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}) \overrightarrow{_{\mathfrak{p}'}} (\mathfrak{g}, \mathfrak{h}')$, where $\mathfrak{p}' = \tilde{\mathfrak{p}} \oplus \mathfrak{k}$ for any $\mathfrak{k}$ such that $\mathfrak{p} = \tilde{\mathfrak{g}} \oplus \mathfrak{k}$ ($\mathfrak{h}'$ is not changed by this operation);*
- *and finally we enlarge $\mathfrak{p}'$ to a maximal admissible enlargement $\mathfrak{p}_1$ for $\hat{\mathfrak{g}}$ (as a result $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}) \overrightarrow{_{\mathfrak{p}_1}} (\mathfrak{g}, \mathfrak{h}_1)$)*

Proof. (of Theorem 0.2)

Let $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$ and $(\mathfrak{g}, \mathfrak{h})$ be obtained from $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ via $\mathfrak{p}$.

First of all, if the pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is not almost horospherical then $\mathfrak{h} \subset \mathfrak{h}_1$, where $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}_1) \overrightarrow{_{\mathfrak{p}}} (\mathfrak{g}, \mathfrak{h}_1)$ and $\tilde{\mathfrak{h}}_1 \supset \tilde{\mathfrak{h}}$ is a non-horospherical subalgebra of $\tilde{\mathfrak{h}}$. Also, if $\mathfrak{p}$ is not maximal admissible for $\tilde{\mathfrak{g}}$ then $\mathfrak{h} \subset \mathfrak{h}_1$, where $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \xrightarrow[\mathfrak{p}_{\max}]{} (\mathfrak{g}, \mathfrak{h}_1)$ and $\mathfrak{p}_{\max} \supset \mathfrak{p}$ is a maximal admissible for $\tilde{\mathfrak{g}}$ parabolic subalgebra.

In both cases it easy to verify that $(\mathfrak{g}, \mathfrak{h}_1)$ is not horospherical. Consequently, each almost horospherical reducible pair is obtained from a cuspidal almost horospherical pair by parabolic induction via a maximal admissible parabolic subalgebra.

Now let $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$ be cuspidal and almost horospherical and $(\mathfrak{g}, \mathfrak{h})$ be obtained from $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ via a maximal admissible for $\tilde{\mathfrak{g}}$ parabolic subalgebra $\mathfrak{p}$. Suppose that $\mathfrak{h}_1 \supset \mathfrak{h}$ is not horospherical.

Let us apply Theorem 1.3 of [8] and study what happens during the application of these three operations. If $\tilde{\mathfrak{h}}_1 \neq \tilde{\mathfrak{h}}$ then $\tilde{\mathfrak{h}}_1$ is horospherical, therefore $\mathfrak{h}_1$ is horospherical as well, and we come to a contradiction. Therefore, $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_1$, so we cannot pass to a deeper parabolic induction since $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is cuspidal. Hence we cannot enlarge the parabolic subalgebra for $\mathfrak{p}$ is maximal admissible, so $\mathfrak{h} = \mathfrak{h}_1$. ∎

Lemma 1.2. *Let $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ be the grading of $\mathfrak{g}$ defined by a subset $\Pi_1 \subset \Pi$ of the system of simple roots. Then $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i \geq 0$.*

Proof. It is enough to prove that for any root γ of a positive Π_1-height that there is a root α of Π_1-height 1 such that $[e_{-\alpha}, e_\gamma] \neq 0$. We will prove it by induction on the Π-height of γ.

The induction base is obvious. Let now $\mathrm{ht}_\Pi \gamma = i + 1$. There exists a root $\delta \in \Pi$ such that $[e_{-\delta}, e_\gamma] \neq 0$. If $\delta \in \Pi_1$ then we can set $\alpha = \delta$ and we are done. Otherwise $\gamma - \delta$ is a root of Π-height i and we can use the induction assumption and find such β that $\mathrm{ht}_{\Pi_1} \beta = 1$ and $[e_{-\beta}, [e_{-\delta}, e_\gamma]] \neq 0$. According to the Jacobi identity, either $\delta + \beta$ is a root and we can set $\alpha = \delta + \beta$, or $[e_{-\beta}, e_\gamma] \neq 0$ and we can set $\alpha = \beta$. ∎

Corollary 1.3. *If* $\mathfrak{h} \subset \mathfrak{g}_{\geq 1}$ *is a subalgebra such that* $\mathfrak{h} + \mathfrak{g}_{\geq 2} = \mathfrak{g}_{\geq 1}$ *then* $\mathfrak{h} = \mathfrak{g}_{\geq 1}$.

Proof. We have

$$\forall b \in \mathfrak{g}_1 \; \exists a \in \mathfrak{g}_{\geq 2} : b + a \in \mathfrak{h}. \tag{1}$$

Using this statement as the base of induction and Lemma 1.2 as the induction step, we prove

$$\forall b \in \mathfrak{g}_i \; \exists a \in \mathfrak{g}_{\geq i+1} : b + a \in \mathfrak{h} \text{ for } i > 0. \tag{2}$$

Let $j = \mathrm{depth}(\Pi_1)$. The formula (2) applied to $i = j$ gives us the inclusion $\mathfrak{g}_j \subset \mathfrak{h}$. Then, using the last inclusion and the formula (2) for $i = j - 1$, we prove that $\mathfrak{g}_{j-1} \subset \mathfrak{h}$ and so on. Finally we show that $\mathfrak{g}_{\geq 1} \subset \mathfrak{h}$, and we are done. ■

Now we can turn to the proof of Theorem 0.3. From here on we will suppose the pair $(\mathfrak{g}, \mathfrak{h})$ to be cuspidal and almost horospherical.

Let $\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{h}_u$ be the Levi decomposition of $\mathfrak{h}$. As it is well known, there is a parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{h}$ can be *properly included* in $\mathfrak{p}$, i.e. $\mathfrak{m} \subset \mathfrak{l}$ and $\mathfrak{h}_u \subset \mathfrak{p}_u$ where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$ is a Levi decomposition of $\mathfrak{p}$. Suppose that $\mathfrak{h}$ cannot be properly included in any proper parabolic subalgebra $\mathfrak{p}$. Then $\mathfrak{h}$ is reductive and maximal; consequently, according to the results of Dynkin, it is semisimple. Thus, we come to the cases 1 or 2 of the theorem.

¿From now on we will suppose that $\mathfrak{p} \neq \mathfrak{g}$.

Lemma 1.4. $\mathfrak{m} \supset [\mathfrak{l}, \mathfrak{l}]$.

Proof. Consider the subalgebra $\mathfrak{h}' = \mathfrak{h} + \mathfrak{p}_u$. Since the pair $(\mathfrak{g}, \mathfrak{h})$ is almost horospherical, either $\mathfrak{h} \supset \mathfrak{p}_u$, or $\mathfrak{h}'$ is horospherical. The first case is impossible since the pair $(\mathfrak{g}, \mathfrak{h})$ is cuspidal. Thus $\mathfrak{h}'$ is horospherical, therefore $\mathfrak{m} \supset [\mathfrak{l}, \mathfrak{l}]$. ■

Now let us replace the algebras by conjugate ones, in order to have $\mathfrak{p} \supset \mathfrak{b}$ and $\mathfrak{l} \supset \mathfrak{t}$. Set $\Pi_0 = \{\alpha \in \Pi | e_{-\alpha} \in \mathfrak{p}\}$, $\Pi_1 = \Pi \backslash \Pi_0$ and consider the grading defined by Π_1. Then $\mathfrak{p} = \mathfrak{p}(\Pi_1)$ and $\mathfrak{l} = \mathfrak{g}_0$. The algebra $\mathfrak{a} = \{a \in \mathfrak{t} | \alpha(a) = 0, \forall \alpha \in \Pi_0\}$ is the direct complement of $[\mathfrak{l}, \mathfrak{l}]$ in $\mathfrak{l}$, i.e. it is the center of $\mathfrak{l}$.

Lemma 1.5. $\mathfrak{h} \supset \mathfrak{g}_{\geq 2}$.

Proof. Suppose the contrary. The set $\mathfrak{h}_u + \mathfrak{g}_{\geq 2}$ is a subalgebra. Since $\mathfrak{h}$ is almost horospherical, we have $\mathfrak{h}_u + \mathfrak{g}_{\geq 2} = \mathfrak{g}_{\geq 1}$, but then according to Corollary 1.3, $\mathfrak{h}_u = \mathfrak{g}_{\geq 1}$, and we come to a contradiction. ■

Let us study the $\mathfrak{m}$-invariant intersection $\mathfrak{h} \cap \mathfrak{g}_1$ now. To do that, let us consider the $\mathfrak{m}$-equivariant decomposition $\mathfrak{g}_1 = \oplus_{\alpha \in \Pi_1} \mathfrak{g}_1(\alpha)$, where

$$\mathfrak{g}_1(\alpha) := \bigoplus_{\beta \in \Delta \cap (\alpha + \langle \Pi_0 \rangle)} \mathbb{C}\, e_\beta$$

is $\mathfrak{m}$-irreducible (see [9]).

Lemma 1.6. *The* $\mathfrak{m}$*-representations* $\mathfrak{g}_1(\alpha)$ *are isomorphic to each other,* $\mathfrak{h}$ *does not contain any of the vector spaces* $\mathfrak{g}_1(\alpha)$*, and* $\mathfrak{g}$ *is simple.*

Proof. Let V be an isotypical component of $\mathfrak{g}_1$ not contained in $\mathfrak{h}$ and $\mathfrak{g}_1 = V \oplus V'$. The vector subspace $\mathfrak{h} \cap \mathfrak{g}_1$ is $\mathfrak{m}$-invariant, therefore $\mathfrak{h} \cap \mathfrak{g}_1 = (\mathfrak{h} \cap V) \oplus (\mathfrak{h} \cap V')$. Hence $\mathfrak{h} + V'$ is a non-horospherical subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Since $\mathfrak{h}$ is almost horospherical, we have $\mathfrak{h} \supset V'$.

Now let $\Pi' \subset \Pi_1$ be the subset of simple roots α such that $\mathfrak{h} \supset \mathfrak{g}_1(\alpha)$, $\mathfrak{p}' = \mathfrak{p}(\Pi')$. Then $\mathfrak{h}$ contains the unipotent radical of $\mathfrak{p}'$. Now it is enough to note that $(\mathfrak{g}, \mathfrak{h})$ is cuspidal. ∎

Lemma 1.7. *1. $\mathfrak{h} \cap \mathfrak{a} = \mathbb{C}h_0$, where h_0 is the element such that $\alpha(h_0) = 1$ for $\alpha \in \Pi_1$ and $\alpha(h_0) = 0$ for $\alpha \in \Pi_0$;*

2. if one fixes isomorphisms $\mathfrak{g}_1(\alpha) \cong \mathfrak{g}_1(\alpha')$ for all $\alpha, \alpha' \in \Pi_1$, then the intersection $\mathfrak{h} \cap \mathfrak{g}_1$ can be given by an equation $\sum c_\alpha x_\alpha = 0$, where $x_\alpha \in \mathfrak{g}_1(\alpha)$, and c_α is a set of non-zero coefficients;

3. replacing $\mathfrak{h}$ by a conjugate subalgebra, one may assume $c_\alpha = 1$ for all α;

Proof. It follows from Lemma 1.6 that any element of $\mathfrak{h} \cap \mathfrak{a}$ must act by the same character on the root vectors e_α for $\alpha \in \Pi_1$. Hence $\mathfrak{h} \cap \mathfrak{a} \subset \mathbb{C}h_0$. On the other hand, $\mathfrak{h} + \mathbb{C}h_0$ is not horospherical, therefore $\mathfrak{h} \cap \mathfrak{a} = \mathbb{C}h_0$, and we have proved the first statement of the lemma.

Observe now that for any $\mathfrak{m}$-invariant subspace V of $\mathfrak{g}_1$ the set $\mathfrak{m} + \mathbb{C}h_0 + V + \mathfrak{g}_{\geq 2}$ is an algebra. The algebra $\mathfrak{h}$ is almost horospherical, therefore $\dim(\mathfrak{g}_1) - \dim(\mathfrak{h} \cap \mathfrak{g}_1) = \dim(\mathfrak{g}_1(\alpha))$ for $\alpha \in \Pi_1$. It proves the second statement of the lemma, and the third one is obvious. ∎

Now we can complete the proof of Theorem 0.3.

Proof. The $\mathfrak{m}$-representation spaces $\mathfrak{g}_1(\alpha_1)$ and $\mathfrak{g}_1(\alpha_2)$ are isomorphic to each other for $\alpha_1, \alpha_2 \in \Pi_1$, therefore the $\mathfrak{t} \cap \mathfrak{m}$-weights of α_1 and α_2 coincide. In other words, the points corresponding to the roots α_1 and α_2 on the Dynkin diagram are located "in the same position" with respect to the subdiagram corresponding to Π_0. Suppose first that those weights are equal to zero. Then the Dynkin diagram splits into two diagrams, corresponding to Π_0 and Π_1. According to Lemma 1.6, one of those diagrams must be empty, hence $\Pi = \Pi_1$. Now Lemma 1.7 leads us directly to the fourth possibility of Theorem 0.3.

Now let us turn to the case when $\Pi_0 \neq \emptyset$.

To start with, let $\#\Pi_1 = 1$. According to Lemma 1.7, $\mathfrak{h} \cap \mathfrak{g}_1 = \{0\}$ and $\mathfrak{t} \subset \mathfrak{h}$. The algebra $\mathfrak{h}$ can be described in the following way: there is a maximal parabolic subalgebra $\mathfrak{g}_{\geq 0}$ in $\mathfrak{g}$, and $\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_{\geq 2}$. If $\operatorname{depth}(\Pi_1) = 1$, then $\mathfrak{h} = \mathfrak{g}_0$, so we come to the third possibility of Theorem 0.3. If $\operatorname{depth}(\Pi_1) = 2$, then we come to a contradiction, since in this case $\mathfrak{h}$ can be included in a semisimple subalgebra $\mathfrak{g}_0 + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. However, the case $\operatorname{depth}(\Pi_1) \geq 3$ can occur only for an exceptional $\mathfrak{g}$, so we come to the sixth possibility of Theorem 0.3.

Now let $\#\Pi_1 = 2$. Then there are two points on the Dynkin diagram located "in the same position" with respect to all the other points; thus $\mathfrak{g} = D_n$, $[\mathfrak{l}, \mathfrak{l}] = A_{n-2}$, and $\Pi_1 = \{\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n\}$. Then, according to Lemma 1.7, $\mathfrak{h} = \mathfrak{g}_{\geq 2} + \mathbb{C}h_{\varepsilon_1 + \ldots + \varepsilon_{n-1}} + [\mathfrak{l}, \mathfrak{l}] + \sum_{i=1,\ldots,n-1} \mathbb{C}(e_{\varepsilon_i + \varepsilon_n} + e_{\varepsilon_i - \varepsilon_n})$, but this algebra is not almost horospherical since $\mathfrak{h} \subset B_{n-1} \subset D_n$. Thus, $\#\Pi_1 \neq 2$.

A Dynkin diagram cannot have vertices of degree more than three, thus $\#\Pi_1 \leq 3$. We have one possibility left: $\#\Pi_1 = 3$, so $\mathfrak{g} = D_4$. It follows that $\mathfrak{l} = \mathfrak{t} + \mathbb{C}e_{\alpha_2} + \mathbb{C}e_{-\alpha_2}$, $\mathfrak{g}_1 = \mathbb{C}e_{\alpha_1} + \mathbb{C}e_{\alpha_3} + \mathbb{C}e_{\alpha_4} + \mathbb{C}e_{\alpha_1+\alpha_2} + \mathbb{C}e_{\alpha_3+\alpha_2} + \mathbb{C}e_{\alpha_4+\alpha_2}$ and $\mathfrak{g}_{\geq 2} = \mathbb{C}e_{\alpha_1+\alpha_3+\alpha_4+\alpha_2} + \mathbb{C}e_{\alpha_1+\alpha_3+\alpha_4+2\alpha_2} + \mathbb{C}e_{\alpha_1+\alpha_3+\alpha_2} + \mathbb{C}e_{\alpha_1+\alpha_4+\alpha_2} + \mathbb{C}e_{\alpha_3+\alpha_4+\alpha_2}$.

According to Lemma 1.7, $2h_{\alpha_1} + 3h_{\alpha_2} + 2h_{\alpha_3} + 2h_{\alpha_4} = h_0 \in \mathfrak{h}$ and $c_1e_{\alpha_1} + c_3e_{\alpha_3} + c_4e_{\alpha_4} \in \mathfrak{h} \Leftrightarrow c_1e_{\alpha_1+\alpha_2} + c_3e_{\alpha_3+\alpha_2} + c_4e_{\alpha_4+\alpha_2} \in \mathfrak{h} \Leftrightarrow c_1 + c_3 + c_4 = 0$. So we come to the fifth possibility of Theorem 0.3.

So now we know that all cuspidal almost horospherical pairs are listed in Theorem 0.3.

It remains to prove the inverse statement: that all pairs listed in Theorem 0.3 are almost horospherical and cuspidal.

Let us first prove that they are almost horospherical. It is clear for the first two possibilities for they are maximal.

Assume the contrary. Let $\mathfrak{h}' \supsetneq \mathfrak{h}$ be a non-horospherical subalgebra.

Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, and let $\mathfrak{h} = \mathfrak{g}_0$ be the third case. The representations $\mathfrak{g}_0 : \mathfrak{g}_{\pm 1}$ are irreducible, therefore $\mathfrak{h}$ contains either $\mathfrak{g}_1$ or $\mathfrak{g}_{-1}$, and we come to a contradiction.

Consider the fourth case. It is easy to see that $\mathfrak{h}' \cap \mathfrak{p} = \mathfrak{h}$. Choose an element $e := \sum_{\alpha\in\Pi} c_\alpha e_\alpha \in \mathfrak{h}$ with all c_α not equal to zero and include it in an $\mathfrak{sl}_2$-triple (e, h, f). Then it follows from the theory of $\mathfrak{sl}_2$ representations that $h = 2h_0$ and the operator $a \to [e, a]$ is injective on the negative components. The algebra $\mathfrak{h}'$ is an (e, h)-invariant subspace of $\mathfrak{g}$, therefore $[e, \mathfrak{h}' \cap \mathfrak{g}_{-1}] = \mathbb{C}h_0$, and so $\mathfrak{h}' \cap \mathfrak{g}_{-1} = \mathbb{C}f$. The operator $a \to [f, a]$ preserves $\mathfrak{h}'$ and is injective on the positive components, so $[f, \mathfrak{g}_2] = \mathbb{C}e$, but this is impossible since e is a highest vector for $\mathfrak{sl}_2$.

Consider the fifth case. Similarly, we have $\mathfrak{h}' \cap \mathfrak{p} = \mathfrak{h}$ and $\mathfrak{h}' \cap \mathfrak{g}_{-1} \neq \{0\}$. Let $x = c_1e_{-\alpha_1} + c_3e_{-\alpha_3} + c_4e_{-\alpha_4} + c_1'e_{-\alpha_1-\alpha_2} + c_3'e_{-\alpha_3-\alpha_2} + c_4'e_{-\alpha_4-\alpha_2}$ be a non-zero element of this intersection. We have $[\mathfrak{h} \cap \mathfrak{g}_1, x] \subset \mathfrak{m}$, thus $[e_{\alpha_1} - e_{\alpha_3}, x] \in \mathfrak{m}$. But the last element is equal to $c_1h_{\alpha_1} + c_3h_{\alpha_3} + te_{-\alpha_2}$ for some number t. Therefore $c_1h_{\alpha_1} + c_3h_{\alpha_3} \in \mathfrak{m}$, so $c_1 = c_3 = 0$. Analogously we prove that the other coefficients are also equal to zero, so $x = 0$ and we come to a contradiction.

Consider the sixth case. Suppose first that $\mathfrak{h}'$ can be properly included into a parabolic subalgebra $\mathfrak{p}' \neq \mathfrak{g}$. We have $\mathfrak{p}' \supset \mathfrak{g}_0$, therefore either $\mathfrak{p}' = \mathfrak{g}_{\leq 0}$ or $\mathfrak{p}' = \mathfrak{g}_{\geq 0}$. The first case is obviously impossible, and in follows from the second one that $\mathfrak{h}' \subset \mathfrak{g}_{\geq 0}$. Hence $\mathfrak{h}' \cap \mathfrak{g}_1 \neq 0$, but the representation $\mathfrak{g}_0 : \mathfrak{g}_1$ is irreducible, and we come to a contradiction. Therefore there is no such subalgebra $\mathfrak{p}'$, so $\mathfrak{h}'$ is reductive. Then $\mathfrak{h}' \supset \mathfrak{g}_{-2}$, and it is enough to note that $[\mathfrak{g}_{-2}, \mathfrak{g}_3] \neq \{0\}$.

Let us now prove that the pairs listed in Theorem 0.3 are cuspidal. Suppose the contrary. Then $\mathfrak{h}$ contains the unipotent radical of a parabolic subalgebra $\mathfrak{p}$. Obviously it is impossible for the first three cases of the theorem, for those subalgebras are reductive. In the sixth case we have $\mathfrak{p} \supset \mathfrak{g}_0$, therefore either $\mathfrak{p} = \mathfrak{g}_{\leq 0}$ or $\mathfrak{p} = \mathfrak{g}_{\geq 0}$, but both those possibilities are impossible for $\mathfrak{p}_u \not\subset \mathfrak{h}$.

Let us consider the two remaining cases. First of all, let us prove that $\mathfrak{p} \supset \mathfrak{t}$. If we consider the fourth case, then $h_0 \in \mathfrak{p}$. It is a semisimple element that can be included in a Cartan subalgebra of $\mathfrak{p}$. But the centralizer of h_0 coincides with $\mathfrak{t}$, hence $\mathfrak{p} \supset \mathfrak{t}$. In the fifth case an analogous proof works for a general linear combination of h_0 and h_{α_2}. Now we know that $\mathfrak{p} \supset \mathfrak{t}$ and $\mathfrak{p} \supset \mathfrak{h}$, therefore

$\mathfrak{p} \supset [\mathfrak{t}, \mathfrak{h}] \supset \mathfrak{b}$. Consequently there must be a simple root α such that $e_\alpha \in \mathfrak{p}_u$ and $e_{-\alpha} \notin \mathfrak{p}$, so $e_\alpha \in \mathfrak{h}$ and $e_{-\alpha} \notin \mathfrak{h}$. A contradiction.

■

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