Journal of Lie Theory
Volume **17** (2007) 163–189

Spectral Multipliers on Damek–Ricci Spaces

Maria Vallarino*

Communicated by M. Cowling

Abstract. Let S be a Damek–Ricci space, and Δ be a distinguished Laplacean on S which is left invariant and selfadjoint in $L^2(\rho)$. We prove that S is a Calderón–Zygmund space with respect to the right Haar measure ρ and the left invariant distance. We give sufficient conditions of Hörmander type on a multiplier m so that the operator $m(\Delta)$ is bounded on $L^p(\rho)$ when $1 < p < \infty$, and of weak type $(1,1)$. *Mathematics Subject Index 2000:* 22E30, 42B15, 42B20, 43A80.
Keywords and phrases: Multipliers, singular integrals, Calderon–Zygmund decomposition, Damek–Ricci spaces.

1. Introduction

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type algebra and let N be the connected and simply connected Lie group associated to $\mathfrak{n}$ (see Section 2 for the details). Let S be a one-dimensional harmonic extension of N. Specifically, let S be the one-dimensional extension of N obtained by making $A = \mathbb{R}^+$ act on N by homogeneous dilations. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be a unit vector. The Lie algebra $\mathfrak{s}$ is solvable.

Let λ and ρ denote left and right invariant Haar measures on S, and d a *left invariant* Riemannian metric on S. It is well known that the right Haar measure of geodesic balls is an exponentially growing function of the radius, so that S is a group of *exponential growth.*

Harmonic extensions of H-type groups, now called Damek–Ricci spaces, were introduced by E. Damek and F. Ricci [9], [10], [11], [12], and include all rank one symmetric spaces of the noncompact type. Most of them are nonsymmetric harmonic manifolds, and provide counterexamples to the Lichnerowicz conjecture. The geometry of these extensions was studied by M. Cowling, A. H. Dooley, A. Korányi and Ricci in [5], [6].

* **Acknowledgement**. Work partially supported by the Progetto Cofinanziato MURST "Analisi Armonica" and the Progetto GNAMPA "Calcolo funzionale per generatori di semi-gruppi e analisi armonica su gruppi"

ISSN 0949–5932 / $2.50

Let $\{E_0,\dots,E_{n-1}\}$ be an orthonormal basis of the algebra $\mathfrak{s}$ such that $E_0=H$, $\{E_1,\dots,E_{m_\mathfrak{v}}\}$ is an orthonormal basis of $\mathfrak{v}$ and $\{E_{m_\mathfrak{v}+1},\dots,E_{n-1}\}$ is an orthonormal basis of $\mathfrak{z}$. Let $X_0,X_1,\dots,X_{n-1}$ be the left invariant vector fields on S which agree with $E_0,E_1,\dots,E_{n-1}$ at the identity. Let Δ be the operator defined by

$$\Delta=-\sum_{i=0}^{n-1}X_i^2\,;$$

the *Laplacean* Δ is *left invariant* and essentially selfadjoint on $C_c^\infty(S)\subset L^2(\rho)$. Therefore there exists a spectral resolution E_Δ of the identity for which

$$\Delta f=\int_0^\infty t\,\mathrm{d}E_\Delta(t)f\qquad\forall f\in\mathrm{Dom}(\Delta).$$

By the spectral theorem, for each bounded measurable function m on $\mathbb{R}^+$ the operator $m(\Delta)$ defined by

$$m(\Delta)f=\int_0^\infty m(t)\,\mathrm{d}E_\Delta(t)f\qquad\forall f\in L^2(\rho),$$

is bounded on $L^2(\rho)$; $m(\Delta)$ is called the *spectral operator* associated to the *spectral multiplier* m. A classical problem is to find conditions on m that ensure that $m(\Delta)$ extends to a bounded operator from $L^1(\rho)$ to the Lorentz space $L^{1,\infty}(\rho)$ and to a bounded operator on $L^p(\rho)$ when $1<p<\infty$. The main result of this paper (see Theorem 4.3 below) is that this holds if m satisfies a Hörmander type condition of suitable order.

An analogue Δ_r of Δ on the solvable groups coming from the Iwasawa decomposition of noncompact connected semisimple Lie groups of finite centre (any rank) was studied by Cowling, S. Giulini, A. Hulanicki and G. Mauceri [7]. They proved that if a bounded measurable function m on $\mathbb{R}^+$ belongs locally to a suitable Sobolev space and satisfies Hörmander conditions of suitable order at infinity, then the operator $m(\Delta_r)$ is bounded from $L^1(\lambda)$ to $L^{1,\infty}(\lambda)$ and on $L^p(\lambda)$ when $1<p<\infty$. A similar result in the case of Damek–Ricci spaces was proved by F. Astengo [2].

We warn the reader that actually these authors studied a Laplacean Δ_r which is right invariant and selfadjoint on $L^2(\lambda)$, whereas Δ is left invariant and selfadjoint on $L^2(\rho)$. However, it is straightforward to check that

$$\Delta f=(\Delta_r\check{f})\check{}\qquad\forall f\in C_c^\infty(S)\,,$$

where $\check{f}(x)=f(x^{-1})$, and hence

$$m(\Delta)f=(m(\Delta_r)\check{f})\check{}\qquad\forall f\in C_c^\infty(S)\,,$$

for every bounded measurable function m on $\mathbb{R}^+$. Since $\check{}$ is an isometry between $L^p(\lambda)$ and $L^p(\rho)$ if $1\le p\le\infty$, and between $L^{1,\infty}(\lambda)$ and $L^{1,\infty}(\rho)$, boundedness results for $m(\Delta)$ on $L^p(\rho)$ may be rephrased as boundedness results for $m(\Delta_r)$ on $L^p(\lambda)$.

Subsequently W. Hebisch and T. Steger [15] sharpened the result in [7] and [2], proving a genuine Hörmander type theorem for spectral multipliers of Δ_r in the special case of solvable groups associated to real hyperbolic spaces.

In this paper we extend the result in [15] to all Damek–Ricci spaces. To be more specific, we prove that if a bounded measurable function m on $\mathbb{R}^+$ satisfies Hörmander conditions both at infinity and locally, then the operator $m(\Delta)$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$ and on $L^p(\rho)$ when $1 < p < \infty$.

The strategy of the proof, which is similar to that of [15, Theorem 2.4], is to show that $m(\Delta)$ may be realized as a singular integral operator, and that such operators are bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$ and on $L^p(\rho)$ when $1 < p < \infty$.

The classical theory of higher dimensional singular integrals was developed by A. Calderón and A. Zygmund. One of the basic results of this theory is the so-called *Calderón–Zygmund decomposition*, where each integrable function f is decomposed as $g + \sum_{j=1}^{\infty} b_j$, where the "good" function g is bounded, and each "bad" function b_j may be unbounded but is supported in a ball and its integral vanishes.

The Calderón–Zygmund theory was extended to *spaces of homogeneous type* by R. Coifman and G. Weiss [8]. These spaces are measured metric spaces (X, μ, ϱ), where balls satisfy the so-called doubling condition, i.e., there exists a constant C such that

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \qquad \forall x \in X \qquad \forall r \in \mathbb{R}^+. \tag{1}$$

The main result of the theory is that an integral operator, which is bounded on $L^2(\mu)$, and whose kernel satisfies the so-called Hörmander condition

$$\sup_{y \in X} \sup_{r > 0} \int_{B(y, \kappa r)^c} |K(x, y) - K(x, z)|\, \mathrm{d}\mu(x) < \infty \qquad \forall z \in B(y, r), \tag{2}$$

extends to a bounded operator on $L^p(\mu)$ when $1 < p < 2$ and to a bounded operator from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. We remark that the doubling condition (1) plays a fundamental rôle in the theory. Hence, it does not apply to (S, ρ, d), because the doubling condition for balls fails.

Recently, Hebisch and Steger [15] realized that a Calderón–Zygmund like theory of singular integrals may be developed on some groups of exponential growth. The main idea is that the rôle of balls in the classical theory may be played by other types of sets.

First, they defined a *Calderón–Zygmund space* (see Section 3 for the details). Roughly speaking, a metric measured space (X, μ, ϱ) is a Calderón–Zygmund space with Calderón–Zygmund constant κ_0, provided that each integrable function f may be decomposed as $g + \sum_{j=1}^{\infty} b_j$, where the "good" function g is bounded, and each "bad" function b_j is unbounded, supported in a set R_j and its integral vanishes. The main difference with the classical theory is that R_j need not be a ball, but there exist a point x_j in X and a positive number r_j such that the following two conditions hold:

(i) $R_j \subseteq B(x_j, \kappa_0\, r_j) \qquad \forall j \in \mathbb{N}$;

(ii) $\mu(R_j^*) \leq \kappa_0\, \mu(R_j)$ where $R_j^* = \{x \in X : \varrho(x, R_j) < r_j\}$.

Clearly, (ii) is a substitute for the doubling condition for balls.

Then Hebisch and Steger proved that an integral operator on a Calderón–Zygmund space that is bounded on $L^2(\mu)$ and whose kernel satisfies the Hörmander condition (2), extends to a bounded operator on $L^p(\mu)$ when $1 < p < 2$, and to a bounded operator from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

In this paper we prove that all Damek–Ricci spaces (S, ρ, d) are Calderón–Zygmund spaces. It is worth pointing out that even for complex hyperbolic spaces this result is new. To do so, we shall use a family of suitable sets in S which we call *admissible sets* and describe in detail in Section 3. It may be worth observing that "small sets" are balls of small radius, while "big sets" are rectangles, i.e. products of dyadic sets in N and intervals in A.

We apply this result to study the $L^p(\rho)$ boundedness of spectral multipliers associated to the Laplacean Δ.

Our paper is organized as follows: in Section 2, we recall the definition of an H-type group N and its Damek–Ricci extension S; then we summarize some results of spherical analysis on S. In Section 3, we prove that S, endowed with the right Haar measure ρ and the left invariant metric d, is a Calderón–Zygmund space. In Section 4, we prove a Hörmander type theorem for spectral multipliers associated to the Laplacean Δ. The proof of this hinges on an L^1-estimate of the gradient of the heat kernel associated to Δ.

Positive constants are denoted by C; these may differ from one line to another, and may depend on any quantifiers written, implicitly or explicitly, before the relevant formula.

The author would like to thank Stefano Meda for his precious help and encouragement.

2. Damek–Ricci spaces

In this section we recall the definition of H-type groups, describe their Damek–Ricci extensions, and recall the main results of spherical analysis on these spaces. For the details see [1], [2], [3], [5], [6], [13].

Let $\mathfrak{n}$ be a Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$ and denote by $|\cdot|$ the corresponding norm. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n}, \mathfrak{z}] = \{0\}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$. According to Kaplan [17], the algebra $\mathfrak{n}$ is of H-type if for every Z in $\mathfrak{z}$ of unit length the map $J_Z : \mathfrak{v} \to \mathfrak{v}$, defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle \qquad \forall X, Y \in \mathfrak{v}\,,$$

is orthogonal. The connected and simply connected Lie group N associated to $\mathfrak{n}$ is called an H-type group. We identify N with its Lie algebra $\mathfrak{n}$ via the exponential map

$$\begin{aligned} \mathfrak{v} \times \mathfrak{z} &\to N \\ (X, Z) &\mapsto \exp(X + Z)\,. \end{aligned}$$

The product law in N is

$$(X,Z)(X',Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']) \qquad \forall X,\, X' \in \mathfrak{v} \quad \forall Z,\, Z' \in \mathfrak{z}\,.$$

The group N is a two-step nilpotent group, hence unimodular, with Haar measure $\mathrm{d}X\,\mathrm{d}Z$. We define the following dilations on N:

$$\delta_a(X,Z)=(a^{1/2}X,aZ)\qquad\forall(X,Z)\in N\quad\forall a\in\mathbb{R}^+\,.$$

The group N is an homogeneous group with homogeneous norm

$$\mathcal{N}(X,Z)=\left(\frac{|X|^4}{16}+|Z|^2\right)^{1/4}\qquad\forall(X,Z)\in N\,.$$

Note that $\mathcal{N}(\delta_a(X,Z))=a^{1/2}\mathcal{N}(X,Z)$. Given (X_0,Z_0) in N and $r>0$, the homogeneous ball centred at (X_0,Z_0) of radius r is

$$B_N((X_0,Z_0),r)=\{(X,Z)\in N:\mathcal{N}((X_0,Z_0)^{-1}(X,Z))<r\}\,.$$

Obviously, $B_N(0_N,r)=\delta_{r^2}B_N(0_N,1)$. Set $Q=(m_{\mathfrak{v}}+2m_{\mathfrak{z}})/2$, where $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$ denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$. The measure of the ball $B_N((X_0,Z_0),r)$ is $r^{2Q}|B_N(0_N,1)|$.

Let S and $\mathfrak{s}$ be as in the Introduction. The map

$$\begin{array}{rcl}\mathfrak{v}\times\mathfrak{z}\times\mathbb{R}^+ & \to & S\\ (X,Z,a) & \mapsto & \exp(X+Z)\exp(\log a\,H)\end{array}$$

gives global coordinates on S. The product in S is given by the rule

$$(X,Z,a)(X',Z',a')=(X+a^{1/2}X',Z+a\,Z'+\frac{1}{2}\,a^{1/2}[X,X'],a\,a')$$

for all (X,Z,a), $(X',Z',a')\in S$. Let e be the identity of the group S. We shall denote by n the dimension $m_{\mathfrak{v}}+m_{\mathfrak{z}}+1$ of S. The group S is nonunimodular: the right and left Haar measures on S are given by

$$\mathrm{d}\rho(X,Z,a)=a^{-1}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a\qquad\text{and}\qquad\mathrm{d}\lambda(X,Z,a)=a^{-(Q+1)}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a\,.$$

Then the modular function is $\delta(X,Z,a)=a^{-Q}$. We denote by $L^p(\rho)$ the space of all measurable functions f such that $\int_S|f|^p\,\mathrm{d}\rho<\infty$ and by $L^{1,\infty}(\rho)$ the Lorentz space of all measurable functions f such that

$$\sup_{t>0}t\,\rho(\{x\in S:|f(x)|>t\})<\infty\,.$$

We equip S with the left invariant Riemannian metric which agrees with the inner product on $\mathfrak{s}$ at the identity e. Denote by d the distance induced by this Riemannian structure. From [3, formula (2.18)], for all (X,Z,a) in S,

$$\cosh^2\left(\frac{d((X,Z,a),e)}{2}\right)=\left(\frac{a^{1/2}+a^{-1/2}}{2}+\frac{1}{8}\,a^{-1/2}|X|^2\right)^2+\frac{1}{4}\,a^{-1}|Z|^2\,.\qquad(3)$$

We denote by $B((X_0,Z_0,a_0),r)$ the ball in S with centre (X_0,Z_0,a_0) and radius r. We write B_r for the ball with centre e and radius r; from [3, formula (1.18)], there exist positive constants γ_1, γ_2 such that, for all r in $(0,1)$,

$$\gamma_1\,r^n\leq\rho(B_r)\leq\gamma_2\,r^n\,,\qquad(4)$$

and for all r in $[1,\infty)$

$$\gamma_1\,\mathrm{e}^{Qr}\leq\rho(B_r)\leq\gamma_2\,\mathrm{e}^{Qr}\,.$$

This shows that S, equipped with the right Haar measure ρ, is a group of exponential growth.

From (3) we can easily deduce various properties of balls in S.

Proposition 2.1. *The following hold:*

(i) there exists a constant c_3 *such that*

$$B(e,\log r)\subset B_N(0_N,c_3\,r)\times(1/r,r)\qquad\forall r\in(1,\infty)\,;$$

(ii) if $b>0$ *and* $B>1/2$*, then there exists a constant* $c_{b,B}$ *such that*

$$B_N(0_N,b\,r^B)\times(1/r,r)\subset B(e,c_{b,B}\,\log r)\qquad\forall r\in[\mathrm{e},+\infty)\,.$$

A radial function on S is a function that depends only on the distance from the identity. If f is radial, then by [3, formula (1.16)],

$$\int_S f\,\mathrm{d}\lambda=\int_0^\infty f(r)\,A(r)\,\mathrm{d}r\,,$$

where

$$A(r)=2^{m_{\mathfrak v}+2m_{\mathfrak z}}\sinh^{m_{\mathfrak v}+m_{\mathfrak z}}\left(\frac{r}{2}\right)\cosh^{m_{\mathfrak z}}\left(\frac{r}{2}\right)\qquad\forall r\in\mathbb{R}^+\,.$$

We shall use repeatedly the fact that

$$A(r)\le C\left(\frac{r}{1+r}\right)^{n-1}\mathrm{e}^{Qr}\qquad\forall r\in\mathbb{R}^+\,.\tag{5}$$

A radial function ϕ is spherical if it is an eigenfunction of the Laplace–Beltrami operator $\mathcal{L}$ (associated to d) and $\phi(e)=1$. For s in $\mathbb{C}$, let ϕ_s be the spherical function with eigenvalue $s^2+Q^2/4$, as in [3, formula (2.6)]. In [2, Lemma 1], it is shown that

$$\phi_0(r)\le C\,(1+r)\,\mathrm{e}^{-Qr/2}\qquad\forall r\in\mathbb{R}^+\,.\tag{6}$$

We shall use the following integration formula on S, whose proof is like that of [7, Lemma 1.3] and [1, Lemma 3]:

Lemma 2.2. *For every radial function* f *in* $C_c^\infty(S)$,

$$\int_S\delta^{1/2}f\,\mathrm{d}\rho=\int_0^\infty\phi_0(r)\,f(r)\,A(r)\,\mathrm{d}r\,.$$

The spherical Fourier transform $\mathcal{H}f$ of an integrable radial function f on S is defined by the formula

$$\mathcal{H}f(s)=\int_S\phi_s\,f\,\mathrm{d}\lambda\,.$$

For "nice" radial functions f on S, an inversion formula and a Plancherel formula hold:

$$f(x)=c_S\int_0^\infty\mathcal{H}f(s)\,\phi_s(x)\,|\mathbf{c}(s)|^{-2}\,\mathrm{d}s\qquad\forall x\in S\,,$$

and

$$\int_S|f|^2\,\mathrm{d}\lambda=c_S\int_0^\infty|\mathcal{H}f(s)|^2\,|\mathbf{c}(s)|^{-2}\,\mathrm{d}s\,,$$

where the constant c_S depends only on $m_{\mathfrak v}$ and $m_{\mathfrak z}$, and $\mathbf{c}$ denotes the Harish-Chandra function. For later developments, we recall from [2, formula (1)] that

$$|\mathbf{c}(s)|^{-2}\le\begin{cases}C\,|s|^2&\forall\,s\in[-1,1]\\ C\,|s|^{n-1}&\forall s\in\mathbb{R}-[-1,1]\,.\end{cases}$$

Let $\mathcal{A}$ denote the Abel transform and let $\mathcal{F}$ denote the Fourier transform on the real line, defined by

$$\mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(r)\, \mathrm{e}^{-isr}\, \mathrm{d}r\,,$$

for each integrable function g on $\mathbb{R}$. It is well known that $\mathcal{H} = \mathcal{F} \circ \mathcal{A}$, hence $\mathcal{H}^{-1} = \mathcal{A}^{-1} \circ \mathcal{F}^{-1}$. For later use, we recall the inversion formula for the Abel transform [3, formula (2.24)]. We define the differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on the real line by

$$\mathcal{D}_1 = -\frac{1}{\sinh r}\frac{\partial}{\partial r}\,, \qquad \mathcal{D}_2 = -\frac{1}{\sinh(r/2)}\frac{\partial}{\partial r}\,. \tag{7}$$

If $m_{\mathfrak{z}}$ is even, then

$$\mathcal{A}^{-1}f(r) = a_S^e\, \mathcal{D}_1^{m_{\mathfrak{z}}/2}\mathcal{D}_2^{m_{\mathfrak{v}}/2} f(r)\,, \tag{8}$$

where $a_S^e = 2^{-(3m_{\mathfrak{v}}+m_{\mathfrak{z}})/2}\pi^{-(m_{\mathfrak{v}}+m_{\mathfrak{z}})/2}$, while if $m_{\mathfrak{z}}$ is odd, then

$$\mathcal{A}^{-1}f(r) = a_S^o \int_r^{\infty} \mathcal{D}_1^{(m_{\mathfrak{z}}+1)/2}\mathcal{D}_2^{m_{\mathfrak{v}}/2} f(s)\, \mathrm{d}\nu(s)\,, \tag{9}$$

where $a_S^o = 2^{-(3m_{\mathfrak{v}}+m_{\mathfrak{z}})/2}\pi^{-n/2}$ and $\mathrm{d}\nu(s) = (\cosh s - \cosh r)^{-1/2} \sinh s\, \mathrm{d}s$.

Now consider the Laplacean Δ on S, as defined in the introduction. The operator Δ has a special relationship with the Laplace–Beltrami operator $\mathcal{L}$. Indeed, denote by $\mathcal{L}_Q$ the shifted operator $\mathcal{L} - Q^2/4$; then by [1, Proposition 2],

$$\delta^{-1/2}\Delta\, \delta^{1/2} f = \mathcal{L}_Q f \tag{10}$$

for all smooth compactly supported radial functions f on S. The spectra of $\mathcal{L}_Q$ on $L^2(\lambda)$ and Δ on $L^2(\rho)$ are both $[0, +\infty)$. Let $E_{\mathcal{L}_Q}$ and E_Δ be the spectral resolution of the identity for which

$$\mathcal{L}_Q = \int_0^{+\infty} t\; \mathrm{d}E_{\mathcal{L}_Q}(t) \qquad \text{and} \qquad \Delta = \int_0^{+\infty} t\; \mathrm{d}E_\Delta(t)\,.$$

For each bounded measurable function m on $\mathbb{R}^+$ the operators $m(\mathcal{L}_Q)$ and $m(\Delta)$, spectrally defined by

$$m(\mathcal{L}_Q) = \int_0^{+\infty} m(t)\, \mathrm{d}E_{\mathcal{L}_Q}(t) \qquad \text{and} \qquad m(\Delta) = \int_0^{+\infty} m(t)\, \mathrm{d}E_\Delta(t)\,,$$

are bounded on $L^2(\lambda)$ and $L^2(\rho)$ respectively. By (10) and the spectral theorem,

$$\delta^{-1/2} m(\Delta)\, \delta^{1/2} f = m(\mathcal{L}_Q) f\,,$$

for smooth compactly supported radial functions f on S. Let $k_{m(\Delta)}$ and $k_{m(\mathcal{L}_Q)}$ denote the convolution kernels of $m(\Delta)$ and $m(\mathcal{L}_Q)$ respectively; then

$$m(\mathcal{L}_Q) f = f * k_{m(\mathcal{L}_Q)} \qquad \text{and} \qquad m(\Delta) f = f * k_{m(\Delta)} \qquad \forall f \in C_c^\infty(S)\,,$$

where $*$ denotes the convolution on S, defined by

$$\begin{aligned} f * g(x) &= \int_S f(xy)\, g(y^{-1})\, \mathrm{d}\lambda(y) \\ &= \int_S f(xy^{-1})\, g(y)\, \mathrm{d}\rho(y)\,, \end{aligned}$$

for all functions f, g in $C_c(S)$ and x in S.

Proposition 2.3. *Let m be a bounded measurable function on $\mathbb{R}^+$. Then $k_{m(\mathcal{L}_Q)}$ is radial and $k_{m(\Delta)} = \delta^{1/2}\, k_{m(\mathcal{L}_Q)}$. The spherical transform $\mathcal{H}k_{m(\mathcal{L}_Q)}$ of $k_{m(\mathcal{L}_Q)}$ is given by*

$$\mathcal{H}k_{m(\mathcal{L}_Q)}(s) = m(s^2) \qquad \forall s \in \mathbb{R}^+ .$$

Proof. See [1], [3]. ∎

3. Calderón–Zygmund decomposition

In this section we prove that Damek–Ricci spaces are Calderón–Zygmund spaces.

Recently Hebisch and Steger [15] gave the following axiomatic definition of a Calderón–Zygmund space.

Definition 3.1. Let (X, μ, ϱ) be a metric measured space. Suppose that f is in $L^1(\mu)$, κ_0 is a positive constant and $\alpha > \kappa_0 \, \|f\|_{L^1(\mu)}/\mu(X)$ (if $\mu(X) = \infty$, then the right hand side is taken to be 0). A *Calderón–Zygmund decomposition* of f at height α with Calderón–Zygmund constant κ_0 is a decomposition $f = g + \sum_{i\in\mathbb{N}} b_i$ where there exist sets R_i, points x_i and positive numbers r_i such that:

(i) $|g| \leq \kappa_0\, \alpha \qquad \mu$-almost everywhere;

(ii) $\mathrm{supp}(b_i) \subseteq R_i$ and $\int b_i \,\mathrm{d}\mu = 0 \qquad \forall i \in \mathbb{N}$;

(iii) $R_i \subseteq B(x_i, \kappa_0\, r_i) \qquad \forall i \in \mathbb{N}$;

(iv) $\sum_i \mu(R_i^*) \leq \kappa_0 \frac{\|f\|_{L^1(\mu)}}{\alpha}$, where $R_i^* = \{x \in X : \varrho(x, R_i) < r_i\}$;

(v) $\sum_i \|b_i\|_{L^1(\mu)} \leq \kappa_0 \, \|f\|_{L^1(\mu)}$.

A *Calderón–Zygmund space* is a metric measured space (X, μ, ϱ) for which there exists a positive constant κ_0 such that each function f in $L^1(\mu)$ has a Calderón–Zygmund decomposition at height α with Calderón–Zygmund constant κ_0 whenever $\alpha > \kappa_0 \, \|f\|_{L^1(\mu)}/\mu(X)$.

Clearly spaces of homogeneous type are Calderón–Zygmund spaces. Note that in this case we may choose R_i as balls and R_i^* as balls of the same centre and dilated radius. It is remarkable that some spaces which are not of homogeneous type are Calderón–Zygmund spaces. Indeed, Hebisch and Steger proved that real hyperbolic spaces are Calderón–Zygmund spaces.

Let S be a Damek–Ricci space, as described in Section 2. We shall prove that (S, ρ, d) is a Calderón–Zygmund space. The first problem is to define suitable sets R_i, as in Definition 3.1. As we have already remarked, we cannot use geodesic balls as in the classical case, because their measure increases exponentially: so we define suitable families of "big admissible sets" and "small admissible sets".

Big admissible sets. To define "big" sets we generalize the idea used by Hebisch and Steger in [15]. They defined admissible sets as "rectangles" which are products of dyadic sets in $\mathbb{R}^n$ and intervals in $\mathbb{R}$. In the context of H-type groups dyadic sets were introduced by M. Christ [4, Theorem 11]. For the readers' convenience we recall their properties in the following theorem.

Theorem 3.2. *Let N be an H-type group, endowed with the homogeneous distance and the Haar measure. There exist subsets Q_α^k of N, where $k \in \mathbb{Z}$ and α is in a countable index set I_k, positive constants η, c_N, C_N and M ($\eta > 1$ and M is an integer) such that:*

(i) $|N - \bigcup_{\alpha \in I_k} Q_\alpha^k| = 0 \qquad \forall k \in \mathbb{Z}$;

(ii) *there are points n_α^k in N such that* $B_N(n_\alpha^k, c_N\,\eta^k) \subseteq Q_\alpha^k \subseteq B_N(n_\alpha^k, C_N\,\eta^k)$;

(iii) $Q_\alpha^k \cap Q_\beta^k = \emptyset$ *if* $\alpha \neq \beta$;

(iv) *each set Q_α^k has at most M subsets of type Q_β^{k-1}*;

(v) $\forall \ell \leq k$ *and β in I_ℓ there is a unique α in I_k such that* $Q_\beta^\ell \subseteq Q_\alpha^k$;

(vi) *if $\ell \leq k$, then either* $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ *or* $Q_\beta^\ell \subseteq Q_\alpha^k$.

We define big admissible sets as products of dyadic sets in N and intervals in A. Roughly speaking, we may think of these sets as left translates of a family of sets containing the identity (see [18]). For technical reasons we cannot do exactly that, because left translates and dilates of dyadic sets in N may not be dyadic sets.

Definition 3.3. A *big admissible set* is a set of the form $Q_\alpha^k \times (a_0/r, a_0\,r)$, where Q_α^k is a dyadic set in N, $a_0 \in A$, $r \geq \mathrm{e}$,

$$a_0^{1/2} r^\beta \leq \eta^k < a_0^{1/2} r^{4\beta}\,, \tag{11}$$

and β is a constant greater than $\max\{3/2, 1/4 + \log\eta, 1 + \log(c_3/c_N)\}$, where c_3, η, c_N are the constants which appear in Proposition 2.1 and Theorem 3.2.

We now investigate some geometric properties of big admissible sets.

Proposition 3.4. *Denote by R the big admissible set $Q_\alpha^k \times (a_0/r, a_0\,r)$, and take c_3, c_N, C_N and η as in Proposition 2.1 and Theorem 3.2 (ii). Then the following hold:*

(i) *there exists a constant $C_{N,\beta}$ such that* $R \subseteq B((n_\alpha^k, a_0), C_{N,\beta} \log r)$;

(ii) $c_N^{2Q}\,|B_N(0_N, 1)|\,(a_0^{1/2} r^\beta)^{2Q}\,\log r \leq \rho(R) \leq C_N^{2Q}\,|B_N(0_N, 1)|\,(a_0^{1/2} r^{4\beta})^{2Q}\,\log r$;

(iii) *let R^* be the set* $\{(n, a) \in S : d((n, a), R) < \log r\}$; *then*

$$\rho(R^*) \leq \Big(\frac{c_3 + C_N}{c_N}\Big)^{2Q} \rho(R)\,.$$

Proof. To prove (i), note that by Theorem 3.2 (ii)

$$R \subseteq B_N(n_\alpha^k, C_N\,\eta^k) \times (a_0/r, a_0\,r)\,,$$

which, in turn, is contained in $B_N(n_\alpha^k, C_N a_0^{1/2} r^{4\beta}) \times (a_0/r, a_0\, r)$ by the admissibility condition (11). By the left invariance of the metric and Proposition 2.1 (ii),

$$\begin{aligned} R &\subseteq (n_\alpha^k, a_0)\cdot [B_N(0_N, C_N\, r^{4\beta}) \times (1/r, r)] \\ &\subseteq (n_\alpha^k, a_0)\cdot [B(e, C_{N,\beta}\, \log r)] \\ &= B((n_\alpha^k, a_0), C_{N,\beta}\, \log r)\,, \end{aligned}$$

as required.

We now prove (ii). Since $\rho(R) = |Q_\alpha^k|\, \log r$, by Theorem 3.2 (ii),

$$c_N^{2Q}\, |B_N(0_N, 1)|\, \eta^{2Qk}\, \log r \le \rho(R) \le C_N^{2Q}\, |B_N(0_N, 1)|\, \eta^{2Qk}\, \log r\,.$$

Since $a_0^{1/2} r^\beta \le \eta^k < a_0^{1/2} r^{4\beta}$, (ii) follows.

To prove (iii), we observe that

$$R^* = \bigcup_{(n,a)\in R} B((n,a), \log r)\,.$$

Using the left invariance of the metric and Proposition 2.1 (i), we deduce that

$$\begin{aligned} B((n,a), \log r) &= (n,a)\cdot [B(e, \log r)] \\ &\subseteq (n,a)\cdot [B_N(0_N, c_3\, r) \times (1/r, r)] \\ &= B_N(n, c_3\, a^{1/2} r) \times (a/r, a\, r) \qquad \forall (n,a) \in R\,. \end{aligned}$$

Since (n,a) is in R and R is admissible, we see that

$$(a/r, ar) \subseteq (a_0/r^2, a_0 r^2)$$

and

$$\begin{aligned} B_N(n, c_3\, a^{1/2} r) &\subseteq B_N(n, c_3\, a_0^{1/2} r^{3/2}) \\ &\subseteq B_N(n_\alpha^k, c_3\, a_0^{1/2} r^\beta + C_N\, \eta^k) \\ &\subseteq B_N(n_\alpha^k, (c_3 + C_N)\, \eta^k)\,. \end{aligned}$$

Thus

$$R^* \subseteq B_N(n_\alpha^k, (c_3 + C_N)\, \eta^k) \times (a_0/r^2, a_0\, r^2)\,.$$

Finally,

$$\begin{aligned} \rho(R^*) &\le \Big(\frac{c_3 + C_N}{c_N}\Big)^{2Q} |B_N(n_\alpha^k, c_N\, \eta^k)|\, \log r \\ &\le \Big(\frac{c_3 + C_N}{c_N}\Big)^{2Q} \rho(R)\,, \end{aligned}$$

as required. ∎

Another useful geometric property is that most big admissible sets may be split up into a finite number of mutually disjoint smaller subsets which are still admissible. More precisely the following lemma holds.

Lemma 3.5. *Let R denote the big admissible set $Q^k_\alpha \times (a_0/r_0, a_0 r_0)$ and let η, M, n^k_α, c_N and C_N be as in Theorem 3.2. The following hold:*

(i) if $\eta^{k-1} \geq a_0^{1/2} r^\beta$, then there exist J mutually disjoint big admissible sets $R_1, \dots, R_J$, where $2 \leq J \leq M$, such that $R = \bigcup_{i=1}^J R_i$ and

$$(c_N/(\eta\, C_N))^{2Q} \rho(R) \leq \rho(R_i) \leq \rho(R) \qquad i = 1, \dots, J\,;$$

(ii) if $\eta^{k-1} < a_0^{1/2} r^\beta$ and $r \geq \mathrm{e}^2$, then there exist two disjoint big admissible sets R_1 and R_2 such that $R = R_1 \cup R_2$ and $\rho(R_i) = \rho(R)/2$, where $i = 1, 2$;

(iii) if $\eta^{k-1} < a_0^{1/2} r^\beta$ and $r < \mathrm{e}^2$, then there exists a constant $\sigma_{N,\beta}$ such that

$$B((n^k_\alpha, a_0), 1) \subseteq R \subseteq B((n^k_\alpha, a_0), \sigma_{N,\beta})\,. \tag{12}$$

Proof. To prove (i), suppose that $\eta^{k-1} \geq a_0^{1/2} r^\beta$. We split up R in the following way: let Q^{k-1}_i be the subsets of Q^k_α as in Theorem 3.2, where $1 \leq i \leq J \leq M$. Define

$$R_i = Q^{k-1}_i \times (a_0/r_0, a_0\, r_0) \qquad i = 1, \dots, J\,.$$

Since $\eta^{k-1} \geq a_0^{1/2} r^\beta$, the sets R_i are admissible. Obviously $R = \bigcup_{i=1}^J R_i$ and $\rho(R_i) \leq \rho(R)$. By Theorem 3.2 (ii)

$$\begin{aligned}
\rho(R_i) &= |Q^{k-1}_i|\, \log r \\
&\geq |B_N(0_N, c_N\, \eta^{k-1})|\, \log r \\
&= \Big|B_N\Big(0_N, (c_N/(\eta\, C_N))\, C_N\, \eta^k\Big)\Big|\, \log r \\
&\geq (c_N/(\eta\, C_N))^{2Q} |B_N(0_N, C_N\, \eta^k)|\, \log r \\
&\geq (c_N/(\eta\, C_N))^{2Q} \rho(R)\,,
\end{aligned}$$

as required.

To prove (ii), suppose that $\eta^{k-1} < a_0^{1/2} r^\beta$ and $r \geq \mathrm{e}^2$. Then by the admissibility condition (11),

$$a_0^{1/2} r^\beta \leq \eta^k < \eta\, a_0^{1/2} r^\beta\,. \tag{13}$$

Define R_1 and R_2 by

$$R_1 = Q^k_\alpha \times (a_0/r, a_0) \qquad \text{and} \qquad R_2 = Q^k_\alpha \times (a_0, a_0\, r)\,.$$

Clearly the centres of R_1 and R_2 are $(n^k_\alpha, a_0/\sqrt{r})$ and $(n^k_\alpha, a_0\sqrt{r})$ respectively. Note that $\sqrt{r} \geq \mathrm{e}$. To prove that R_1 and R_2 are admissible, we use (13):

$$\begin{aligned}
(a_0/\sqrt{r})^{1/2} (\sqrt{r})^\beta &\leq a_0^{1/2} r^\beta \\
&\leq \eta^k\,; \\
(a_0/\sqrt{r})^{1/2} (\sqrt{r})^{4\beta} &= \eta^{-1} r^{\beta - 1/4}\, \eta\, a_0^{1/2} r^\beta \\
&> \eta^{-1} \mathrm{e}^{\beta - 1/4}\, \eta^k \\
&> \eta^k\,.
\end{aligned}$$

This proves that R_1 is admissible. The proof of the admissibility of R_2 is similar and is omitted. Obviously $R = R_1 \cup R_2$ and $\rho(R_i) = \rho(R)/2$, $i = 1, 2$, as required.

We now consider (iii). Suppose that $\eta^k \le \eta\, a_0^{1/2} r^\beta$ and $\mathrm{e} \le r < \mathrm{e}^2$. By the admissibility condition (11) and the left invariance of the metric,

$$\begin{aligned} R &\subseteq B_N(n_\alpha^k, C_N\, \eta^k) \times (a_0/r, a_0\, r) \\ &\subseteq B_N(n_\alpha^k, C_N\, a_0^{1/2} r^{4\beta}) \times (a_0/r, a_0 r) \\ &= (n_\alpha^k, a_0) \cdot [B_N(0_N, C_N\, r^{4\beta}) \times (1/r, r)] \,. \end{aligned}$$

Since $r < \mathrm{e}^2$, we conclude from Proposition 2.1 (ii) that

$$\begin{aligned} R &\subseteq (n_\alpha^k, a_0) \cdot [B_N(0_N, C_N\, \mathrm{e}^{8\beta}) \times (1/\mathrm{e}^2, \mathrm{e}^2)] \\ &\subseteq B((n_\alpha^k, a_0), \sigma_{N,\beta}) \,, \end{aligned}$$

where $\sigma_{N,\beta}$ depends only on β and C_N. Similarly, (11) and the left invariance of the metric imply that

$$R \supseteq B((n_\alpha^k, a_0), 1) \,,$$

as required. ∎

For later developments it is useful to distinguish big admissible sets that satisfy condition (i) or (ii) in Lemma 3.5, which may be split up into a finite number of smaller big admissible sets, and big admissible sets that satisfy condition (iii) in Lemma 3.5, which cannot be split up in that way.

Definition 3.6. A big admissible set $Q_\alpha^k \times (a_0/r, a_0\, r)$ is said to be *nondivisible* if $\eta^{k-1} < a_0^{1/2} r^\beta$ and $r < \mathrm{e}^2$, and to be *divisible* otherwise.

Next we show that there exists a partition of S consisting of big admissible sets whose measure is as large as desired.

Lemma 3.7. *For all positive σ, there exists a partition $\mathcal{P}$ of S which consists of big admissible sets whose measure is greater than σ.*

Proof. First, we choose $r_0 \ge \mathrm{e}$ and $k_0 \in \mathbb{Z}$ such that $r_0^\beta \le \eta^{k_0} < r_0^{4\beta}$ and $r_0^{2\beta Q} \log r_0 > \sigma / c_N^{2Q}\, |B_N(0_N, 1)|$. Set $R_\alpha^0 = Q_\alpha^{k_0} \times (1/r_0, r_0)$, where $\alpha \in I_{k_0}$. The R_α^0 are big admissible sets and

$$\begin{aligned} \rho(R_\alpha^0) &= |Q_\alpha^{k_0}|\, \log r_0 \\ &\ge c_N^{2Q}\, |B(0_N, 1)|\, \eta^{2k_0 Q} \log r_0 \\ &\ge c_N^{2Q}\, |B(0_N, 1)|\, r_0^{2\beta Q} \log r_0 \\ &> \sigma \qquad \forall \alpha \in I_{k_0} \,. \end{aligned}$$

Then the sets R_α^0, where $\alpha \in I_{k_0}$, form a partition of the strip $N \times (1/r_0, r_0)$, consisting of big admissible sets whose measure is greater than σ.

Next suppose that a partition of a strip $N \times (a_n/r_n, a_n\, r_n)$ which consists of admissible sets whose measure is greater than σ has been chosen. Then we choose $r_{n+1} \ge \mathrm{e}$ and $k_{n+1} \in \mathbb{Z}$ such that

$$(a_{n+1}^{1/2}\, r_{n+1}^\beta)^{2Q} \log r_{n+1} > \frac{\sigma}{c_N^{2Q}\, |B(0_N, 1)|} \qquad \text{and} \qquad a_{n+1}^{1/2}\, r_{n+1}^\beta \le \eta^{k_1} < a_{n+1}^{1/2}\, r_{n+1}^{4\beta} \,,$$

where $a_{n+1} = a_n\, r_n\, r_{n+1}$. Set $R_\alpha^{n+1} = Q_\alpha^{k_{n+1}} \times (a_n\, r_n, a_{n+1}\, r_{n+1})$, where $\alpha \in I_{k_{n+1}}$. Then the R_α^{n+1} are big admissible sets whose measure is greater than σ, which partition the strip $N \times (a_n\, r_n, a_{n+1}\, r_{n+1})$.

By iterating this process we obtain a partition of $N \times (r_0, \infty)$. Similarly, we define a partition of $N \times (0, 1/r_0)$ consisting of big admissible sets with the required property. ■

Small admissible sets. A *small admissible set* is a ball with radius less than $1/2$. Note that

$$\begin{aligned} \rho(B((n_0, a_0), r)) &= \delta^{-1}((n_0, a_0))\, \rho(B_r) \\ &= a_0^Q\, \rho(B_r)\,. \end{aligned}$$

By (4), there exist positive constants γ_1, γ_2 such that

$$\gamma_1\, a_0^Q\, r^n \le \rho(B((n_0, a_0), r)) \le \gamma_2\, a_0^Q\, r^n \qquad \forall r \in (0, 1/2)\,.$$

Set $\gamma = 1 + 2(2\,\mathrm{e}^Q\, \gamma_2/\gamma_1)^{1/n}$. Define $B^*(x_0, r) = B(x_0, \gamma\, r)$. Since $r < 1/2$, there exists a constant C^* such that

$$\rho(B^*(x_0, r)) \le C^*\, \rho(B(x_0, r))\,. \tag{14}$$

The balls of small radius satisfy the following covering lemma.

Lemma 3.8. *Let B_1 and B_2 be balls of radii less than $1/2$. If $B_1 \cap B_2 \neq \emptyset$ and $\rho(B_1) \le 2\,\rho(B_2)$, then $B_1 \subseteq B_2^*$.*

Proof. Let x_i and r_i denote the centre and the radius of the ball B_i, $i = 1, 2$. Since $B_1 \cap B_2 \neq \emptyset$, we have that $d(x_1, x_2) < r_1 + r_2 < 1$. The condition $\rho(B_1) \le 2\,\rho(B_2)$ implies that $\delta(x_1)^{-1}\, \gamma_1\, r_1^n \le 2\,\delta(x_2)^{-1}\, \gamma_2\, r_2^n$. Thus

$$r_1 \le (2\,\delta(x_1 x_2^{-1})\, \gamma_2/\gamma_1)^{1/n}\, r_2\,.$$

Since $x_1 x_2^{-1}$ is in $B(e, 1)$ we have that $\delta(x_1 x_2^{-1}) \le \mathrm{e}^Q$ and then

$$r_1 \le (2\,\mathrm{e}^Q\, \gamma_2/\gamma_1)^{1/n}\, r_2\,.$$

It follows that

$$\begin{aligned} B(x_1, r_1) &\subseteq B(x_2, 2r_1 + r_2) \\ &\subseteq B\Big(x_2, \big(1 + 2(2\,\mathrm{e}^Q\, \gamma_2/\gamma_1)^{1/n}\big) r_2\Big) \\ &= B(x_2, \gamma\, r_2) \\ &= B_2^*\,, \end{aligned}$$

as required. ■

Let $\mathcal{R}^0$ denote the family of all balls of radius less than $1/2$ and let $M^{\mathcal{R}^0}$ be the noncentred maximal operator

$$M^{\mathcal{R}^0} f(x) = \sup_{x \in B} \frac{1}{\rho(B)} \int_B |f|\, \mathrm{d}\rho \qquad \forall x \in S\,,$$

where the supremum is taken over all balls in $\mathcal{R}^0$. From the covering Lemma 3.8 it follows that $M^{\mathcal{R}_0}$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$.

We now prove a geometric lemma concerning intersection properties between “small balls” and “big nondivisible sets”.

Lemma 3.9. *Let B be a ball of radius R, such that $1/2 \leq R \leq \gamma/2$, where $\gamma = 1 + 2(2\,\mathrm{e}^Q\,\gamma_2/\gamma_1)^{1/n}$. Let $\{F_\ell\}_\ell$ be a family of mutually disjoint nondivisible big admissible sets. Then:*

(i) if $B \cap F_\ell \neq \emptyset$, then $\rho(B) \geq 2^{-n}\,(\gamma_1/\gamma_2)\,\mathrm{e}^{-Q(2\sigma_{N,\beta}+\gamma/2)}\,\rho(F_\ell)$;

(ii) the ball B intersects at most $(\gamma_2/\gamma_1)\,\mathrm{e}^{Q(1+\sigma_{N,\beta}+\gamma/2)}$ sets of the family $\{F_\ell\}_\ell$, where $\sigma_{N,\beta}$ is the constant which appears in Lemma 3.5.

Proof. Let x_0 be the centre of B. Note that $B(x_0,1/2) \subseteq B \subseteq B(x_0,\gamma/2)$. By (12), there exist points y_ℓ such that $B(y_\ell,1) \subseteq F_\ell \subseteq B(y_\ell,\sigma_{N,\beta})$.

To prove (i), note that

$$\rho(B) \geq \gamma_1\,\delta^{-1}(x_0)\,R^n \geq \gamma_1\,\delta^{-1}(x_0)\,(1/2)^n\,,$$

while

$$\rho(F_\ell) \leq \delta^{-1}(y_\ell)\,\rho(B(e,\sigma_{N,\beta})) \leq \gamma_2\,\delta^{-1}(y_\ell)\,\mathrm{e}^{Q\,\sigma_{N,\beta}}\,.$$

If $B \cap F_\ell \neq \emptyset$, then $d(x_0,y_\ell) < \gamma/2 + \sigma_{N,\beta}$, and so $\delta(y_\ell x_0^{-1}) \geq \mathrm{e}^{-Q(\sigma_{N,\beta}+\gamma/2)}$. Therefore

$$\rho(B)/\rho(F_\ell) \geq 2^{-n}\,(\gamma_1/\gamma_2)\,\mathrm{e}^{-Q(2\sigma_{N,\beta}+\gamma/2)}\,,$$

as required in (i).

To prove (ii), note that if $\ell \neq k$, then $B(y_\ell,1) \cap B(y_k,1) = \emptyset$, since $F_\ell \cap F_k = \emptyset$. Now let $\mathcal{I} = \{\ell : B \cap F_\ell \neq \emptyset\}$. Obviously, if ℓ is in $\mathcal{I}$, then $B(y_\ell,1) \subseteq B(x_0,\gamma/2+1+\sigma_{N,\beta})$, so that

$$\bigcup_{\ell\in\mathcal{I}} B(y_\ell,1) \subseteq B(x_0,\gamma/2+1+\sigma_{N,\beta})\,.$$

Now consider the left invariant measure of the sets above:

$$\sharp\mathcal{I}\,\cdot\,\lambda(B(e,1)) \leq \lambda(B(e,\gamma/2+1+\sigma_{N,\beta}))\,.$$

Then B intersects at most

$$\begin{aligned}\sharp\mathcal{I} &\leq \lambda(B(e,\gamma/2+1+\sigma_{N,\beta}))/\lambda(B(e,1))\\ &\leq (\gamma_2/\gamma_1)\,\mathrm{e}^{Q(1+\sigma_{N,\beta}+\gamma/2)}\end{aligned}$$

sets of the family $\{F_\ell\}_\ell$. ∎

We now prove our first main theorem.

Theorem 3.10. *A Damek–Ricci space (S,ρ,d) is a Calderón–Zygmund space.*

Proof. Let f be in $L^1(\rho)$ and $\alpha > 0$. We want to define a Calderón–Zygmund decomposition of f at height α.

Let $\mathcal{P}$ be a partition of S consisting of big admissible sets of measure greater than $\|f\|_{L^1(\rho)}/\alpha$ (it exists by Lemma 3.7). For each R in $\mathcal{P}$,

$$\frac{1}{\rho(R)}\int_R |f|\,\mathrm{d}\rho < \alpha\,.$$

We split up each divisible set R in $\mathcal{P}$ into big admissible disjoint subsets R_i, where $1 \le i \le J \le M$, as in Lemma 3.5. If

$$\frac{1}{\rho(R_i)}\int_{R_i}|f|\,\mathrm{d}\rho \ge \alpha\,,$$

then we stop. Otherwise, if R_i is divisible, then we split up R_i and stop when we find a subset E such that

$$\frac{1}{\rho(E)}\int_{E}|f|\,\mathrm{d}\rho \ge \alpha\,.$$

By iterating this process, we obtain a family $\{E_i\}_i$ of stopping sets. The sets E_i have the following properties:

(i) E_i are mutually disjoint big admissible sets;

(ii) $\displaystyle\frac{1}{\rho(E_i)}\int_{E_i}|f|\,\mathrm{d}\rho \ge \alpha$;

(iii) for each set E_i, there exists a set E_i' such that

$$\frac{1}{\rho(E_i')}\int_{E_i'}|f|\,\mathrm{d}\rho < \alpha$$

and

$$\rho(E_i') \le \max\{2,(\eta\, C_N/c_N)^{2Q}\}\,\rho(E_i)\,.$$

Then

$$\begin{aligned}\frac{1}{\rho(E_i)}\int_{E_i}|f|\,\mathrm{d}\rho &\le \max\{2,(\eta\, C_N/c_N)^{2Q}\}\,\frac{1}{\rho(E_i')}\int_{E_i'}|f|\,\mathrm{d}\rho\\ &< \max\{2,(\eta\, C_N/c_N)^{2Q}\}\,\alpha\,;\end{aligned}$$

(iv) the complement of $\bigcup_i E_i$ is the union of mutually disjoint nondivisible big admissible sets $\{F_\ell\}_\ell$ such that $\int_{F_\ell}|f|\,\mathrm{d}\rho < \alpha\,\rho(F_\ell)$.

Define h, g_f, and b_f^i by $h = f\,\chi_{(\cup_i E_i)^c}$,

$$g_f = \sum_i\Big(\frac{1}{\rho(E_i)}\int_{E_i} f\,\mathrm{d}\rho\Big)\,\chi_{E_i} \qquad \text{and} \qquad b_f^i = \Big(f - \frac{1}{\rho(E_i)}\int_{E_i} f\,\mathrm{d}\rho\Big)\,\chi_{E_i}\,.$$

By (iii), $|g_f| \le \max\{2,(\eta\, C_N/c_N)^{2Q}\}\,\alpha$. Each function b_f^i is supported in E_i and its integral vanishes. The sum of the L^1-norms of the functions b_f^i is

$$\begin{aligned}\sum_i \|b_f^i\|_{L^1(\rho)} &\le 2\sum_i\int_{E_i}|f|\,\mathrm{d}\rho\\ &\le 2\,\|f\|_{L^1(\rho)}\,.\end{aligned}$$

By Proposition 3.4 (i), there are x_i and r_i such that $E_i \subseteq B(x_i, C_{N,\beta}\,\log r_i)$. Moreover, write $E_i^* = \{x \in S : d(x,E_i) < \log r_i\}$. Then $\rho(E_i^*) \le \big((c_3 + C_N)/c_N\big)^{2Q}\,\rho(E_i)$.

Thus

$$\begin{aligned}\sum_i \rho(E_i^*) &\leq \Big(\frac{c_3+C_N}{c_N}\Big)^{2Q} \sum_i \rho(E_i)\\ &\leq \Big(\frac{c_3+C_N}{c_N}\Big)^{2Q} \frac{1}{\alpha}\sum_i \int_{E_i} |f| \,\mathrm{d}\rho\\ &\leq \Big(\frac{c_3+C_N}{c_N}\Big)^{2Q} \frac{\|f\|_{L^1(\rho)}}{\alpha}\,.\end{aligned}$$

We now decompose the function h. Let $O^0_\alpha = \{x \in S : M^{\mathcal{R}^0}h(x) > \alpha\}$. For each point $x \in O^0_\alpha$ we choose a ball B_x in $\mathcal{R}^0$ such that $\int_{B_x} |h| \,\mathrm{d}\rho > \alpha\, \rho(B_x)$ and

$$\rho(B_x) > \frac{1}{2}\sup\Big\{\rho(B) : B \in \mathcal{R}^0,\, x \in B,\, \frac{1}{\rho(B)}\int_B |h| > \alpha\Big\}\,. \tag{15}$$

Now we select a disjoint subfamily of $\{B_x\}_x$. We choose B_{x_1} such that $\rho(B_{x_1}) > (1/2)\sup\{\rho(B_x) : x \in O^0_\alpha\}$. Next, suppose that $B_{x_1},\dots,B_{x_n}$ have been chosen. Then $B_{x_{n+1}}$ is chosen, disjoint from $B_{x_1},\dots,B_{x_n}$, such that

$$\rho(B_{x_{n+1}}) > \frac{1}{2}\sup\{\rho(B_x) : x \in O^0_\alpha,\, B_x \cap B_{x_i} = \emptyset,\, i = 1,\dots,n\}\,.$$

Then $\bigcup_j B_{x_j} \subseteq O^0_\alpha \subseteq \bigcup_j B^*_{x_j}$, where $B^*_{x_j} = B(c_{x_j}, \gamma\, r_{x_j})$, $\gamma = 1 + 2(2\,\mathrm{e}^Q\,\gamma_2/\gamma_1)^{1/n}$ and γ_1, γ_2 are the constants which appear in (4). Indeed, each set B_{x_j} is contained in O^0_α by construction. Moreover, for each point $x \in O^0_\alpha$ either $B_x = B_{x_{j_0}} \subset B^*_{x_{j_0}}$ for some index j_0 or $B_x \neq B_{x_j}$ for all j. In this case there exists an index j_0 such that $B_x \cap B_{x_{j_0}} \neq \emptyset$ and $\rho(B_x) \leq 2\,\rho(B_{x_{j_0}})$. By Lemma 3.8, $x \in B_x \subseteq B^*_{x_{j_0}}$.

Now define

$$G_j = B^*_{x_j} \cap \big(\bigcup_{k<j} G_k\big)^c \cap \big(\bigcup_{\ell>j} B_{x_\ell}\big)^c \cap O^0_\alpha\,.$$

It is easy to check that the sets G_j are mutually disjoint and that $B_{x_j} \subseteq G_j \subseteq B^*_{x_j}$, so that their measures are comparable. Moreover $\bigcup_j G_j = O^0_\alpha$. Indeed, on the one hand, $\bigcup_j G_j \subseteq O^0_\alpha$ by construction; on the other hand, if x is in O^0_α, then there exists an index j_0 such that x is in $B^*_{x_{j_0}}$. Now either x is in B_{x_ℓ} for some index $\ell > j_0$ (and then x is in G_ℓ) or x is in G_k for some index $k < j_0$ or x is in G_{j_0}.

We claim that

$$\frac{1}{\rho(G_j)}\int_{G_j} |h| \,\mathrm{d}\rho \leq C^*\, 2^n\, (\gamma_2/\gamma_1)^2\, \mathrm{e}^{Q(3\,\sigma_{N,\beta}+\gamma+1)}\,\alpha\,, \tag{16}$$

where C^* is the constant which appears in (14). To see this we first observe that

$$\frac{1}{\rho(G_j)}\int_{G_j} |h| \,\mathrm{d}\rho \leq C^*\,\frac{1}{\rho(B^*_{x_j})}\int_{B^*_{x_j}} |h| \,\mathrm{d}\rho\,.$$

To estimate this average we shall distinguish two cases. First, suppose that $B^*_{x_j}$ is in $\mathcal{R}^0$. Since $\rho(B_{x_j}) \leq 2\,\rho(B^*_{x_j})$, by (15)

$$\frac{1}{\rho(B^*_{x_j})}\int_{B^*_{x_j}} |h| \,\mathrm{d}\rho \leq \alpha\,.$$

Next, if $B^*_{x_j}$ is not in $\mathcal{R}^0$, then $1/2 \le \gamma\, r_{x_j} \le \gamma/2$. Hence we may apply Lemma 3.9 to the ball $B^*_{x_j}$ and the family $\{F_\ell\}_\ell$ of nondivisible big admissible sets. Let $\mathcal{I} = \{\ell : B^*_{x_j} \cap F_\ell \neq \emptyset\}$. Since h is supported in $\bigcup_\ell F_\ell$, by Lemma 3.9,

$$\begin{aligned}
\frac{1}{\rho(B^*_{x_j})}\int_{B^*_{x_j}} |h|\,\mathrm{d}\rho &= \sum_{\ell\in\mathcal{I}} \frac{1}{\rho(B^*_{x_j})}\int_{B^*_{x_j}\cap F_\ell} |h|\,\mathrm{d}\rho \\
&\le \sum_{\ell\in\mathcal{I}} 2^n\,(\gamma_2/\gamma_1)\,\mathrm{e}^{Q(2\sigma_{N,\beta}+\gamma/2)}\,\frac{1}{\rho(F_\ell)}\int_{F_\ell}|h|\,\mathrm{d}\rho \\
&\le \sharp\mathcal{I}\cdot 2^n\,(\gamma_2/\gamma_1)\,\mathrm{e}^{Q(2\sigma_{N,\beta}+\gamma/2)}\,\alpha \\
&\le 2^n\,(\gamma_2/\gamma_1)^2\,\mathrm{e}^{Q(3\sigma_{N,\beta}+\gamma+1)}\,\alpha\,.
\end{aligned}$$

The claim (16) follows from the last three estimates.

We now define the decomposition of h:

$$g_h = h\,\chi_{(O^0_\alpha)^c} + \sum_j\Big(\frac{1}{\rho(G_j)}\int_{G_j} h\,\mathrm{d}\rho\Big)\,\chi_{G_j}\,, \qquad b^j_h = \Big(h - \frac{1}{\rho(G_j)}\int_{G_j} h\,\mathrm{d}\rho\Big)\,\chi_{G_j}\,.$$

By (16), $|g_h| \le C^*\,2^n\,(\gamma_2/\gamma_1)^2\,\mathrm{e}^{Q(3\,\sigma_{N,\beta}+\gamma+1)}\,\alpha$ on each set G_j and $|g_h| = |h| \le \alpha$ on $(O^0_\alpha)^c$. Each function b^j_h is supported in G_j and its integral vanishes. The sum of the L^1-norms of the functions b^j_h is

$$\begin{aligned}
\sum_j \|b^j_h\|_{L^1(\rho)} &\le 2\sum_j\int_{G_j}|h|\,\mathrm{d}\rho \\
&\le 2\int_{O^0_\alpha}|h|\,\mathrm{d}\rho \\
&\le 2\,\|h\|_{L^1(\rho)} \\
&\le 2\,\|f\|_{L^1(\rho)}\,.
\end{aligned}$$

Now $G_j \subseteq B^*_{x_j}$ and $G^*_j = \{x\in S : d(x,G_j) < r_{x_j}\} \subseteq B(c_{x_j},(\gamma+1)r_{x_j})$; then there exists a constant C^{**} such that $\rho(G^*_j) \le C^{**}\,\rho(G_j)$. Thus

$$\begin{aligned}
\sum_j \rho(G^*_j) &\le C^{**}\sum_j\rho(G_j) \\
&\le C^{**}\,\rho(O^0_\alpha) \\
&\le \frac{C^{**}}{\alpha}\,|||M^{\mathcal{R}^0}|||_{L^1(\rho);\,L^{1,\infty}(\rho)}\,\|h\|_{L^1(\rho)} \\
&\le \frac{C^{**}}{\alpha}\,|||M^{\mathcal{R}^0}|||_{L^1(\rho);\,L^{1,\infty}(\rho)}\,\|f\|_{L^1(\rho)}\,,
\end{aligned}$$

since $M^{\mathcal{R}^0}$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$.

Then $f = g_f + g_h + \sum_i b^i_f + \sum_j b^j_h$ is a Calderón–Zygmund decomposition of the function f at height α. The Calderón–Zygmund constant of the space is

$$\begin{aligned}
\kappa_0 \;=\; \max\Big\{&2,\,(\eta\,C_N/c_N)^{2Q},\,C_{N,\beta},\,\gamma,\,\Big(\frac{c_3+C_N}{c_N}\Big)^{2Q}, \\
&C^*\,2^n\,(\gamma_2/\gamma_1)^2\,\mathrm{e}^{Q(3\,\sigma_{N,\beta}+\gamma+1)},\,C^{**}\,|||M^{\mathcal{R}^0}|||_{L^1(\rho);\,L^{1,\infty}(\rho)}\Big\}\,.
\end{aligned}$$

∎

4. The multiplier theorem

In this section we prove our main result.

Let ψ be a function in $C_c^\infty(\mathbb{R}^+)$, supported in $[1/4,4]$, such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\lambda) = 1 \qquad \forall \lambda \in \mathbb{R}^+ .$$

Let m be a bounded measurable function on $\mathbb{R}^+$; we define $\|m\|_{0,s}$ and $\|m\|_{\infty,s}$ thus:

$$\begin{aligned} \|m\|_{0,s} &= \sup_{t<1} \|m(t\cdot)\,\psi(\cdot)\|_{H^s(\mathbb{R})}\,, \\ \|m\|_{\infty,s} &= \sup_{t\geq 1} \|m(t\cdot)\,\psi(\cdot)\|_{H^s(\mathbb{R})}\,, \end{aligned}$$

where $H^s(\mathbb{R})$ denotes the L^2-Sobolev space of order s on $\mathbb{R}$. Let $K_{m(\Delta)}$ and $k_{m(\Delta)}$ denote the integral kernel and the convolution kernel of the operator $m(\Delta)$ respectively. It is easy to check that $K_{m(\Delta)}(x,y) = k_{m(\Delta)}(y^{-1}x)\,\delta(y)$ for all $x,y \in S$.

Our aim is to find sufficient conditions on m that ensure the boundedness of $m(\Delta)$ from $L^1(\rho)$ to $L^{1,\infty}(\rho)$ and on $L^p(\rho)$ when $1<p<\infty$. To do it we apply a boundedness theorem for integral operators on Calderón–Zygmund spaces proved by Hebisch and Steger [15, Theorem 2.1], which we state for the readers' convenience.

Theorem 4.1. *Let (X,μ,d) be a Calderón–Zygmund space. Let T be a linear operator bounded on $L^2(\mu)$ such that $T=\sum_{j\in\mathbb{Z}} T_j$, where*

(i) the series converges in the strong topology of $L^2(\mu)$;

(ii) every T_j is an integral operator with kernel K_j;

(iii) there exist positive constants a, A, ε and $c>1$ such that

$$\int_X |K_j(x,y)|\,(1+c^j d(x,y))^{\varepsilon}\,\mathrm{d}\mu(x) \leq A \qquad \forall y \in X; \tag{17}$$

$$\int_X |K_j(x,y)-K_j(x,z)|\,\mathrm{d}\mu(x) \leq A\,(c^j d(y,z))^a \qquad \forall y,z \in X . \tag{18}$$

Then T extends from $L^1(\mu)\cap L^2(\mu)$ to an operator of weak type $(1,1)$ and to a bounded operator on $L^p(\mu)$ when $1<p\leq 2$.

We shall apply Theorem 4.1 to the operator $m(\Delta)$. The proof of (17) hinges on a weighted L^2-estimate (see Lemma 4.2 below) for the kernel of spectral operators associated to Δ with respect to the weight

$$w(x) = \delta^{-1/2}(x)\,\mathrm{e}^{Q\,d(x,e)/2} \qquad \forall x \in S . \tag{19}$$

Lemma 4.2. *There exists a constant C such that for all r in $[1,\infty)$ the following hold:*

(i) $\int_{B_r} w^{-1}\,\mathrm{d}\rho \le C\,r^2$ *;*

(ii) for all compactly supported functions f *on* $\mathbb{R}^+$

$$\int_{B_r} |k_{f(\Delta)}|^2\,w\,\mathrm{d}\rho \le C\,r\int_{B_r} |k_{f(\Delta)}|^2\,\mathrm{d}\rho\,.$$

Proof. From Lemma 2.2, we see that

$$\begin{aligned}\int_{B_r} w^{-1}\,\mathrm{d}\rho &= \int_{B_r} \delta^{1/2}(x)\,\mathrm{e}^{-Qd(x,e)/2}\,\mathrm{d}\rho(x)\\ &= \int_0^r \phi_0(t)\,\mathrm{e}^{-Qt/2}\,A(t)\,\mathrm{d}t\,,\end{aligned}$$

which, by (5) and (6), is bounded above by

$$C\int_0^r (1+t)\,\mathrm{e}^{-Qt/2}\mathrm{e}^{-Qt/2}\Big(\frac{t}{1+t}\Big)^{n-1}\mathrm{e}^{Qt}\,\mathrm{d}t \le C\,r^2 \qquad \forall r\in[1,\infty)\,.$$

This proves (i).

To prove (ii), let f be compactly supported on $\mathbb{R}^+$ and let $k_{f(\mathcal{L}_Q)}$ denote the convolution kernel of the operator $f(\mathcal{L}_Q)$. By Proposition 2.3, $k_{f(\Delta)} = \delta^{1/2}\,k_{f(\mathcal{L}_Q)}$. We split up the ball B_r into annuli $A_R = \{x\in S : R-1 < d(x,e) < R\}$, $R = 1,\dots,[r]$ and $C_r = \{x\in S : [r] < d(x,e) < r\}$ and estimate the integral of $|k|^2\,w$ on A_R and C_r separately. We start with the integral on A_R. By Lemma 2.2,

$$\begin{aligned}\int_{A_R} |k_{f(\Delta)}|^2\,w\;\mathrm{d}\rho &= \int_{A_R} |\delta^{1/2}(x)\,k_{f(\mathcal{L}_Q)}(x)|^2\,\delta^{-1/2}(x)\,\mathrm{e}^{Qd(x,e)/2}\,\mathrm{d}\rho(x)\\ &= \int_{R-1}^R \phi_0(t)\,|k_{f(\mathcal{L}_Q)}(t)|^2\,\mathrm{e}^{Qt/2}\,A(t)\,\mathrm{d}t\,,\end{aligned}$$

which, by (6), is bounded above by

$$\begin{aligned}C\int_{R-1}^R (1+t)\,|k_{f(\mathcal{L}_Q)}(t)|^2\,A(t)\,\mathrm{d}t &\le C\,r\int_{A_R} |k_{f(\mathcal{L}_Q)}|^2\,\mathrm{d}\lambda\\ &= C\,r\int_{A_R} |k_{f(\Delta)}|^2\,\mathrm{d}\rho\,.\end{aligned}$$

The proof of the estimate

$$\int_{C_r} |k_{f(\Delta)}|^2\,w\,\mathrm{d}\rho \le C\,r\int_{C_r} |k_{f(\Delta)}|^2\,\mathrm{d}\rho$$

is similar and is omitted. We sum these two estimates to obtain (ii). ∎

We now state our main result.

Theorem 4.3. *Let* S *be a Damek–Ricci space. Suppose that* $s_0 > 3/2$ *and that* $s_\infty > \max\{3/2, n/2\}$. *Let* m *be a bounded measurable function on* $\mathbb{R}^+$ *such that* $\|m\|_{0,s_0} < \infty$ *and* $\|m\|_{\infty,s_\infty} < \infty$. *Then* $m(\Delta)$ *is bounded from* $L^1(\rho)$ *to* $L^{1,\infty}(\rho)$ *and on* $L^p(\rho)$, *for all* p *in* $(1,\infty)$.

Structure of the proof. Take a sufficiently small positive ε that $s_0 > 3/2 + \varepsilon$ and $s_\infty > \max\{3/2, n/2\} + \varepsilon$, and a bounded measurable function m on $\mathbb{R}^+$ such that $\|m\|_{0,s_0} < \infty$ and $\|m\|_{\infty,s_\infty} < \infty$. Since m is bounded, $m(\Delta)$ is bounded on $L^2(\rho)$ by the spectral theorem. Now define

$$m_j(\lambda) = m(2^j\lambda)\,\psi(\lambda) \qquad \forall j \in \mathbb{Z} \quad \forall \lambda \in \mathbb{R}^+ .$$

By the spectral theorem the operators $m_j(2^{-j}\Delta)$ are bounded on $L^2(\rho)$; it is straightforward to check that

$$m(\Delta) = \sum_{j\in\mathbb{Z}} m_j(2^{-j}\Delta)$$

in the strong topology of $L^2(\rho)$. To prove the theorem it suffices to show that the integral kernels $K_{m_j(2^{-j}\Delta)}$ of the operators $m_j(2^{-j}\Delta)$ satisfy estimates (17) and (18) of Theorem 4.1. More precisely, we need to show that there exists a constant C such that, for all y in S,

$$\int_S |K_{m_j(2^{-j}\Delta)}(x,y)|(1+2^{j/2}d(x,y))^\varepsilon \,\mathrm{d}\rho(x) \leq \begin{cases} C\,\|m\|_{0,s_0} & \forall j < 0 \\ C\,\|m\|_{\infty,s_\infty} & \forall j \geq 0 . \end{cases} \tag{20}$$

The proof of this estimate is very much the same as the proof of [14, Theorem 1.2], the main difference being that we replace the weight function used by Hebisch by w (see (19)) and [14, Lemma 1.9] by Lemma 4.2. Furthermore, we need to establish that there exists a constant C such that for all y, z in S

$$\int_S |K_{m_j(2^{-j}\Delta)}(x,y) - K_{m_j(2^{-j}\Delta)}(x,z)|\,\mathrm{d}\rho(x) \leq \begin{cases} C\,2^{j/2}\,d(y,z)\,\|m\|_{0,s_0} & \forall j < 0 \\ C\,2^{j/2}\,d(y,z)\,\|m\|_{\infty,s_\infty} & \forall j \geq 0 . \end{cases}$$

The proof of this inequality is like the proof of [15, Theorem 2.4] and hinges on an L^1-estimate of the gradient of the heat kernel associated to Δ which is established in Proposition 4.7. Then the operators $m_j(2^{-j}\Delta)$ satisfy the hypothesis of Theorem 4.1. It follows that $m(\Delta)$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$ and on $L^p(\rho)$, for all p in $(1,2)$. The boundedness of $m(\Delta)$ on $L^p(\rho)$ for p in $(2,\infty)$ follows by a duality argument. The proof of the theorem is complete, except for the L^1-estimate of the gradient of the heat kernel, which is established below. ■

An L^1-estimate of the gradient of the heat kernel. For all x in S define $|x| = d(x,e)$ and $\mathrm{Ch}_Q(x) = \cosh^{-Q}(|x|/2)$. Let h_t and q_t denote the heat kernels associated to the operators Δ and $\mathcal{L}_Q$ respectively. By Proposition 2.3, $h_t = \delta^{1/2}\,q_t$ and $\mathcal{H}q_t(s) = \mathrm{e}^{-ts^2} = \mathcal{F}h_t^{\mathbb{R}}(s)$ for all $s \in \mathbb{R}^+$, where $h_t^{\mathbb{R}}$ denotes the heat kernel on $\mathbb{R}$. Then

$$\begin{aligned} h_t(x) &= \delta^{1/2}(x)\,(\mathcal{A}^{-1}\circ\mathcal{F}^{-1})\,(\mathcal{F}h_t^{\mathbb{R}})(|x|) \\ &= \delta^{1/2}(x)\,\mathcal{A}^{-1}(h_t^{\mathbb{R}})(|x|) \qquad \forall x \in S . \end{aligned} \tag{21}$$

To prove Proposition 4.7 we shall need various technical results which we prove in Lemmata 4.4 to 4.6. In particular, in Lemma 4.4, we recall various estimates of $h_t^{\mathbb{R}}$ and its derivatives (see [3, Proposition 5.22]).

Lemma 4.4. *For all r in $\mathbb{R}^+$ and t in $\mathbb{R}^+$, the following hold:*

(i) for all positive integers j, there exists a constant C, independent of t and r, such that

$$r^j\, h_t^{\mathbb{R}}(r) \leq C\, t^{j/2}\, h_{2t}^{\mathbb{R}}(r)\,;$$

(ii) for all nonnegative integers p and q,

$$\mathcal{D}_1^q\, \mathcal{D}_2^p\, (h_t^{\mathbb{R}})(r) = \sum_{j=1}^{p+q} t^{-j}\, a_j(r)\, h_t^{\mathbb{R}}(r)\,,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the differential operators defined in (7),

$$a_j(r) = \cosh^{-(p+2q)}(r/2)\, (\alpha_j\, r^j + f_j(r))\,,$$

f_j, f_j' are bounded functions on $\mathbb{R}^+$ and α_j are constants.

In the next two lemmata we prove various integral estimates.

Lemma 4.5. *The following hold for all $i \in \{0, \ldots, n-1\}$:*

(i) $|X_i(|\cdot|)| \leq 1$*;*

(ii) $t^{-1/2} \int_S h_{2t}^{\mathbb{R}}(|(x)|)\, |X_i(\delta^{1/2}\ \mathrm{Ch}_Q)(x)|\, \mathrm{d}\rho(x) \leq C\, t^{-1/2} \qquad \forall t \in \mathbb{R}^+$*;*

(iii) $t^{-1} \int_{B_1^c} \delta^{1/2}(x)\ \mathrm{Ch}_Q(x)\, h_t^{\mathbb{R}}(|x|)\, \mathrm{d}\rho(x) \leq C\, t^{-1/2} \qquad \forall t \in \mathbb{R}^+$.

Proof. For the proof of (i), see [16].

To prove (ii), recall that $\delta^{1/2}((X,Z,a)) = a^{-Q/2}$ and so, by (3),

$$\mathrm{Ch}_Q((X,Z,a)) = 2^Q\, a^{Q/2}\, [(a+1+|X|^2/4)^2 + |Z|^2]^{-Q/2}\,.$$

Thus

$$(\delta^{1/2}\ \mathrm{Ch}_Q)((X,Z,a)) = 2^Q\, [(a+1+|X|^2/4)^2 + |Z|^2]^{-Q/2}\,.$$

By differentiating along the vector field X_0, we see that

$$\begin{aligned} |X_0(\delta^{1/2}\ \mathrm{Ch}_Q)(X,Z,a)| &\leq C\, \frac{a\,(a+1+|X|^2/4)}{[(a+1+|X|^2/4)^2 + |Z|^2]^{Q/2+1}} \\ &\leq C\, \frac{a\,(a+1+|X|^2/4)^{-Q-1}}{[1+(a+1+|X|^2/4)^{-2}\, |Z|^2]^{Q/2+1}}\,. \end{aligned}$$

Since $h_{2t}^{\mathbb{R}}(r) = C\, t^{-1/2}\, \mathrm{e}^{-r^2/8t}$ and $|\log a| < |(X,Z,a)|$,

$$\begin{aligned} & t^{-1/2} \int_S h_{2t}^{\mathbb{R}}(|(X,Z,a)|)\, |X_0(\delta^{1/2}\ \mathrm{Ch}_Q)(X,Z,a)|\, \mathrm{d}\rho(X,Z,a) \\ \leq\ & C\, t^{-1} \int_{\mathbb{R}^+} e^{-\frac{(\log a)^2}{8t}} \int_N \frac{a\,(a+1+|X|^2/4)^{-Q-1}}{[1+(a+1+|X|^2/4)^{-2}\, |Z|^2]^{Q/2+1}}\, a^{-1}\, \mathrm{d}X\, \mathrm{d}Z\, \mathrm{d}a\,, \end{aligned}$$

which, on changing variables ($W=(a+1+|X|^2/4)^{-1}\,Z$), becomes

$$
\begin{aligned}
& C\,t^{-1}\int_{\mathbb{R}^+} a\,\mathrm{e}^{-\frac{(\log a)^2}{8t}}\int_{\mathfrak{v}}\frac{1}{(a+1+|X|^2/4)^{Q+1+m_{\mathfrak{z}}}}\,\mathrm{d}X\int_{\mathfrak{z}}\frac{\mathrm{d}W}{(1+|W|^2)^{Q/2+1}}\,a^{-1}\,\mathrm{d}a\\
\le\ & C\,t^{-1}\int_{\mathbb{R}^+}\mathrm{e}^{-\frac{(\log a)^2}{8t}}\int_{\mathfrak{v}}(a+1+|X|^2/4)^{-m_{\mathfrak{v}}/2-1}\,\mathrm{d}X\,\mathrm{d}a\\
\le\ & C\,t^{-1}\int_{\mathbb{R}^+}\mathrm{e}^{-\frac{(\log a)^2}{8t}}\,(a+1)^{-m_{\mathfrak{v}}/2-1}\int_{\mathfrak{v}}(1+(a+1)^{-1/2}|X|^2/4)^{-m_{\mathfrak{v}}/2-1}\,\mathrm{d}X\,\mathrm{d}a\\
\le\ & C\,t^{-1}\int_{\mathbb{R}^+}\mathrm{e}^{-\frac{(\log a)^2}{8t}}\,(a+1)^{-1}\int_{\mathfrak{v}}(1+|X|^2/4)^{-m_{\mathfrak{v}}/2-1}\,\mathrm{d}X\,\mathrm{d}a\\
\le\ & C\,t^{-1}\int_{\mathbb{R}^+}\mathrm{e}^{-\frac{(\log a)^2}{8t}}\,a^{-1}\,\mathrm{d}a\,.
\end{aligned}
$$

The last integral, on changing variables ($s=\log a$), becomes

$$
C\,t^{-1}\int_{\mathbb{R}} e^{-s^2/8t}\,\mathrm{d}s = C\,t^{-1/2}\,,
$$

as required. The proof of (ii) when $i=1,\dots,n-1$ is similar, and is omitted.

To prove (iii), we use Lemma 2.2 and apply (5) and (6):

$$
\begin{aligned}
& t^{-1}\int_{B_1^c}\delta^{1/2}(x)\ \mathrm{Ch}_Q(x)\,h_t^{\mathbb{R}}(|x|)\,\mathrm{d}\rho(x)\\
\le\ & C\,t^{-3/2}\int_1^\infty \phi_0(r)\ \cosh^{-Q}(r/2)\,\mathrm{e}^{-r^2/4t}\,A(r)\,\mathrm{d}r\\
\le\ & C\,t^{-3/2}\int_1^\infty r\,\mathrm{e}^{-Qr/2}\,\mathrm{e}^{-Qr/2}\,\mathrm{e}^{-r^2/4t}\,\mathrm{e}^{Qr}\,\mathrm{d}r\\
\le\ & C\,t^{-3/2}\int_1^\infty r\,\mathrm{e}^{-r^2/4t}\,\mathrm{d}r\\
\le\ & C\,t^{-1/2}\,,
\end{aligned}
$$

as required. ■

Lemma 4.6. *Let F_1 be the function defined by $F_1(s)=(\alpha_1\,s+f_1(s))\,h_t^{\mathbb{R}}(s)$ for all s in $\mathbb{R}^+$, where α_1 and f_1 are as in Lemma 4.4. Set*

$$
I(r,t)=\int_1^\infty F_1(2\,\mathrm{arc\,cosh}(\cosh(r/2)\,v)\ \frac{\mathrm{d}v}{v^Q(2v^2-2)^{1/2}}\,.
$$

Then the following hold:

(i) $|I(r,t)|\le C\,t^{1/2}\,h_{2t}^{\mathbb{R}}(r)\qquad \forall t\in\mathbb{R}^+\quad \forall r\in[1,\infty)$;

(ii) $|I'(r,t)\le C\,h_t^{\mathbb{R}}(r)\qquad \forall t\in\mathbb{R}^+\quad \forall r\in[1,\infty)$.

Proof. First we observe that by Lemma 4.4 (i) and (ii),

$$
\begin{aligned}
|F_1(s)| &\le C\,s\,h_t^{\mathbb{R}}(s)\\
&\le C\,t^{1/2}h_{2t}^{\mathbb{R}}(s)\,,
\end{aligned}
$$

and

$$\begin{aligned} |F_1'(s)| &\leq C\,(1+s/2t)\,h_t^{\mathbb{R}}(s) \\ &\leq C\,h_t^{\mathbb{R}}(s)\,. \end{aligned}$$

We now prove (i):

$$\begin{aligned} |I(r,t)| &\leq C\,t^{1/2}\,h_{2t}^{\mathbb{R}}(r)\int_1^\infty \frac{\mathrm{d}v}{v^Q(v^2-1)^{1/2}} \\ &\leq C\,t^{1/2}\,h_{2t}^{\mathbb{R}}(r)\,, \end{aligned}$$

as required.

To prove (ii), note that

$$\begin{aligned} &|I'(r,t)| \\ \leq\;& \int_1^\infty |F_1'(2\operatorname{arc\,cosh}(\cosh(r/2)v))|\,\frac{\sinh(r/2)v}{2(\cosh^2(r/2)v^2-1)^{1/2}}\,\frac{\mathrm{d}v}{v^Q(v^2-1)^{1/2}} \\ \leq\;& C\,h_t^{\mathbb{R}}(r)\int_1^\infty \frac{\mathrm{d}v}{v^Q(v^2-1)^{1/2}} \\ \leq\;& C\,h_t^{\mathbb{R}}(r)\,, \end{aligned}$$

as required. ■

Now we may prove the L^1-estimate of the gradient of the heat kernel h_t.

Proposition 4.7. *For all t in $\mathbb{R}^+$,*

$$\int_S |\nabla h_t|\,\mathrm{d}\rho \leq C\,t^{-1/2}\,.$$

Proof. Denote by q_t the heat kernel associated to the operator $\mathcal{L}_Q$. From Proposition 2.3, $h_t=\delta^{1/2}\,q_t$, so

$$|\nabla h_t| \leq C\,\delta^{1/2}\,(|q_t|+|\nabla q_t|) \qquad \forall t\in\mathbb{R}^+\,.$$

It is well known ([3, Theorem 5.9], [3, Corollary 5.49]) that q_t is radial and

$$\begin{aligned} |q_t(x)| &\leq C\,t^{-1}\,(1+|x|)\Big(1+\frac{1+|x|}{t}\Big)^{(n-3)/2}\,\mathrm{e}^{-Q\,|x|/2}\,h_t^{\mathbb{R}}(|x|) \\ |\nabla q_t(x)| &\leq C\,t^{-1}\,|x|\Big(1+\frac{1+|x|}{t}\Big)^{(n-1)/2}\,\mathrm{e}^{-Q\,|x|/2}\,h_t^{\mathbb{R}}(|x|)\,, \end{aligned} \tag{22}$$

for every t in $\mathbb{R}^+$ and x in S. Our purpose is to estimate

$$\begin{aligned} \int_S |\nabla h_t|\,\mathrm{d}\rho &\leq C\int_S \delta^{1/2}\,(|q_t|+|\nabla q_t|)\,\mathrm{d}\rho \\ &= C\int_0^\infty \phi_0(r)\,(|q_t(r)|+|\nabla q_t(r)|)\,A(r)\,\mathrm{d}r\,. \end{aligned} \tag{23}$$

We study the cases where $t<1$ and $t\geq 1$ separately.

In the case where $t<1$, it suffices to use the pointwise estimates (22) of q_t and its gradient in (23).

In the case where $t\geq 1$, by using (22) in (23), we estimate the integral of $|\nabla h_t|$ on the unit ball. The estimate on the complement of the unit ball is more difficult. We have already (21) noted that

$$h_t(x)=\delta^{1/2}(x)\,(\mathcal{A}^{-1}\circ\mathcal{F}^{-1})(\mathcal{F}h_t^{\mathbb{R}})(|x|)=\delta^{1/2}(x)\,\mathcal{A}^{-1}(h_t^{\mathbb{R}})(|x|)\,.$$

By the inversion formula for the Abel transform (8) and (9), if $m_{\mathfrak{z}}$ is even, then

$$h_t(x)=C\,\delta^{1/2}(x)\,\mathcal{D}_1^{m_{\mathfrak{z}}/2}\,\mathcal{D}_2^{m_{\mathfrak{v}}/2}(h_t^{\mathbb{R}})(|x|)\,,$$

while if $m_{\mathfrak{z}}$ is odd, then

$$h_t(x)=C\,\delta^{1/2}(x)\int_{|x|}^{\infty}\mathcal{D}_1^{(m_{\mathfrak{z}}+1)/2}\mathcal{D}_2^{m_{\mathfrak{v}}/2}(h_t^{\mathbb{R}})(s)\,\mathrm{d}\nu(s)\,, \tag{24}$$

for all x in S. We now consider the cases where $m_{\mathfrak{z}}$ is even and odd separately.

In the case where $m_{\mathfrak{z}}$ is odd, we use Lemma 4.4, taking q and p to be $(m_{\mathfrak{z}}+1)/2$ and $m_{\mathfrak{v}}/2$, and (24), to deduce that

$$\begin{aligned} h_t(x) &= C\,\delta^{1/2}(x)\sum_{j=1}^{n/2}t^{-j}\int_{|x|}^{\infty}a_j(s)\,h_t^{\mathbb{R}}(s)\,\mathrm{d}\nu(s)\\ &= C\,\delta^{1/2}(x)\sum_{j=1}^{n/2}H_j(|x|,t)\,. \end{aligned}$$

We estimate the gradient of each summand $\delta^{1/2}H_j$. When $0\leq i\leq n-1$ and $j\geq 1$, we see that

$$X_i(\delta^{1/2}H_j)(x)=C\,\delta^{1/2}(x)\,H_j(|x|,t)+C\,\delta^{1/2}(x)\,H_j'(|x|,t)\,X_i(|\cdot|)(x)\,,$$

and since $|X_i(|\cdot|)|\leq 1$,

$$|X_i(\delta^{1/2}H_j)(x)|\leq C\,\delta^{1/2}(x)\,(|H_j(|x|,t)|+|H_j'(|x|,t)|)\,. \tag{25}$$

First suppose that $j\geq 2$. Integration by parts shows that

$$H_j(r,t)=-2t^{-j}\int_r^{\infty}\partial_s(a_jh_t^{\mathbb{R}})(s)\,(\cosh s-\cosh r)^{1/2}\,\mathrm{d}s\,,$$

and

$$H_j'(r,t)=-t^{-j}\int_r^{\infty}\partial_s(a_jh_t^{\mathbb{R}})(s)\,\frac{\sinh r\,\mathrm{d}s}{(\cosh s-\cosh r)^{1/2}}\qquad \forall r\in[1,\infty)\,.$$

By Lemma 4.4 (i) and (ii)

$$\begin{aligned} |\partial_s(a_jh_t^{\mathbb{R}})(s)| &\leq \Big|a_j'(s)-\frac{s}{2t}\,a_j(s)\Big|\,h_t^{\mathbb{R}}(s)\\ &\leq C\,\Big(\frac{s^j}{2t}+s^{j-1}\Big)\,s\,\mathrm{e}^{-(Q+1)s/2}\,h_t^{\mathbb{R}}(s)\\ &\leq C\,t^{(j-1)/2}\,s\,\mathrm{e}^{-(Q+1)s/2}\,h_{2t}^{\mathbb{R}}(s)\,. \end{aligned}$$

By using this in (25), we see that

$$|X_i(\delta^{1/2}H_j)(x)| \leq C\,\delta^{1/2}(x)\,t^{-(j+1)/2}\int_{|x|}^{\infty} s\,\mathrm{e}^{-(Q+1)s/2}\,h_{2t}^{\mathbb{R}}(s)\,\mathrm{d}\nu(s)\,.$$

Since $j \geq 2$, clearly $t^{-j/2} \leq t^{-1}$ and then the integral above is bounded by

$$C\,t^{-1/2}\,\delta^{1/2}(x)\,t^{-1}\int_{|x|}^{\infty} a_1(s)\,h_{2t}^{\mathbb{R}}(s)\,\mathrm{d}\nu(s)\ \leq C\,t^{-1/2}\,h_{2t}(x)\,.$$

We integrate on the complement of the unit ball:

$$\begin{aligned} \textstyle\int_{B_1^c} |X_i(\delta^{1/2}H_j)(x)|\,\mathrm{d}\rho(x) &\leq C\,t^{-1/2}\textstyle\int_{B_1^c} h_{2t}(x)\,\mathrm{d}\rho(x) \\ &\leq C\,t^{-1/2} \qquad \forall j \geq 2 \quad i = 0,\dots,n-1\,. \end{aligned} \tag{26}$$

If $j = 1$, then the estimate is more delicate. Note that

$$\begin{aligned} H_1(r,t) &= t^{-1}\int_r^{\infty} a_1(s)\,h_t^{\mathbb{R}}(s)\,\mathrm{d}\nu(s) \\ &= t^{-1}\int_r^{\infty} \cosh^{-(Q+1)}(s/2)\,(\alpha_1 s + f_1(s))\,h_t^{\mathbb{R}}(s)\,\mathrm{d}\nu(s) \\ &= t^{-1}\int_r^{\infty} \cosh^{-(Q+1)}(s/2)\,F_1(s)\,\mathrm{d}\nu(s)\ , \end{aligned}$$

where $F_1(s) = (\alpha_1 s+f_1(s))\,h_t^{\mathbb{R}}(s)$ and α_1, f_1 are as in Lemma 4.4 (ii). By changing variables ($u = \cosh(s/2)\cosh^{-1}(r/2)$), the integral for H_1 transforms into

$$t^{-1}\cosh^{-Q}(r/2)\int_1^{\infty} F_1(2\,\mathrm{arc\,cosh}(\cosh(r/2)v)\ \frac{\mathrm{d}v}{v^Q(2v^2-2)^{1/2}}\,.$$

Define

$$I(r,t) = \int_1^{\infty} F_1(2\,\mathrm{arc\,cosh}(\cosh(r/2)v)\ \frac{\mathrm{d}v}{v^Q(2v^2-2}\,.$$

Thus

$$H_1(r,t) = t^{-1}\ \cosh^{-Q}(r/2)\,I(r,t)\,.$$

Then, when $i = 0,\dots,n-1$,

$$\begin{aligned} &X_i(\delta^{1/2}H_1)(x) \\ &\quad = t^{-1}\,X_i(\delta^{1/2}\ \mathrm{Ch}_Q)(x)\,I(|x|,t) + t^{-1}\,\delta^{1/2}(x)\ \mathrm{Ch}_Q(x)\,I'(|x|,t)\,X_i(|\cdot|)(x)\,. \end{aligned}$$

From Lemma 4.4 (i) and Lemma 4.6 we deduce that

$$\begin{aligned} |X_i(\delta^{1/2}H_1)(x)| &\leq t^{-1}|X_i(\delta^{1/2}\ \mathrm{Ch}_Q)(x)|\,|I(|x|,t)| + t^{-1}\,\delta^{1/2}\ \mathrm{Ch}_Q(x)\,|I'(|x|,t)| \\ &\leq C\,t^{-1/2}\,h_{2t}^{\mathbb{R}}(|x|)\,|X_i(\delta^{1/2}\,\mathrm{Ch}_Q)(x)| + C\,t^{-1}\,\delta^{1/2}\ \mathrm{Ch}_Q(x)\,h_t^{\mathbb{R}}(|x|)\,. \end{aligned}$$

We now integrate the last expression on the complement of the unit ball and apply Lemma 4.5 (ii) and (iii):

$$\begin{aligned} \int_{B_1^c} |X_i(\delta^{1/2}H_1)|\,\mathrm{d}\rho &\leq C\,t^{-1/2}\int_S h_{2t}^{\mathbb{R}}(|(x)|)\,|X_i(\delta^{1/2}\,\mathrm{Ch}_Q)(x)|\,\mathrm{d}\rho(x) \\ &\qquad + C\,t^{-1}\int_{B_1^c} \delta^{1/2}(x)\ \mathrm{Ch}_Q(x)\,h_t^{\mathbb{R}}(|x|)\,\mathrm{d}\rho(x) \\ &\leq C\,t^{-1/2} \qquad \forall i = 0,\dots,n-1\,. \end{aligned} \tag{27}$$

We sum (26) and (27), to conclude that

$$\int_{B_1^c} |\nabla h_t(x)| \, \mathrm{d}\rho(x) \;\leq\; C \sum_{i=0}^{n-1} \sum_{j=1}^{n/2} \int_{B_1^c} |X_i(\delta^{1/2} H_j)(x)| \, \mathrm{d}\rho(x) \leq C\, t^{-1/2}\,,$$

as required.

In the case where $m_{\mathfrak{z}}$ is even, the argument is similar but easier, since the inversion formula for the Abel transform is simpler. We omit the details. ■

References

[1] Astengo F., *The maximal ideal space of a heat algebra on solvable extensions of H-type groups*, Boll. Un. Mat. Ital. A(7) **9** (1995), 157–165.

[2] —, *Multipliers for a distinguished Laplacean on solvable extensions of H-type groups*, Monatsh. Math. **120** (1995), 179–188.

[3] Anker J. P., E. Damek , and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1996), 643–679.

[4] Christ M., *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601–628.

[5] Cowling M., A. H. Dooley, A. Korányi, and F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math. **87** (1991), 1–41.

[6] —, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. **8** (1998), 199–237.

[7] Cowling M., S. Giulini, A. Hulanicki, and G. Mauceri, *Spectral multipliers for a distinguished laplacian on certain groups of exponential growth*, Studia Math. **111** (1994), 103–121.

[8] Coifman R. R., and G. Weiss, *Analyse harmonique non commutative sur certains espaces homogenes*, Lecture Notes in Mathematics **242**, Springer (1971).

[9] Damek E., *Curvature of a semidirect extension of a Heisenberg type nilpotent group*, Colloq. Math. **53** (1987), 249–253.

[10] —, *Geometry of a semidirect extension of a Heisenberg type nilpotent group*, Colloq. Math. **53** (1987), 255–268.

[11] Damek E., and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139–142.

[12] —, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. **2** (1992), 213–248.

[13] Faraut J., *Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques*, Les Cours du C.I.M.P.A. (1983).

[14] Hebisch W., *Analysis of Laplacians on solvable Lie groups*, Notes for ICMS conference, Edinburgh (1998).

[15] Hebisch W., and T. Steger, *Multipliers and singular integrals on exponential growth groups*, Math. Z. **245** (2003), 37–61.

[16] Hunt G. A., *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264–293.

[17] Kaplan A., *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258** (1975), 145–159.

[18] Vallarino M., *A maximal function on harmonic extensions of H-type groups*, Ann. Math. Blaise Pascal **13** (2006), 81–95.

Maria Vallarino
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano Bicocca
Via R. Cozzi, 53
20125 Milano (Italy)
maria.vallarino@unimib.it

Received December 22, 2004
and in final form October 24, 2006