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Derivations from the Even Parts into the Odd Parts for Lie Superalgebras W and S

Wende Liu and Baoling Guan*

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Abstract. Let $\mathcal{W}$ and $\mathcal{S}$ denote the even parts of the generalized Witt superalgebra W and the special superalgebra S over a field of characteristic $p > 3$, respectively. In this note, using the method of reduction on $\mathbb{Z}$-gradations, we determine the derivation space $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ from $\mathcal{W}$ into $W_{\bar{1}}$ and the derivation space $\mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ from $\mathcal{S}$ into $W_{\bar{1}}$. In particular, the derivation space $\mathrm{Der}(\mathcal{S}, S_{\bar{1}})$ is determined.
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0. Introduction

The underlying field $\mathbb{F}$ is assumed of characteristic $p > 3$ throughout. We shall study the derivations from the even parts of the generalized Witt superalgebra W and the special superalgebra S into the odd part of W, where $W_{\bar{1}}$ is viewed as modules for $W_{\bar{0}}$ and $S_{\bar{0}}$ by means of the adjoint representation. The motivation came from the following observation. Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra. Then $L_{\bar{0}}$ is a Lie algebra and $L_{\bar{1}}$ is an $L_{\bar{0}}$-module. Two questions arise naturally: Does the derivation algebra of the even part of L coincide with the even part of the superderivation algebra of L? Does the derivation space from $L_{\bar{0}}$ into $L_{\bar{1}}$ coincide with the odd part of the superderivation algebra of L? For the generalized Witt superalgebra and the special superalgebra the first question was answered affirmatively in [2, Remarks 3.2.12 and 4.3.8]. In this note, the second question will also be answered affirmatively for these two Lie superalgebras of Cartan type (Remarks 2.16 and 3.20). Speaking accurately, we shall determine the derivation spaces from the even parts of the generalized Witt superalgebra W and the special superalgebra S into the odd part of W, respectively (Theorems 2.15 and 3.18). As a direct consequence, the derivation space from the even part into the odd part for the special superalgebra is determined (Theorem 3.19). The authors would like

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1. Basics

In this note we adopt the notation and concepts used in [2], but here, for convenience and completeness, we repeat certain necessary symbols and notions.

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the field of two elements. For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by $\mathrm{p}(a) = \theta$ the *parity of a homogeneous element* $a \in V_\theta, \theta \in \mathbb{Z}_2$. We assume throughout that the notation $\mathrm{p}(x)$ implies that x is a $\mathbb{Z}_2$-homogeneous element.

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. A linear mapping $D : \mathfrak{g} \to V$ is called a *derivation* from $\mathfrak{g}$ into V if $D(xy) = x \cdot D(y) - y \cdot D(x)$ for all $x, y \in \mathfrak{g}$. A derivation $D : \mathfrak{g} \to V$ is called *inner* if there is $v \in V$ such that $D(x) = x \cdot v$ for all $x \in \mathfrak{g}$. Following [5, p. 13], denote by $\mathrm{Der}(\mathfrak{g}, V)$ the *derivation space* from $\mathfrak{g}$ into V. Then $\mathrm{Der}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, V)$. Assume in addition that $\mathfrak{g}$ and V are finite-dimensional and that $\mathfrak{g} = \oplus_{r \in \mathbb{Z}} \mathfrak{g}_r$ is $\mathbb{Z}$-graded and $V = \oplus_{r \in \mathbb{Z}} V_r$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module. Then $\mathrm{Der}(\mathfrak{g}, V) = \oplus_{r \in \mathbb{Z}} \mathrm{Der}_r(\mathfrak{g}, V)$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module by setting

$$\mathrm{Der}_r(\mathfrak{g}, V) := \{D \in \mathrm{Der}(\mathfrak{g}, V) \mid D(\mathfrak{g}_i) \subset V_{r+i} \text{ for all } i \in \mathbb{Z}\}.$$

If $\mathfrak{g} = \oplus_{-r \le i \le s} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie algebra, then $\oplus_{-r \le i \le 0} \mathfrak{g}_i$ is called the *top of* $\mathfrak{g}$ (with respect to the gradation).

We note that the derivations from $\mathfrak{g}$ into V are just the 1-cocycles and that the inner derivations from $\mathfrak{g}$ into V are just 1-coboundaries. Thus $\mathrm{Der}(\mathfrak{g}, V)/\mathrm{InnDer}(\mathfrak{g}, V)$ is isomorphic to $H^1(\mathfrak{g}, V)$, the first cohomology group of $\mathfrak{g}$ relative to the module V, where $\mathrm{InnDer}(\mathfrak{g}, V)$ stands for the space of inner derivations from $\mathfrak{g}$ into V.

In the below we review the notions of modular Lie superalgebras W and S of Cartan-type and their gradation structures. In addition to the standard notation $\mathbb{Z}$, we write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of nonnegative integers. Henceforth, we will let m, n denote two fixed integers in $\mathbb{N} \setminus \{1, 2\}$ without notice. For $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| := \sum_{i=1}^m \alpha_i$. Let $\mathcal{O}(m)$ denote the *divided power algebra over* $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^m\}$. For $\varepsilon_i := (\delta_{i1}, \ldots, \delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to x_i, $i = 1, \ldots, m$. Let $\Lambda(n)$ be the *exterior superalgebra over* $\mathbb{F}$ in n variables $x_{m+1}, \ldots, x_{m+n}$. Denote the tensor product by $\mathcal{O}(m, n) := \mathcal{O}(m) \otimes_{\mathbb{F}} \Lambda(n)$. Obviously, $\mathcal{O}(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathcal{O}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Moreover, $\mathcal{O}(m, n)$ is super-commutative. For $g \in \mathcal{O}(m), f \in \Lambda(n)$, we write gf for $g \otimes f$. The following formulas hold in $\mathcal{O}(m, n)$:

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m;$$

$$x_k x_l = -x_l x_k \quad \text{for } k, l = m+1, \ldots, m+n;$$

$$x^{(\alpha)} x_k = x_k x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m,\ k = m+1, \ldots, m+n,$$

where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^{m}\binom{\alpha_i+\beta_i}{\alpha_i}$. Put $Y_0 := \{1, 2, \ldots, m\}$, $Y_1 := \{m+1, \ldots, m+n\}$ and $Y := Y_0 \cup Y_1$. For convenience, we adopt the notation $r' := r + m$ for $r \in Y_1$. Thus, $Y_1 := \{1', 2', \ldots, n'\}$. Set

$$\mathbb{B}_k := \{\langle i_1, i_2, \ldots, i_k\rangle | m+1 \le i_1 < i_2 < \cdots < i_k \le m+n\}$$

and $\mathbb{B} := \mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k\rangle \in \mathbb{B}_k$, set $|u| := k$, $|\emptyset| := 0$, $x^{\emptyset} := 1$, and $x^u := x_{i_1}x_{i_2}\ldots x_{i_k}$; we use also u to stand for the index set $\{i_1, i_2, \ldots, i_k\}$. Clearly, $\left\{x^{(\alpha)}x^u \mid \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}\right\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(m,n)$. Let $D_1, D_2, \ldots, D_{m+n}$ be the linear transformations of $\mathcal{O}(m,n)$ such that

$$D_r(x^{(\alpha)}x^u) = \begin{cases} x^{(\alpha-\varepsilon_r)}x^u, & r \in Y_0 \\ x^{(\alpha)}\partial x^u/\partial x_r, & r \in Y_1. \end{cases}$$

Then $D_1, D_2, \ldots, D_{m+n}$ are superderivations of the superalgebra $\mathcal{O}(m,n)$. Let

$$W(m,n) = \Big\{ \sum_{r\in Y} f_r D_r \mid f_r \in \mathcal{O}(m,n), r \in Y \Big\}.$$

Then $W(m,n)$ is a Lie superalgebra, which is contained in $\mathrm{Der}(\mathcal{O}(m,n))$. Obviously, $\mathrm{p}(D_i) = \tau(i)$, where

$$\tau(i) := \begin{cases} \bar{0}, & i \in Y_0 \\ \bar{1}, & i \in Y_1. \end{cases}$$

One may verify that

$$[fD, gE] = fD(g)E - (-1)^{\mathrm{p}(fD)\mathrm{p}(gE)}gE(f)D + (-1)^{\mathrm{p}(D)\mathrm{p}(g)}fg[D, E]$$

for $f, g \in \mathcal{O}(m,n)$, $D, E \in \mathrm{Der}\,\mathcal{O}(m,n)$. Let

$$\underline{t} := (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m, \quad \pi := (\pi_1, \pi_2, \ldots, \pi_m)$$

where $\pi_i := p^{t_i} - 1, i \in Y_0$. Let $\mathbb{A} := \mathbb{A}(m;\underline{t}) = \{\alpha \in \mathbb{N}_0^m \mid \alpha_i \le \pi_i, i \in Y_0\}$. Then

$$\mathcal{O}(m,n;\underline{t}) := \mathrm{span}_{\mathbb{F}}\left\{x^{(\alpha)}x^u \mid \alpha \in \mathbb{A}, u \in \mathbb{B}\right\}$$

is a finite-dimensional subalgebra of $\mathcal{O}(m,n)$ with a natural $\mathbb{Z}$-gradation $\mathcal{O}(m,n;\underline{t}) = \bigoplus_{r=0}^{\xi}\mathcal{O}(m,n;\underline{t})_r$ by putting

$$\mathcal{O}(m,n;\underline{t})_r := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha| + |u| = r\}, \quad \xi := |\pi| + n.$$

Set

$$W(m,n;\underline{t}) := \Big\{ \sum_{r\in Y} f_r D_r \mid f_r \in \mathcal{O}(m,n;\underline{t}), r \in Y \Big\}.$$

Then $W(m,n;\underline{t})$ is a finite-dimensional simple Lie superalgebra (see [6]). Obviously, $W(m,n;\underline{t})$ is a free $\mathcal{O}(m,n;\underline{t})$-module with $\mathcal{O}(m,n;\underline{t})$-basis $\{D_r \mid r \in Y\}$. We note that $W(m,n;\underline{t})$ possesses a *standard* $\mathbb{F}$-*basis* $\{x^{(\alpha)}x^u D_r \mid \alpha \in \mathbb{A}, u \in \mathbb{B}, r \in Y\}$. Let $r, s \in Y$ and $D_{rs} : \mathcal{O}(m,n;\underline{t}) \to W(m,n;\underline{t})$ be the linear mapping such that

$$D_{rs}(f) = (-1)^{\tau(r)\tau(s)}D_r(f)D_s - (-1)^{(\tau(r)+\tau(s))\mathrm{p}(f)}D_s(f)D_r \quad \text{for } f \in \mathcal{O}(m,n;\underline{t}).$$

Then the following equation holds:

$$[D_k, D_{rs}(f)] = (-1)^{\tau(k)\tau(r)} D_{rs}(D_k(f)) \quad \text{for } k, r, s \in Y;\ f \in \mathcal{O}(m, n; \underline{t}).$$

Put

$$S(m, n; \underline{t}) := \mathrm{span}_{\mathbb{F}}\{D_{rs}(f) \mid r, s \in Y;\ f \in \mathcal{O}(m, n; \underline{t})\}.$$

Then $S(m, n; \underline{t})$ is a finite-dimensional simple Lie superalgebra (see [6]). Let $\mathrm{div} : W(m, n; \underline{t}) \to \mathcal{O}(m, n; \underline{t})$ be the *divergence* such that

$$\mathrm{div}\Big(\sum_{r\in Y} f_r D_r\Big) = \sum_{r\in Y} (-1)^{\tau(r)\mathrm{p}(f_r)} D_r(f_r).$$

$$\overline{S}(m, n; \underline{t}) := \{D \in W(m, n; \underline{t}) \mid \mathrm{div}(D) = 0\}.$$

Then $\overline{S}(m, n; \underline{t})$ is a subalgebra of $W(m, n; \underline{t})$ and $S(m, n; \underline{t})$ is a subalgebra of $\overline{S}(m, n; \underline{t})$. The $\mathbb{Z}$-gradation of $\mathcal{O}(m, n; \underline{t})$ induces naturally a $\mathbb{Z}$-gradation structure of $W(m, n; \underline{t}) = \oplus_{i=-1}^{\xi-1} W(m, n; \underline{t})_i$, where

$$W(m, n; \underline{t})_i := \mathrm{span}_{\mathbb{F}}\{f D_s \mid s \in Y,\ f \in \mathcal{O}(m, n; \underline{t})_{i+1}\}.$$

In addition, $S(m, n; \underline{t})$ and $\overline{S}(m, n; \underline{t})$ are all $\mathbb{Z}$-graded subalgebras of $W(m, n; \underline{t})$. In the following sections, $W(m, n; \underline{t})$, $S(m, n; \underline{t})$, $\overline{S}(m, n; \underline{t})$, and $\mathcal{O}(m, n; \underline{t})$ will be denoted by $W, S, \overline{S}$, and $\mathcal{O}$, respectively. In addition, the the even parts of W, S and $\overline{S}$ will be denoted by $\mathcal{W}$, $\mathcal{S}$ and $\overline{\mathcal{S}}$, respectively.

2. Generalized Witt superalgebras

View $W_{\bar{1}}$ as a $\mathcal{W}$-module by means of the adjoint representation. In this section, the main purpose is to characterize the derivation space $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$. Note that the $\mathbb{Z}$-gradation of W induces a $\mathbb{Z}$-gradation of $\mathcal{W} = \oplus_{i\geq -1}\mathcal{W}_i$. We know that gradation structures provide a powerful tool for the study of (super)derivation algebras of Lie (super)algebra; in particular, the top of a $\mathbb{Z}$-graded Lie (super)algebra plays a predominant role (c.f. [1, 4, 2]). Following [5], we call $\mathcal{T} := \mathrm{span}_{\mathbb{F}}\{\Gamma_i \mid i \in Y\}$ the canonical torus of $\mathcal{W}$. In the following, we first reduce every nonnegative $\mathbb{Z}$-homogeneous derivation ϕ in $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ to be vanishing on $\mathcal{W}_{-1}$; that is, we find an inner derivation $\mathrm{ad}x$ such that $(\phi - \mathrm{ad}x)(\mathcal{W}_{-1}) = 0$, where $x \in W_{\bar{1}}$. In addition, we reduce the derivations in $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ to be vanishing on the canonical torus of $\mathcal{W}$. In next step, based on these results, we shall reduce the derivations in $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ to be vanishing on the top $\mathcal{W}_{-1} \oplus \mathcal{W}_0$.

Set

$$\mathcal{G} := \mathrm{span}_{\mathbb{F}}\{x^u D_i \mid i \in Y, u \in \mathbb{B}(n), \mathrm{p}(x^u D_i) = \bar{1}\}.$$

Note that $\mathcal{G} = C_{W_{\bar{1}}}(\mathcal{W})$. Then $\mathcal{G}$ is a $\mathbb{Z}$-graded subspace of $W_{\bar{1}}$.

In the sequel we adopt the following notation. Let P be a proposition. Define $\delta_P := 1$ if P is true and $\delta_P := 0$, otherwise. Put $\Gamma_i := x_i D_i$ for $i \in Y$ and $\Gamma := \sum_{i\in Y}\Gamma_i$, $\Gamma' := \sum_{i\in Y_1}\Gamma_i$ and $\Gamma'' := \sum_{i\in Y_0}\Gamma_i$. We call $\mathrm{ad}\,\Gamma$ *the degree derivation of* W (or $\mathcal{W}$), and $\mathrm{ad}\,\Gamma'$ and $\mathrm{ad}\,\Gamma''$ *the semi-degree derivations* of W (or $\mathcal{W}$). The following simple facts will be frequently used in this note:

$$\mathrm{ad}\,\Gamma(E) = rE \quad \text{for all } E \in W_r,\ r \in \mathbb{Z};$$

$$\operatorname{ad}\Gamma'(x^{(\alpha)}x^u D_j) = (|u| - \delta_{j\in Y_1})x^{(\alpha)}x^u D_j \quad \text{for all } \alpha \in \mathbb{A}(m;\underline{t}),\ u \in \mathbb{B}(n),\ j \in Y;$$
$$\operatorname{ad}\Gamma''(x^{(\alpha)}x^u D_j) = (|\alpha| - \delta_{j\in Y_0})x^{(\alpha)}x^u D_j \quad \text{for all } \alpha \in \mathbb{A}(m;\underline{t}),\ u \in \mathbb{B}(n),\ j \in Y.$$

In particular, each standard $\mathbb{F}$-basis element $x^{(\alpha)}x^u D_j$ of W is an eigenvector of the degree derivation and the semi-degree derivation of W.

Similar to [2, Lemma 2.1.1], we have the following

Lemma 2.1. *Suppose that $\mathcal{L}$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$ and $\mathcal{L}_{-1} = \mathcal{W}_{-1}$. Let $E \in \mathcal{L}$ and $\phi \in \operatorname{Der}(\mathcal{L}, W_{\bar{1}})$ satisfying $\phi(\mathcal{W}_{-1}) = 0$. Then $\phi(E) \in \mathcal{G}$ if and only if $[E, \mathcal{W}_{-1}] \subseteq \ker\phi$.*

Analogous to [5, Proposition 8.2, p. 192], we have the following

Lemma 2.2. *Let $k \leq n$, $f_1, \ldots, f_k \in \Lambda(n)$ be nonzero elements and $\Gamma_{q_i} := x_{q_i} D_{q_i}$, $q_i \in Y_1$, $1 \leq i \leq k$. Suppose that*

(a) $\Gamma_{q_i}(f_j) = \Gamma_{q_j}(f_i)$ *for* $1 \leq i, j \leq k$;

(b) $\Gamma_{q_i}(f_i) = f_i$ *for* $1 \leq i \leq k$.

Then there is $f \in \Lambda(n)$ such that $\Gamma_{q_i}(f) = f_i$ for $1 \leq i \leq k$.

Analogous to [2, Lemma 2.1.6], we have

Lemma 2.3. *Suppose that $\mathcal{L}$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$ satisfying $\mathcal{L}_{-1} = \mathcal{W}_{-1}$. Let $\phi \in \operatorname{Der}(\mathcal{L}, W_{\bar{1}})$ with $\operatorname{zd}(\phi) = t \geq 0$. Then there is $E \in (W_{\bar{1}})_t$ such that*

$$\big(\phi - \operatorname{ad}E\big)(\mathcal{L}_{-1}) = 0.$$

In view of Lemma 2.3, every nonnegative $\mathbb{Z}$-homogeneous derivation from $\mathcal{W}$ into W may be reduced to be vanishing on $\mathcal{W}_{-1}$. Thus, next step is to reduce such derivations to be vanishing on the top $\mathcal{W}_{-1} \oplus \mathcal{W}_0$. To that end, we first consider the canonical torus of $\mathcal{W}$, that is, $\mathcal{T} := \operatorname{span}_{\mathbb{F}}\{\Gamma_i \mid i \in Y\}$.

The following lemma will simplify our consideration, it tells us that in order to reduce derivations on the canonical torus it suffices to reduce these derivations on $\mathcal{T}' := \operatorname{span}_{\mathbb{F}}\{\Gamma_j \mid j \in Y_1\}$.

Lemma 2.4. *Suppose that $\phi \in \operatorname{Der}_t(\mathcal{W}, W_{\bar{1}})$ with $t \geq 0$ and $\phi(\mathcal{W}_{-1} \cup \mathcal{T}') = 0$. Then $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.*

Proof. (i) First consider the case $t > 0$. From $\phi(\mathcal{W}_{-1}) = 0$ and Lemma 2.1 we have $\phi(\Gamma_i) \in \mathcal{G}_t$ for all $i \in Y_0$. Thus one may assume that

$$\phi(\Gamma_i) = \sum_{k\in Y,\, u\in\mathbb{B}_{t+1}} c_{u,k} x^u D_k \quad \text{where } c_{u,k} \in \mathbb{F}. \tag{2.1}$$

For arbitrary $l \in Y$ and $v \in \mathbb{B}_{t+1}$, noticing that $t+1 > 1$, one may find $j \in v\backslash\{l\}$. Clearly,

$$[\Gamma_j,\ \phi(\Gamma_i)] = \sum_{k\in Y_0,\, u\in\mathbb{B}_{t+1}} c_{u,k}[\Gamma_j,\ x^u D_k]. \tag{2.2}$$

Note that

$$[\Gamma_j,\ x^u D_k] = (\delta_{j\in u} - \delta_{jk})x^u D_k; \tag{2.3}$$

in particular,

$$[\Gamma_j,\ x^v D_l] = x^v D_l. \tag{2.4}$$

On the other hand, since $[\Gamma_i,\ \Gamma_j] = 0$ and $\phi(\Gamma_j) = 0$, one may easily see that

$$[\Gamma_j,\ \phi(\Gamma_i)] = 0.$$

From (2.2)–(2.4) and the equation above, we obtain that $c_{v,l} = 0$. It follows from (2.1) that $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.

(ii) Let us consider the case $t = 0$. In view of Lemma 2.1, we have $\phi(\Gamma_i) \in \mathcal{G}_0$. Obviously, $[\Gamma_j, \Gamma_i] = 0$ for all $j \in Y_1, i \in Y_0$. Therefore,

$$[\Gamma_j, \phi(\Gamma_i)] = 0 \quad \text{for all } j \in Y_1.$$

Note that each standard basis element of $W_{\bar{1}}$ is an eigenvectors of $\mathrm{ad}\Gamma_j$ for $j \in Y_1$. It follows from the equation displayed above that

$$\phi(\Gamma_i) = \sum_{k \in Y_1} c_{i,k}\Gamma_k \quad \text{where } c_{i,k} \in \mathbb{F}. \tag{2.5}$$

For $j, l \in Y_1$, by Lemma 2.1, one gets $\phi(x_j D_l) \in \mathcal{G}_0$. Assume that $[\Gamma_i, \phi(x_j D_l)] = \sum_{k \in Y_1} \lambda_k x_k D_i$. From the equation $[\Gamma_i, x_j D_l] = 0$, we obtain that

$$[\phi(\Gamma_i), x_j D_l] = -\sum_{k \in Y_1} \lambda_k x_k D_i.$$

Then it follows from (2.5) that $c_{ij} = c_{il}$ for $j, l \in Y_1$. Write $c_{ij} := c_i$ for all $j \in Y_1$. Then (2.5) shows that

$$\phi(\Gamma_i) = c_i \Gamma' \quad \text{for } i \in Y_0.$$

We want to show that $c_i = 0$ for all $i \in Y_0$. Suppose that we are given $i \in Y_0$, $j, l \in Y_1$. Clearly, $[x_j x_l D_i, \Gamma_i] = x_j x_l D_i$. Applying ϕ to this equation, we have

$$\begin{aligned} [\phi(x_j x_l D_i), \Gamma_i] - \phi(x_j x_l D_i) &= -[x_j x_l D_i, \phi(\Gamma_i)] \\ &= -[x_j x_l D_i, c_i \Gamma'] \\ &= 2c_i x_j x_l D_i. \end{aligned}$$

By Lemma 2.1, it is easily seen that $\phi(x_j x_l D_i) \in \mathcal{G}_1$. Thus one may assume that $\phi(x_j x_l D_i) = \sum_{k \in Y, u \in \mathbb{B}_2} c_{u,k} x^u D_k$. Note that $[x^u D_k, x_i D_i] = \delta_{ki} x^u D_i$. It follows that

$$\sum_{k \in Y,\, u \in \mathbb{B}_2} (\delta_{ki} - 1) c_{u,k} x^u D_k = 2c_i x_j x_l D_i.$$

A comparison of the coefficients of $x_j x_l D_i$ in the equation above yields that $2c_i = 0$ for $i \in Y_0$. Since $\mathrm{char}\mathbb{F} \neq 2$, we have $c_i = 0$ for all $i \in Y_0$. So far, we have proved that $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.

Now, by (i) and (ii), we obtain the desired result. ■

We first consider the odd positive $\mathbb{Z}$-homogeneous derivations.

Lemma 2.5. *Suppose that $\phi \in \mathrm{Der}_t(\mathcal{W}, W_{\bar{1}})$ where $\mathrm{zd}(\phi) = t \geq 1$ is odd. If $\phi(\mathcal{W}_{-1}) = 0$, then there is $z \in \mathcal{G}_t$ such that $(\phi - \mathrm{ad}z)(\mathcal{T}') = 0$.*

Proof. Using Lemma 2.1 and noting that t is odd, we may assume that

$$\phi(\Gamma_i) = \sum_{r\in Y_1} f_{ri}D_r \quad \text{where } i \in Y_1,\ f_{ri} \in \Lambda(n). \tag{2.6}$$

Applying ϕ to the equation that $[\Gamma_i, \Gamma_j] = 0$ for $i, j \in Y_1$, we have

$$\sum_{r\in Y_1} (\Gamma_i(f_{rj}) - \Gamma_j(f_{ri}))D_r + f_{ji}D_j - f_{ij}D_i = 0.$$

Consequently,

$$\Gamma_i(f_{rj}) = \Gamma_j(f_{ri}) \quad \text{whenever } r \neq i, j; \tag{2.7}$$

$$\Gamma_j(f_{ii}) = \Gamma_i(f_{ij}) - f_{ij} \quad \text{whenever } i \neq j. \tag{2.8}$$

For $r, i \in Y_1$, one may assume that $f_{ri} = \sum_{|u|=t+1} c_{uri}x^u$, $c_{uri} \in \mathbb{F}$. By (2.7), we have

$$c_{uri}\delta_{j\in u} = c_{urj}\delta_{i\in u} \quad \text{whenever } r \neq i, j.$$

This implies that

$$c_{uri} \neq 0 \text{ and } \ j \in u \Longleftrightarrow c_{urj} \neq 0 \text{ and } \ i \in u.$$

Let $r \neq i$ and assume that $c_{uri} \neq 0$. Then the implication relation above shows that $i \in u$. Accordingly,

$$\Gamma_i(f_{ri}) = f_{ri} \quad \text{whenever } r \neq i. \tag{2.9}$$

For any fixed $r \in Y_1$, Lemma 2.2 ensures that there is $\overline{f}_r \in \Lambda(n)$ such that

$$\Gamma_i(\overline{f}_r) = f_{ri} \quad \text{for all } i \in Y_1\backslash\{r\}. \tag{2.10}$$

Assert that

$$\Gamma_i(f_{ii}) = 0 \quad \text{for all } i \in Y_1. \tag{2.11}$$

Using (2.8) and noticing the fact that $\Gamma_i^2 = \Gamma_i$, we obtain that

$$\Gamma_j\Gamma_i(f_{ii}) = \Gamma_i\Gamma_j(f_{ii}) = \Gamma_i^2(f_{ij}) - \Gamma_i(f_{ij}) = 0 \quad \text{for } j \neq i.$$

Note that $\mathrm{zd}(f_{ii}) = t + 1 \geq 2$ and $\Gamma_i(x^u) = \delta_{i\in u}x^u$. (2.11) follows.

For $r \in Y_1$, put $f_r := -f_{rr} + \Gamma_r(\overline{f}_r)$. Obviously, $f_r \in \Lambda(n)$. It follows from (2.11) that

$$\Gamma_r(f_r) - f_r = -\Gamma_r(f_{rr}) + \Gamma_r^2(\overline{f}_r) + f_{rr} - \Gamma_r(\overline{f}_r) = f_{rr}. \tag{2.12}$$

For $i \in Y_1\backslash r$, by (2.8) and (2.10) we obtain that

$$\begin{aligned}\Gamma_i(f_r) &= -\Gamma_i(f_{rr}) + \Gamma_i\Gamma_r(\overline{f}_r) = -\Gamma_i(f_{rr}) + \Gamma_r\Gamma_i(\overline{f}_r)\\ &= -(\Gamma_r(f_{ri}) - f_{ri}) + \Gamma_r(f_{ri}) = f_{ri}.\end{aligned}$$

Let $z' := -\sum_{r\in Y_1} f_rD_r$. A combination of (2.12) and the equation above yields that for $i \in Y_1$,

$$\begin{aligned}[z', \Gamma_i] &= -\sum_{r\in Y_1}[f_rD_r, \Gamma_i] = \sum_{r\in Y_1}\Gamma_i(f_r)D_r - f_iD_i\\ &= \sum_{r\in Y_1\backslash i}\Gamma_i(f_r)D_r + (\Gamma_i(f_i) - f_i)D_i\\ &= \sum_{r\in Y_1\backslash i} f_{ri}D_r + f_{ii}D_i = \phi(\Gamma_i).\end{aligned}$$

Let z be the t-component of z'. Since $\mathrm{zd}(\phi) = t$, one gets $[z, \Gamma_i] = \phi(\Gamma_i)$ for all $i \in Y_1$. Putting $\psi := \phi - \mathrm{ad}z$, then $\psi \in \mathrm{Der}_t(\mathcal{W}, W_{\bar{1}})$ and $\psi(\Gamma_i) = 0$ for all $i \in Y_1$. ∎

Lemma 2.6. *Suppose that* $\phi \in \mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ *and* $\mathrm{zd}(\phi) = t \geq 0$ *is even. If* $\phi(\mathcal{W}_{-1}) = 0$ *then there is* $z \in \mathcal{G}_t$ *such that* $(\phi - \mathrm{ad}z)(\mathcal{T}') = 0$.

Proof. Since $\mathrm{zd}(\phi) = t$ is even, by Lemma 2.1, one may assume that

$$\phi(\Gamma_i) = \sum_{r \in Y_0} f_{ri} D_r \quad \text{where } i \in Y_1,\ f_{ri} \in \Lambda(n). \tag{2.13}$$

Analogous to the proof of Lemma 2.5, one may easily show that

$$\Gamma_i(f_{rj}) = \Gamma_j(f_{ri}) \quad \text{for all } i, j \in Y_1. \tag{2.14}$$

Suppose that

$$f_{ri} = \sum_{u \in \mathbb{B}_{t+1}} c_{u,r,i} x^u \quad \text{where } c_{u,r,i} \in \mathbb{F}. \tag{2.15}$$

Then we obtain from (2.14) and (2.15) that

$$c_{u,r,i} \delta_{j \in u} = c_{u,r,j} \delta_{i \in u} \quad \text{for all } i, j \in Y_1,\ r \in Y_0.$$

Consequently, for $i, j \in Y_1,\ r \in Y_0$ and $u \in \mathbb{B}_{t+1}$,

$$c_{u,r,i} \neq 0 \text{ and } j \in u \Longleftrightarrow c_{u,r,j} \neq 0 \text{ and } i \in u. \tag{2.16}$$

Let us complete the proof of this lemma. Assume that $\mathrm{zd}(\phi) = t \geq 2$. If $c_{u,r,i} \neq 0$ for $i \in Y_1$, one may pick $j \in u \backslash i$. By (2.16), we have $i \in u$. Assume that $\mathrm{zd}(\phi) = 0$. Then (2.16) implies that there is at most one nonzero summand $c_{\langle i \rangle, r, i} x_i$ in the right-hand side of (2.15). Summarizing, every nonzero summand in the right-hand side of (2.15) possesses the factor x_i. Therefore,

$$\Gamma_i(f_{ri}) = f_{ri} \quad \text{for all } i \in Y_1,\ r \in Y_0. \tag{2.17}$$

For any fixed $r \in Y_0$, by (2.14) and (2.17), $\{f_{r,m+1}, f_{r,m+2}, \ldots, f_{r,m+n}\}$ fulfills the conditions of Lemma 2.2. Hence, there is $f_r \in \Lambda(n)$ such that

$$\Gamma_i(f_r) = f_{ri} \quad \text{for } i \in Y_1.$$

Let $z' := -\sum_{r \in Y_0} f_r D_r$. Then (2.13) and the equation above show that $[z', \Gamma_i] = \phi(\Gamma_i)$ for $i \in Y_1$. Let z be t-component of z'. Then $z \in \mathcal{G}_t$ and $(\phi - \mathrm{ad}z)(\Gamma_i) = 0$ for all $i \in Y_1$. ∎

Now we come to the following main result.

Proposition 2.7. *Let* ϕ *be a homogeneous derivation from* $\mathcal{W}$ *into* $W_{\bar{1}}$ *with nonnegative* $\mathbb{Z}$*-degree* t. *Then* ϕ *can be reduced to be vanishing on* $\mathcal{W}_{-1}$ *and the canonical torus of* $\mathcal{W}$; *that is, there is* $E \in (W_{\bar{1}})_t$ *such that* $(\phi - \mathrm{ad}E)\mid_{\mathcal{W}_{-1} + \mathcal{T}} = 0$.

Proof. By Lemma 2.3, there is $E' \in (W_{\bar{1}})_t$ such that $(\phi - \mathrm{ad}E')(\mathcal{W}_{-1}) = 0$. Then by Lemmas 2.4—2.6, there is $E'' \in \mathcal{G}_t$ such that $(\phi - \mathrm{ad}E' - \mathrm{ad}E'')(\mathcal{T}) = 0$. Putting $E := E' + E''$, then $(\phi - \mathrm{ad}E)(\mathcal{W}_{-1} + \mathcal{T}) = 0$. ■

In the following, using Proposition 2.7, we first reduce every nonnegative $\mathbb{Z}$-homogeneous derivation from $\mathcal{W}$ into $W_{\bar{1}}$ to be vanishing on the top $\mathcal{W}_{-1} \oplus \mathcal{W}_0$ of $\mathcal{W}$; then we determine the $\mathbb{Z}$-homogeneous components $\mathrm{Der}_t(\mathcal{W}, W_{\bar{1}})$ for $t \geq 0$.

Proposition 2.8. *Let $\phi \in \mathrm{Der}_t(\mathcal{W}, W_{\bar{1}})$ with $t \geq 0$. Then there is $E \in (W_{\bar{1}})_t$ such that $(\phi - \mathrm{ad}E) |_{\mathcal{W}_{-1} \oplus \mathcal{W}_0} = 0$.*

Proof. By Proposition 2.7, without loss of generality we may assume that $\phi(\mathcal{W}_{-1} + \mathcal{T}) = 0$.

(i) We first consider $\phi(x_k D_i)$ where $i, k \in Y_0$ with $i \neq k$. By Lemma 2.1, $\phi(x_k D_i) \in \mathcal{G}_t$. Assume that $\phi(x_k D_i) = \sum_{r \in Y, u \in \mathbb{B}_{t+1}} c_{u,r} x^u D_r$ where $c_{u,r} \in \mathbb{F}$. If t is even, then $\phi(x_k D_i) = \sum_{r \in Y_0, u \in \mathbb{B}_{t+1}} c_{u,r} x^u D_r$. Note that $[\Gamma_j, \phi(x_k D_i)] = 0$ for arbitrary $j \in Y_1$. It follows that $\phi(x_k D_i) = 0$. If t is odd, then $\phi(x_k D_i) = \sum_{r \in Y_1, u \in \mathbb{B}_{t+1}} c_{u,r} x^u D_r$. Then

$$\phi(x_k D_i) = \phi([\Gamma_k, x_k D_i]) = [\Gamma_k, \phi(x_k D_i)] = 0.$$

(ii) We next consider $\phi(x_k D_l)$ where $k, l \in Y_1$ with $k \neq l$.

(a) Suppose that t is even. Just as in (i) one may assume that

$$\phi(x_k D_l) = \sum_{r \in Y_0, u \in \mathbb{B}_{t+1}} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}.$$

Then, from the equation that $[\Gamma_i, \phi(x_k D_l)] = 0$ for all $i \in Y_0$, one gets $c_{u,r} = 0$ for all $r \in Y_0$, $u \in \mathbb{B}_{t+1}$. Hence, $\phi(x_k D_l) = 0$.

(b) Suppose that t is odd. We proceed in two cases $t \geq 3$ and $1 \leq t \leq 2$ to show that $\phi(x_k D_l) = 0$ for $k, l \in Y_1$. Suppose that $t \geq 3$. By Lemma 2.1, one may assume that

$$\phi(x_k D_l) = \sum_{r \in Y_1,\ |u| \geq 4} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \tag{2.18}$$

Given $v \in \mathbb{B}$ with $| v | \geq 4$ and $s \in Y_1$, choose $q \in v \setminus \{k, l, s\}$. Then $[\Gamma_q, x^v D_s] = x^v D_s$. On the other hand, since $[\Gamma_q, x_k D_l] = 0$, we have $[\Gamma_q, \phi(x_k D_l)] = 0$. Note that each standard basis element of W is an eigenvector of Γ_q and $[\Gamma_q, x^v D_s] = x^v D_s$. It follows from (2.18) that $c_{v,s} = 0$. Therefore, $\phi(x_k D_l) = 0$.

Finally we consider the case $t = 1$. Clearly, $[\Gamma', x_k D_l] = 0$. Consequently, $[\Gamma', \phi(x_k D_l)] = 0$. On the other hand, by Lemma 2.1, $\phi(x_k D_l) \in \mathcal{G}$. Thus

$$0 = [\Gamma', \phi(x_k D_l)] = \phi(x_k D_l).$$

The proof is complete. ■

In order to determine the homogeneous derivation subspace $\mathrm{Der}_t(\mathcal{W}, W_{\bar{1}})$ for $t \geq 0$, we need the generator set of $\mathcal{W}$ (see [2, Proposition 2.2.1]).

Lemma 2.9. *$\mathcal{W}$ is generated by $\mathcal{P} \cup \mathcal{N} \cup \mathcal{M}$, where*

$$\mathcal{P} := \{x_k x_l D_i \mid k, l \in Y_1,\ i \in Y_0\},$$
$$\mathcal{N} := \{x_k x_l D_i \mid k \in Y_0,\ l, i \in Y_1\},$$
$$\mathcal{M} := \{x^{(k\varepsilon_i)} D_j \mid 0 \leq k \leq \pi_i,\ i, j \in Y_0\}.$$

Now we can determine the nonnegative $\mathbb{Z}$-homogeneous derivations:

Proposition 2.10. *Let $t \in \mathbb{N}_0$. Then $\mathrm{Der}_t(\mathcal{W}, W_{\bar{1}}) = \mathrm{ad}(W_{\bar{1}})_t$.*

Proof. If suffices to show the inclusion "$\subset$". Let $\phi \in \mathrm{Der}_t(\mathcal{W}, W)$. By Proposition 2.8, one may assume that $\phi(\mathcal{W}_{-1} \oplus \mathcal{W}_0) = 0$. In the following we consider the application of ϕ to $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{M}$, respectively.

(i) First consider $\mathcal{P}$. Let $i, j \in Y_1,\ k \in Y_0$. In view of Lemma 2.1, we have $\phi(x_i x_j D_k) \in \mathcal{G}_{t+1}$. Clearly, $[\Gamma'', x_i x_j D_k] = -x_i x_j D_k$. Applying ϕ, we obtain that

$$[\Gamma'', \phi(x_i x_j D_k)] = -\phi(x_i x_j D_k).$$

It follows that $\phi(x_i x_j D_k)$ is of the form:

$$\phi(x_i x_j D_k) = \sum_{r \in Y_0,\ u \in \mathbb{B}_{t+2}} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \tag{2.19}$$

Since $n > 2$, picking $l \in Y_1 \setminus \{i, j\}$, one gets $[x_i x_j D_k, x_l D_l] = 0$. It follows that $[\phi(x_i x_j D_k), x_l D_l] = 0$. Furthermore,

$$\Big[\sum_{r \in Y_0,\ u \in \mathbb{B}_{t+2}} c_{u,r} x^u D_r, x_l D_l\Big] = 0 \quad \text{for all } l \neq i, j. \tag{2.20}$$

Now it follows from (2.19) and (2.20) that $c_{u,r} = 0$ unless $u = \{i, j\}$. Thus (2.19) gives

$$\phi(x_i x_j D_k) = \sum_{\substack{r \in Y_0, u \in \mathbb{B} \\ u = \{i,j\}}} c_{u,r} x^u D_r. \tag{2.21}$$

If $t > 0$ then (2.21) implies that $\phi(x_i x_j D_k) = 0$; if $t = 0$, we also obtain from (2.21) that $\phi(x_i x_j D_k) = 0$, since $\phi \in \mathrm{Der}(\mathcal{W}, W_{\bar{1}})$.

(ii) Let us show that $\phi(\mathcal{N}) = 0$. Let $i \in Y_0,\ j, k \in Y_1$. By Lemma 2.1, $\phi(x_i x_j D_k) \in \mathcal{G}_{t+1}$. Since $[\Gamma'', x_i x_j D_k] = x_i x_j D_k$, as in (i) one may assume that

$$\phi(x_i x_j D_k) = \sum_{r \in Y_0, u \in \mathbb{B}_{t+2}} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}.$$

Then

$$-\phi(x_i x_j D_k) = [\Gamma'', \phi(x_i x_j D_k)] = \phi(x_i x_j D_k).$$

Since $\mathrm{char}\mathbb{F} \neq 2$, it follows that

$$\phi(x_i x_j D_k) = 0 \quad \text{for all } i \in Y_0,\ j, k \in Y_1;$$

that is, $\phi(\mathcal{N}) = 0$.

(iii) Just as in the proof of [2, Lemma 3.1.4], one may show that $\phi(\mathcal{M}) = 0$.

Now, Lemma 2.9 shows that $\phi = 0$. ■

In view of Proposition 2.10, in order to determine the derivation space $\mathrm{Der}(\mathcal{W}, W_{\bar{1}})$ it suffices to determine the negative $\mathbb{Z}$-homogeneous derivations. We first consider the derivations of $\mathbb{Z}$-degree -1.

Lemma 2.11. *Suppose that* $\varphi \in \mathrm{Der}_{-1}(\mathcal{W}, W_{\bar{1}})$ *and* $\varphi(\mathcal{W}_0) = 0$*. Then* $\varphi = 0$.

Proof. We first assert that $\varphi(\mathcal{N}) = \varphi(\mathcal{P}) = 0$. Given $i, j \in Y_1, k \in Y_0$, by Lemma 2.1, $\varphi(x_i x_j D_k) \in \mathcal{G}_0$. Thus one may assume that $\varphi(x_i x_j D_k) = \sum_{r\in Y_1,\ s\in Y_0} c_{r,s} x_r D_s,\ c_{r,s} \in \mathbb{F}$. Then, since $[\Gamma', x_i x_j D_k] = 2x_i x_j D_k$, we have

$$\varphi(x_i x_j D_k) = [\Gamma', \varphi(x_i x_j D_k)] = 2\varphi(x_i x_j D_k).$$

It follows that $\varphi(x_i x_j D_k) = 0$; that is, $\varphi(\mathcal{P}) = 0$. Similarly, applying φ to the equation $[\Gamma'', x_k x_i D_j] = x_k x_i D_j$ gives

$$-\varphi(x_k x_i D_j) = [\Gamma'', \varphi(x_k x_i D_j)] = \varphi(x_k x_i D_j).$$

It follows that $\varphi(x_k x_i D_j) = 0$. Hence, $\varphi(\mathcal{N}) = 0$.

It remains to show that $\varphi(\mathcal{M}) = 0$. Given $k \in Y_0$, just as in the proof of [2, Lemma 3.2.6], one may prove by induction on r that

$$\varphi(x^{(r\varepsilon_k)} D_k) = 0 \quad \text{for all } r \in \mathbb{N}.$$

From this one may easily prove that $\varphi(\mathcal{M}) = 0$. Summarizing, by Lemma 2.9, $\varphi = 0$. ∎

Now we can determine the derivations from $\mathcal{W}$ into $W_{\bar{1}}$ of $\mathbb{Z}$-degree -1.

Proposition 2.12. $\mathrm{Der}_{-1}(\mathcal{W}, W_{\bar{1}}) = \mathrm{ad}(W_{\bar{1}})_{-1}$.

Proof. The inclusion "$\supset$" is clear. Let $\phi \in \mathrm{Der}_{-1}(\mathcal{W}, W)$. For $i \in Y_0, k \in Y_1$, applying ϕ to the equation that $[\Gamma_i, \Gamma_k] = 0$, we have $[\phi(\Gamma_i), \Gamma_k] + [\Gamma_i, \phi(\Gamma_k)] = 0$. As $\phi(\Gamma_i),\ \phi(\Gamma_k) \in W_{-1} \cap W_{\bar{1}}$, we have $[\Gamma_i, \phi(\Gamma_k)] = 0$ and therefore, $[\phi(\Gamma_i), \Gamma_k] = 0$ for all $k \in Y_1$. This implies that $\phi(\Gamma_i) = 0$ for $i \in Y_0$. It follows that $\phi(x_i D_j) = 0$ for all $i, j \in Y_0$.

For $k \in Y_1$, just as in the proof of [2, Proposition 3.2.7], one may prove that there are $c_k \in \mathbb{F}$ such that $\phi(\Gamma_k) = c_k D_k$ and $\phi(x_k D_l) = c_k D_l$ for all $k, l \in Y_1$. By Lemma 2.11, $\phi = \sum_{r\in Y_1} c_r D_r \in \mathrm{ad}(W_{\bar{1}})_{-1}$. ∎

Analogous to [2, Lemma 3.2.8], we also have the following

Lemma 2.13. *Let* $\phi \in \mathrm{Der}_{-q}(\mathcal{W}, W_{\bar{1}})$ *with* $q > 1$*. If* $\phi(x^{(q\varepsilon_i)} D_i) = 0$ *for all* $i \in Y_0$*, then* $\phi = 0$.

Proposition 2.14. *Suppose that* $q > 1$*. Then* $\mathrm{Der}_{-q}(\mathcal{W}, W_{\bar{1}}) = 0$.

Proof. Let $\phi \in \mathrm{Der}_{-t}(\mathcal{W}, W_{\bar{1}})$. In view of Lemma 2.13, it is sufficient to show that

$$\phi(x^{(q\varepsilon_i)} D_i) = 0 \quad \text{for all } i \in Y_0.$$

Note that $[\Gamma', x^{(q\varepsilon_i)} D_i] = 0$ and $\phi(x^{(q\varepsilon_i)} D_i) \in (W_{\bar{1}})_{-1}$. It follows that

$$0 = [\Gamma', \phi(x^{(q\varepsilon_i)} D_i)] = -\phi(x^{(q\varepsilon_i)} D_i).$$

The proof is complete. ∎

Theorem 2.15. $\mathrm{Der}(\mathcal{W}, W_{\bar{1}}) = \mathrm{ad}(W_{\bar{1}})$.

Proof. By Propositions 2.10, 2.12 and 2.14, "$\subset$" holds. The converse inclusion is clear. ■

Remark 2.16. By [7, Theorem 2], [2, Theorem 3.2.11] and Theorem 2.15, the even part and the odd part of the superderivation algebra of the finite-dimensional generalized Witt superalgebra W coincide with the derivation algebra of the even part of W and the derivation space of the even part into the odd part of W, respectively; that is, $(\mathrm{Der}\, W)_{\bar{0}} = \mathrm{Der}(W_{\bar{0}})$, $(\mathrm{Der}\, W)_{\bar{1}} = \mathrm{Der}(W_{\bar{0}}, W_{\bar{1}})$.

Remark 2.17. As noted in Section 1, by Theorem 2.15, we have

$$H^1(\mathcal{W}, W_{\bar{1}}) = 0.$$

3. Special superalgebras

Recall the canonical torus $\mathcal{T}_{\mathcal{S}}$ of $\mathcal{S}$ (c.f. [2]). Clearly,

$$\{x_r D_r - x_s D_s | \tau(r) = \tau(s); r, s \in Y\} \cup \{x_r D_r + x_s D_s | \tau(r) \neq \tau(s); r, s \in Y\}$$

is an $\mathbb{F}$-basis of $\mathcal{T}_{\mathcal{S}}$ consisting of toral elements.

The following fact is simple.

Lemma 3.1. $\mathcal{S}_0 = \mathrm{span}_{\mathbb{F}}\{\mathcal{T}_{\mathcal{S}} \cup \{x_r D_s \mid \tau(r) = \tau(s), r \neq s; r, s \in Y\}\}$.

Put

$$\mathcal{Q} := \{D_{ij}(x^{(r\varepsilon_j)}) \mid i, j \in Y_0, r \in \mathbb{N}_0\};$$

$$\mathcal{R} := \{D_{il}(x^{(2\varepsilon_i)} x_k) \mid i \in Y_0, k, l \in Y_1\} \cup \{D_{ij}(x_i x^v) \mid i, j \in Y_0, v \in \mathbb{B}_2\}.$$

Lemma 3.2. [2, Proposition 2.2.3] *$\mathcal{S}$ is generated by $\mathcal{Q} \cup \mathcal{R} \cup \mathcal{S}_0$.*

In the following we consider the top of $\mathcal{S}$.

Lemma 3.3. *Suppose that $\phi \in \mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ with $\mathrm{zd}(\phi) \geq 0$ and that $\phi(\mathcal{S}_{-1} + \mathcal{S}_0) = 0$. Then*

(i) $\phi(D_{il}(x^{(2\varepsilon_i)} x_k)) = 0$ *for all* $i \in Y_0$, $k, l \in Y_1$.

(ii) $\phi(D_{ij}(x_i x^v)) = 0$ *for all* $i, j \in Y_0$ *and* $v \in \mathbb{B}_2$.

(iii) $\phi(D_{ij}(x^{(a\varepsilon_i)})) = 0$ *for all* $i, j \in Y_0$ *and all* $a \in \mathbb{N}$.

Proof. (i) The proof is similar to the one of [2, Lemma 4.1.1]. Our discussion here for $\mathrm{zd}(\phi)$ odd is completely analogous to one in [2, Lemma 4.1.1] for $\mathrm{zd}(\phi)$ even; and, the discussion here for $\mathrm{zd}(\phi)$ even is completely analogous to one in [2, Lemma 4.1.1] for $\mathrm{zd}(\phi)$ odd.

Similar to [2, Lemmas 4.1.2, 4.1.3], one may prove (ii) and (iii) in the same way. ■

As a direct consequence of Lemmas 3.2 and 3.3, we have the following.

Corollary 3.4. *Suppose that* $\phi \in \mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ *with* $\mathrm{zd}(\phi) \geq 0$ *and that* $\phi(\mathcal{S}_{-1} + \mathcal{S}_0) = 0$. *Then* $\phi = 0$.

In order to describe the derivations of nonnegative degree we first give two technical lemmas which will simplify our discussion.

Lemma 3.5. *Suppose that* $\phi \in \mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$ *and* $\phi(\mathcal{S}_{-1}) = 0$.
(i) *If* $t = n-1$ *is even, then* $\phi(\Gamma_{1'} - \Gamma_k) = 0$ *for all* $k \in Y_1 \setminus 1'$.
(ii) *If* $t = n-1$ *is odd, then there is* $\lambda \in \mathbb{F}$ *such that*

$$(\phi - \lambda \mathrm{ad}(x^{\omega} D_{1'}))(\Gamma_{1'} - \Gamma_k) = 0 \textit{ for all } k \in Y_1 \setminus 1'.$$

(iii) *If* $t > n-1$, *then* $\phi = 0$.

Proof. (i) The proof is completely analogous to the one of [2, Lemmas 4.2.1(i)].

(ii) The proof is completely analogous to the one of [2, Lemmas 4.2.1(ii)].

(iii) Using Lemma 2.1 and induction on r one may easily prove that $\phi(S_r) = 0$ for all $r \in \mathbb{N}$. ■

Analogous to [2, Lemmas 4.2.2], we have

Lemma 3.6. *Suppose that* $\phi \in \mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ *and* $\mathrm{zd}(\phi) \geq 0$ *is even.*
(i) *If* $\mathrm{zd}(\phi) < n-1$ *and*

$$\phi(\Gamma_{1'} - \Gamma_{2'}) = \phi(\Gamma_{1'} - \Gamma_{3'}) = \cdots = \phi(\Gamma_{1'} - \Gamma_{n'}) = 0,$$

then

$$\phi(\Gamma_1 - \Gamma_2) = \phi(\Gamma_1 - \Gamma_3) = \cdots = \phi(\Gamma_1 - \Gamma_m) = 0; \quad \phi(\Gamma_1 + \Gamma_{1'}) = 0.$$

(ii) *If* $\mathrm{zd}(\phi) = n-1$, *then there are* $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ *such that*

$$\Big(\phi - \mathrm{ad}\Big(\sum_{i \in Y_0} \lambda_i x^{\omega} D_i\Big)\Big)(\Gamma_1 - \Gamma_j) = 0 \quad \textit{for all } j \in Y_0 \setminus 1;$$

$$\Big(\phi - \mathrm{ad}\Big(\sum_{i \in Y_0} \lambda_i x^{\omega} D_i\Big)\Big)(\Gamma_1 + \Gamma_{1'}) = 0.$$

Recall the canonical torus of $\mathcal{S}$

$$\mathcal{T}_{\mathcal{S}} := \mathrm{span}_{\mathbb{F}}\{\Gamma_1 - \Gamma_2, \ldots, \Gamma_1 - \Gamma_m, \ldots, \Gamma_1 + \Gamma_{1'}, \Gamma_{1'} - \Gamma_{2'}, \ldots, \Gamma_{1'} - \Gamma_{n'}\}.$$

As a direct consequence of Lemma 3.5 (iii) and Lemma 3.6, we have the following fact:

Corollary 3.7. *Suppose that* $\phi \in \mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ *is a nonnegative even* $\mathbb{Z}$*-homogeneous derivation such that* $\phi(\mathcal{S}_{-1}) = 0$ *and* $\phi(\Gamma_k - \Gamma_{1'}) = 0$ *for all* $k \in Y_1 \setminus 1'$. *Then there is* $E \in \mathcal{G}$ *such that* $\phi - \mathrm{ad}E$ *vanishes on the canonical torus* $\mathcal{T}_{\mathcal{S}}$.

Now we prove the following two key lemmas. First, consider the even $\mathbb{Z}$-homogeneous derivations.

Lemma 3.8. *Suppose that* $\phi \in \mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$, *where* $t \geq 0$ *is even. If* $\phi(\mathcal{S}_{-1}) = 0$, *then there is* $D \in \mathcal{G}_t$ *such that*

$$(\phi - \mathrm{ad}D)(\Gamma_k - \Gamma_{1'}) = 0 \textit{ for all } k \in Y_1 \setminus 1'.$$

Proof. By Lemma 3.5 (i) and (iii) it suffices to consider the setting $t < n-1$. By Lemma 2.1, one may assume that

$$\phi(\Gamma_k - \Gamma_{1'}) = \sum_{r \in Y_0} f_{rk} D_r \text{ where } k \in Y_1 \setminus 1';\ f_{rk} \in \Lambda(n). \tag{3.1}$$

Write

$$f_{rk} = \sum_{|u|=t+1} c_{u,r,k} x^u \quad \text{where } c_{u,r,k} \in \mathbb{F}. \tag{3.2}$$

Discussing just as in the proof of [2, Lemma 4.2.4], we may obtain that

$$\sum_{|u|=t+1} (\delta_{k\in u} - \delta_{1'\in u}) c_{u,r,l} x^u = \sum_{|u|=t+1} (\delta_{l\in u} - \delta_{1'\in u}) c_{u,r,k} x^u.$$

Since $\{x^u \mid u \in \mathbb{B}\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$, it follows that

$$(\delta_{k\in u} - \delta_{1'\in u}) c_{u,r,l} = (\delta_{l\in u} - \delta_{1'\in u}) c_{u,r,k} \quad \text{for } r \in Y_0,\ k, l \in Y_1 \setminus 1'. \tag{3.3}$$

Suppose that $c_{u,r,k}$ is any a nonzero coefficient in (3.2), where $|u| = t+1 < n$, $r \in Y_0$ and $k \in Y_1 \setminus 1^{'}$. Note that $|u| \geq 1$. We proceed in two steps to show that $\delta_{k\in u} + \delta_{1'\in u} = 1$.

Case (i): $|u| \geq 2$. If $1^{'} \notin u$, one may find $l \in u \backslash k$. Then (3.3) shows that $\delta_{k\in u} = 1$; that is, $k \in u$. If $1^{'} \in u$, noting that $|u| \leq n-1$, one may find $l \in Y_1 \setminus u$. Then (3.3) shows that $\delta_{k\in u} = 0$; that is, $k \notin u$. Summarizing, for any nonzero coefficient $c_{u,r,k}$ in (3.2), we have $\delta_{k\in u} + \delta_{1'\in u} = 1$.

Case (ii): $|u| = 1$. Since $|u| = 1$, the case of $k \in u$ and $1^{'} \in u$ does not occur. If $k \notin u$ and $1^{'} \notin u$, then there is $l \in u$, since $|u| = 1$. Then by (3.3), we get $c_{u,r,k} = 0$, this is a contradiction. Hence, we have $\delta_{k\in u} + \delta_{1'\in u} = 1$.

Then, just like in the proof of [2, Lemma 4.2.4], we can rewrite (3.2) as follows

$$f_{rk} = \sum_{1'\in u, k\notin u} c_{u,r,k} x^u + \sum_{1'\notin u, k\in u} c_{u,r,k} x^u.$$

Now, following the corresponding part of the proof for [2, Lemma 4.2.4], one may find $D \in \mathcal{G}_t$ such that $(\phi - \mathrm{ad}D)(\Gamma_k - \Gamma_{1'}) = 0$ for all $k \in Y_1 \setminus 1'$. The proof is complete. ■

Let us consider the case of odd $\mathbb{Z}$-degree.

Lemma 3.9. *Let* $\phi \in \mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$ *where* $t > 0$ *is odd. If* $\phi(\mathcal{S}_{-1}) = 0$, *then there is* $D \in \mathcal{G}_t$ *such that*

$$(\phi - \mathrm{ad}D)(\Gamma_1 + \Gamma_k) = 0 \textit{ for all } k \in Y_1. \tag{3.4}$$

Proof. Deleting the part (ii) in the proof of [2, Lemma 4.2.5] gives the proof of this lemma. ■

For our purpose, we need still the following three reduction lemmas.

Lemma 3.10. *Suppose that $\phi \in \mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$ and $\phi(\mathcal{S}_{-1}) = 0$, where $t > 0$ is odd. If $\phi(\Gamma_1 + \Gamma_k) = 0$ for all $k \in Y_1$, then $\phi(\mathcal{S}_0) = 0$.*

Proof. Following parts (i) and (ii) in the proof of [2, lemma 4.2.6], one may show that $\phi(\Gamma_1 - \Gamma_i) = 0$ and $\phi(x_i D_j) = 0$ for all $i, j \in Y_0$ with $i \neq j$.

To show that $\phi(x_k D_l) = 0$ for $k, l \in Y_1$ with $k \neq l$, just as in the part (iii) of the proof of [2, Lemma 4.2.6], it suffices to consider separately two cases $\mathrm{zd}(\phi) = 1$ and $\mathrm{zd}(\phi) \geq 3$. Now Lemma 3.1 ensures that $\phi(\mathcal{S}_0) = 0$. ∎

Analogous to [2, Lemma 4.2.7], one may prove the following

Lemma 3.11. *Suppose that $\phi \in \mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ is a nonnegative even $\mathbb{Z}$-homogeneous derivation and $\phi(\mathcal{S}_{-1} + \mathcal{T}_{\mathcal{S}}) = 0$. Then $\phi(\mathcal{S}_0) = 0$.*

Now we are able to characterize the homogeneous derivation space of nonnegative $\mathbb{Z}$-degree. Using Lemmas 3.8–3.11, Corollaries 3.4 and 3.7, and Proposition 2.3, one may prove the following result (cf. [2, Proposition 4.2.9]).

Proposition 3.12. $\mathrm{Der}_t(\mathcal{S}, W_{\bar{1}}) = \mathrm{ad}(W_{\bar{1}})_t$ *for* $t \geq 0$.

As an application of Proposition 3.12, we have:

Proposition 3.13. $\mathrm{Der}_t(\mathcal{S}, S_{\bar{1}}) = \mathrm{ad}(\overline{S}_{\bar{1}})_t$ *for* $t \geq 0$.

Proof. Since S is an ideal of $\overline{S}$ (see [7, p. 139]), $\mathrm{ad}(\overline{S}_{\bar{1}})_t \subset \mathrm{Der}_t(\mathcal{S}, S_{\bar{1}})$. Let $\phi \in \mathrm{Der}_t(\mathcal{S}, S_{\bar{1}})$. View ϕ as a derivation of $\mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$. Then by Proposition 3.12, there is $D \in (W_{\bar{1}})_t$ such that $\phi = \mathrm{ad}D \in \mathrm{Der}_t(\mathcal{S}, W_{\bar{1}})$, and therefore, $\phi = \mathrm{ad}D \in \mathrm{Der}_t(\mathcal{S}, S_{\bar{1}})$. Let $\mathrm{Nor}_{W_{\bar{1}}}(\mathcal{S}, S_{\bar{1}}) := \{x \in W_{\bar{1}} \mid [x, \mathcal{S}] \subset S_{\bar{1}}\}$. Clearly, $D \in \mathrm{Nor}_{W_{\bar{1}}}(\mathcal{S}, S_{\bar{1}})_t$. Therefore, it suffices to show that $\mathrm{Nor}_{W_{\bar{1}}}(\mathcal{S}, S_{\bar{1}})_t \subset (\overline{S}_{\bar{1}})_t$. Let E be an arbitrary element of $\mathrm{Nor}_{W_{\bar{1}}}(\mathcal{S}, S_{\bar{1}})_t$. If $t = 0$ then $E \in W_{\bar{1}} \cap W_0$, which implies that $\mathrm{div}(E) = 0$ and therefore, $E \in \overline{S}_{\bar{1}}$. Now suppose that $t > 0$. Note that $\mathrm{div}([E, \mathcal{S}_{-1}]) = 0$. It follows that $D_i(\mathrm{div}(E)) = 0$ for all $i \in Y_0$. This implies that $\mathrm{div}(E) \in \Lambda(n)_t$. Similarly, $[E, \mathcal{S}] \subset S_{\bar{1}}$ implies that $[\mathrm{div}(E), \mathcal{S}] = 0$. In particular, $[\mathrm{div}(E), \mathcal{T}_{\mathcal{S}}] = 0$. Since $\mathrm{div}(E) \in \Lambda(n)_t$, one gets $[\mathrm{div}(E), \mathcal{T}] = 0$. Keeping in mind that $\mathrm{div}(E) \in \Lambda(n)_{\bar{1}}$, one may easily deduce that $\mathrm{div}(E) = 0$ (cf. [2, Proposition 4.2.10]). ∎

In the following we first determine the negative $\mathbb{Z}$-homogeneous derivations from $\mathcal{S}$ into $W_{\bar{1}}$. This combining with Proposition 3.12 will give the structure of the derivation space $\mathrm{Der}(\mathcal{S}, W_{\bar{1}})$. The following lemma tells us that a $\mathbb{Z}$-degree -1 derivation from $\mathcal{S}$ into $W_{\bar{1}}$ is completely determined by its action on $\mathcal{S}_0$.

Lemma 3.14. *Suppose that $\phi \in \mathrm{Der}_{-1}(\mathcal{S}, W_{\bar{1}})$ and that $\phi(\mathcal{S}_0) = 0$. Then $\phi = 0$.*

Proof. We first show that $\phi(\mathcal{R}) = 0$. By the definition of D_{il},

$$D_{il}(x^{(2\varepsilon_i)} x_k) = x_i x_k D_l + \delta_{kl} x^{(2\varepsilon_i)} D_i \quad \text{for all } i \in Y_0,\ k, l \in Y_1. \tag{3.5}$$

We shall use the following simple fact (by Lemma 2.1):

$$\phi(\mathcal{S}_1) \subseteq \mathcal{G}_0.$$

We may assume that

$$\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = \sum_{k\in Y_1, r\in Y_0} c_{k,r}x_kD_r. \tag{3.6}$$

Given $i \in Y_0$, $k, l \in Y_1$, take $j \in Y_0 \setminus i$. Then

$$[\Gamma_i - \Gamma_j, D_{il}(x^{(2\varepsilon_i)}x_k)] = D_{il}(x^{(2\varepsilon_i)}x_k).$$

Applying ϕ to the equation above and then combining that with (3.6), one may obtain by a comparison of coefficients that

$$c_{k,r} = 0 \quad \text{for } k \in Y_1,\ r \in Y_0 \setminus j.$$

Hence, by (3.6), we may obtain

$$\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = \sum_{k\in Y_1} c_{k,j}x_kD_j. \tag{3.7}$$

Case (i): $k \neq l$. Then by (3.5), we have $D_{il}(x^{(2\varepsilon_i)}x_k) = x_ix_kD_l$ and

$$[\Gamma_i + \Gamma_k, D_{il}(x^{(2\varepsilon_i)}x_k)] = 2D_{il}(x^{(2\varepsilon_i)}x_k). \tag{3.8}$$

Applying ϕ to (3.8) and using (3.7), one may obtain by comparing coefficients that

$$c_{s,j} = 0 \quad \text{for } s \in Y_1.$$

Consequently, $\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = 0$.

Case (ii): $k = l$. Then by (3.5), we have

$$D_{ik}(x^{(2\varepsilon_i)}x_k) = x^{(2\varepsilon_i)}D_i + x_ix_kD_k.$$

and

$$[\Gamma_i + \Gamma_k, D_{ik}(x^{(2\varepsilon_i)}x_k)] = D_{ik}(x^{(2\varepsilon_i)}x_k).$$

Applying ϕ to the equation above and using (3.7), one may obtain by comparing coefficients that

$$c_{s,j} = 0 \quad \text{for } s \in Y_1 \setminus k.$$

Hence, By (3.7), we may obtain

$$\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = c_{k,j}x_kD_j. \tag{3.9}$$

For $i \in Y_0$, $k, l \in Y_1$, choose $q \in Y_1 \setminus k$. Then $m\Gamma_q + \Gamma'' \in \mathcal{S}_0$ and

$$[m\Gamma_q + \Gamma'', D_{ik}(x^{(2\varepsilon_i)}x_k)] = D_{ik}(x^{(2\varepsilon_i)}x_k).$$

Applying ϕ and using (3.9), one gets

$$\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = [m\Gamma_q + \Gamma'', \phi(D_{ik}(x^{(2\varepsilon_i)}x_k))] = -\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)).$$

Consequently, $\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = 0$ for all $i \in Y_0, k \in Y_1$.

We want to show that $\phi(D_{ij}(x_i x_k x_l)) = 0$ for $i, j \in Y_0, k, l \in Y_1$. By a same argument, we can also obtain that

$$\phi(D_{ij}(x_i x_k x_l)) = c_{kj} x_k D_j \quad \text{for } k \in Y_1, j \in Y_0 \setminus i. \tag{3.10}$$

Note that $n\Gamma_q + \Gamma' \in \mathcal{S}_0$ for $q \in Y_0$ and that

$$[n\Gamma_q + \Gamma', D_{ij}(x_i x_k x_l)] = (2 - n\delta_{q,j}) D_{ij}(x_i x_k x_l) \quad \text{for } i, j \in Y_0, k, l \in Y_1. \tag{3.11}$$

Applying ϕ to (3.11) and using (3.10), one gets

$$c_{kj} = 0 \quad \text{for } k \in Y_1, j \in Y_0 \setminus i.$$

Consequently, $\phi(D_{ij}(x_i x_k x_l)) = 0$.

It remains to show that $\phi(\mathcal{Q}) = 0$. But this can be verified completely analogous to the proof of [2, Lemma 4.3.1]. ■

Using Lemma 3.14 we can determine the derivations of $\mathbb{Z}$-degree -1.

Proposition 3.15. $\mathrm{Der}_{-1}(\mathcal{S}, W_{\bar{1}}) = \mathrm{ad}(W_{\bar{1}})_{-1}$. *In particular,* $\mathrm{Der}_{-1}(\mathcal{S}, S_{\bar{1}}) = \mathrm{ad}(\overline{S}_{\bar{1}})_{-1}$.

Proof. Let $\phi \in \mathrm{Der}_{-1}(\mathcal{S}, W_{\bar{1}})$. Let $k \in Y_1$. Assume that

$$\phi(\Gamma_k + \Gamma_1) = \sum_{r \in Y_1} c_{kr} D_r \quad \text{where } c_{kr} \in \mathbb{F}. \tag{3.12}$$

Let $l \in Y_1 \setminus k$. Then $[\Gamma_k + \Gamma_1, \phi(\Gamma_l + \Gamma_1)] = [\Gamma_l + \Gamma_1, \phi(\Gamma_k + \Gamma_1)]$. By (3.12), $c_{kl} = 0$ whenever $k, l \in Y_1$ with $k \neq l$. It follows that $\phi(\Gamma_k + \Gamma_1) = c_{kk} D_k$ where $c_{kk} \in \mathbb{F}$. Obviously, $[\Gamma_k + \Gamma_1, x_k D_l] = x_k D_l$ for $k, l \in Y_1$ with $k \neq l$. Then

$$c_{kk} D_l + [\Gamma_k + \Gamma_1, \phi(x_k D_l)] = \phi(x_k D_l).$$

Since $\phi(x_k D_l) \in (W_{\bar{1}})_{-1}$, it follows that $\phi(x_k D_l) = c_{kk} D_l$. Put $\psi := \phi - \sum_{r \in Y_1} c_{rr} \mathrm{ad} D_r$. Then

$$\psi(\Gamma_k + \Gamma_1) = \psi(x_k D_l) = 0 \quad \text{for } k, l \in Y_1,\ k \neq l. \tag{3.13}$$

We next show that

$$\psi(x_i D_j) = 0 \quad \text{for } i, j \in Y_0,\ i \neq j. \tag{3.14}$$

Take $r \in Y_0 \setminus \{i, j\}$. Then $[\psi(\Gamma_r + \Gamma_q), x_i D_j] + [\Gamma_r + \Gamma_q, \psi(x_i D_j)] = 0$. Since $\psi(\Gamma_r + \Gamma_q) \in (W_{\bar{1}})_{-1}$, we have $[\psi(\Gamma_r + \Gamma_q), x_i D_j] = 0$. Consequently, $[\Gamma_q, \psi(x_i D_j)] = 0$ for all $q \in Y_1$. Hence $\psi(x_i D_j) = 0$, since $\psi(x_i D_j) \in (W_{\bar{1}})_{-1}$.

In the same way we can verify that

$$\psi(\Gamma_1 - \Gamma_j) = 0 \quad \text{for all } j \in Y_0 \setminus 1. \tag{3.15}$$

By (3.13)–(3.15), we have $\psi(\mathcal{S}_0) = 0$. It follows from Lemma 3.14 that $\psi = 0$ and hence $\phi \in \mathrm{ad}(W_{\bar{1}})_{-1}$. This completes the proof. ■

To compute the derivations of $\mathbb{Z}$-degree less than -1 from $\mathcal{S}$ into $W_{\bar{1}}$, we establish the following lemma.

Lemma 3.16. *Suppose that $\phi \in \mathrm{Der}_{-t}(\mathcal{S}, W_{\bar{1}})$ with $t > 1$ and that $\phi(D_{ij}(x^{((t+1)\varepsilon_i)})) = 0$ for all $i, j \in Y_0$. Then $\phi = 0$.*

Proof. First claim that $\phi(\mathcal{Q}) = 0$. To that aim, we proceed by induction on q to show that

$$\phi(D_{ij}(x^{(q\varepsilon_i)})) = 0 \quad \text{for all } i, j \in Y_0 \text{ with } i \neq j. \tag{3.16}$$

Without loss of generality suppose that $q > t + 1$ in the following. By inductive hypothesis and Lemma 2.1, $\phi(D_{ij}(x^{(q\varepsilon_i)})) \in \mathcal{G}_{q-t-2}$. Thus one may write

$$\phi(D_{ij}(x^{(q\varepsilon_i)})) = \sum_{r \in Y, |u| = q-t-1} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \tag{3.17}$$

If $q - t \geq 3$, proceeding just as Case (i) in the proof of [2, Lemma 4.3.3], one may show that $\phi(D_{ij}(x^{(q\varepsilon_i)})) = 0$. Suppose that $q - t < 3$. Note that $q > t + 1$. Then rewrite (3.17) as

$$\phi(D_{ij}(x^{(q\varepsilon_i)})) = \sum_{l \in Y_1, r \in Y_0} c_{l,r} x_l D_r \quad \text{where } c_{l,r} \in \mathbb{F}. \tag{3.18}$$

For any fixed coefficient c_{l_0,r_0} in (3.18). Choose $k \in Y_1 \setminus l_0$, $s \in Y_0 \setminus \{r_0, i\}$, since $n, m \geq 3$.

If $s = j$, then $[\Gamma_s + \Gamma_k, D_{ij}(x^{(q\varepsilon_i)})] = -D_{ij}(x^{(q\varepsilon_i)})$. Applying ϕ to the equation above and then combining that with (3.18), one may obtain by a comparison of coefficients of $x_{l_0} D_{r_0}$ that

$$c_{l_0,r_0} = 0 \quad \text{for } l_0 \in Y_1, r_0 \in Y_0.$$

Consequently, $\phi(D_{ij}(x^{(q\varepsilon_i)})) = 0$.

If $s \neq j$, then $[n\Gamma_s + \Gamma', D_{ij}(x^{(q\varepsilon_i)})] = 0$. Applying ϕ and then combining that with (3.18), one may obtain by a comparison of coefficients of $x_{l_0} D_{r_0}$ that

$$c_{l_0 r_0} = 0 \quad \text{for } l_0 \in Y_1, r_0 \in Y_0.$$

Consequently, $\phi(D_{ij}(x^{(q\varepsilon_i)})) = 0$. Thus (3.16) holds for all q and therefore, $\phi(\mathcal{Q}) = 0$.

We next prove that $\phi(\mathcal{R}) = 0$. Since $\mathcal{R} \subseteq \mathcal{S}_1$, $\mathrm{zd}(\phi) \leq -2$, it suffices to consider the case that $\mathrm{zd}(\phi) = -2$. Note that $\phi(\mathcal{S}_1) \subset \mathcal{S}_{-1}$. For $k, l \in Y_1, i \in Y_0$, take $q \in Y_0 \setminus \{i, j\}$. Then $n\Gamma_q + \Gamma' \in \mathcal{S}_0$ and $[n\Gamma_q + \Gamma', D_{il}(x^{(2\varepsilon_i)} x_k)] = 0$. Since $\phi(D_{il}(x^{(2\varepsilon_i)} x_k)) \in (W_{\bar{1}})_{-1}$, it follows that

$$\phi(D_{il}(x^{(2\varepsilon_i)} x_k)) = -[n\Gamma_q + \Gamma', \phi(D_{il}(x^{(2\varepsilon_i)} x_k))] = 0 \quad \text{for all } i \in Y_0,\ k, l \in Y_1.$$

Obviously,

$$[n\Gamma_q + \Gamma', D_{ij}(x_i x_k x_l)] = 2D_{ij}(x_i x_k x_l).$$

Applying ϕ, one gets

$$2\phi(D_{ij}(x_i x_k x_l)) = [n\Gamma_q + \Gamma', \phi(D_{ij}(x_i x_k x_l))] = -\phi(D_{ij}(X_i x_k x_l)),$$

since $\phi(D_{ij}(x_i x_k x_l)) \in (W_{\bar{1}})_{-1}$. The assumption $p \neq 3$ ensures that $\phi(D_{ij}(x_i x_k x_l)) = 0$. By Lemma 3.2, $\phi = 0$, completing the proof. ∎

Finally, we are to determine the homogeneous derivations of $\mathbb{Z}$-degree less than -1 from $\mathcal{S}$ into $W_{\bar{1}}$.

Proposition 3.17. $\mathrm{Der}_{-t}(\mathcal{S}, W_{\bar{1}}) = 0$ *for* $t > 1$. *In particular,* $\mathrm{Der}_{-t}(\mathcal{S}, S_{\bar{1}}) = 0$ *for* $t > 1$.

Proof. Let $\phi \in \mathrm{Der}_{-t}(\mathcal{S}, W_{\bar{1}})$. Assert that

$$\phi(D_{ij}(x^{((t+1)\varepsilon_i)})) = 0 \quad \text{for all } i, j \in Y_0.$$

Recall $\Gamma' = \sum_{r \in Y_1} \Gamma_r$. Choose $q \in Y_0 \setminus \{i, j\}$, since $m \geq 3$. Clearly, $n\Gamma_q + \Gamma' \in \mathcal{S}_0$. Then

$$[n\Gamma_q + \Gamma', D_{ij}(x^{((t+1)\varepsilon_i)})] = 0.$$

Applying ϕ, one gets

$$0 = [n\Gamma_q + \Gamma', \phi(D_{ij}(x^{((t+1)\varepsilon_i)}))] = -\phi(D_{ij}(x^{((t+1)\varepsilon_i)})),$$

since $\phi(D_{ij}(x^{((t+1)\varepsilon_i)})) \in (W_{\bar{1}})_{-1}$. Consequently, $\phi(D_{ij}(x^{((t+1)\varepsilon_i)})) = 0$. By Lemma 3.16, $\phi = 0$. The proof is complete. ■

Now we can describe the derivation spaces $\mathrm{Der}(\mathcal{S}, W_{\bar{1}})$ and $\mathrm{Der}(\mathcal{S}, S_{\bar{1}})$.

Theorem 3.18. $\mathrm{Der}(\mathcal{S}, W_{\bar{1}}) = \mathrm{ad} W_{\bar{1}}$.

Proof. This is a direct consequence of Propositions 3.12, 3.15 and 3.17. ■

Theorem 3.19. $\mathrm{Der}(\mathcal{S}, S_{\bar{1}}) = \mathrm{ad}\overline{S}_{\bar{1}}$.

Proof. This is a direct consequence of Propositions 3.13, 3.15 and 3.17. ■

Remark 3.20. By [7, Theorem 3] and Theorem 3.19 above, the odd part of the superderivation algebra of the finite-dimensional special superalgebra S coincides with the derivation space from the even part into the odd part of S; that is, $(\mathrm{Der}\, S)_{\bar{1}} = \mathrm{Der}(S_{\bar{0}}, S_{\bar{1}})$.

Remark 3.21. By Theorem 3.18, we have $H^1(\mathcal{S}, W_{\bar{1}}) = 0$. By Theorem 3.19 and [3, Proposition 2.8], we have $H^1(\mathcal{S}, \mathcal{S}_{\bar{1}}) = 0$ if n is even; and $\dim H^1(\mathcal{S}, \mathcal{S}_{\bar{1}}) = m$ if n is odd.

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Wende Liu
Department of Mathematics
Harbin Normal University
Harbin 150080, China
wendeliu@hrbnu.edu.cn

Baoling Guan
Department of Mathematics
Harbin Normal Univerisity
Harbin 150080, China, and
Department of Mathematics
Qiqihar University
Qiqihar 161006, China

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