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On the L_0-Module Structure for the Generalized Witt Algebra and the Special Lie Algebra

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Abstract. For $n > 1$, the L_0-module structure of the restricted simple Lie algebra $W(n,1)$ and $S(n,1)$ is determined completely.
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1. Introduction

Assume that $\mathbb{F}$ is an algebraically closed field with characteristic $p \geq 3$. Let $L = \bigoplus_{i=-1}^{s} L_i$ be a restricted Lie algebra W or S. With the adjoint action, L is an L_0-module and each L_i is an L_0-submodule. L_{-1} and L_s are two simple W_0-modules. For each $0 < i < s$, the L_0-module structure of L_i remains unknown. But we often need to know such structures in the nonrestricted representations of L, where the p-characters can be put in some convenient form. That is, we need to find certain "nice" representatives in each orbit of the p-characters under the action of the automorphism groups (see [2]). With this motivation, we determine the L_0-module structure of L in this paper. By introducing a gradation, we study the structure of the submodules of each L_i.

The paper is organized as follows. In Section 2, we give the preliminaries. In Section 3, we determine the maximal vectors of W_l and study the submodules of W_l. Then in Section 4, we prove that if $p \nmid (n+l)$, W_l is the direct sum of its two proper simple submodules. If $p|(n+l)$, we determine the composition series of W_l. In Section 5, we use the results for W to determine the S_0-module structure of S.

2. Preliminaries

In this section we describe the simple restricted Lie algebra of Cartan type, drawing most of the notation and results from [8]. Assume $n \geq 2$. Consider n-tuples

$$a = (a_1, a_2, \ldots, a_n) \quad b = (b_1, b_2, \ldots, b_n)$$

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in $\mathbb{Z}^n$. We write $a \leqslant b$ if $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$ and we write $a < b$ if $a \leqslant b$ but $a \neq b$. If $a, b \geq 0$, define $\binom{a}{b} = \Pi_i \binom{a_i}{b_i}$, where $\binom{a_i}{b_i}$ is the usual binomial coefficient with the convention that $\binom{a_i}{b_i} = 0$ unless $b_i \leqslant a_i$. Set $\mathfrak{C} := \{a \in \mathbb{Z}^n | 0 \leqslant a \leqslant \tau\}$, where $\tau := (p-1, \ldots, p-1)$. The *divided power algebra* $\mathfrak{A} = \mathfrak{A}(n, 1)$ is the associative $\mathbb{F}-$algebra having $\mathbb{F}-$basis $\{x^{(a)} | a \in \mathfrak{C}\}$ and multiplication subject to the rule

$$x^{(a)} x^{(b)} = \begin{cases} \binom{a+b}{a} x^{(a+b)}, & a + b \leqslant \tau \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathfrak{A}$ is $\mathbb{Z}$-graded by $\mathfrak{A}_k = \langle x^{(a)} | a \in \mathfrak{C}, |a| = k \rangle$, where $|a| = \sum_{i=1}^n a_i$. For $j \in \{1, \ldots, n\}$ we consider $D_j \in \mathrm{Der}_{\mathbb{F}}(\mathfrak{A})$ given by $D_j(x^{(a)}) = x^{(a - \epsilon_j)}$. Then we have the *simple restricted* Witt algebra $W = W(n, 1) = \sum_{j=1}^n \mathfrak{A} D_j$ and W inherits a gradation from $\mathfrak{A}$ by means of $W_i = \sum_{j=1}^n \mathfrak{A}_{i+1} D_j$. Consequently, $W = \oplus_{i=-1}^{s_W} W_i$ with $s_W = n(p-1) - 1 = |\tau| - 1$. In particular, $W_{-1} = \sum_{i=1}^n \mathbb{F} D_i$.

Suppose $n \geq 3$. We introduce the mappings

$$D_{ij} : \begin{cases} \mathfrak{A} \to W(n, 1), \\ f \mapsto D_j(f) D_i - D_i(f) D_j. \end{cases}$$

Then the simple *restricted special Lie algebra* is

$$S = S(n, 1) = \langle D_{ij}(f) | f \in \mathfrak{A}, 1 \leqslant i < j \leqslant n \rangle.$$

$S = \oplus_{i=-1}^{s_S} S \cap W_i$ is graded with $s_S = n(p-1) - 2 = |\tau| - 2$. In particular, $S_{-1} = W_{-1}$.

Identifying $GL(n)$ with the central extension of the Chevalley group of $W_0 = \mathfrak{gl}(n, \mathbb{F})$, each W_l is then naturally a $GL(n)$-module. Besides, every $GL(n)$-submodule is a W_0-submodule. Set $M = \mathfrak{A}_1$. The action of W_0 on M identifies W_0 with $\mathfrak{gl}(M)$. Then the adjoint action of W_0 on W_{-1} identifies W_{-1} with M^*. For each $0 \leqslant l \leqslant s_W$, we then get a $\mathfrak{gl}(M)$-module($GL(n)$-module) isomorphism $\mathfrak{A}_{l+1} \otimes M^* \to W_l$: $x^{(a)} \otimes D_j \to x^{(a)} D_j$.

Let $X^+(n)$ denote the set of n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \cdots \geq \lambda_n$. Then $X^+(n)$ can be identified with dominant weights for the root system of $GL(n)$. Let $L_n(\lambda)$ denote the simple $GL(n)$-module with highest weight λ.

Fix $\lambda \in X^+(n)$ and $1 \leqslant i \leqslant n$. We say that i is λ-removable if either $i = n$ or $1 \leqslant i < n$ and $\lambda_i > \lambda_{i+1}$. We say that i is λ-addable if either $i = 1$ or $1 < i \leqslant n$ and $\lambda_i < \lambda_{i-1}$. For each $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$, we set $res(a, b) = b - a \in \mathbb{F}_p$. We say that i is normal for λ if i is λ-removable and there is a decreasing injection from the set of

$$\lambda - addable \quad j \quad with \quad i < j \leqslant n \quad and \quad res(i, \lambda_i) = res(j, \lambda_j + 1)$$

into the set of

$$\lambda - removable \quad j' \quad with \quad i < j \leqslant n \quad and \quad res(i, \lambda_i) = res(j', \lambda_{j'}).$$

We say that i is good for λ if i is normal for λ and there is no j that is normal for λ with $1 \leqslant j < i$ with $res(j, \lambda_j) = res(i, \lambda_i)$ ([1]).

Since the structure of $W(1, 1)$ is quite clear, we focus only on $W(n, 1)$ with $n \geq 2$ throughout the paper.

3. Two W_0-submodules of W_l

For each $0 < l < s_W$, let $l+1 = k(p-1)+r$, $0 < r \leqslant p-1$. Let $\bar{a}$ denote the n-tuple

$$(p-1, \ldots, \underset{k}{p-1}, r, 0, \ldots, 0).$$

Clearly we have $\mathfrak{A}_{l+1} = L_n(\bar{a})$. In the following, we denote the Lie multiplication $[x, y]$ simply by $x \cdot y$.

Definition 3.1. Let $v \in W_l$. If $x_i D_j \cdot v = 0$ whenever $1 \leqslant i < j \leqslant n$, and for every $1 \leqslant i \leqslant n$, $x_i D_i \cdot v = c_i v$ for some $c_i \in \mathbb{F}$, then v is called a maximal vector of weight $(c_1, \ldots, c_n)$.

Let U denote the unipotent subgroup of $GL(n)$ generated by elements in the set

$$\{1 + t x_i D_j | t \in \mathbb{F}, i < j\},$$

and let B denote the Borel subgroup of $GL(n)$ with unipotent radical U. For each W_l as a $GL(n)$-module as described above, we can also define the maximal vector $v \in W_l$ relative to B (see [3, Sec. 31.2]). It turns out at the end for $v \in W_l$, the two definitions of v being maximal agree.

Lemma 3.2. *In each W_l, $0 < l < s_W$, there are two maximal vectors (up to scalar multiple); namely:*

$$v_1 = x^{(\bar{a})} D_n, \quad v_2 = \begin{cases} r x^{(\bar{a})} D_{k+1} + \sum_{i > k+1} x^{(\bar{a} - \epsilon_{k+1} + \epsilon_i)} D_i, & \text{if } k+1 < n \\ x^{(\bar{a})} D_{n-1} - x^{(\bar{a} - \epsilon_{n-1} + \epsilon_n)} D_n, & \text{if } k+1 = n. \end{cases}$$

Proof. By definition, there are two removable i that are normal for $\bar{a}$, namely,

$$i = n \quad \text{and} \quad i = \begin{cases} k+1, & \text{if } k+1 < n \\ n-1, & \text{if } k+1 = n. \end{cases}$$

Then by [1, Th.5.9 (1)], there are two maximal vectors in W_l. It is easy to check that both v_1 and v_2 are maximal. ∎

Set $V_i =: u(W_0) v_i$, $i = 1, 2$, where $u(W_0)$ is the reduced enveloping algebra of W_0. Both V_1 and V_2 are W_0-submodules of W_l. Let $W_0^- =: \sum_{j>i} \mathbb{F} x_j D_i$. Then by the PBW theorem $V_i = u(W_0^-) v_i$, $i = 1, 2$.

Let

$$\tilde{A} =: \{(a_1, a_2, \ldots, a_n) | -1 \leqslant a_i \leqslant p-1, i = 1, \ldots, n\}.$$

We introduce an $\tilde{A}-$gradation on W (denoted $\mathfrak{G}$) as follows: $\mathfrak{G}(x^{(a)} D_i) = a - \epsilon_i \in \tilde{A}$. $\tilde{A}$ is a completely ordered set with the order

$$(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n) \quad \text{if} \quad a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i < b_i$$

for some $i \geq 1$. We write $a \preccurlyeq b$ if $a \prec b$ or $a = b$. Let

$$(W_l)_\alpha = \langle x^{(a)} D_i \in W_l | a - \epsilon_i = \alpha \rangle$$

and let $\bar{A}_l$ denote the set $\{a \in \tilde{A} | (W_l)_a \neq 0\}$. Then $W_l = \oplus_{\alpha \in \bar{A}_l} (W_l)_\alpha$. Note that the $\bar{A}_l$-gradation of W_l is just the weight space decomposition of W_l under the maximal torus of all diagonal matrices in $GL(n)$.

Clearly we have

$$\mathfrak{G}(v_1) = \bar{a} - \epsilon_n, \mathfrak{G}(v_2) = \begin{cases} \bar{a} - \epsilon_{k+1}, & \text{if } k+1 < n \\ \bar{a} - \epsilon_{n-1}, & \text{if } k+1 = n. \end{cases}$$

Then $\mathfrak{G}(v_2) \prec \mathfrak{G}(v_1)$. If $x_i D_j \cdot x^{(a)} D_k \neq 0$ for some $i < j$, then

$$\mathfrak{G}(x_i D_j \cdot x^{(a)} D_k) = a - \epsilon_k + \epsilon_i - \epsilon_j \succ a - \epsilon_k = \mathfrak{G}(x^{(a)} D_k).$$

Corollary 3.3. $v_1 \notin V_2$.

For each $a \in \bar{A}_l$ with $a \geq 0$, set $\mathfrak{T}(a) = \{i | a_i < p-1\}$. We denote by $t(a)$ the cardinality of the set $\mathfrak{T}(a)$ and by i_t the greatest index in $\mathfrak{T}(a)$. Then we have $(W_l)_a = \langle x^{(a+\epsilon_i)} D_i | i \in \mathfrak{T}(a) \rangle$.

If $a \in \bar{A}_l$ with $a \ngeq 0$, so that there is a unique i such that $a_i = -1$, then we obtain $(W_l)_a = \mathbb{F} x^{(a+\epsilon_i)} D_i$.

3.1. The structure of $V_i, i = 1, 2$, for $l < (n-1)(p-1) - 1$.

Theorem 3.4. *Assume $l = k(p-1) + r - 1 < (n-1)(p-1) - 1$, $0 < r \leqslant p-1$. Let $a \in \bar{A}_l$.*

(1) If $a \geq 0$, then $t(a) \geq 2$, and $(V_1)_a$ has a basis

$$\{x^{(a+\epsilon_i)} D_i - x^{(a+\epsilon_{i_t})} D_{i_t} | i \in \mathfrak{T}(a) \setminus \{i_t\}\}.$$

(2) If $a \ngeq 0$, then $(V_1)_a = (W_l)_a$.

Proof. The proof is by induction on $a \in \bar{A}_l$. We divide it into three steps.

Step 1. *The theorem holds for all $b \succcurlyeq \bar{a} - \epsilon_{k+1}$.*

Suppose $b \succ \bar{a} - \epsilon_{k+1}$. Then $b \ngeq 0$ and there is $i > k+1$ such that $b_i = -1$. Thus we have either $b = \bar{a} - \epsilon_i$ or $b = \bar{a} - \epsilon_{k+1} + \epsilon_j - \epsilon_i$ for some $j \in \{k+2, \ldots, i-1\}$.

In the former case,

$$(W_l)_b = \mathbb{F} x^{(\bar{a})} D_i = \begin{cases} \mathbb{F} x^{(\bar{a})} D_n, & \text{if } i = n \\ \mathbb{F} x_n D_i \cdot x^{(\bar{a})} D_n, & \text{if } i < n, \end{cases}$$

which gives $(W_l)_b = (V_1)_b$.

In the latter case,

$$(W_l)_b = \mathbb{F}(x_j D_{k+1})(x_n D_i) \cdot v_1 = (V_1)_b.$$

The greatest $a \geq 0$ is $\bar{a} - \epsilon_{k+1}$, for which we shall now show that

$$(V_1)_{\bar{a}-\epsilon_{k+1}} = \langle x^{(\bar{a})} D_{k+1} - x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)} D_i | i > k+1 \rangle. \quad (\star)$$

By the PBW theorem, we have

$$(V_1)_{\bar{a}-\epsilon_{k+1}} \subseteq \sum_{k+1 < j < i \leqslant n} x_i D_j \cdot (V_1)_{\bar{a}-\epsilon_{k+1}+\epsilon_j-\epsilon_i} + \sum_{i > k+1} x_i D_{k+1} \cdot (V_1)_{\bar{a}-\epsilon_i}.$$

We have just shown that

$$(V_1)_{\bar{a}-\epsilon_{k+1}+\epsilon_j-\epsilon_i} = \mathbb{F}x^{(\bar{a}-\epsilon_{k+1}+\epsilon_j)}D_i \quad \text{for} \quad k+1<j<i\leqslant n,$$

$$(V_1)_{\bar{a}-\epsilon_i} = \mathbb{F}x^{(\bar{a})}D_i \quad \text{for} \quad i>k+1.$$

Since

$$x_iD_j \cdot x^{(\bar{a}-\epsilon_{k+1}+\epsilon_j)}D_i = x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i - x^{(\bar{a}-\epsilon_{k+1}+\epsilon_j)}D_j$$

and

$$x_iD_{k+1} \cdot x^{(\bar{a})}D_i = x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i - x^{(\bar{a})}D_{k+1},$$

$$(V_1)_{\bar{a}-\epsilon_{k+1}} \subseteq \langle x^{(\bar{a})}D_{k+1} - x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i | i>k+1\rangle.$$

On the other hand, since $x^{(\bar{a})}D_i = -x_nD_i \cdot v_1 \in V_1$ for every $k+1<i<n$,

$$x^{(\bar{a})}D_{k+1} - x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i$$

$$= -x_iD_{k+1} \cdot x^{(\bar{a})}D_i \in (V_1)_{\bar{a}-\epsilon_{k+1}}.$$

Thus we have

$$(V_1)_{\bar{a}-\epsilon_{k+1}} \supseteq \langle x^{(\bar{a})}D_{k+1} - x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i | i>k+1\rangle.$$

This completes $(\star)$ and hence Step 1.

Step 2. *Assume the theorem is true for each* $b \succ a$. *We prove it for the case* $0 \leqslant a \prec \bar{a}-\epsilon_{k+1}$.

Suppose $a+\epsilon_i-\epsilon_j \geq 0$ for some $i<j$. Then $(V_1)_{a+\epsilon_i-\epsilon_j}$ has basis vectors each in the form

$$x^{(a+\epsilon_i-\epsilon_j+\epsilon_s)}D_s - x^{(a+\epsilon_i-\epsilon_j+\epsilon_m)}D_m, s<m$$

by the induction assumption. We see that

$$x_jD_i \cdot [x^{(a+\epsilon_i-\epsilon_j+\epsilon_s)}D_s - x^{(a+\epsilon_i-\epsilon_j+\epsilon_m)}D_m]$$

$$= a_j[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_m)}D_m] + \delta_{js}[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_i)}D_i]$$

$$+\delta_{mj}[x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_m)}D_m]$$

$$\in \langle x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} | s \in \mathfrak{T}(a)\setminus\{i_t\}\rangle.$$

Suppose $a+\epsilon_i-\epsilon_j \ngeq 0$ for some $i<j$. Then we must have $a_i<p-1$ and $a_j=0$, so that $(V_1)_{a+\epsilon_i-\epsilon_j} = \mathbb{F}x^{(a+\epsilon_i)}D_j$ by the induction assumption. In this case

$$x_jD_i \cdot x^{(a+\epsilon_i)}D_j = x^{(a+\epsilon_j)}D_j - x^{(a+\epsilon_i)}D_i$$

$$\in \langle x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} | s \in \mathfrak{T}(a)\setminus\{i_t\}\rangle.$$

Using the PBW theorem, we conclude that

$$(V_1)_a = \sum_{j>i} x_jD_i \cdot (V_1)_{a+\epsilon_i-\epsilon_j}$$

$$\subseteq \langle x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} | s \in \mathfrak{T}(a)\setminus\{i_t\}\rangle.$$

To complete this step, it remains to show that, for each fixed $s \in \mathfrak{T}(a) \setminus \{i_t\}$,

$$(*) \quad x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \in V_1.$$

We split the proof of $(*)$ into five cases.

Case 1. $i_t < n$. In this case $a_n = p-1$. Since $p \geq 3$ and $k+1 < n$, there exists $m < n$ such that

$$(a+\epsilon_m-\epsilon_n)_s < p-1 \quad \text{and} \quad (a+\epsilon_m-\epsilon_n)_{i_t} < p-1.$$

By the induction hypothesis,

$$x^{(a+\epsilon_m-\epsilon_n+\epsilon_s)}D_s - x^{(a+\epsilon_m-\epsilon_n+\epsilon_{i_t})}D_{i_t} \in V_1.$$

Then

$$\begin{aligned}&x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t}\\ &= -x_nD_m \cdot [x^{(a+\epsilon_m-\epsilon_n+\epsilon_s)}D_s - x^{(a+\epsilon_m-\epsilon_n+\epsilon_{i_t})}D_{i_t}] \in V_1.\end{aligned}$$

Case 2. $i_t = n$, $a_n = 0$. Let $b =: a+\epsilon_s-\epsilon_n \succ a$. Then $b \ngeq 0$. By the induction hypothesis, we obtain $(V_1)_b = (W_l)_b = \mathbb{F}x^{(b+\epsilon_n)}D_n$. Hence we have

$$-x_nD_s \cdot x^{(b_s+\epsilon_n)}D_n = x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_n)}D_n \in V_1.$$

Case 3. $i_t = n$, $a_n \neq 0$, $a_s < p-2$. Since $k+1 < n$, $t(a) \geq 3$. Since $b = a+\epsilon_s-\epsilon_n \succcurlyeq a$, the induction assumption gives

$$x^{(a+2\epsilon_s-\epsilon_n)}D_s - x^{(a+\epsilon_s)}D_n \in V_1.$$

Applying x_nD_s, we get

$$\begin{aligned}&x_nD_s \cdot [x^{(a+2\epsilon_s-\epsilon_n)}D_s - x^{(a+\epsilon_s)}D_n]\\ &= (a_n+1)[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_n)}D_n] \in V_1,\end{aligned}$$

which gives $(*)$, since $a_n + 1 \neq 0$.

Case 4. $i_t = n$, $a_n \neq 0$, $a_j < p-2$ for some $j \in \mathfrak{T}(a) \setminus \{s, n\}$. By the induction assumption, we get

$$x^{(a+\epsilon_j-\epsilon_n+\epsilon_s)}D_s - x^{(a+\epsilon_j)}D_n \in V_1.$$

Applying x_nD_j, we have

$$\begin{aligned}&x_nD_j \cdot [x^{(a+\epsilon_j-\epsilon_n+\epsilon_s)}D_s - x^{(a+\epsilon_j)}D_n]\\ &= a_n[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_n)}D_n] + [x^{(a+\epsilon_j)}D_j - x^{(a+\epsilon_n)}D_n] \in V_1.\end{aligned}$$

Using the conclusion in the preceding case and the assumption $a_n \neq 0$, one gets $(*)$.

Case 5. $i_t = n$, $a_n \neq 0$, $a_j = p-2$ for every $j \in \mathfrak{T}(a) \setminus \{n\}$. In this case we have $t(a) \geq 4$, since $l < (n-1)(p-1)-1$. Taking $\alpha, \beta \in \mathfrak{T}(a) \setminus \{s, n\}$ with $\alpha < \beta$, the induction hypothesis then yields

$$x^{(a+\epsilon_\alpha-\epsilon_\beta+\epsilon_s)}D_s - x^{(a+\epsilon_\alpha-\epsilon_\beta+\epsilon_n)}D_n \in V_1.$$

Applying $x_\beta D_\alpha$, one gets $(*)$. This completes Step 2.

Step 3. *Assume the theorem is true for each $b \succ a$. We prove it for the case $0 \nleq a \prec \bar{a} - \epsilon_{k+1}$.*

Let $a_i = -1$.

Suppose $i < n$. If $a_s = 0$ for every $s > i$, then $0 \nleq a + \epsilon_i - \epsilon_n \preccurlyeq \bar{a} - \epsilon_n$. By induction hypotheses, $(V_1)_{a+\epsilon_i-\epsilon_n} = \mathbb{F}x^{(a+\epsilon_i)}D_n$. So we get

$$(V_1)_a = \mathbb{F}x_n D_i \cdot x^{(a+\epsilon_i)}D_n = (W_l)_a.$$

If there exists $s > i$ such that $0 < a_s < p-1$, then $0 \leqslant a + \epsilon_i - \epsilon_s \preccurlyeq \bar{a} - \epsilon_n$. By (1), we have $x^{(a+2\epsilon_i-\epsilon_s)}D_i - x^{(a+\epsilon_i)}D_s \in (V_1)_{a+\epsilon_i-\epsilon_s}$. Since

$$x_s D_i \cdot [x^{(a+2\epsilon_i-\epsilon_s)}D_i - x^{(a+\epsilon_i)}D_s] = (a_s+1)x^{(a+\epsilon_i)}D_i \in (V_1)_a,$$

and $a_s + 1 \neq 0$, $(V_1)_a = (W_l)_a$.

If $a_s = p-1$ for every $s > i$, then by the assumption $l < (n-1)(p-1)-1$, we get $a_t < p-1$ for some $t < i$, so that $0 \nleq a + \epsilon_t - \epsilon_n$. By the induction hypotheses, $x^{(a+\epsilon_t+\epsilon_i-\epsilon_n)}D_i \in (V_1)_{a+\epsilon_t-\epsilon_n}$. This gives us

$$x^{(a+\epsilon_i)}D_i = -(x_n D_t) \cdot x^{(a+\epsilon_t+\epsilon_i-\epsilon_n)}D_i \in (V_1)_a,$$

and hence $(V_1)_a = (W_l)_a$.

If $i = n$, so that $a_n = -1$, then since $a \prec \bar{a} - \epsilon_{k+1}$, there is $s \leqslant k+1$ such that $a_s < \bar{a}_s$. Hence we must have $a_t > 0$ for some $t > s$. So the induction assumption yields $x^{(a+\epsilon_s-\epsilon_t+\epsilon_n)}D_n \in (V_1)_{a+\epsilon_s-\epsilon_t}$. It follows that

$$x^{(a+\epsilon_n)}D_n = a_t^{-1}x_t D_s \cdot x^{(a+\epsilon_s-\epsilon_t+\epsilon_n)}D_n \in (V_1)_a,$$

so that $(V_1)_a = (W_l)_a$. ∎

For each $0 \leqslant a \in \bar{A}_l$, set

$$\mathfrak{v}_a =: \sum_{i=1}^{n}(a_i+1)x^{(a+\epsilon_i)}D_i = \sum_{i\in\mathfrak{T}(a)}(a_i+1)x^{(a+\epsilon_i)}D_i.$$

Theorem 3.5. *Assume $l \leqslant (n-1)(p-1)-1$. Let $l+1 = k(p-1)+r$, $0 < r \leqslant p-1$. Then for $a \in \bar{A}_l$, we have:*

(1) If $a \ngeq 0$, then $(V_2)_a = 0$.

(2) If $a \geq 0$, then $(V_2)_a = \mathbb{F}\mathfrak{v}_a$.

Proof. We proceed by induction on a. Recall that

$$v_2 = rx^{(\bar{a})}D_{k+1} + \sum_{i>k+1} x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i$$

and $\mathfrak{G}(v_2) = \bar{a} - \epsilon_{k+1}$. Clearly we have $v_2 = \mathfrak{v}_{\bar{a}-\epsilon_{k+1}}$ and $(V_2)_{\bar{a}-\epsilon_{k+1}} = \mathbb{F}v_2$. Also by definition, $(V_2)_b = 0$ for every $b \succ \bar{a} - \epsilon_{k+1}$.

Assume both (1) and (2) are true for all $b \succ a$, and consider the case $a \prec \bar{a} - \epsilon_{k+1}$.

Suppose $a \geq 0$. Since $0 \leqslant a \prec \bar{a} - \epsilon_{k+1}$, there is $i < j$ such that

$$0 \leqslant a + \epsilon_i - \epsilon_j \preccurlyeq \bar{a} - \epsilon_{k+1}.$$

For any such i, j, let $b =: a + \epsilon_i - \epsilon_j$. Then the induction assumption shows that $(V_2)_b = \mathbb{F}\mathfrak{v}_b$. Notice that

$$x_j D_i \cdot \mathfrak{v}_b = \sum_{m=1}^{n} (b_m + 1) x_j D_i \cdot x^{(b+\epsilon_m)} D_m$$

$$= \sum_{m=1}^{n} (b_m + 1)[(b_j + 1 + \delta_{mj}) x^{(b+\epsilon_m-\epsilon_i+\epsilon_j)} D_m - \delta_{mj} x^{(b+\epsilon_m)} D_i]$$

$$= \sum_{m \neq i,j} (b_m + 1)(b_j + 1) x^{(b+\epsilon_m-\epsilon_i+\epsilon_j)} D_m$$

$$+ (b_j + 1)(b_j + 2) x^{(b+2\epsilon_j-\epsilon_i)} D_j - (b_j + 1) x^{(b+\epsilon_j)} D_i + (b_i + 1)(b_j + 1) x^{(b+\epsilon_j)} D_i$$

$$= (b_j + 1) \sum_{m=1}^{n} (a_m + 1) x^{(a+\epsilon_m)} D_m = (b_j + 1)\mathfrak{v}_a$$

and $b_j + 1 \neq 0$.

If there is $i < j$ such that $a + \epsilon_i - \epsilon_j \ngeq 0$ or $a + \epsilon_i - \epsilon_j \npreceq \bar{a} - \epsilon_{k+1}$, which means again $a + \epsilon_i - \epsilon_j \ngeq 0$, then the induction assumption yields $(V_2)_{a+\epsilon_i-\epsilon_j} = 0$.

Using the PBW Theorem, one gets

$$(V_2)_a = \sum_{j>i} x_j D_i \cdot (V_2)_{a+\epsilon_i-\epsilon_j} = \mathbb{F}\mathfrak{v}_a,$$

which completes the proof of (2).

Suppose $a \ngeq 0$. If $(V_2)_{a+\epsilon_i-\epsilon_j} \neq 0$ for some $i < j$, then the induction assumption yields

$$b =: a + \epsilon_i - \epsilon_j \geq 0, \quad a_i = -1, \quad \text{and} \quad (V_2)_b = \mathbb{F}\mathfrak{v}_b.$$

Note that $b_i = 0$. Then we have

$$x_j D_i \cdot \mathfrak{v}_b = x_j D_i \cdot (b_i + 1) x^{(b+\epsilon_i)} D_i + x_j D_i \cdot (b_j + 1) x^{(b+\epsilon_j)} D_j$$

$$= (b_j + 1) x^{(b+\epsilon_j)} D_i - (b_j + 1) x^{(b+\epsilon_j)} D_i = 0.$$

Hence we have by the PBW Theorem that $(V_2)_a = 0$. So (1) follows. ■

3.2. The structure of $V_i, i = 1, 2$, for $l > (n-1)(p-1) - 1$.

Theorem 3.6. *Assume $l = (n-1)(p-1) + r - 1$, $0 < r < p - 1$. Let $a \in \bar{A}_l$.*

(1) $(V_1)_a = \mathbb{F}\mathfrak{v}_a$.

(2) If $t(a) = 1$, $(V_2)_a = 0$.

(3) If $t(a) \geq 2$, $(V_2)_a = \langle x^{(a+\epsilon_i)} D_i - x^{(a+\epsilon_{i_t})} D_{i_t} | i \in \mathfrak{T}(a) \setminus \{i_t\}\rangle$.

Proof. We prove (1)-(3) by induction on a.

The greatest $a \in \bar{A}_l$ with $t(a) = 1$ is $\bar{a} - \epsilon_n$, for which we have $(V_1)_{\bar{a}-\epsilon_n} = \mathbb{F}v_1$ and $v_1 = r^{-1}\mathfrak{v}_{\bar{a}-\epsilon_n}$. Also, we have $(V_2)_{\bar{a}-\epsilon_n} = 0$. The second greatest $a \in \bar{A}_l$ is $\bar{a} - \epsilon_{n-1}$, for which we have $t(a) \geq 2$ and

$$\begin{aligned}(V_2)_a &= \mathbb{F}v_2 = \mathbb{F}(x^{(\bar{a})}D_{n-1} - x^{(\bar{a}-\epsilon_{n-1}+\epsilon_n)}D_n) \\ &= \mathbb{F}(x^{(a+\epsilon_{n-1})}D_{n-1} - x^{(a+\epsilon_n)}D_n).\end{aligned}$$

We see that

$$(V_1)_{\bar{a}-\epsilon_{n-1}} = \mathbb{F}x_nD_{n-1} \cdot v_1 = \mathbb{F}((r+1)x^{(\bar{a}-\epsilon_{n-1}+\epsilon_n)}D_n - x^{(\bar{a})}D_{n-1}) = \mathbb{F}\mathfrak{v}_{\bar{a}-\epsilon_{n-1}}.$$

So we conclude that (1)-(3) are true for all $b \succcurlyeq \bar{a} - \epsilon_{n-1}$.

Assume that (1)-(3) are true for every $b \succ a$, and consider the case $a \prec \bar{a} - \epsilon_{n-1}$.

Notice that we have obtained, in the proof of Theorem 3.5,

$$x_jD_i \cdot \mathfrak{v}_{a+\epsilon_i-\epsilon_j} = a_j\mathfrak{v}_a \quad \text{if} \quad j > i.$$

Using the PBW theorem and the induction assumption, we get

$$(V_1)_a = \sum_{j>i} x_jD_i \cdot (V_1)_{a+\epsilon_i-\epsilon_j} = \sum_{j>i} x_jD_i \cdot \mathbb{F}\mathfrak{v}_{a+\epsilon_i-\epsilon_j} = \mathbb{F}\mathfrak{v}_a.$$

This completes the proof of (1).

If $t(a) = 1$, then we have $(W_l)_a = \mathbb{F}x^{(a+\epsilon_i)}D_i$ for some $i < n$ and $a_j = p-1$ for every $j \neq i$. Since $l < n(p-1)-1$, $a_i < p-2$. Let $j > i$. Then $t(a+\epsilon_i-\epsilon_j) = 2$, and the induction hypothesis gives

$$(V_2)_{a+\epsilon_i-\epsilon_j} = \mathbb{F}(x^{(a+2\epsilon_i-\epsilon_j)}D_i - x^{(a+\epsilon_i)}D_j).$$

Since

$$x_jD_i \cdot (x^{(a+2\epsilon_i-\epsilon_j)}D_i - x^{(a+\epsilon_i)}D_j) = a_jx^{(a+\epsilon_i)}D_i + x^{(a+\epsilon_i)}D_i = 0,$$

$$(V_2)_a = \sum_{j>i} x_jD_i \cdot (V_2)_{a+\epsilon_i-\epsilon_j} = 0.$$

This completes the proof of (2).

If $t(a) \geq 2$, then the induction assumption says that it is sufficient to assume $t(b) \geq 2$, and hence

$$(V_2)_b = \langle x^{(b+\epsilon_s)}D_s - x^{(b+\epsilon_t)}D_t | s,t \in \mathfrak{T}(b)\}$$

for each $b = a + \epsilon_i - \epsilon_j$ with $i < j$. In the light of the proof of Theorem 1, one gets

$$(V_2)_a = \sum_{j>i} x_jD_i \cdot (V_2)_{a+\epsilon_i-\epsilon_j} \subseteq \langle x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} | s \in \mathfrak{T}(a) \setminus \{i_t\}\rangle.$$

To completes the proof of (3), it remains to show that

$$x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \in (V_2)_a$$

for every fixed $s \in \mathfrak{T}(a) \setminus \{i_t\}$.

a) If $t(a) \geq 3$, we take $i \in \mathfrak{T}(a) \setminus \{s, i_t\}$. It's no loss of generality to assume $i < s$. Note that $a_m > 0$ for each $m \in \mathfrak{T}(a)$, since $l > (n-1)(p-1)-1$. So we have $a + \epsilon_i - \epsilon_{i_t} \geq 0$, $a + \epsilon_i - \epsilon_s \geq 0$. Then the induction hypothesis gives

$$x^{(a+\epsilon_i-\epsilon_{i_t}+\epsilon_s)}D_s - x^{(a+\epsilon_i)}D_{i_t} \in (V_2)_{a+\epsilon_i-\epsilon_{i_t}},$$

$$x^{(a+\epsilon_i)}D_s - x^{(a+\epsilon_i-\epsilon_s+\epsilon_{i_t})}D_{i_t} \in (V_2)_{a+\epsilon_i-\epsilon_s}.$$

Since

$$\begin{aligned} &x_{i_t}D_i \cdot [x^{(a+\epsilon_i-\epsilon_{i_t}+\epsilon_s)}D_s - x^{(a+\epsilon_i)}D_{i_t}] \\ &= a_{i_t}[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t}] + [x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_{i_t})}D_{i_t}] \in (V_2)_a \end{aligned}$$

and

$$\begin{aligned} &x_sD_i \cdot [x^{(a+\epsilon_i)}D_s - x^{(a+\epsilon_i-\epsilon_s+\epsilon_{i_t})}D_{i_t}] \\ &= (a_s+1)[x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t}] - [x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_{i_t})}D_{i_t}] \in (V_2)_a, \end{aligned}$$

with $p|(a_s + a_{i_t} + 1)$ or, equivalently $a_s + a_{i_t} = p-1$, one would get $l = \sum a_i \leqslant (n-1)(p-1)-1$, a contradiction. Therefore $p \nmid (a_s + a_{i_t} + 1)$, which implies that

$$x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \in (V_2)_a.$$

b) When $t(a) = 2$, we have $\mathfrak{T}(a) = \{s, i_t\}$.

Suppose $a_s < p-2$. Then $a + \epsilon_s - \epsilon_{i_t} \geq 0$, since $l > (n-1)(p-1)-1$. So we get

$$x^{(a+2\epsilon_s-\epsilon_{i_t})}D_s - x^{(a+\epsilon_s)}D_{i_t} \in (V_2)_{a+\epsilon_s-\epsilon_{i_t}}$$

by the induction assumption. It follows that

$$\begin{aligned} &x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \\ &= (a_{i_t}+1)^{-1}x_{i_t}D_s \cdot [x^{(a+2\epsilon_s-\epsilon_{i_t})}D_s - x^{(a+\epsilon_s)}D_{i_t}] \in (V_2)_a. \end{aligned}$$

Suppose $a_s = p-2$. Then we get $s < n-1$, since $a \prec \bar{a} - \epsilon_{n-1}$.

If $i_t = n$, we get

$$x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_n)}D_n = -x_{n-1}D_s \cdot v_2 \in (V_2)_a.$$

If $i_t < n$ and $a_{i_t} < p-2$, the induction hypothesis shows that

$$x^{(a+\epsilon_{i_t}-\epsilon_n+\epsilon_s)}D_s - x^{(a+2\epsilon_{i_t}-\epsilon_n)}D_{i_t} \in (V_2)_{a+\epsilon_{i_t}-\epsilon_n},$$

since $a + \epsilon_{i_t} - \epsilon_n \geq 0$. It follows that

$$\begin{aligned} &x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \\ &= -(x_nD_{i_t}) \cdot [x^{(a+\epsilon_{i_t}-\epsilon_n+\epsilon_s)}D_s - x^{(a+2\epsilon_{i_t}-\epsilon_n)}D_{i_t}] \in (V_2)_a. \end{aligned}$$

If $i_t = n-1$ and $a_{i_t} = p-2$, we have

$$x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} = x_nD_s \cdot v_2 \in (V_2)_a.$$

If $i_t < n-1$ and $a_{i_t} = p-2$, the induction assumption says that

$$x^{(a+\epsilon_s-\epsilon_{n-1}+\epsilon_{i_t})}D_{i_t} - x^{(a+\epsilon_s)}D_{n-1} \in (V_2)_{a+\epsilon_s-\epsilon_{n-1}},$$

since $a+\epsilon_s-\epsilon_{n-1} \geq 0$. Therefore

$$\begin{aligned} & x^{(a+\epsilon_s)}D_s - x^{(a+\epsilon_{i_t})}D_{i_t} \\ = {} & (x_{n-1}D_s)\cdot[x^{(a+\epsilon_s-\epsilon_{n-1}+\epsilon_{i_t})}D_{i_t} - x^{(a+\epsilon_s)}D_{n-1}] \in (V_2)_a. \end{aligned}$$

This completes the proof of (3).

■

4. The W_0 – module structure of W_l

4.1. $p \nmid (n+l)$**.**

Theorem 4.1. *If $p \nmid (n+l)$, $W_l = V_1 \oplus V_2$.*

Proof. Let

$$\bar{a} = (p-1,\ldots,p-1,r,0,\ldots,0), 0 < r \leqslant p-1.$$

By the proof of Lemma 3.2, there are two normal i. If $k+1 < n$, then $res(k+1,r) = res(n,0)$ if and only if $r-k-1 = -n(\mathrm{mod} p)$, which means $p|(n+l)$ (note that $l = r-k-1(\mathrm{mod} p)$). If $p \nmid (n+l)$, we have by definition that both $k+1$ and n are good for $\bar{a}$. If $k = n-1$, then since $res(n-1,p-1) \neq res(n,r)$, both normal i are good.

By [1, Th. $\acute{A}(2)$], both V_1 and V_2 are simple $GL(n)$-modules and $W_l = V_1 \oplus V_2$. Since V_i, $i = 1,2$ contains no maximal vectors other than v_i, it is a simple W_0-module. ■

The decomposition and the simplicity of $V_i, i = 1,2$ for $p > l-1$ are also given in [6, Sec.10]. For $n = 2$, the theorem is also given in [9].

Corollary 4.2. *If $p \nmid (n+l)$, then V_1 and V_2 are the only simple W_0- submodules of W_l.*

Proof. Let M be a simple W_0-submodule of W_l, and let $v \in M$ be a maximal vector. Then we have either $v = v_1$ or $v = v_2$. It follows that $V_1 \subseteq M$ or $V_2 \subseteq M$. Since M is simple we have $M = V_1$ or $M = V_2$. ■

4.2. $p|(n+l)$**.**

If $l = (n-1)(p-1)+r-1$, $0 < r < p-1$, then $n+l = r(\mathrm{mod} p)$ and hence $p \nmid (n+l)$. Assume $p|(n+l)$. Then we must have $l \leqslant (n-1)(p-1)-1$. Let $l+1 = k(p-1)+r$, $0 < r \leqslant p-1$. Then $k+1 < n$.

Since $n+l = n+k(p-1)+r-1 = n-(k+1)+r(\mathrm{mod} p)$,

$$r = -[n-(k+1)](\mathrm{mod} p).$$

Therefore,

$$\begin{aligned} v_2 &= rx^{(\bar{a})}D_{k+1} + \sum_{i>k+1} x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i \\ &= \sum_{i>k+1}(x^{(\bar{a}-\epsilon_{k+1}+\epsilon_i)}D_i - x^{(\bar{a})}D_{k+1}). \end{aligned}$$

4..0.1 $l < (n-1)(p-1)-1$

If $l < (n-1)(p-1)-1$, then by Theorem 3.4 we have $v_2 \in (V_1)_{\bar{a}-\epsilon_{k+1}}$ and hence $V_2 \subseteq V_1$. Since v_1 and v_2 are the only maximal vectors in W_l, V_2 is the unique simple W_0-submodule. Set $\overline{W}_l =: W_l/V_2$. Let $l+1 = k(p-1)+r$, $0 < r \leqslant p-1$. By assumption, $k+1 < n$. Then the structure of V_i $(i = 1, 2)$ is given by Theorem 3.4 and Theorem 3.5.

Lemma 4.3. *Let $\bar{v} \in \overline{W}_l$ be a maximal vector for some $v \in (W_l)_a \setminus V_2$. Then $a \not\geq 0$.*

Suppose $a \geq 0$. Let $\mathfrak{T}(a) = \{i_1, \ldots, i_t\}$. Since $k+1 < n$, $t(a) \geq 2$. We can write $v = \sum_{m=1}^{t-1} c_m x^{(a+\epsilon_{i_m})} D_{i_m}$. By definition, $x_i D_j \cdot v \in (V_2)_{a+\epsilon_i-\epsilon_j}$ for every $i < j$, that is,

$$x_i D_j \cdot v = \sum_{m=1}^{t-1} c_m (a_i + 1 + \delta_{i_m i}) x^{(a+\epsilon_i-\epsilon_j+\epsilon_{i_m})} D_{i_m} - c_m \delta_{i_m i} x^{(a+\epsilon_i)} D_j$$

$$= c\mathfrak{v}_{a+\epsilon_i-\epsilon_j}, \quad c \quad \text{in} \quad \mathbb{F}.$$

This implies that $x_i D_j \cdot v = 0$ in the following cases:

(1) $i \in \mathfrak{T}(a) \setminus \{i_t\}$, $j \notin \mathfrak{T}(a)$. In this case the term $x^{(a+\epsilon_i-\epsilon_j+\epsilon_{i_t})} D_{i_t}$ appears in $\mathfrak{v}_{a+\epsilon_i-\epsilon_j}$ by definition, but not on the left of the last equality. So we have $c = 0$.

(2) $i, j, n \in \mathfrak{T}(a)$, $a_j \neq 0$, $j < n$. Note that $a + \epsilon_i - \epsilon_j \geq 0$. Since the term $x^{(a+\epsilon_i-\epsilon_j+\epsilon_n)} D_n$ appears on the right of the last equality by definition, but not on the left, we have $c = 0$.

(3) $a_i = p-2$, $a_j = 0$. Since $a + \epsilon_i - \epsilon_j \not\geq 0$, Theorem 3.5 shows that $x_i D_j \cdot v \in (V_2)_{a+\epsilon_i-\epsilon_j} = 0$.

Proof of Lemma 2. Suppose $a \geq 0$. If $i_s \in \mathfrak{T}(a)$ and $j \notin \mathfrak{T}(a)$ for some $i_s < j$, since $t(a) \geq 2$, we may assume $i_s < i_t$. Then from $x_{i_s} D_j \cdot v = 0$, we would get

$$c_m (a_{i_s} + 1) = 0 \quad \text{if} \quad m \neq s, \quad \text{and} \quad c_s = 0.$$

This gives us $c_m = 0$ for $m = 1, \ldots, t-1$, a contradiction. Therefore $\mathfrak{T}(a) = \{s, \ldots, n\}$ for some $s \leqslant n-1$. Accordingly, $v = \sum_{m=s}^{n-1} c_m x^{(a+\epsilon_m)} D_m$.

In case $a_i < p-2$ and $a_j \neq 0$ for some $s \leqslant i < j < n$, $a + \epsilon_i - \epsilon_j \geq 0$. We would get, from $x_i D_j \cdot v = 0$,

$$c_m (a_i + 1) = 0 \quad \text{if} \quad m \neq i, j,$$

$$c_m (a_i + 2) = 0 \quad \text{if} \quad m = i,$$

$$c_j (a_i + 1) - c_i = 0 \quad \text{if} \quad m = j,$$

which implies that $c_s = \cdots = c_{n-1} = 0$, a contradiction. Therefore, we must have

$$a = (p-1, \ldots, p-1, \underset{s}{p-2}, \ldots, p-2, r', 0, \ldots 0, a_n), 0 < r' < p-1.$$

Suppose $s \leqslant n-3$. Taking $s \leqslant i < j < n$, we would have, from $x_i D_j \cdot v = 0$, $c_m(a_i + 1) = 0$ if $m \neq i, j$. Thus $c_m = 0$ for each $m \neq i, j$. For each $m \in \{s, \ldots, n-1\}$, there are $i, j \in \{s, \ldots, n-1\}$ such that $m \neq i, j$, since $s \leqslant n-3$, so we get $c_s = \cdots = c_{n-1} = 0$, a contradiction.

Thus we have $s \geq n-2$, and v can be put in the form

$$v = c_{n-2}c^{(a+\epsilon_{n-2})}D_{n-2} + c_{n-1}x^{(a+\epsilon_{n-1})}D_{n-1}.$$

We only sketch the rest of the proof, and leave the details to the interested reader.

Suppose $a_{n-2} < p-2$. From $x_{n-2}D_n \cdot \bar{v} = 0$, one would get $a_n = p-1$, a contradiction.

Suppose $a_{n-2} = p-2$. From $x_{n-2}D_{n-1} \cdot \bar{v} = 0$, one would get that v can be put in the form $v = x^{(a+\epsilon_{n-2})}D_{n-2} - x^{(a+\epsilon_{n-1})}D_{n-1}$. Then from $x_{n-1}D_n \cdot \bar{v} = 0$, one gets

$$a = (p-1, \ldots, p-1, p-2, p-2, 1),$$

which gives $l = (n-1)(p-1)-1$, contrary to the assumption.

Suppose $a_{n-2} = p-1$. Then

$$a = (p-1, \ldots, p-1, a_{n-1}, a_n).$$

¿From $x_{n-1}D_n \cdot \bar{v} = 0$, one would get $a_n = p-1$, a contradiction.

Lemma 4.4. *Assume $p|(n+l)$ and let $l < (n-1)(p-1)-1$. Then $\overline{V}_1 = V_1/V_2$ is a simple W_0- submodule of $\overline{W}_l$.*

Proof. First we show that $\overline{W_l}$ has a unique maximal vector $\bar{v}_1$. Assume $v \in (W_l)_a \setminus V_2$ such that $\bar{v} \in \overline{W}_l$ is maximal. By Lemma 4.3, $a_m = -1$ for some $1 \leqslant m \leqslant n$. Since $(V_1)_a = \mathbb{F}x^{(a+\epsilon_m)}D_m$, we can write $v = x^{(a+\epsilon_m)}D_m$.

If $m < n$, we would have $0 \neq x_mD_n \cdot v \in V_2$; that is,

$$\begin{aligned} x_mD_n \cdot x^{(a+\epsilon_m)}D_m &= (a_m+2)x^{(a+2\epsilon_m-\epsilon_n)}D_m - x^{(a+\epsilon_m)}D_n \\ &= x^{(a+2\epsilon_m-\epsilon_n)}D_m - x^{(a+\epsilon_m)}D_n = c\mathfrak{v}_{a+\epsilon_m-\epsilon_n}, \end{aligned}$$

for some $c \in \mathbb{F}$. By comparing both sides of the last equality, we have $a_i = p-1$ whenever $i \neq m$, and hence $l = (n-1)(p-1)-1$, a contradiction. Then we get $m = n$.

Since $(V_2)_a = 0$ for each $a \not\geq 0$, the assumption that $\bar{v}$ is maximal shows that $x_iD_j \cdot v = 0$ whenever $i < j$. Therefore, v is maximal in W_l, hence $v = v_1$, as asserted.

Since $\overline{V}_1 \subseteq \overline{W}_l$, $\bar{v}_1$ is the only maximal vector in $\overline{V}_1$. Since $\bar{v}_1$ generates $\overline{V}_1$, $\overline{V}_1$ is simple. ■

Theorem 4.5. *If $p|(n+l)$ and if $l < (n-1)(p-1)-1$, then W_l has a unique composition series $W_l \supseteq V_1 \supseteq V_2 \supseteq 0$.*

Proof. Set $\widetilde{W_l} =: W_l/V_1$. By Theorem 3.4, $(\widetilde{W_l})_a = 0$ for each $a \not\geq 0$. By the discussion preceding Lemma 4.3, V_2 is the unique simple submodule of W_l. ¿From the proof of Lemma 4.4, $\overline{V}_1$ is the unique simple submodule of $\overline{W}_l$. So it is sufficient to show that $\widetilde{W_l}$ is simple.

For $a \geq 0$, let $\mathfrak{T}(a) = \{i_1, \ldots, i_t\}$. By Theorem 1(1), we have

$$\widetilde{x^{(a+\epsilon_{i_1})}}D_{i_1} = \cdots = \widetilde{x^{(a+\epsilon_{i_t})}}D_{i_t},$$

and hence $\dim(\widetilde{W_l})_a = 1$.

Let $l = k(p-1) + r - 1$, $0 < r \leqslant p-1$. Let v denote $\widetilde{x^{(\bar{a})}D_{k+1}}$. If $b \succ \bar{a} - \epsilon_{k+1}$, then $b \not\geq 0$ and hence $(\widetilde{W_l})_b = 0$. This implies that v is maximal.

To prove that $\widetilde{W_l}$ is generated by v we use induction on a. The greatest a with $(\widetilde{W_l})_a \neq 0$ is $\mathfrak{G}(v) = \bar{a} - \epsilon_{k+1}$, for which we have $(\widetilde{W_l})_{\bar{a}-\epsilon_{k+1}} = \mathbb{F}v$.

Assume that $(\widetilde{W_l})_b$ is generated by v for every $b \succ a$, and consider the case $0 \leqslant a \prec \bar{a} - \epsilon_{k+1}$. Since $a \prec \bar{a} - \epsilon_{k+1}$, there exists $j > i_1$ such that $a_j \neq 0$. Let $b =: a + \epsilon_{i_1} - \epsilon_j \succ a$. Then the induction assumption says that $\widetilde{x^{(b+\epsilon_j)}D_j}$ is generated by v. Since

$$\begin{aligned} &x_j D_{i_1} \cdot x^{(b+\epsilon_j)} D_j \\ &= (b_j + 2) x^{(b+2\epsilon_j - \epsilon_{i_1})} D_j - x^{(b+\epsilon_j)} D_{i_1} \\ &= (a_j + 1)[x^{(a+\epsilon_j)} D_j - x^{(a+\epsilon_{i_1})} D_{i_1}] + a_j x^{(a+\epsilon_{i_1})} D_{i_1} \\ &= a_j x^{(a+\epsilon_{i_1})} D_{i_1} (\mathrm{mod} V_1), \end{aligned}$$

$\widetilde{x^{(a+\epsilon_{i_1})}D_{i_1}}$ is generated by v.

We claim that v is the unique maximal vector in $\widetilde{W_l}$. To establish the claim, it suffices to show that, for each $0 \leqslant a \prec \bar{a} - \epsilon_{k+1}$, there are i and j with $i < j$ such that $x_i D_j \cdot (\widetilde{W_l})_a \neq 0$.

For each $0 \leqslant a \prec \bar{a} - \epsilon_{k+1}$, we have $0 \leqslant a + \epsilon_i - \epsilon_j \preccurlyeq \bar{a} - \epsilon_{k+1}$ for some $i < j$.

(1) If $a_i < p-2$, then

$$\begin{aligned} x_i D_j \cdot x^{(a+\epsilon_i)} D_i &= (a_i + 2) x^{(a+2\epsilon_i - \epsilon_j)} D_i - x^{(a+\epsilon_i)} D_j \\ &= (a_i + 1) x^{(a+2\epsilon_i - \epsilon_j)} D_i + [x^{(a+2\epsilon_i - \epsilon_j)} D_i - x^{(a+\epsilon_i)} D_j] \\ &= (a_i + 1) x^{(a+2\epsilon_i - \epsilon_j)} D_i (\mathrm{mod} V_1). \end{aligned}$$

(2) If $a_i = p-2$, then $x_i D_j \cdot x^{(a+\epsilon_i)} D_i = -x^{(a+\epsilon_i)} D_j$.

By Theorem 3.4, $x_i D_j \cdot x^{(a+\epsilon_i)} D_i \notin V_1$ in both cases above, i.e., $x_i D_j \cdot \widetilde{x^{(a+\epsilon_i)}D_i} \neq 0$. So the claim holds. Consequently, $\widetilde{W_l}$ is simple.

■

For the case $l = 1$, the theorem is also discussed in [6].

4.3. $l = (n-1)(p-1) - 1$.

Assume $l = (n-1)(p-1) - 1$ and let

$$\bar{a}_i := (p-1, \ldots, p-1, \underset{i}{0}, p-1, \ldots, p-1).$$

Let v_1^i denote $x^{(\bar{a}_i)} D_i$ and V_1^i denote the submodule of W_l generated by v_1^i. Set $\widetilde{W_l} =: W_l / V_2$.

Proposition 4.6. *(1) V_2 is the unique simple submodule of W_l.*

(2) $\tilde{v}_1^i$ is maximal in $\tilde{W}_l$ for every $1 \leqslant i \leqslant n$.

(3) $\widetilde{V_1^i} =: V_1^i / V_2$ is a $1-$dimensional submodule of $\widetilde{W_l}$.

Proof. (1) Since $\mathfrak{G}(v_2) \prec \mathfrak{G}(v_1)$, V_2 is the unique simple submodule of W_l (note that $v_1 = x^{(\bar{a})}D_n$ is denoted here also by v_1^n).

(2) (3) Assume $s \neq t$. Then

$$x_s D_t \cdot v_1^i = \begin{cases} 0, & \text{if } s \neq i \\ x^{(\bar{a}_i+\epsilon_i-\epsilon_t)}D_i - x^{(\bar{a}_i)}D_t, & \text{if } s = i. \end{cases}$$

Since $x^{(\bar{a}_i+\epsilon_i-\epsilon_t)}D_i - x^{(\bar{a}_i)}D_t = \mathfrak{v}_{\bar{a}-\epsilon_t}$, Theorem 3.5(2) says that $x_iD_t \cdot v_1^i \in V_2$. Then $\tilde{v}_1^i$ is maximal and $\widetilde{V}_1^i$ is a $1-$dimensional submodule. ■

Set $W_l' =: \sum_{i=1}^n V_1^i$ and let $\overline{W_l} =: W_l/W_l'$. Since every $a \ngeq 0$ with $(W_l)_a \neq 0$ is in the form $\bar{a} - \epsilon_i (= \mathfrak{G}(v_1^i))$ for some $1 \leqslant i \leqslant n$, we have $(\overline{W}_l)_a \neq 0$ only if $a \geq 0$.

Lemma 4.7. *(1) If $n = 2$, $\overline{W_l}$ contains a unique maximal vector $\bar{w}_1 = \overline{x^{(\bar{a})}D_{n-1}}$.*

(2) For $n \geq 3$, let b denote $(p-1, \ldots, p-1, \underset{n-2}{p-2}, p-2, 1)$. Then $\overline{W_l}$ contains two maximal vectors, namely:

$$\bar{w}_1 = \overline{x^{(\bar{a})}D_{n-1}}, \quad \bar{w}_2 = \overline{x^{(b+\epsilon_{n-2})}D_{n-2}} - \overline{x^{(b+\epsilon_{n-1})}D_{n-1}}.$$

Proof. Clearly, $\overline{x^{(\bar{a})}D_{n-1}} \in \overline{W_l}$ is maximal. Suppose $v \in (W_l)_a$ such that $\bar{v}$ is maximal. Then $a \geq 0$. Assume $\mathfrak{T}(a) = \{i_1, \ldots, i_t\}$. Then $t(a) = t \geq 2$. Since $\dim(W_l')_a = 1$, v can be put in the form

$$v = \sum_{m=1}^{t-1} c_m x^{(a+\epsilon_{i_m})}D_{i_m}.$$

If $\mathfrak{G}(x_iD_j \cdot v) \ngeq 0$ for some $i < j$, then we must have $a_i = p-2$ and $a_j = 0$. So $\mathfrak{T}(a) = \{i, j\}$ and $v = cx^{(a+\epsilon_i)}D_i$.

Suppose $j < n$. Then since $x_jD_n \cdot v = cx^{(a+\epsilon_i+\epsilon_j-\epsilon_n)}D_i \in W_l'$ for some $c \in \mathbb{F}$ and $a + \epsilon_j - \epsilon_n \geq 0$,

$$cx^{(a+\epsilon_i+\epsilon_j-\epsilon_n)}D_i \in (V_2)_{a+\epsilon_j-\epsilon_n} = \mathbb{F}\mathfrak{v}_{a+\epsilon_j-\epsilon_n}.$$

Using the definition of $\mathfrak{v}_{a+\epsilon_j-\epsilon_n}$, one gets $c = 0$, a contradiction. Then we must have $j = n$.

Suppose $i < n-1$. Then

$$x_iD_{n-1} \cdot v = -cx^{(\bar{a})}D_{n-1} \in (V_2)_{\bar{a}-\epsilon_{n-1}} = \mathbb{F}\mathfrak{v}_{\bar{a}-\epsilon_{n-1}}.$$

By the definition of $\mathfrak{v}_{\bar{a}-\epsilon_{n-1}}$, one gets $c = 0$, a contradiction. So we get $i = n-1$, and hence $v = x^{(\bar{a})}D_{n-1}$.

Assume $\mathfrak{G}(x_iD_j \cdot v) \geq 0$ for all $i < j$. Then $x_iD_j \cdot v \in (V_2)_{a+\epsilon_i-\epsilon_j}$ whenever $i < j$. Applying a similar argument as that used in the proof of Lemma 4.3, we have, in the case $n \geq 3$, $a = (p-1, \ldots, p-1, \underset{n-2}{p-2}, p-2, 1)$. Let $w_2 = x^{(a+\epsilon_{n-2})}D_{n-2} - x^{(a+\epsilon_{n-1})}D_{n-1}$. By a direct computation we get $\bar{w}_2$ is maximal in $\overline{W}_l$. Note that $\dim(\overline{W}_l)_a = 1$ implies that $\bar{w}_2$ is the only maximal vector with grading a. ■

Note that $\bar{w}_2 = -(x_n D_{n-1})(x_{n-1} D_{n-2}) \cdot \bar{w}_1 - x_n D_{n-2} \cdot \bar{w}_1$. Let $\bar{B}$ denote the submodule of $\overline{W}_l$ generated by $\bar{w}_2$ (we use B to denote the pre-image of $\bar{B}$ under the canonical epimorphism from W_l to $\overline{W}_l$). Since $\mathfrak{G}(\bar{w}_2) \prec \mathfrak{G}(\bar{w}_1)$, $\bar{B}$ is simple.

Identify $W(n-1,1)$ with the Lie subalgebra of $W(n,1)$ in the canonical way. Let $\mathfrak{gl}(n-1,\mathbb{F})$ denote

$$W(n-1,1)_0 = \langle x_i D_j | 1 \leqslant i, j \leqslant n-1 \rangle$$

and let $W(n-1,1)_0^-$ denote

$$\langle x_j D_i | 1 \leqslant i < j \leqslant n-1 \rangle.$$

Then for each $0 \leqslant i \leqslant p-2$, $(x_n D_{n-1})^i \bar{w}_2$ is maximal in the $\mathfrak{gl}(n-1,\mathbb{F})$-module $\overline{W}_l$.

For $a \geq 0$, since $(W_l)_a$ has a basis $\{x^{(a+\epsilon_i)} D_i | i \in \mathfrak{T}(a)\}$, $(\overline{W}_l)_a$ has a basis $\{\overline{x^{(a+\epsilon_i)} D_i} | i \in \mathfrak{T}(a) \setminus \{i_t\}\}$, so that $\overline{W}_l$ has a basis

$$\mathfrak{Z} = \cup_{a \geq 0} \{\overline{x^{(a+\epsilon_i)} D_i} | i \in \mathfrak{T}(a) \setminus \{i_t\}\}.$$

For $0 \leqslant s \leqslant p-1$, set $(\overline{W}_l)_s =: \langle \overline{x^{(a)} D_i} \in \mathfrak{Z} | a_n = s \rangle$. Then $\overline{W}_l = \oplus_{s=0}^{p-1} (\overline{W}_l)_s$. For $0 \leqslant s \leqslant p-2$, since $(\overline{W}_l)_s$ has a basis $\{\overline{x^{(a)} D_i} | i < n, a_n = s\}$, there is a $\mathfrak{gl}(n-1,\mathbb{F})$-module isomorphism:

$$\psi_s : (\overline{W}_l)_s \underset{\overline{x^{(a)} D_i} \to x^{(a_1,\ldots,a_{n-1})} D_i}{\longrightarrow} W_{l-s} (\subseteq W(n-1,1)).$$

It is easy to see that $\psi_{s+1}((x_n D_{n-1})^s \bar{w}_2) = (s+1)! v_2$ for $0 \leqslant s < p-2$.

We now determine the structure of $\bar{B}$. Let $b = (p-1, \ldots, p-1, p-2, p-2, 1)$. Notice that

$$(x_n D_{n-1})^{p-2} \bar{w}_2 = (p-1)! \overline{\mathfrak{v}_{b-(p-2)\epsilon_{n-1}+(p-2)\epsilon_n}} = 0.$$

Thus, by the PBW Theorem,

$$\bar{B} = \sum_{0 \leqslant c_i \leqslant p-1, 0 \leqslant s < p-2} (x_n D_1)^{c_1} \ldots (x_n D_{n-2})^{c_{n-2}} u(W(n-1,1)_0^-) (x_n D_{n-1})^s \bar{w}_2.$$

For each $0 \leqslant s < p-2$, let $\bar{B}_s$ denote

$$u(W(n-1,1)_0^-)(x_n D_{n-1})^s \bar{w}_2 (\subseteq (\overline{W}_l)_{s+1}).$$

Notice that $l - (s+1) > (n-2)(p-1) - 1$. Using the isomorphism ψ_{s+1} and Theorem 3.6(3) for $W(n-1,1)_{l-(s+1)}$, we have

Corollary 4.8. *For each $0 \leqslant s < p-2$, $(\bar{B}_s)_a \neq 0$ if and only if $a \geq 0$ and $t(a) \geq 3$. In this case*

$$(\bar{B}_s)_a = \langle \overline{x^{(a+\epsilon_i)} D_i} - \overline{x^{(a+\epsilon_j)} D_j} | i, j \in \mathfrak{T}(a) \setminus \{n\} \rangle.$$

Noting that $a_n < p-1$ implies that $a_m \neq 0$ for each $m \in \mathfrak{T}(a)$, we have, for $i<j<n$ and $1 \leqslant m \leqslant n-2$,

$$x_n D_m \cdot (\overline{x^{(a+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_j)}D_j})$$
$$= (a_n+1)(\overline{x^{(a+\epsilon_i+\epsilon_n-\epsilon_m)}D_i} - \overline{x^{(a+\epsilon_j+\epsilon_n-\epsilon_m)}D_j}). \quad (*)$$

Hence $x_n D_m \cdot \bar{B}_s \subseteq \bar{B}_{s+1}$ for $0 \leqslant s < p-3$. So we obtain

$$\bar{B} = \oplus_{0 \leqslant j < p-2} \bar{B}_j \oplus \sum_{i \leqslant n-2} x_n D_i \cdot \bar{B}_{p-3}.$$

Let $\bar{B}_{p-2}$ denote $\sum_{i \leqslant n-2} x_n D_i \cdot \bar{B}_{p-3}$.

Corollary 4.9. *For $s = p-2$, $(\bar{B}_s)_a \neq 0$ if and only if $a \geq 0$ and $t(a) \geq 3$. In this case*

$$(\bar{B}_s)_a = \langle \overline{x^{(a+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_j)}D_j} | i,j \in \mathfrak{T}(a) \setminus \{i_t\}\rangle.$$

Proof. Note that $n \notin \mathfrak{T}(a)$ for any $a \geq 0$ with $(\bar{B}_{p-2})_a \neq 0$, since $\bar{B}_{p-2} \subseteq (\overline{W}_l)_{p-1}$.

Assume $a \geq 0$ and $t(a) = 2$. Let $\mathfrak{T}(a) = \{i,j\}$, $i<j$. Then $j<n$. Since $l = (n-1)(p-1)-1$, $a_i + a_j = p-2$. So we have $a_i + 1 = -(a_j+1)(\mathrm{mod} p)$.

Since

$$\begin{aligned} \mathfrak{v}_a &= (a_i+1)x^{(a+\epsilon_i)}D_i + (a_j+1)x^{(a+\epsilon_j)}D_j \\ &= (a_i+1)(x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_j)}D_j), \end{aligned}$$

Corollary 4.8 and the formula $(*)$ above give $x_n D_m \cdot \bar{B}_{p-3} = 0$ for all $0 \leqslant m \leqslant n-2$. So we get $(\bar{B}_{p-2})_a = 0$.

Assume $a \geq 0$ and $t(a) \geq 3$. For any $i,j \in \mathfrak{T}(a)$, there is $s \in \mathfrak{T}(a)$ such that $s \neq i,j$. By Corollary 4.8,

$$\overline{x^{(a+\epsilon_s-\epsilon_n+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_s-\epsilon_n+\epsilon_j)}D_j} \in (\bar{B}_{p-3})_{a+\epsilon_s-\epsilon_n}.$$

Applying $x_n D_s$, we get $\overline{x^{(a+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_j)}D_j} \in (\bar{B}_{p-2})_a$, so that

$$\langle \overline{x^{(a+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_j)}D_j} | i,j \in \mathfrak{T}(a) \setminus \{i_t\}\rangle \subseteq (\bar{B}_{p-2})_a.$$

On the other hand, using the formula $(*)$ above, we see that

$$(\bar{B}_{p-2})_a \subseteq \langle \overline{x^{(a+\epsilon_i)}D_i} - \overline{x^{(a+\epsilon_j)}D_j} | i,j \in \mathfrak{T}(a) \setminus \{i_t\}\rangle.$$

■

For $n \geq 3$, let $\widehat{W_l}$ denote $\overline{W}_l/\bar{B}$. Since $(\overline{W}_l/\overline{B})_a \cong (\overline{W}_l)_a$ for $a \geq 0$ and $t(a) = 2$, we get

$$\dim(\widehat{W_l})_a = \begin{cases} 1, & \text{if } a \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.10. *(1) If $n \geq 3$, $\widehat{W_l}$ is simple.*

(2) If $n = 2$, $\overline{W}_l$ is simple.

Proof. (1) We shall divide the proof into two steps.

Step1. $\widehat{W_l}$ *is generated by* $\widehat{x^{(\bar a)}D_{n-1}}$.

We proceed by induction on a. The greatest a with $(\widehat{W_l})_a \neq 0$ is $\bar a - \epsilon_{n-1}$. Clearly,

$$(\widehat{W_l})_{\bar a-\epsilon_{n-1}} = \mathbb{F}\widehat{x^{(\bar a)}D_{n-1}}.$$

Assume that $(\widehat{W_l})_b$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$ for every $b \succ a$. Consider the case $0 \leqslant a \prec \bar a - \epsilon_{n-1}$. Let $\mathfrak{T}(a) = \{i_1, \dots, i_t\}$. Then $(\widehat{W_l})_a = \mathbb{F}\widehat{x^{(a+\epsilon_{i_j})}D_{i_j}}$, $j = 1, \dots, t-1$.

Suppose $t(a) \geq 3$. Since $l = (n-1)(p-1)-1$, $a_{i_t} \neq 0$. For each $m < t$, taking $i_s \in \mathfrak{T}(a) \setminus \{i_m, i_t\}$, we have that $\widehat{x^{(\bar a)}D_{n-1}}$ generates $(\widehat{W_l})_{a+\epsilon_{i_s}-\epsilon_{i_t}}$ by the induction assumption. Taking

$$\widehat{x^{(a+\epsilon_{i_m}+\epsilon_{i_s}-\epsilon_{i_t})}D_{i_m}} \in (\widehat{W_l})_{a+\epsilon_{i_s}-\epsilon_{i_t}},$$

since

$$x_{i_t}D_{i_s} \cdot x^{(a+\epsilon_{i_m}+\epsilon_{i_s}-\epsilon_{i_t})}D_{i_m} = a_{i_t}x^{(a+\epsilon_{i_m})}D_{i_m},$$

$(\widehat{W_l})_a$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$.

Suppose $t(a) = 2$. Let $\mathfrak{T}(a) = \{i, j\}$, $i < j$. Then either $a_i < p-2$ or $a_j < p-2$, say, $a_j < p-2$. If $j < n$, then $a + \epsilon_j - \epsilon_n \succ a$, so the induction assumption says that $\widehat{x^{(a+\epsilon_i+\epsilon_j-\epsilon_n)}D_i}$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$. Since

$$x_nD_j \cdot \widehat{x^{(a+\epsilon_i+\epsilon_j-\epsilon_n)}D_i} = a_n\widehat{x^{(a+\epsilon_i)}D_i}$$

and $a_n \neq 0$, $(\widehat{W_l})_a = \mathbb{F}\widehat{x^{(a+\epsilon_i)}D_i}$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$.

In case $j = n$ and $i < n-1$, since

$$x^{(a+\epsilon_i)}D_i = -x_{n-1}D_i \cdot x^{(\bar a)}D_{n-1},$$

$(\widehat{W_l})_a$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$.

In case $j = n$ and $i = n-1$, the assumption $a \prec \bar a - \epsilon_{n-1}$ shows that $a_i < p-2$, then similarly we have that $(\widehat{W_l})_a$ is generated by $\widehat{x^{(\bar a)}D_{n-1}}$.

Step 2. $\widehat{W_l}$ *has a unique maximal vector* $\widehat{x^{(\bar a)}D_{n-1}}$.

It is sufficient to show that, for each $a \prec \bar a - \epsilon_{n-1}$ with $(\widehat{W_l})_a \neq 0$, $x_iD_j \cdot (\widehat{W_l})_a \neq 0$ for some $i < j$.

Suppose $t(a) \geq 3$. Let $i, j \in \mathfrak{T}(a)$ with $i < j$. Note that

$$x_iD_j \cdot x^{(a+\epsilon_i)}D_i = (a_i+2)x^{(a+2\epsilon_i-\epsilon_j)}D_i - x^{(a+\epsilon_i)}D_j.$$

If $t(a+\epsilon_i-\epsilon_j) \geq 3$, then we have, by Corollary 4.8,

$$x_iD_j \cdot \widehat{x^{(a+\epsilon_i)}}D_i = (a_i+2)\widehat{x^{(a+2\epsilon_i-\epsilon_j)}}D_i - \widehat{x^{(a+\epsilon_i)}}D_j = (a_i+1)\widehat{x^{(a+2\epsilon_i-\epsilon_j)}}D_i \neq 0.$$

If $t(a+\epsilon_i-\epsilon_j) = 2$, then we must have $a_i = p-2$, and hence $x_iD_j \cdot \widehat{x^{(a+\epsilon_i)}}D_i = -\widehat{x^{(a+\epsilon_i)}}D_j \neq 0$.

Suppose $t(a) = 2$. Let $\mathfrak{T}(a) = \{i, j\}$, $i < j$. Since $\bar{B}_a = 0$, $B_a = (W_l')_a = \mathbb{F}\mathfrak{v}_a$. In case $j < n$, we have

$$x_iD_j \cdot \overline{x^{(a+\epsilon_i)}D_i} = \overline{\mathfrak{v}_{a+\epsilon_i-\epsilon_j}} + (a_i+1)\overline{x^{(a+\epsilon_i)}D_j}$$

$$= (a_i+1)\overline{x^{(a+\epsilon_i)}D_j} \notin \bar{B},$$

i.e., $x_iD_j \cdot \widehat{x^{(a+\epsilon_i)}D_i} \neq 0$. In a similar fashion, one can check that:

In case $j = n$ and $i < n-1$, $x_iD_{n-1} \cdot \widehat{x^{(a+\epsilon_i)}D_i} \neq 0$. In case $i = n-1$ and $j = n$, so that by assumption $a_i < p-2$,

$$x_iD_n \cdot \widehat{x^{(a+\epsilon_i)}D_i} = (p-1-a_n)\widehat{x^{(a+\epsilon_i)}D_n} \neq 0.$$

This completes the proof of (1).

(2) Let $n = 2$. By Lemma 4.7, $\overline{W}_l$ has a unique maximal vector $\overline{x^{(\bar{a})}D_{n-1}}$. Applying a similar argument as that for the case $t(a) = 2$ in Step 1 above, we have that $\overline{x^{(\bar{a})}D_{n-1}}$ generates $\overline{W}_l$, so $\overline{W}_l$ is simple. ■

5. The S_0-module structure of S

In this section, we determine the $S_0 -$ module structure of S. By [8], $S(n,1)$ is generated by S_{-1} and S_1. For every $i > 0$, since $W_{i-1} = [W_{-1}, W_i]$, we have $S_i \subsetneq W_i$. For each $l \geq 0$, W_l is a $S_0 -$ module, and each $W_0 -$ submodule of W_l is also a $S_0 -$ submodule. On the other hand, each $S_0 -$ submodule of W_l, with the adjoint action of x_nD_n given in W, is extended to a $W_0 -$ submodule (note that $W_0 = S_0 \oplus \mathbb{F}x_nD_n$).

Theorem 5.1. *(1) If $0 < l < (n-1)(p-1)-1$, then $S_l = V_1$.*

(2) If $l > (n-1)(p-1)-1$, then $S_l = V_2$.

Proof. (1) Assume $0 < l < (n-1)(p-1)-1$. Let $l+1 = k(p-1)+r$, $0 < r \leqslant p-1$, and denote $\bar{a} = (p-1, \ldots, \underset{k}{p-1}, r, \ldots, 0)$. Since

$$v_1 = x^{(\bar{a})}D_n = \begin{cases} -D_{k+1,n}x^{(\bar{a}+\epsilon_{k+1})}, & \text{if } r < p-1 \\ -D_{k+2,n}x^{(\bar{a}+\epsilon_{k+2})}, & \text{if } r = p-1, \end{cases}$$

we have $v_1 \in S_l$, and hence $V_1 \subseteq S_l$.

By Theorem 4.1 and Theorem 4.5, V_1 is a maximal W_0-submodule of W_l, so we get $S_l = V_1$.

(2) Assume $l > (n-1)(p-1)-1$. Let $l+1 = (n-1)(p-1)+r$, $0 < r \leqslant p-1$, and let $\bar{a} = (p-1, \ldots, p-1, r)$. Since

$$D_{n,n-1}x^{(\bar{a}+\epsilon_n)}$$

$$= x^{(\bar{a})}D_{n-1} - x^{(\bar{a}-\epsilon_{n-1}+\epsilon_n)}D_n = v_2 \in S_l,$$

so that $V_2 \subseteq S_l$, we have, by Theorem 4.1, $S_l = V_2$. ■

For $l = (n-1)(p-1)-1$, we have $V_2 \subseteq S_l$, since S_l is a W_0-submodule of W_l and V_2 is the unique simple submodule of W_l.

Lemma 5.2. *Let $l=(n-1)(p-1)-1$. If $a \not\geq 0$, $(S_l)_a = 0$.*

Proof. Recall $\bar{a}_i = \tau-(p-1)\epsilon_i$ and $v_1^i = x^{(\bar{a}_i)}D_i$. Suppose $v_1^i \in S_l$. Then we have

$$v_1^i = \sum_{s<t} C_{st}^a D_{st} x^{(a)} = \sum_{i\notin\{s,t\}} C_{st}^a D_{st} x^{(a)} + \sum_{t\neq i} C_{ti}^a (x^{(a-\epsilon_t)}D_i - x^{(a-\epsilon_i)}D_t).$$

Notice that each term $x^{(a-\epsilon_t)}D_i$ in the second summation is not equal to $x^{(\bar{a}_i)}D_i$, since $t \neq i$. Also, the term $x^{(\bar{a}_i)}D_i$ does not appear in the first summation . This leads to $v_1^i = 0$, a contradiction. ■

For each $a \geq 0$, since $l=(n-1)(p-1)-1$, $t(a) \geq 2$. Let $\mathfrak{X} = \oplus_{a\geq 0}\mathfrak{X}_a$, where $\mathfrak{X}_a =: \langle x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_j)}D_j | i,j \in \mathfrak{T}(a)\rangle$. We see that, for $i,j \in \mathfrak{T}(a)$, $x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_j)}D_j = D_{ji}x^{(a+\epsilon_i+\epsilon_j)} \in S_l$. Hence $\mathfrak{X} \subseteq S_l$.

Lemma 5.3. *$\mathfrak{X}$ is W_0-submodule (hence a S_0-submodule) of W_l.*

Proof. For $i \neq j$, it is easy to see that $x_iD_j \cdot \mathfrak{X}_a \subseteq \mathfrak{X}_{a+\epsilon_i-\epsilon_j}$ if $a \geq 0$ and $a+\epsilon_i-\epsilon_j \geq 0$.

If $a \geq 0$, $a+\epsilon_i-\epsilon_j \not\geq 0$, then we must have $a_i = p-2$ and $a_j = 0$. In this case $\mathfrak{X}_a = \mathbb{F}(x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_j)}D_j)$. Then $x_iD_j \cdot (x^{(a+\epsilon_i)}D_i - x^{(a+\epsilon_j)}D_j) = 0$. So we have $x_iD_j \cdot \mathfrak{X} \subseteq \mathfrak{X}$ whenever $i \neq j$. Since $x_iD_i \cdot \mathfrak{X}_a \subseteq \mathfrak{X}_a$ for each $1 \leqslant i \leqslant n$, the lemma holds. ■

Lemma 5.4. *Let $l=(n-1)(p-1)-1$. For any $0 \leqslant a \in \bar{A}_l$, $(S_l)_a \subsetneq (W_l)_a$.*

Proof. Suppose there were $0 \leqslant a \in \bar{A}_l$ such that $(S_l)_a = (W_l)_a$. Let b be the greatest such a. If there is $i<j$ with $i \in \mathfrak{T}(b)$ and $j \notin \mathfrak{T}(b)$, take $q \in \mathfrak{T}(b)\setminus i$, then

$$x_iD_j \cdot x^{(b+\epsilon_q)}D_q = (b_i+1)x^{(b+\epsilon_i-\epsilon_j+\epsilon_q)}D_q \in (S_l)_{b+\epsilon_i-\epsilon_j}.$$

So the fact $\mathfrak{X} \subseteq S_l$ implies that $(S_l)_{b+\epsilon_i-\epsilon_j} = (W_l)_{b+\epsilon_i-\epsilon_j}$, a contradiction. Hence we have $\mathfrak{T}(b) = \{s, s+1, \ldots, n\}$.

Suppose $s \leqslant n-2$. Then $b_i \neq 0$ for all $i \in \mathfrak{T}(b)$, since $l=(n-1)(p-1)-1$. Notice that

$$x_sD_{s+1} \cdot x^{(b+\epsilon_n)}D_n = (a_s+1)x^{(b+\epsilon_s-\epsilon_{s+1}+\epsilon_n)}D_n \in (S_l)_{b+\epsilon_s-\epsilon_{s+1}}$$

implies that $(S_l)_{b+\epsilon_s-\epsilon_{s+1}} = (W_l)_{b+\epsilon_s-\epsilon_{s+1}}$, a contradiction. So we have $\mathfrak{T}(b) = \{n-1, n\}$. Suppose $b_{n-1} < p-2$. Then $b_n > 0$, and

$$x_{n-1}D_n \cdot x^{(b+\epsilon_n)}D_n = (b_{n-1}+1)x^{(b+\epsilon_{n-1})}D_n \in (S_l)_{b+\epsilon_{n-1}-\epsilon_n}$$

gives $(S_l)_{b+\epsilon_{n-1}-\epsilon_n} = (W_l)_{b+\epsilon_{n-1}-\epsilon_n}$, a contradiction. Suppose $b_{n-1} = p-2$. Then we get $b = \bar{a} - \epsilon_{n-1}$, so that

$$(S_l)_b = (W_l)_b = \langle x^{(\bar{a})}D_{n-1}, x^{(\bar{a}-\epsilon_{n-1}+\epsilon_n)}D_n\rangle.$$

Which leads to $v_1^n = -x_{n-1}D_n \cdot x^{(\bar{a}-\epsilon_{n-1}+\epsilon_n)}D_n \in S_l$, a contradiction. ■

Corollary 5.5. *Let $l=(n-1)(p-1)-1$. Then $S_l = \mathfrak{X}$.*

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