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A Counterexample in the Dimension Theory of Homogeneous Spaces of Locally Compact Groups

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Abstract. We construct a locally compact group G and a closed subgroup H such that such that the quotient space G/H is connected and has weight $w(G/H) = 2^{\aleph_0}$ but fails to contain a cube $\mathbb{I}^{w(G/H)}$ of the same weight. This proves as incorrect an assertion made in Theorem 4.2 of K. H. Hofmann and S. A. Morris: Transitive actions of compact groups and topological dimension, J. of Algebra **234** (2000), 454–479.
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Introduction

In [5], it is shown that an infinite dimensional quotient space G/H of a compact group G and a closed subgroup H, firstly, contains a homeomorphic copy of the cube $\mathbb{I}^{w(G/H)}$, $\mathbb{I} = [0,1]$ of the same weight and that, secondly, the dimension $\dim \mathfrak{g}/\mathfrak{h}$ of the quotient vector space of the respective Lie algebras is the largest cardinal $\mathfrak{a}$ such that G/H contains a copy of $\mathbb{I}^{\mathfrak{a}}$, thereby allowing the definition of a transfinite dimension for this class of spaces. It is further argued that these results extend to infinite dimensional connected quotient spaces G/H of *locally* compact groups as well. The example we shall present will show that the first of these assertions is wrong for locally compact groups while the results in the compact case and remain intact.

We intend to publish a paper in which is is shown that the first result is also true if the weight $w(G/H)$ is outside the interval $]\aleph_0, 2^{\aleph_0}]$.

The Example

For the verification of the properties of the construction we use the following lemma, based on a result due to Gerlits.

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Lemma 1. *Let A be an uncountable set, X a Hausdorff space of countable weight, and $f\colon \mathbb{I}^A \to X$. Then $\dim f^{-1}\big(f(p)\big) = \infty$ for all $p \in \mathbb{I}^A$.*

Proof. Suppose that the fiber $F \overset{\text{def}}{=} f^{-1}\big(f(p)\big) \subseteq \mathbb{I}^A$ is finite dimensional for some $p = (p_a)_{a\in A} \in \mathbb{I}^A$. Then for each subset $B \subseteq A$ the copy

$$\{(r_a)_{a\in A} \in \mathbb{I}^A : r_a = p_a \text{ for } a \in B\}$$

of $\mathbb{I}^{A\setminus B}$ that is contained in F is finite dimensional as well. Since A is uncountable, this implies that B is uncountable. Then [4], Corollary 4 proves that the weight of X is uncountable, and that is a contradiction. ∎

We now proceed to construct the example and denote the circle group by $\mathbb{T} \overset{\text{def}}{=} \mathbb{R}/\mathbb{Z}$, the two element group by $\mathbb{Z}(2) \overset{\text{def}}{=} \mathbb{Z}/2\mathbb{Z}$, and the continuum cardinality by $\mathfrak{c} \overset{\text{def}}{=} 2^{\aleph_0}$. Let G be the infinite dimensional locally compact group $\mathbb{T}^{\aleph_0} \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{Z}(2)^{\mathfrak{c}}$. Then $G_0 = \mathbb{T}^{\aleph_0} \times \mathrm{SL}(2,\mathbb{R}) \times \{0\}$ and $G/G_0 \cong \mathbb{Z}(2)^{\mathfrak{c}}$ is compact. The Lie group $\mathrm{SL}(2,\mathbb{R})$ contains a discrete copy of the free group on 2 generators, e.g. the one generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

see e.g. [3] or [6]. The commutator group of a free group of 2 generators is free on countably many generators (cf. [1], AI, p. 145, Ex.14) and so $\mathrm{SL}(2,\mathbb{R})$ contains a discrete copy of the free group F of countably infinite rank. On the other hand $\mathbb{Z}(2)^{\mathfrak{c}}$ is separable (cf. [2], TGI, p. 95, Ex. 5), and so there is a homomorphism $\varphi\colon F \to Z(2)^{\mathfrak{c}}$ with a dense image. Since F is discrete, its graph $\mathrm{gr}(\varphi)$ is discrete in $F \times Z(2)^{\mathfrak{c}}$ and therefore $\{0\} \times \mathrm{gr}(\varphi)$ is closed in G. Thus

$$G/H \cong \mathbb{T}^{\aleph_0} \times \frac{\mathrm{SL}(2,\mathbb{R}) \times \mathbb{Z}(2)^{\mathfrak{c}}}{\mathrm{gr}(\varphi)}$$

is a locally compact homogeneous space.

Proposition 2. *G/H has the following properties*

(i) *G/H is connected,*
(ii) *$\dim G/H = \infty$,*
(iii) *$w(G/H) = \mathfrak{c}$, and*
(iv) *G/H does not contains any copies of $\mathbb{I}^{\mathfrak{c}}$.*

Proof. (i) $G_0H = \mathbb{T}^{\aleph_0} \times \mathrm{SL}(2,\mathbb{R}) \times \operatorname{im}\varphi$ is dense in G whence G_0H/H is dense in G/H on the one hand and is connected as a continuous image of G_0. Therefore G/H is connected.

(ii) Since H is a discrete subgroup of G, the factor space G/H and G are locally homeomorphic. Thus G/H is infinite dimensional.

(iii) The local weights of the locally homeomorphic spaces G and G/H agree and the local weight of G is $w(G) = \mathfrak{c}$. So $\mathfrak{c} \le w(G/H) \le w(G) = \mathfrak{c}$. Hence G/H has the weight of the continuum.

(iv) Suppose that G/H contains a copy C of the cube $\mathbb{I}^{w(G/H)} = \mathbb{I}^{\mathfrak{c}}$. Let $\pi\colon G/H \to \mathbb{T}^{\aleph_0}$ be the obvious projection onto the first factor. Then $\pi^{-1}\big(\pi(p)\big)$ is locally homeomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{Z}(2)^{\mathfrak{c}}$ and is therefore 3-dimensional. Define $f \overset{\text{def}}{=} \pi|C : C \to \mathbb{T}^{\aleph_0}$. Then

$$(\forall c \in C)\ \dim f^{-1}\big(f(c)\big) \leq \dim \pi^{-1}\big(\pi(c)\big) \leq 3$$

on the one hand, while Lemma 1 applies to the function f on the other, showing that

$$(\forall c \in C)\ \dim f^{-1}\big(f(c)\big) = \infty.$$

This amply provides a contradiction to the supposition, and thus G/H does not contain homeomorphic copies of $\mathbb{I}^{w(G/H)}$. ∎

The statement of Proposition 2 remains intact if the cardinal $\mathfrak{c}$ is replaced by any cardinal $\mathfrak{a}$ satisfying $\aleph_0 < \mathfrak{a} \leq \mathfrak{c}$; the proof permits that generalisation without additional complications.

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