Actions of Vanishing Homogeneity Rank on Quaternionic-Kähler Projective Spaces

Lucio Bedulli and Anna Gori

Communicated by E. B. Vinberg

Abstract. We classify isometric actions of compact Lie groups on quaternionic-Kähler projective spaces with vanishing homogeneity rank. We also show that in general they are not quaternion-coisotropic.
Mathematics Subject Classification 2000: 53C26, 57S15.
Key Words and Phrases: Quaternionic-Kähler manifolds, homogeneity rank, polar actions.

1. Introduction

Let M be a smooth manifold endowed with a smooth action of a compact Lie group G. We denote by $c(G, M)$ the cohomogeneity of the action, i.e. the codimension of the principal orbits in M, and by H a principal isotropy subgroup. In [6, p. 194] Bredon proved the following inequality for the dimension of the fixed point set of a maximal torus T in G:

$$\dim M^T \leqslant c(G, M) - \operatorname{rk} G + \operatorname{rk} H$$

whenever M^T is nonempty. Drawing on this fact, Püttmann introduced in [21] the *homogeneity rank* of (G, M) as the integer

$$\operatorname{hrk}(G, M) := \operatorname{rk} G - \operatorname{rk} H - c(G, M).$$

In this paper we are interested in studying actions on quaternionic projective spaces and there are at least two reasons to consider actions with *vanishing* homogeneity rank.
A first motivation comes from the following proposition which can be deduced from [21].

Proposition 1.1. *Let M be a compact manifold with positive Euler characteristic acted on by a compact Lie group G. Then $\operatorname{hrk}(G, M) \leqslant 0$.*

Indeed quaternionic projective spaces (and more generally positive quaternionic-Kähler manifolds, see [15]) have positive Euler characteristic, thus the actions we aim to classify are those with *maximal* homogeneity rank and this fact turns out to have remarkable consequences on the geometry of the action.

Furthermore, in the symplectic framework, Hamiltonian actions with vanishing homogeneity rank have a precise geometric meaning. Let the compact Lie group G act on the symplectic manifold (M, ω) in a Hamiltonian fashion, then $\mathrm{hrk}(G, M) = 0$ if and only if every principal orbit $\mathcal{O}$ of the G-action is coisotropic, i.e. the symplectic orthogonal $(T_p\mathcal{O})^\omega$ is contained in $T_p\mathcal{O}$ (such an action is said to be *coisotropic*). If further M is compact and admits a G-invariant ω-compatible complex structure J, then (M, ω, J) turns out to be a projective algebraic *spherical variety*, that is the Borel subgroup of $G^\mathbb{C}$ has an open orbit in M (see [10]). Coisotropic actions on symplectic and Kähler manifolds have been extensively studied starting from [8] and have been classified on Hermitian symmetric spaces in [20], [4] and [3]. Linear actions with vanishing homogeneity rank have been considered by several authors: the classification in the complex case can be deduced from [11] and [5], (while absolutely irreducible real representations with vanishing homogeneity rank of compact Lie groups have been classified in [7]).

It is therefore rather natural to look for relations analogous to those found in the complex/symplectic framework in the quaternionic setting.

Let M be a complete quaternionic Kähler manifold with positive scalar curvature and Q the rank-3 sub-bundle of $\mathrm{End}\,TM$ defining the quaterionic Kähler structure. We say that a submanifold N of M is *quaternion-coisotropic* if for every $p \in N$ and $J \in Q_p$ we have $J(T_pN)^\perp \subseteq T_pN$. The reason to consider the previous definition is the fact that the principal orbits of *polar* actions on compact symmetric quaternionic-Kähler manifolds are indeed quaternion-coisotropic [23] (in the same way as polar actions on Kähler manifolds have coisotropic principal orbits [19]).

A first result of this paper is an example of an action on the quaternionic projective space with vanishing homogeneity rank which is not quaternion-cosotropic (Example 3.3). We further determine in our main theorem all the compact Lie subgroups of $\mathrm{Sp}(n)$ acting with vanishing homogeneity rank on the quaternionic projective space (see Definition 4.2 for the meaning of *minimal* action).

Theorem 1.2. *Let $\rho : G \to \mathrm{Sp}(V)$ be a n-dimensional quaternionic representation of a compact connected Lie group. Then ρ induces a minimal vanishing homogeneity rank action of G on $\mathbb{P}_\mathbb{H}(V) \simeq \mathbb{HP}^{n-1}$ if and only if one of the following is satisfied:*

1. $G = Sp(1)^{n-1}$ *and* $\rho = \rho_s \oplus \ldots \oplus \rho_s \oplus 1$, *where* $\rho_s : \mathrm{Sp}(1) \to \mathrm{Sp}(\mathbb{H})$ *is the standard representation and 1 is the trivial representation on* $\mathbb{H}$;

2. $G = H \times Sp(1)^r$ *and* $\rho = \sigma \oplus \rho_s \oplus \ldots \oplus \rho_s$, *where* $\sigma : H \to \mathrm{Sp}(W)$ *is one of the following* $4(n-r)$-*dimensional quaternionic representation:*

   (a) $H = S(\mathrm{U}(k) \times \mathrm{U}(n-r-k)) \subset \mathrm{SU}(n-r) \subset \mathrm{Sp}(n-r)$ *and* k *is odd;*

   (b) $H = S(\mathrm{U}(1)\mathrm{Sp}(k) \times \mathrm{U}(n-r-k)) \subset S(\mathrm{U}(2k) \times \mathrm{U}(n-r-2k)) \subset \mathrm{Sp}(n-r)$;

    (c) $H \times Sp(1) \curvearrowright W \otimes_{\mathbb{H}} \mathbb{H}$ is orbit equivalent to the isotropy representation of a quaternionic-Kähler symmetric space;

    (d) $H = \mathrm{Spin}(7) \otimes \mathrm{Sp}(1) \subset \mathrm{SO}(8) \otimes \mathrm{Sp}(1) \subset \mathrm{Sp}(8)$.

Note that many of these actions turn out to be non polar.

The paper is organized as follows. In Section 2 we prove several lemmas about the homogeneity rank necessary for the proof of the main theorem. Results about polar actions on Wolf spaces and an example of a vanishing homogeneity rank action which is not quaternion-coisotropic are provided in Section 3, while Section 4 is devoted to the classification actions on $\mathbb{HP}^{n-1}$ with vanishing homogeneity rank.

Finally in the Appendix one can find some tables we refer to in the course of the classification. These are taken from [12], [11], [5] and [7].

Notation. Lie groups and their Lie algebras will be indicated by capital and gothic letters respectively. Moreover the notation for the fundamental weights Λ_i of the simple Lie algebras follows the standard conventions as in [18], and we often refer to irreducible representations through the corresponding maximal weights.

2. Homogeneity rank of compact Lie group actions

In this section we are going to prove several results about the homogeneity rank which will be useful in the classification of actions with vanishing homogeneity rank on quaternionic projective spaces. On the other hand these statements have an autonomous interest since they hold in general for actions of compact Lie groups.

    The following lemma allows us to by-pass (sometimes) the computation of the principal isotropy subgroup.

Lemma 2.1. *Let G be a compact connected Lie group acting on a compact manifold M. Take $p \in M$ and denote by δ the difference $\mathrm{rk}\,G - \mathrm{rk}\,G_p$ and by Σ the slice representation at p. Then $\mathrm{hrk}(G, M) = \mathrm{hrk}(G_p, \Sigma) + \delta$. In particular if the G-orbit through p has positive Euler characteristic, then the action of G on M has vanishing homogeneity rank if and only if the slice representation at p does.*

Proof. Since the action of G on M is proper, it is known that at every point the slice representation has the same cohomogeneity as that of the action of G on M. Let Σ be the slice for the action at p. Let $q \in \Sigma$ be principal both for the G-action on M and the G_p-action on Σ (which is equivalent to the slice representation). Obviously $(G_p)_q = G_q = G_{\mathrm{princ}}$. Thus

$$
\begin{aligned}
\mathrm{hrk}(G, \Sigma) &= \mathrm{rk}\,G_p - \mathrm{rk}\,(G_p)_q - c(G_p, \Sigma) \\
&= \mathrm{rk}\,G - \delta - \mathrm{rk}\,G_q - c(G, M) = \mathrm{hrk}(G, M) - \delta,
\end{aligned}
$$

and the conclusion follows. The last statement is a consequence of the well known fact that the homogeneous space G/G_p has positive Euler characteristic if and only if $\mathrm{rk}\,G = \mathrm{rk}\,G_p$.   ■

    In order to achieve the classification, the following result proved in [7, Proposition 2] will be essential.

Proposition 2.2. *Let G' be a closed subgroup of the Lie group G acting smoothly and properly on M. Then $\mathrm{hrk}(G', M) \leqslant \mathrm{hrk}(G, M)$.*

One of its immediate consequences is the following property of the homogeneity rank of reducible representations.

Lemma 2.3. *Let $\rho_i \colon G \to \mathrm{GL}(V_i)$ $(i = 1, 2)$ be two finite-dimensional representations of the compact Lie group G. Then $\mathrm{hrk}(G, V_1 \oplus V_2) \leqslant \mathrm{hrk}(G, V_1) + \mathrm{hrk}(G, V_2)$.*

Proof. Let v_i be a principal point of (G, V_i) for $i = 1, 2$. Denote by $\mathcal{O}_i = G/H_i$ the corresponding orbits.
Now consider the action of G on $V_1 \oplus V_2$. The slice representation at $(v_1, 0)$ is $V_2 \oplus U$ where (G, V_2) is the original action and U is a trivial G-module of dimension $c(G, V_1)$. Now $(v_2, 0)$ is obviously principal for the slice representation, so that a principal isotropy subgroup H of $(G, V_1 \oplus V_2)$ is $(H_1, V_2)_{\mathrm{princ}}$. Thus we have $c(G, V) = c(G, V_1) + c(H_1, V_2)$ and therefore

$$
\begin{aligned}
\mathrm{hrk}(G, V) &= \mathrm{rk}\, G - \mathrm{rk}\, H - c(G, V) \\
&= \mathrm{rk}\, G - \mathrm{rk}\, (H_1, V_2)_{\mathrm{princ}} - c(H_1, V_2) - c(G, V_1) \\
&= \mathrm{rk}\, G - \mathrm{rk}\, H_1 + \mathrm{hrk}(H_1, V_2) - c(G, V_1) \\
&= \mathrm{hrk}(G, V_1) + \mathrm{hrk}(H_1, V_2) \leqslant \mathrm{hrk}(G, V_1) + \mathrm{hrk}(G, V_2)
\end{aligned}
$$

This concludes the proof. ∎

Another important tool in the classification carried out in Section 4 will be the following proposition which generalizes, in the case of positive Euler characteristic, the *Restriction Lemma* given in [10] for complex G-invariant submanifolds of Hamiltonian isometric actions on compact Kähler manifolds.

Proposition 2.4. *Let G be a compact connected Lie group acting by isometries on a compact Riemannian manifold M. Let Y be a compact G-invariant submanifold of M such that $\chi(Y) > 0$. If $\mathrm{hrk}(G, M) = 0$, then $\mathrm{hrk}(G, Y) = 0$.*

Proof. Let $\nu_M Y$ be the normal bundle to Y in M. Since Y is compact, we can use the invariant version of the tubular neighborhood theorem (see e.g. [6, p. 306]) to get a G-equivariant diffeomorphism of an open G-invariant neighborhood U of the zero section of $\nu_M Y$ onto an open G-invariant neighborhood W of Y in M. Now, since W is open in M and G acts with vanishing homogeneity rank on M, the G-action on W has vanishing homogeneity rank too, hence also $\mathrm{hrk}(G, U) = 0$. Consider now the restriction of the natural projection $\pi_{|U} \colon U \to Y$. Let $y \in Y$ be such that

1. y is principal for the action of G on Y;

2. $F := \pi_{|U}^{-1}(y) \subset U$ has non-empty intersection with M_{princ}.

Now consider the action of G_y on F and take $x \in F$ such that

1. $x \in M_{\mathrm{princ}}$;

2. x is principal for the action of G_y on F.

Since the action of G_y on $F \cong \nu_M(Y)_y$ is linear, the homogeneity rank of this action is non-positive ([21, corollary 1.2]), i.e.

$$\dim F \geqslant \dim G_y - \dim G_x + \operatorname{rk} G_y - \operatorname{rk} G_x.$$

Thus we can compute

$$
\begin{aligned}
c(G, Y) &= \dim Y - \dim G + \dim G_y = \dim X - \dim F - \dim G + \dim G_y \\
&\leqslant \dim X - (\dim G_y - \dim G_x + \operatorname{rk} G_y - \operatorname{rk} G_x) - \dim G + \dim G_y \\
&= c(G, X) - \operatorname{rk} G_y + \operatorname{rk} G_x = \operatorname{rk} G - \operatorname{rk} G_y,
\end{aligned}
$$

so that $\operatorname{hrk}(G, Y) \geqslant 0$. On the other hand the positive Euler characteristic of Y obstructs actions with positive homogeneity rank (Proposition 1.1) and the claim follows. ∎

In the case of the quaternionic projective space we deduce also the following useful consequence

Corollary 2.5. *Let G_1 and G_2 be closed subgroups respectively of $\operatorname{Sp}(n_1)$ and $\operatorname{Sp}(n_2)$. Assume that the action of $G = G_1 \times G_2$ on $\mathbb{P}_{\mathbb{H}}(\mathbb{H}^{n_1} \oplus \mathbb{H}^{n_2}) \simeq \mathbb{HP}^{n_1+n_2-1}$ has vanishing homogeneity rank. Then G_i acts on $\mathbb{P}_{\mathbb{H}}(\mathbb{H}^{n_i}) \simeq \mathbb{HP}^{n_i-1}$ with homogeneity rank zero.*

Proof. Simply take two non-zero vectors v_1 and v_2 respectively in $\mathbb{H}^{n_1}$ and $\mathbb{H}^{n_2}$ and consider the orbits $\mathcal{O}_i = G \cdot [v_i] \simeq \mathbb{HP}^{n_i-1}$. Now apply Proposition 2.4 to the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. ∎

3. Quaternion-coisotropic actions and the vanishing of the homogeneity rank

In order to introduce the right notion of "coisotropic" actions in the quaternionic setting, it is necessary to fix some notation. Let (M, g) be a Riemannian manifold and ∇ its Levi-Civita connection. A quaternionic-Kähler structure on M is a ∇-parallel rank 3 subbundle Q of $\operatorname{End} TM$, which is *locally* generated by a triple of locally defined anticommuting g-orthogonal almost complex structures $(J_1, J_2, J_3 = J_1 J_2)$. Recall that a quaternionic-Kähler manifold is automatically Einstein, hence if its scalar curvature is positive it is automatically compact. Here we consider only *positive* quaternionic-Kähler manifolds.

A submanifold N of M will be called *quaternion-coisotropic* if for every $p \in N$ and $J \in Q_p$ we have $J(T_p N)^\perp \subseteq T_p N$. Trying to seek the analogy with the symplectic context, it is rather natural to consider the following situation.

Definition 3.1. Let (M, g, Q) be a quaternionic-Kähler manifold. We say that the action of a compact Lie group of isometries of M is *quaternion-coisotropic* if the principal orbits are quaternion-coisotropic submanifolds of M.

Recall that an isometric action of a compact Lie group G on a Riemannian manifold M is said to be polar if there is an embedded submanifold Σ (a *section*) which meets all principal orbits orthogonally. In [23] it is proved, using the classification results of [19] and [13], that quaternion-coisotropic actions generalize polar actions on Wolf spaces [23, Theorem 4.10] in the same manner as coisotropic actions generalize polar actions on compact Kähler manifolds ([20]). The classification of polar actions on quaternion projective space has been obtained by Podestà and Thorbergsson up to orbit equivalence. Recall that two isometric actions of two Lie groups G and G' on a Riemannian manifold M are called *orbit equivalent* if there exists an isometry of M which maps G-orbits onto G'-orbits. Here we restate the classification theorem because in the statement of [19] a (trivial) case is missing.

Theorem 3.2.   *[19] The isometric action of a compact Lie group G on $\mathbb{HP}^{n-1}$ is polar if and only if it is orbit equivalent to the action induced by a n-dimensional quaternionic representation $\rho_1 \oplus \ldots \oplus \rho_k$ where ρ_i is the isotropy representation of a quaternionic-Kähler symmetric space of rank one for $i = 1, \ldots, k-1$ and ρ_k is one of the following:*

1. *the isotropy representation of a quaternionic-Kähler symmetric space of arbitrary rank;*

2. *the trivial representation on a 1-dimensional quaternionic module $\mathbb{H}$.*

Note that the missing case (this including a trivial module) is easily seen to be quaternion-coisotropic.

In spite of these analogies, the parallel with the symplectic setting does not go further, indeed we have the following

Example 3.3.   Consider the action of $G = \mathrm{U}(k) \times \mathrm{U}(n-k) \subset \mathrm{U}(n) \subset \mathrm{Sp}(n)$ on $M = \mathbb{HP}^{n-1}$. It is not hard to see that, for $k \geqslant 3$, the Lie algebra of principal isotropy subgroup is isomorphic to $\mathfrak{u}(k-2) \oplus \mathfrak{u}(n-k-2)$ whence the cohomogeneity of the action is 4 and the homogeneity rank vanishes. Suppose now that the principal orbits are quaternion-coisotropic and consider the lifted action of G on the twistor space $Z = \mathbb{CP}^{2n-1}$ (It is the S^2-bundle associated to Q, see [22]). In general, when we lift an isometric action with $\mathrm{hrk} = 0$ of a compact Lie group on a positive quaternionic-Kähler manifold, three cases may occur according to the cohomogeneity of the action of a principal isotropy subgroup G_p on the twistor line $Z_p \simeq \mathbb{CP}^1$:

1. The action of G_p on Z_p is transitive. In this case $c(G, Z) = c(G, M)$. Furthermore a G-principal orbit of Z is G/G_z. If we take into account the homogeneous fibration $G/G_z \to G/G_p$ where $G_p/G_z = S^2$ and the fact that S^2 has positive Euler characteristic, we have $\mathrm{rk}\, G_z = \mathrm{rk}\, G_p$, and we can compute

$$
\begin{aligned}
\mathrm{hrk}(G, Z) &= \mathrm{rk}\, G - \mathrm{rk}\, G_z - c(G, Z) \\
&= \mathrm{rk}\, G - \mathrm{rk}\, G_p - c(G, M) = 0
\end{aligned}
$$

2. The action of G_p on Z_p has cohomogeneity one. Again, if z is principal for the G_p-action on Z_p, then it is principal for the G-action on Z. Furthermore

the homogeneous fibration $G/G_z \to G/G_p$ has fibre $S^1 = G_p/G_z$, hence $\operatorname{rk} G_z = \operatorname{rk} G_p - 1$. Now, taking into account that $c(G, Z) = c(G, M) + 1$, by a dimensional computation we obtain that also in this case the G-action on Z is coisotropic.

3. The connected component of the identity of G_p acts trivially on Z_p. In this case the G-action on Z is no more coisotropic.

One easily verifies that in our case the homogenity rank of the lifted action is -2, that is the G-action on Z is not coisotropic. This implies that the connected component H of G_p acts trivially on the twistor line $Z_p \simeq \mathbb{CP}^1$. Now denote by ν the normal space to the G-orbit at p which is acted on trivially by H. On the other hand, if J_1, J_2, J_3 are three generators of the algebra Q_p, these are fixed by H thanks to the argument above. Thus H pointwisely fixes the three 4-dimensional mutually orthogonal subspaces $J_1\nu$, $J_2\nu$ and $J_3\nu$ of $T_p G \cdot p$. But this is impossible since we claim that a subspace of $T_p G \cdot p$ fixed by H has dimension 8. Indeed in correspondence to a principal point we have the following reductive decomposition:

$$\mathfrak{u}(k) \oplus \mathfrak{u}(n-k) = \mathfrak{u}(k-2) \oplus \mathfrak{u}(n-k-2) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{k-2}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{n-k-2}).$$

Hence we can identify the tangent space to the principal orbit with

$$\mathfrak{u}(2) \oplus \mathfrak{u}(2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{k-2}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{n-k-2})$$

on which $\mathfrak{u}(k-2) \oplus \mathfrak{u}(n-k-2)$ acts. Then H fixes the two copies of $\mathfrak{u}(2)$, thus has dimension 8, as claimed.

4. Actions with vanishing homogeneity rank on $\mathbb{HP}^{n-1}$: proof of Theorem 1.2

The entire section is devoted to prove Theorem 1.2. The strategy of the classification is based upon the following lemma which is a consequence of Proposition 2.2.

Lemma 4.1. *Let G be a compact Lie group acting by isometries on a compact quaternionic-Kähler manifold M. If G' is a closed subgroup of G acting on M with $\operatorname{hrk}(G', M) = 0$, the same is true for G.*

Proof. Every compact quaternionic-Kähler manifold has positive Euler characteristic (see [15, Theorem 0.3]). As already observed this forces the homogeneity rank to be non-positive. The claim follows applying Proposition 2.2. ∎

Thus it is natural to state the following

Definition 4.2. A G-action with vanishing homogeneity rank on a manifold M is said to be *minimal* if no closed subgroup G' of G acts on M with vanishing homogeneity rank.

From now on we fix $M = \mathbb{HP}^{n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(1)\mathrm{Sp}(n-1)$ and we aim to classify minimal subgroups G of $\mathrm{Sp}(n)$ with $\mathrm{hrk}(G, M) = 0$.

Since the identity component of $\mathrm{Iso}(\mathbb{HP}^{n-1}, g)$ is $\mathrm{Sp}(n)$, we go through all the closed subgroups of it, starting from the maximal ones and then analysing only the subgroups of those giving rise to vanishing homogeneity rank actions.

We proceed, in some sense, by strata: the first level is made by the maximal connected subgroups of $\mathrm{Sp}(n)$, then we pass to the maximal connected subgroups of the groups of the previous level and so on. To this end we enclose in the appendix the list of maximal connected subgroups of classical Lie groups. Note that not all the images of irreducible representations of simple Lie groups are maximal, but this fact will not interfere with our proof.

Before starting the classification we state two more lemmas holding for manifolds with positive Euler characteristic:

Lemma 4.3. *Cohomogeneity one G-actions on a manifold M with $\chi(M) > 0$ have* $\mathrm{hrk}(G, M) = 0$.

Proof. In this case $\mathrm{rk}\, G - \mathrm{rk}\, G_{\mathrm{princ}} \leqslant 1$ and cannot be zero since otherwise the homogeneity rank would be odd which is impossible since $\mathrm{hrk}(G, M)$ has the same parity as $\dim_{\mathbb{R}} M$ (see [21, Theorem A]). ∎

The following is immediate.

Lemma 4.4. *A necessary condition for an action of G on M with $\chi(M) > 0$ to have vanishing homogeneity rank is that*

$$\dim G + \mathrm{rk}\, G \geqslant \dim M. \tag{1}$$

Due to the characterization of Lemma 2.1 of actions with vanishing homogeneity rank in terms of representations, we will make extensive use of the classification results on (irreducible) representations with $\mathrm{hrk} = 0$.

Vanishing homogeneity rank representations admitting an invariant complex structure are exactly *multiplicity free* (or *coisotropic*) representations, which have been classified by Kac [11] (the irreducible ones) and by Benson and Ratcliff [5] and Leahy [14] (the reducible ones). Their results are resumed in Tables 1,2 and 3,4 respectively. These tables need some explanation. First of all in Table 3 it is meant that $\mathrm{U}(1) \times G$ acts on $V = V_1 \oplus V_2$ as follows

$$(z, g) \cdot (v_1, v_2) = g(z^a v_1, z^b v_2),$$

where $a, b \in \mathbb{Z}$ are fixed, and in Table 4 $\mathrm{U}(1) \times \mathrm{U}(1) \times G$ acts on $V = V_1 \oplus V_2$ as follows

$$(z_1, z_2, g) \cdot (v_1, v_2) = g(z_1 v_1, z_2 v_2).$$

Moreover the classification of reducible multiplicity free representations refers only to *indecomposable* ones. Recall that a representation $\rho : G \to V$ of a Lie group G is indecomposable if we cannot write V as the direct sum $V = V_1 \oplus V_2$ of proper $\rho(G)$-invariant subspaces in such a way that $\rho(G) = \rho_1(G) \times \rho_2(G)$, where ρ_i denotes the restriction to V_i.

Vanishing homogeneity rank irreducible representations admitting no invariant complex structure (absolutely irreducible real representations) have been classified in

Theorem 4.5. *[7] An absolutely irreducible representation ρ of a compact connected Lie group G has vanishing homogeneity rank if and only if it is either orbit-equivalent to the isotropy representation af a non-Hermitian symmetric space of inner type or is one of the representations listed in Table 5.*

Finally in order to clarify our procedure we make one more observation. When considering the irreducible representations of simple Lie groups one must often check the dimensional condition (1) or a variation of it. This is made easier by the fact that if $(c_1, \ldots, c_n)$ are the coefficients of the maximal weights of the representation of a rank n simple Lie group, then the function

$$(c_1, \ldots, c_n) \mapsto \deg(\rho_{(c_1, \ldots, c_n)}),$$

is strictly monotonic, i.e. if ρ and ρ' are two irreducible representations of a simple compact Lie group with highest weights λ and λ', given by $(c_1, \ldots, c_n)$ and $(c'_1, \ldots, c'_n)$ respectively, and if $c_i \leqslant c'_i$ for all i and $c_i < c'_i$ for at least one i, then $\deg \rho < \deg \rho'$ (see [17]). Then, in many cases it is sufficient to test the dimensional condition for the fundamental representations, and go further only if the condition is satisfied.

4.1. Maximal subgroups of $\mathrm{Sp}(n)$

4.1.1. $G = \mathrm{U}(n)$. The action of $\mathrm{U}(n)$ on $\mathbb{HP}^{n-1}$, has cohomogeneity 1, thus has vanishing homogeneity rank.

4.1.2. $G = \mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$ $(1 \leqslant k \leqslant n)$. These subgroups act by cohomogeneity 1 on $\mathbb{HP}^{n-1}$, thus hrk $= 0$.

4.1.3. $G = \mathrm{SO}(p) \otimes \mathrm{Sp}(q)$ $(n = pq, \ p \geqslant 3, q \geqslant 1)$.For $q \geqslant 2$ we can compute the slice representation at the quaternionic line ℓ spanned by a pure element of $\mathbb{R}^p \otimes \mathbb{R}^{4q}$. The algebra of the stabilizer is $\mathfrak{o}(p-1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(q-1)$ acting on

$$\mathbb{R}^{p-1} \otimes (U \oplus \mathbb{R}^{4(q-1)}) \tag{2}$$

where U can be seen as the 3-dimensional vector space of the imaginary quaternions on which $\mathrm{Sp}(1)$ acts by conjugation (see [12, p. 590]). Note that $\mathrm{Sp}(1)$ acts also on $\mathbb{R}^{4(q-1)} \cong \mathbb{H}^{q-1}$ by right multiplication. If p is odd the G-orbit of ℓ has positive Euler characteristic so we can easily rule out this case by observing that the irreducible factor $\mathbb{R}^{p-1} \otimes \mathbb{H}^{q-1}$, regarded as a *complex* representation, does not appear in Table 1.

So we are left to consider the cases in which p is even. To get rid of the action on the slice of the unitary quaternions, let us consider the stabilizer of a principal element of $\mathbb{R}^{p-1} \otimes U$: such an element is of the form $v_1 \otimes i + v_2 \otimes j + v_3 \otimes k$, where v_1, v_2, v_3 are linear independent elements of $\mathbb{R}^{p-1}$. Here the algebra of the stabilizer H is $\mathfrak{o}(p-4) \oplus \mathfrak{sp}(q-1)$ and the slice contains as a direct summand the tensor product of the standard representations $V = \mathbb{R}^{p-4} \otimes \mathbb{R}^{4(q-1)}$. Since at this level $\delta = \mathrm{rk}\, G - \mathrm{rk}\, H = 3$, applying Lemma 2.1, in order to exclude also this case it is enough to show that $\mathrm{hrk}(H, V) \leqslant -4$. Indeed this is easy to verify once we subdivide into three more subcases and we compute explicitly the principal isotropy of H on V. If $q \geqslant p-4$ then $\mathfrak{h}_{\mathrm{princ}} = \mathfrak{sp}(q-p+3)$; if $p-6 \leqslant q \leqslant p-3$

then $\mathfrak{h}_{\mathrm{princ}}$ is trivial; if $q \leqslant p - 8$ then $\mathfrak{h}_{\mathrm{princ}} = \mathfrak{sp}(p - q - 6)$ (see e.g. [9, p. 202]) and in all these cases $\mathrm{hrk}(H, V) \leqslant -4$ (note that the equality holds only if $q = 1$). For $q = 1$ this is the action on $\mathbb{HP}^{p-1}$ induced by the isotropy representation of the quaternionic-Kaehler symmetric space $\mathrm{SO}(p + 4)/\mathrm{SO}(p) \times \mathrm{SO}(4)$, thus it is polar by Theorem 3.2. To determine whether it has vanishing homogeneity rank or not we have to distinguish according to the parity of p. If p is odd the G-orbit of ℓ has positive Euler characteristic, the slice representation at ℓ is real and appears in the classification of Theorem 4.5 since it is orbit equivalent to the isotropy representation of the real Grassmannian of 3-planes in $\mathbb{R}^{p+2}$. Thus it has vanishing homogeneity rank. When p is even, at the first step the slice is given by $\mathbb{R}^{p-1} \otimes U$ and with easy computations we find that the principal isotropy is $\mathfrak{h}_{\mathrm{princ}} = \mathfrak{sp}(p - 4)$, $c = 3$ hence $\mathrm{hrk} = 0$.

4.1.4. $G = \rho(H)$ *with ρ complex irreducible representation of quaternionic type of the simple Lie group H.* In this case the dimensional condition that should be satisfied becomes $\dim H + \mathrm{rk}\, H \geqslant 2 \deg \rho - 4$. Going through all the representations of this type, the following cases remain:

1. the representation of $\mathrm{SU}(6)$ on $\Lambda^3 \mathbb{C}^6$;

2. the representation of $\mathrm{Sp}(3)$ with maximal weight $(0, 0, 1)$;

3. the spin representation of $\mathrm{Spin}(11)$;

4. the two half-spin representations of $\mathrm{Spin}(12)$;

5. the standard representation of E_7 on $\mathbb{C}^{56}$.

6. the standard representation of $\mathrm{SU}(2)$;

Except for $\mathrm{SU}(2)$, that gives rise to a homogeneity rank zero action, since it has cohomogeneity one on $\mathbb{HP}^1 \simeq S^4$, the other cases can be treated using the fact that all of them admit a totally complex orbit (see [2] and also [1]). These totally complex submanifolds are Hermitian symmetric spaces, and therefore have positive Euler characteristic. Thus we compute the slice representation on these orbits, obtaining:

1. $\mathrm{SU}(3) \times \mathrm{SU}(3) \cdot \mathrm{U}(1)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}$;

2. $\mathrm{SU}(3) \cdot \mathrm{U}(1)$ on $S^2(\mathbb{C}^3) \otimes \mathbb{C}$;

3. $\mathrm{SU}(5) \cdot \mathrm{U}(1)$ on $\Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}$;

4. $\mathrm{SU}(6) \cdot \mathrm{U}(1)$ on $\Lambda^2(\mathbb{C}^6) \otimes \mathbb{C}$;

5. $E_6 \cdot \mathrm{U}(1)$ on $\mathbb{C}^{27} \otimes \mathbb{C}$.

These all give rise to vanishing homogeneity rank actions, since they are all multiplicity free (see Tables 1 and 2).
Let us remark that all of these actions on the quaternionic projective space are polar.

4.2.   The subgroups of $U(n) \subset Sp(n)$**.** First note that the maximal compact connected subgroups of $U(n)$ are $SU(n)$ and those of the form $Z \cdot H$ where Z is the center of $U(n)$ and H is a maximal compact connected subgroup of $SU(n)$ (see Table 7).

Certainly $SU(n)$ has vanishing homogeneity rank on $\mathbb{HP}^{n-1}$ since it has the same orbits of $U(n)$, so let us go through the remaining cases.

4.2.1. $G = Z \cdot S(U(k) \times U(n-k)) = U(k) \times U(n-k)$. We start by computing the slice representation at the class of the identity in $Sp(n)/Sp(1)Sp(n-1)$. The stabilizer is given by the intersection of G with $Sp(1)Sp(n-1)$. In this way we get $U(1) \times U(k-1) \times U(n-k)$ acting on the slice

$$\Sigma = (\mathbb{C}^* \otimes (\mathbb{C}^{k-1})^*) \oplus (\mathbb{C}^* \otimes (\mathbb{C}^{n-k})^*) \oplus (\mathbb{C}^* \otimes \mathbb{C}^{n-k}).$$

Now it is immediate to see that the principal isotropy group is isomorphic to $U(k-2) \times U(n-k-2)$ so that the cohomogeneity is 4 and the action has vanishing homogeneity rank.

Remark 4.6.   *Observe that the slice representation we just considered is complex, indecomposable and has vanishing homogeneity rank, though it does not appear in the classification of Benson and Ratcliff (Tables 3 and 4). In fact they consider only representations (G, V) which are indecomposable for the* semisimple *part of G.*

4.2.2. $G = Z \cdot Sp(k)$ with $n = 2k$. Proceeding as before we determine the orbit through the class of the identity in $Sp(n)/Sp(1)Sp(n-1)$. Again we get an orbit with positive Euler characteristic, more precisely the Lie algebra of the isotropy is $\mathfrak{z} \oplus \mathfrak{u}(1) \oplus \mathfrak{sp}(k-1)$ and the slice representation is given by $\mathbb{H}^{k-1} \oplus \mathbb{C}$, where the 1-dimensional factor $\mathfrak{z}$ acts (non-trivially) only on $\mathbb{C}$ and $\mathfrak{u}(1)$ acts by scalar multiplication on $\mathbb{H}^{k-1}$. Thus the algebra of the principal isotropy is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{sp}(k-2)$ and $\mathrm{hrk}(G, \mathbb{HP}^{n-1}) = 0$.

Note that the action of the center here is essential: Once the action of Z is removed, there is a trivial module in the slice representation. Therefore $Sp(k) \subset SU(2k) \subset Sp(2k)$ does not have $\mathrm{hrk} = 0$ on $\mathbb{HP}^{n-1}$.

4.2.3. $G = Z \cdot SO(n)$. First consider the totally complex orbit of $U(n) \supset SO(n)$ which is $\mathbb{CP}^{n-1}$ canonically embedded in $\mathbb{HP}^{n-1}$. This orbit in its turn contains a Lagrangian G-orbit ($\mathbb{RP}^{n-1}$ canonically embedded). Here the 1-dimensional factor of the isotropy $\mathfrak{z} \oplus \mathfrak{o}(n-1)$ acts on the slice $\mathbb{R}^{n-1} \oplus \mathbb{C}^{2(n-1)} \otimes \mathbb{C}^*$ only on the second module. From this one easily sees that $\mathfrak{g}_{\mathrm{princ}} \simeq \mathfrak{o}(n-4)$ and the cohomogeneity is therefore 5. Thus $\mathrm{hrk}(G, \mathbb{HP}^{n-1}) = -2$ and the action has non-zero homogeneity rank.

4.2.4. $G = Z \cdot SU(p) \otimes SU(q)$ ($n = pq$ and $p, q \geqslant 2$). Here G acts on $\mathbb{P}_{\mathbb{H}}(\mathbb{C}^p \otimes \mathbb{C}^q \oplus (\mathbb{C}^p \otimes \mathbb{C}^q)^*)$. The orbit through the quaternionic line spanned by a pure element of $\mathbb{C}^p \otimes \mathbb{C}^q$ is the product of two complex projective spaces $\mathbb{CP}^{p-1} \times \mathbb{CP}^{q-1}$ and therefore has positive Euler characteristic. So we are in a position to apply the criterion deriving from Lemma 2.1. The slice representation contains the module

$\mathbb{C}^{p-1} \otimes \mathbb{C}^{q-1} \oplus (\mathbb{C}^{p-1} \otimes \mathbb{C}^{q-1})^*$ on which $\mathfrak{z} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(p-1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(q-1)$ acts. If $p \geqslant 3$ this module does not appear in the classification in Tables 3 and 4, thus the corresponding action has non-zero homogeneity rank. The case $p = 2$ is left to consider: If $q \leqslant 5$ the dimensional condition (1) is not even satisfied, if $q \geqslant 6$ it is easy to find directly that the principal isotropy is $\mathfrak{su}(q-4)$, so that the homogeneity rank is -2.

4.2.5. $G = Z \cdot \rho(H)$ *with ρ irreducible representation of complex type of the simple Lie group H.* If G acts with vanishing homogeneity rank on $\mathbb{HP}^{n-1}$ then, by Proposition 2.4, it acts *coisotropically* on the G-invariant totally complex submanifold $L = \mathbb{CP}^{n-1} = \mathrm{U}(n)/\mathrm{U}(1) \times \mathrm{U}(n-1)$ and, since Z acts trivially on L this is in turn equivalent to the fact that the representation of $\rho(H)^{\mathbb{C}} \times \mathbb{C}^*$ on $\mathbb{C}^n$ is multiplicity free. Using Tables 1 and taking only the representations of complex type we get the standard representation of $\mathrm{SU}(n)$, the representations of $\mathrm{SU}(n)$ on $\Lambda^2(\mathbb{C}^n)$ with $n \geqslant 5$ and on $S_0^2(\mathbb{C}^n)$, the half-spin representation of $\mathrm{Spin}(10)$, the standard representation of E_6 on $\mathbb{C}^{26}$. We have to consider those representations of complex type satisfying the dimensional condition that in this case becomes $\dim H + \mathrm{rk}\, H \geqslant 4 \deg \rho - 6$. The only remaining case is the first one and has already been treated in subsection 4.2.

4.3.   **The subgroups of** $\mathrm{U}(k) \times \mathrm{U}(n-k) \subset \mathrm{U}(n)$**.** Except the diagonal subgroup (when $2k = n$), the maximal compact connected subgroups of $\mathrm{U}(k) \times \mathrm{U}(n-k)$ are $\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$ and those of the form $H \times \mathrm{U}(n-k)$ where H is a maximal compact connected subgroup of $\mathrm{U}(k)$. For the subgroups of this form we can apply Corollary 2.5 arguing that H must necessarily act with vanishing homogeneity rank on $\mathbb{HP}^{k-1}$. Thus for H we have only two possibilities: either $H = \mathrm{U}(k_1) \times \mathrm{U}(k_2)$ (with $k_1 + k_2 = k$) or $H = Z \cdot \mathrm{Sp}(k/2)$ (when k is even).

4.3.1. $H = \mathrm{U}(k_1) \times \mathrm{U}(k_2)$. We can exploit the previous computations and consider the orbit $\mathbb{CP}^{k_1-1} \subset \mathbb{CP}^{k-1} \subset \mathbb{CP}^{n-1} \subset \mathbb{HP}^{n-1}$; so the slice representation is given by
$$\mathbb{C}^* \otimes ((\mathbb{C}^{k_1-1})^* \oplus \mathbb{C}^{k_2} \oplus (\mathbb{C}^{k_2})^* \oplus \mathbb{C}^{n-k} \oplus (\mathbb{C}^{n-k})^*).$$
on which $\mathrm{U}(1) \times \mathrm{U}(k_1 - 1) \times \mathrm{U}(k_2) \times \mathrm{U}(n-k)$ acts. Analogously to a previous case it is easy to see that the principal isotropy group is isomorphic to $\mathrm{U}(k_1 - 2) \times \mathrm{U}(k_2 - 2) \times \mathrm{U}(n-k-2)$ so that the cohomogeneity is 8 and the action has homogeneity rank equal to -2.

4.3.2. $H = Z \cdot \mathrm{Sp}(k/2)$. We can compute the slice representation at the class of the identity in
$$\mathrm{Sp}(n)/\mathrm{Sp}(1)\mathrm{Sp}(n-1).$$
The intersection of $\mathfrak{g}$ with $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n-1)$ is $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{sp}(k/2-1) \oplus \mathfrak{u}(n-k)$ acting on the slice
$$\mathbb{C}^{n-k} \oplus (\mathbb{C}^{n-k})^* \oplus \mathbb{H}^{k/2-1} \oplus \mathbb{C},$$
where one of the two 1-dimensional copies of $\mathfrak{u}(1)$ acts on every module and the other only on the first two modules. Now it is immediate to see that the principal isotropy subalgebra is isomorphic to $\mathfrak{sp}(k/2-2) \oplus \mathfrak{u}(n-k-2)$ so that the cohomogeneity is 5 and the action has vanishing homogeneity rank.

4.3.3. $G = \mathrm{U}(k)_\Delta \subset \mathrm{U}(k) \times \mathrm{U}(k)$ *with* $n = 2k$. In order to conclude that $\mathrm{U}(k)_\Delta$ has non-zero homogeneity rank on $\mathbb{HP}^{n-1}$ it is sufficient to observe that $\mathrm{U}(k)_\Delta \subset \mathrm{Sp}(k)_\Delta \subset \mathrm{Sp}(k) \times \mathrm{Sp}(k)$ and that the action of $\mathrm{Sp}(k)_\Delta$ on $\mathbb{HP}^{n-1}$ is equivalent to that of $\mathrm{Sp}(k) \subset U(2k)$ since the standard representation of $\mathrm{Sp}(k)$ on $\mathbb{C}^{2k}$ is self-dual.

4.4. **The subgroups of** $G = Z(\mathrm{U}(n)) \cdot \mathrm{Sp}(k) \subset \mathrm{U}(n)$ **(with** $n = 2k$**).** Now we are going to show that the action of $Z(\mathrm{U}(n)) \cdot \mathrm{Sp}(k) \subset \mathrm{U}(n)$ is minimal as vanishing homogeneity rank action. The maximal compact connected subgroups of G other than $\mathrm{Sp}(k)$ (that we have considered in a previous step) are of the form $Z \cdot H$ where H is a maximal compact connected subgroup of $\mathrm{Sp}(k)$.

4.4.1. $H = U(k)$. As for this subgroup the conclusion follows immediately from the observation that $Z \cdot \mathrm{U}(k)$ is contained in $Z \cdot \mathrm{SO}(2k)$ which does not act with vanishing homogeneity rank on $\mathbb{HP}^{n-1}$.

4.4.2. $H = \mathrm{SO}(p) \otimes \mathrm{Sp}(q)$ *with* $2pq = n$. If $Z \cdot H$ acts with vanishing homogeneity rank on $\mathbb{HP}^{n-1}$, then it should act coisotropically on the totally complex $\mathrm{U}(2pq)$-orbit $\mathbb{CP}^{2pq-1}$, but this is not the case as one can deduce from Tables 1,2 3 and 4.

4.4.3. $H = \rho(H') \subset \mathrm{Sp}(k)$ *where* ρ *is an irreducible representation of quaternionic type of the simple Lie group* H'. We can argue as in subsection 4.2.5, that is we apply our version of restriction lemma combined with the classification of Kac. In this way we find no proper subgroup H of $\mathrm{Sp}(k)$.

4.4.4. $H = \mathrm{Sp}(r) \times \mathrm{Sp}(k-r)$ *with* $1 \leqslant r \leqslant k-1$. Here it is sufficient to note that $Z(U(2k)) \cdot H$ is a subgroup of $Z(\mathrm{U}(r)) \cdot \mathrm{Sp}(r) \times Z(\mathrm{U}(k-r)) \cdot \mathrm{Sp}(k-r)$ whose action on $\mathbb{HP}^{n-1}$ has non-zero homogeneity rank.

4.5. **The subgroups of** $Z(U(k)) \cdot \mathrm{Sp}(r) \times \mathrm{U}(n-k) \subset \mathrm{U}(n)$ **with** $k = 2r$. Now we prove that the vanishing homogeneity rank action of $Z(U(2r)) \cdot \mathrm{Sp}(r) \times \mathrm{U}(n-2r)$ is minimal. Since the action of $Z(U(2r)) \cdot \mathrm{Sp}(r)$ is minimal, by Proposition 2.4, the only subgroups we need to consider are of the form $Z(U(2r)) \cdot \mathrm{Sp}(r) \times H$, where H is a maximal compact connected subgroup of $\mathrm{U}(n-2r)$ acting with vanishing homogeneity rank on $\mathbb{HP}^{n-2r-1}$. There are three possibilities for H: $H_1 = \mathrm{U}(k_1) \times \mathrm{U}(k_2)$ with $k_1 + k_2 = n - 2r$, $H_2 = Z(U(n-2r)) \cdot Sp(\frac{n-2r}{2})$ (when n is even), $H_3 = \mathrm{SU}(n-2r)$. The subgroup $Z(U(2r)) \cdot \mathrm{Sp}(r) \times H_1$ is contained in $\mathrm{U}(2r) \times \mathrm{U}(k_1) \times \mathrm{U}(k_2)$, hence its action has non-zero homogeneity rank.
The subgroup $Z(U(2r)) \cdot \mathrm{Sp}(r) \times H_2$ need to be treated explicitly, finding the intersection of it with $\mathrm{Sp}(1)\mathrm{Sp}(n-1)$. In this way we get the isotropy subalgebra $\mathfrak{l} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{sp}(r-1) \oplus \mathfrak{sp}(n/2-r)$ acting on the slice

$$\mathbb{H}^{r-1} \oplus \mathbb{H}^{n/2-r} \oplus \mathbb{H}^{n/2-r} \oplus \mathbb{C} \oplus \mathbb{C}.$$

Since the abelian subalgebra of $\mathfrak{l}$ acts on the 1-dimensional modules, this action has vanishing homogeneity rank on each *irreducible* submodule, nevertheless it is easy to see that the principal isotropy is $\mathfrak{sp}(r-2) \oplus \mathfrak{sp}(n/2-r-2)$. Therefore the cohomogeneity is 8 and $\mathrm{hrk}(Z(U(2r)) \cdot \mathrm{Sp}(r) \times H_2, \mathbb{HP}^{n-1}) = -2$.

As for $Z(U(2r)) \cdot \mathrm{Sp}(r) \times H_3$ it is sufficient to observe that it induces on the quaternionic projective space the same action of $\mathrm{Sp}(r) \times \mathrm{U}(n - 2r)$, which has non-zero homogeneity rank.

This concludes the analysis of the subgroups of $\mathrm{U}(n) \subset \mathrm{Sp}(n)$.

4.6. **The subgroups of $G = \rho(H)$ with ρ irreducible representation of quaternionic type of the simple Lie group H.** We have to examine only those subgroups that in case 4.1.4 give rise to vanishing homogeneity rank actions. We exclude all of them simply noting that none of the subgroups of maximal dimension satisfy the dimensional condition (1). The list of subgroups of maximal dimension is given in [16] and can be found also in [12].

4.7. **The subgroups of $G = \mathrm{SO}(n) \otimes \mathrm{Sp}(1)$.** Now we prove that the action of $\mathrm{SO}(n) \otimes \mathrm{Sp}(1)$ is minimal except for $n = 8$.

A maximal compact connected subgroups of G is conjugate to one of the form $H_1 \otimes H_2$ where H_1 is either a compact connected maximal subgroup of $\mathrm{SO}(n)$ or $\mathrm{SO}(n)$ itself, and H_2 is either $\mathrm{Sp}(1)$ or $\mathrm{U}(1)$. The subgroup $\mathrm{SO}(n) \otimes U(1)$ is the same as $Z(\mathrm{U}(n)) \cdot \mathrm{SO}(n) \subset \mathrm{U}(n)$ that we have already excluded (see case 4.2.3), so let us turn to the case $H_1 \otimes \mathrm{Sp}(1)$ and look at Table 6 for maximal subgroups of $\mathrm{SO}(n)$.

4.7.1. $H_1 = \mathrm{U}(k)$ *where* $n = 2k$. It is easy to find the slice representation at the quaternionic line ℓ spanned by a pure element of $\mathbb{R}^k \otimes \mathbb{R}^4$ starting from (2). The stabilizer subalgebra is $\mathfrak{u}(k-1) \oplus \mathfrak{sp}(1)$ acting on

$$\mathbb{C}^{k-1} \otimes_{\mathbb{R}} \mathbb{R}^3 \oplus \mathbb{R}^3$$

where $\mathbb{R}^3$ stands for the adjoint representation of $\mathfrak{o}(3) \simeq \mathfrak{sp}(1)$. It follows immediately that the principal isotropy subalgebra is isomorphic to $\mathfrak{u}(k-4)$ if $n \geqslant 5$, otherwise it is trivial. In any case the homogeneity rank is -4.

4.7.2. $H_1 = \mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(n-k))$. The isotropy subalgebra at $\ell \in \mathbb{HP}^{n-1}$ is $\mathfrak{o}(k-1) \oplus \mathfrak{o}(n-k) \oplus \mathfrak{sp}(1)$ acting on

$$\mathbb{R}^{k-1} \otimes \mathbb{R}^3 \oplus \mathbb{R}^{n-k} \otimes \mathbb{R}^3 \oplus \mathbb{R}^{n-k}.$$

Here, in the general case, we are not allowed to skip the computation of the principal isotropy subalgebra. Nevertheless it is not hard to find that it is isomorphic to $\mathfrak{o}(k-4) \oplus \mathfrak{o}(n-k-4)$ for $k, n-k \geqslant 6$ so that $c = 13$ and $\mathrm{hrk} = -8$. If either k or $n-k$ are smaller than 6, a similar argument leads to the same conclusion. The remaining low-dimensional cases can be excluded using (1).

4.7.3. $H_1 = \mathrm{SO}(p) \otimes \mathrm{SO}(q)$ *with* $n = pq$. The isotropy subalgebra at $\ell \in \mathbb{HP}^{n-1}$ is $\mathfrak{o}(p-1) \oplus \mathfrak{o}(q-1) \oplus \mathfrak{sp}(1)$ acting on

$$\Sigma = (\mathbb{R}^{p-1} \otimes \mathbb{R}^{q-1}) \oplus (\mathbb{R}^{p-1} \otimes \mathbb{R}^{q-1} \otimes \mathbb{R}^3) \oplus (\mathbb{R}^{p-1} \otimes \mathbb{R}^3) \oplus (\mathbb{R}^{q-1} \otimes \mathbb{R}^3).$$

Let us distinguish three subcases according to the parity of p and q. If p and q are odd then the orbit through ℓ has positive Euler characteristic but the real

irreducible module $\mathbb{R}^{p-1} \otimes \mathbb{R}^{q-1} \otimes \mathbb{R}^3$ has negative homogeneity rank (it does not appear in the classification of Theorem 4.5).

If only one among p and q is even (say p), then the orbit has no more positive Euler characteristic but, with the notations of Lemma 2.1, we have $\delta = 1$. Thus it is sufficient to show that $\mathrm{hrk}(G_\ell, \Sigma) \leqslant -2$. Thanks to Lemma 2.3

$$
\begin{aligned}
\mathrm{hrk}(G_\ell, \Sigma) \;\leqslant\;\; & \mathrm{hrk}(\mathrm{O}(p-1) \times \mathrm{O}(q-1) \times \mathrm{O}(3), \mathbb{R}^{p-1} \otimes \mathbb{R}^{q-1} \otimes \mathbb{R}^3) + \\
& \mathrm{hrk}(\mathrm{O}(p-1) \times \mathrm{O}(3), \mathbb{R}^{p-1} \otimes \mathbb{R}^3) \leqslant -2 \,.
\end{aligned}
$$

If both p and q are even, we have $\delta = 2$, but

$$
\begin{aligned}
\mathrm{hrk}(G_\ell, \Sigma) \;\leqslant\;\; & \mathrm{hrk}(\mathrm{O}(p-1) \times \mathrm{O}(q-1) \times \mathrm{O}(3), \mathbb{R}^{p-1} \otimes \mathbb{R}^{q-1} \otimes \mathbb{R}^3) + \\
& \mathrm{hrk}(\mathrm{O}(p-1) \times \mathrm{O}(3), \mathbb{R}^{p-1} \otimes \mathbb{R}^3) + \\
& \mathrm{hrk}(\mathrm{O}(q-1) \times \mathrm{O}(3), \mathbb{R}^{q-1} \otimes \mathbb{R}^3) \leqslant -3 \,.
\end{aligned}
$$

4.7.4. $H_1 = \mathrm{Sp}(p) \otimes \mathrm{Sp}(q)$ *with* $n = 4pq \geqslant 8$. This action has no orbit of positive Euler characteristic. If $p, q \geqslant 2$ the isotropy subalgebra at $\ell \in \mathbb{HP}^{n-1}$ is $\mathfrak{sp}(p-1) \oplus \mathfrak{sp}(q-1) \oplus \mathfrak{sp}(1)$ acting on

$$
(U \otimes \mathbb{R}^3) \oplus (\mathbb{H}^{p-1} \otimes \mathbb{R}^3) \oplus (\mathbb{H}^{q-1} \otimes \mathbb{R}^3) \oplus M \otimes \mathbb{R}^3) \oplus M \oplus U \,,
$$

where $M = \mathcal{M}(p-1, q-1, \mathbb{H})$ and U is the adjoint representation of $\mathfrak{sp}(1)$. Here $\delta = 2$ but $\mathrm{hrk}(\mathrm{Sp}(p-1) \times \mathrm{Sp}(1), \mathbb{H}^{p-1} \otimes \mathbb{R}^3) = -8$. Thus the action has non-zero homogeneity rank.

Obviously this module appears in the slice even when $q = 1$, so we get no new vanishing homogeneity rank actions.

4.7.5. $H_1 = \rho(K)$ *with* ρ *irreducible representation of real type of the simple Lie group* K. We here use again the dimensional condition (1) that in this situation becomes

$$
\dim K + \mathrm{rk}\, K \geqslant 4 \deg \rho - 8.
$$

Kollross in lemma 2.6 in [12] lists all the representations σ of real type of Lie groups L such that $2 \dim L \geqslant \deg \sigma - 2$. This condition is always looser than ours. Counting the dimensions for the groups and the representations from this list, we have that only the spin representation of $K = \mathrm{Spin}(7)$ and the standard representations of $\mathrm{SO}(n)$ satisfy the condition. The latter correspond to the case treated in subsection 4.7. Let us compute $\mathrm{hrk}(\mathrm{Spin}(7) \times \mathrm{Sp}(1), \mathbb{HP}^7)$. As usually we consider the orbit through the quaternionic line ℓ spanned by a pure tensor of $\mathbb{R}^8 \otimes \mathbb{R}^4$. It turns out to be the seven-dimensional sphere $\mathrm{Spin}(7)/\mathrm{G}_2$ and the slice representation is the tensor product of the standard representation of G_2 with the adjoint representation of $\mathrm{Sp}(1)$. It is well known (see e.g. [7, p. 11]) that this irreducible representation has trivial principal isotropy and from this follows that $\mathrm{hrk}(\mathrm{Spin}(7) \times \mathrm{Sp}(1), \mathbb{HP}^7) = 0$.

4.8. The subgroups of $\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$. We analyse this case with the aid of the following lemma:

Lemma 4.7. *Let* $G \subseteq \mathrm{Sp}(N)$ *be a compact Lie group acting with vanishing homogeneity rank on* $\mathbb{HP}^{N-1}$. *Then* $\widetilde{G} = G \times \mathrm{Sp}(n)$ *acts with vanishing homogeneity rank on* $\mathbb{HP}^{N+n-1} = \mathbb{P}_{\mathbb{H}}(\mathbb{H}^N \oplus \mathbb{H}^n)$.

Proof.   If v is taken in $\mathbb{H}^n$, the $\widetilde{G}$-orbit through $[v]$ in $\mathbb{HP}^{N+n-1} = \mathbb{P}_\mathbb{H}(\mathbb{H}^N \oplus \mathbb{H}^n)$ is $\mathbb{HP}^{n-1}$. Therefore the action of $\widetilde{G}$ has homogeneity rank zero if and only if the slice representation at this quaternionic orbit has vanishing homogeneity rank. Note that the last factor of the isotropy subgroup $G \times \mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1)$ acts trivially on the slice $\Sigma_{[v]} \simeq \mathbb{H}^N$. Consider now the natural projection of $\mathbb{H}^N \setminus \{0\}$ on $\mathbb{HP}^{N-1}$. This is an equivariant fibration with fiber $\mathbb{H}$. Thus arguing as in Proposition 2.4 we deduce that

$$\mathrm{hrk}(G \times \mathrm{Sp}(1), \mathbb{H}^N) = \mathrm{hrk}(G, \mathbb{HP}^{N-1}) + \mathrm{hrk}(\mathrm{Sp}(1), \mathbb{H})$$

and the claim follows since both the homogeneity ranks in the right hand side of the equality vanish.   ∎

As a consequence, combining the previous lemma with Proposition 2.4 we obtain the following

Corollary 4.8.   *The group $G \subseteq \mathrm{Sp}(n)$ acts on $\mathbb{HP}^{n-1}$ with vanishing homogeneity rank if and only if $G \times \mathrm{Sp}(N) \subseteq \mathrm{Sp}(n) \times \mathrm{Sp}(N)$ on $\mathbb{HP}^{n+N-1}$ does.*

The previous corollary avoid the analysis of those subgroups of $\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$ of the form $H_1 \times H_2$ where either H_1 or H_2 equals $\mathrm{Sp}(k)$ or $\mathrm{Sp}(n-k)$. Except for the diagonal action of $\mathrm{Sp}(k)_\Delta$ when $k = n-k$ (which has already been excluded), it is therefore sufficient to analyse all the subgroups $H_1 \times H_2$, where $H_1 \subsetneq \mathrm{Sp}(k)$ acts on $\mathbb{HP}^{k-1}$ and $H_2 \subsetneq \mathrm{Sp}(n-k)$ acts on $\mathbb{HP}^{n-k-1}$ both with vanishing homogeneity rank.

The cases that we shall consider are given by all possible combinations of the following:

$$\begin{aligned}
H_1 &= \mathrm{U}(k), \mathrm{Sp}(k_1) \times \mathrm{Sp}(k_2) \,\text{with}\, k_1 + k_2 = k,\\
&\quad \mathrm{SO}(k) \otimes \mathrm{Sp}(1), \mathrm{Spin}(7) \otimes \mathrm{Sp}(1), \rho(H_1)\\
H_2 &= \mathrm{U}(n-k), \mathrm{Sp}(l_1) \times \mathrm{Sp}(l_2) \,\text{with}\, l_1 + l_2 = n-k,\\
&\quad \mathrm{SO}(n-k) \otimes \mathrm{Sp}(1), \mathrm{Spin}(7) \otimes \mathrm{Sp}(1), \rho(H_2)
\end{aligned}$$

Where $\rho(H_1) \otimes \sigma$ and $\rho(H_2) \otimes \sigma$ are orbit equivalent to isotropy representations of a quaternionic-Kähler symmetric space, where σ is the standard representation of $\mathrm{Sp}(1)$.

The case $\mathrm{U}(k) \times \mathrm{U}(n-k)$ has already been treated, the cases in which one of the factor is either $\mathrm{Sp}(k_1) \times \mathrm{Sp}(k_2)$ or $\mathrm{Sp}(l_1) \times \mathrm{Sp}(l_2)$ give rise to vanishing homogeneity rank actions thanks to Lemma 4.7.

The remaining cases can be all excluded with a common argument: we treat explicitly one of them and then we explain how to generalize.

Consider for example $G = \mathrm{E}_7 \times \mathrm{Spin}(11)$ acting on $\mathbb{P}_\mathbb{H}(\mathbb{H}^{28} \oplus \mathbb{H}^{16})$. Let $\mathrm{E}_7/\mathrm{E}_6 \cdot \mathrm{U}(1) \subseteq \mathbb{HP}^{27} \subseteq \mathbb{HP}^{43}$ be the maximal totally complex orbit of G. The factor $\mathrm{U}(1) \times \mathrm{Spin}(11)$ of the isotropy acts on the second module of the slice $\mathbb{C}^{27} \oplus \mathbb{H}^{16}$ with non vanishing homogeneity rank, since it is neither the isotropy representation of a symmetric space of inner type nor it appears in Table 5.

Observe now that all of the factors of the products $H_1 \times H_2$ we are considering admit a totally complex orbit (see [2]). All the cases can therefore be excluded in the same manner taking at a first step a maximal totally complex orbit for the group H_1, and then observing that the slice representation contains a module on

which the isotropy acts with non vanishing homogeneity rank.

The classification is now complete. In fact once one goes further the only possibility that can occur is the product of three factors $G_1 \times G_2 \times G_3$ where all of $G_i \neq \mathrm{Sp}(n_i)$ (otherwise this case can be treated with the aid of Lemma 4.7), where each G_i gives rise to vanishing homogeneity rank action on $\mathbb{HP}^{n_i-1}$. This case can be easily excluded applying Proposition 2.4 to the product of two of the factors.

References

[1]   Alekseevsky, D. V., and S. Marchiafava, *Hermitian and Kähler submanifolds of a quaternionic Kähler manifold*, Osaka J. Math. **38** (2001), 869–904.

[2]   Bedulli, L., A. Gori, and F. Podestà, *Maximal totally complex submanifolds of $\mathbb{HP}^n$: homogeneity and normal holonomy*, Preprint, `arXiv:0810.0173`.

[3]   Biliotti, L., *Coisotropic and polar actions on compact irreducible Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **358** (2006), 3003–3022.

[4]   Biliotti, L., and A. Gori, *Coisotropic and polar actions on complex Grassmannians*, Trans. Amer. Math. Soc. **357** (2005), 1731–1751.

[5]   Benson, C., and G. Ratcliff, *A classification of multiplicity free actions*, J. Algebra **181** (1996), 152–186.

[6]   Bredon, G., Introduction to compact transformation groups, *Academic Press New York - London, 1972.*

[7]   Gorodski, C., and F. Podestà, *Homogeneity rank of real representations of compact Lie groups*, J. Lie Theory **15** (2005), 63–77.

[8]   Guillemin, V., and S. Sternberg, *Multiplicity-free spaces*, J. Differential Geom. **19** (1984), 31–56.

[9]   Hsiang, W.-C., and W.-Y. Hsiang, *Differentiable actions of compact connected classical groups II*, Ann. of Math. (2) **92** (1970), 189–223.

[10]   Huckleberry, A. T., and T. Wurzbacher, *Multiplicity-free complex manifolds*, Math. Annalen **286** (1990), 261–280.

[11]   Kac, V. G., *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190–213.

[12]   Kollross, A., *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612.

[13]   Kollross, A., *Polar actions on symmetric spaces*, J. Differential Geom. **77** (2007), 425–482.

[14]   Leahy, A., *A classification of multiplicity-free representations*, J. Lie Theory **8** (1998), 367–391.

[15]   LeBrun, C., and S. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), 109–132.

[16]   Mann, L. N., *Gaps in the dimensions of transformation groups*, Illinois. J. Math. **10** (1966), 533–546.

[17] Onishchik, A. L., *Inclusion relations among transitive compact transformation groups*, Amer. Math. Soc. Translations, ser. 2, **50** (1966), 5–58.

[18] Onishchik, A. L., and E. B. Vinberg, "Lie Groups and Lie Algebras III," Encyclopaedia of Math. Sciences, **41**, Springer-Verlag, 1994.

[19] Podestà, F., and G. Thorbergsson, *Polar actions on rank-one symmetric spaces*, J. Differential Geom. **53** (1999), 131–175.

[20] —, *Polar and coisotropic actions on Kähler manifolds*, Trans. Amer. Math. Soc. **354** (2002), 1759–1781.

[21] Püttmann, T., *Homogeneity rank and atoms of actions*, Ann. Global Anal. Geom. **22** (2002), 375–399.

[22] Salamon, S., *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.

[23] Tebege, S., *Polar actions on Hermitian and quaternion-Kähler symmetric spaces*, Geom. Dedicata **129** (2007), 155–171.

Appendix: Tables

We collect several tables that have been used during the classification. For the notation see the explanation given before Theorem 4.5.

Table 1: Irreducible representations $\rho : G \to \mathrm{SU}(V)$ of compact connected Lie groups G such that $\mathrm{U}(1) \times \rho(G)$ acts irreducibly and coisotropically on V.

G	ρ	V	cond.
$\mathrm{SU}(n)$	Λ_1	$\mathbb{C}^n$	$n \geq 1$
$\mathrm{SO}(n)$	Λ_1	$\mathbb{C}^n$	$n \geq 3$
$\mathrm{Sp}(n)$	Λ_1	$\mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{SU}(n)$	$2\Lambda_1$	$S^2\mathbb{C}^n$	$n \geq 2$
$\mathrm{SU}(n)$	Λ_2	$\Lambda^2\mathbb{C}^n$	$n \geq 4$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$n, m \geq 2$
$\mathrm{SU}(2) \times \mathrm{Sp}(n)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^2 \otimes \mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{SU}(3) \times \mathrm{Sp}(n)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^3 \otimes \mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{SU}(n) \times \mathrm{Sp}(2)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^n \otimes \mathbb{C}^4$	$n \geq 4$
$\mathrm{Spin}(7)$	Λ_3	$\mathbb{C}^8$	
$\mathrm{Spin}(9)$	Λ_4	$\mathbb{C}^{16}$	
$\mathrm{Spin}(10)$	Λ_4	$\mathbb{C}^{16}$	
E_6	Λ_1	$\mathbb{C}^{27}$	
G_2	Λ_1	$\mathbb{C}^7$	

Table 2: Irreducible coisotropic representations in which the scalars are removable.

G	ρ	V	cond.
$\mathrm{SU}(n)$	Λ_1	$\mathbb{C}^n$	$n \geq 2$
$\mathrm{Sp}(n)$	Λ_1	$\mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{SU}(2m + 1))$	Λ_2	$\Lambda^2\mathbb{C}^{2m+1}$	$m \geq 2$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$n, m \geq 2, \ n \neq m$
$\mathrm{SU}(n) \times \mathrm{Sp}(2)$	$\Lambda_1 \otimes \Lambda_1$	$\mathbb{C}^n \otimes \mathbb{C}^4$	$n \geq 5$
$\mathrm{Spin}(10)$	Λ_4	$\mathbb{C}^{16}$	

Table 3: Reducible representations $\rho : G \rightarrow \mathrm{SU}(V)$ of compact connected Lie groups G such that the action of $\mathrm{U}(1) \times \rho(G)$ on $V = V_1 \oplus V_2$ defined by $(z, g) \cdot (v_1, v_2) = g(z^a v_1, z^b v_2)$ is coisotropic.

G	ρ	V	cond.
$\mathrm{SU}(n)$	$\Lambda_1 \oplus \Lambda_1$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$n \geq 3, \ a \neq b$
$\mathrm{SU}(n)$	$\Lambda_{n-1} \oplus \Lambda_1$	$\mathbb{C}^{n*} \oplus \mathbb{C}^n$	$n \geq 3 \ a \neq -b$
$\mathrm{SU}(2m)$	$\Lambda_1 \oplus \Lambda_2$	$\mathbb{C}^{2m} \oplus \Lambda^2 \mathbb{C}^{2m}$	$m \geq 2, \ b \neq 0$
$\mathrm{SU}(2m+1)$	$\Lambda_1 \oplus \Lambda_2$	$\mathbb{C}^{2m+1} \oplus \Lambda^2 \mathbb{C}^{2m+1}$	$m \geq 2, \ a \neq -mb$
$\mathrm{SU}(2m)$	$\Lambda_{2m-1} \oplus \Lambda_2$	$\mathbb{C}^{2m*} \oplus \Lambda^2 \mathbb{C}^{2m}$	$m \geq 2, \ b \neq 0$
$\mathrm{SU}(2m+1)$	$\Lambda_{2m} \oplus \Lambda_2$	$\mathbb{C}^{2m+1*} \oplus \Lambda^2 \mathbb{C}^{2m+1}$	$m \geq 2, \ a \neq mb$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$(\Lambda_1 \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)$	$2 \leq n < m, \ a \neq 0$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$(\Lambda_1 \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)$	$m \geq 2, \ n \geq m+2, a \neq b$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$(\Lambda_{n-1} \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^{n*} \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)$	$2 \leq n < m, \ a \neq 0$
$\mathrm{SU}(n) \times \mathrm{SU}(m)$	$(\Lambda_{n-1} \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^{n*} \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)$	$m \geq 2, n \geq m+2, \ a \neq -b$
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(n)$	$(\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1)$	$(\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n)$	$n \geq 3, \ a \neq 0$
$\mathrm{SU}(n) \times \mathrm{SU}(2) \times \mathrm{SU}(m)$	$(\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1)$	$(\mathbb{C}^n \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^m)$	$n, m \geq 3$
$\mathrm{SU}(n) \times \mathrm{SU}(2) \times \mathrm{Sp}(m)$	$(\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1)$	$(\mathbb{C}^n \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2m})$	$n \geq 3, m \geq 2, b \neq 0$

Table 4: Reducible representations $\rho : G \rightarrow \mathrm{SU}(V)$ of compact connected Lie groups G such that the action of $\mathrm{U}(1) \times \mathrm{U}(1) \times \rho(G)$ on $V = V_1 \oplus V_2$ defined by $(z_1, z_2, g) \cdot (v_1, v_2) = g(z_1 v_1, z_2 v_2)$ is coisotropic.

G	ρ	V	cond.
$\mathrm{SU}(2)$	$\Lambda_1 \oplus \Lambda_1$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	
$\mathrm{Sp}(n)$	$\Lambda_1 \oplus \Lambda_1$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{Spin}(8)$	$\Lambda_3 \oplus \Lambda_1$	$\mathbb{C}^8 \oplus \mathbb{C}^8$	
$\mathrm{SU}(n) \times \mathrm{SU}(n)$	$(\Lambda_1 \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^n)$	$n \geq 2$
$\mathrm{SU}(n) \times \mathrm{SU}(n)$	$(\Lambda_{n-1} \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^{n*} \oplus (\mathbb{C}^n \otimes \mathbb{C}^n)$	$n \geq 2$
$\mathrm{SU}(n+1) \times \mathrm{SU}(n)$	$(\Lambda_1 \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1} \otimes \mathbb{C}^n)$	$n \geq 2$
$\mathrm{SU}(n+1) \times \mathrm{SU}(n)$	$(\Lambda_n \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^{n+1*} \oplus (\mathbb{C}^{n+1} \otimes \mathbb{C}^n)$	$n \geq 2$
$\mathrm{SU}(2) \times \mathrm{Sp}(n)$	$(\Lambda_1 \otimes 1) \oplus (\Lambda_1 \otimes \Lambda_1)$	$\mathbb{C}^2 \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2n})$	$n \geq 2$
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Sp}(n)$	$(\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1)$	$(\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2n})$	
$\mathrm{Sp}(n) \times \mathrm{SU}(2) \times \mathrm{Sp}(m)$	$(\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1)$	$(\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2m})$	$n, m \geq 2$

Table 5: Homogeneity rank zero real irreducible representations which are not orbit-equivalent to the isotropy representation of a non-Hermitian symmetric space of inner type.

G	ρ	V	cond.
$\mathrm{Sp}(1) \times \mathrm{Sp}(n)$	$3\Lambda_1 \otimes_{\mathbb{H}} \Lambda_1$	$S^3 \mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^{2n}$	$n \geq 2$
$\mathrm{SO}(4) \times \mathrm{Spin}(7)$	$\Lambda_1 \otimes_{\mathbb{R}} \Lambda_3$	$\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^8$	
$\mathrm{Sp}(1) \times \mathrm{Spin}(11)$	$\Lambda_1 \otimes_{\mathbb{H}} \Lambda_5$	$\mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^{32}$	

Table 6: Maximal subgroups of $\mathrm{SO}(n)$

$\mathrm{SO}(k) \times \mathrm{SO}(n-k)$	$1 \le k \le n-1$
$\mathrm{U}(m)$	$2m = n$
$\mathrm{SO}(p) \otimes \mathrm{SO}(q)$	$pq = n,\ 3 \le p \le q$
$\mathrm{Sp}(p) \otimes \mathrm{Sp}(q)$	$4pq = n$
$\rho(H)$	H simple, $\rho \in \mathrm{Irr}_{\mathbb{R}}$, $\deg \rho = n$

Table 7: Maximal subgroups of $\mathrm{SU}(n)$

$\mathrm{SO}(n)$	
$\mathrm{Sp}(m)$	$2m = n$
$\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$	$1 \le k \le n-1$
$\mathrm{SU}(p) \otimes \mathrm{SU}(q)$	$pq = n,\ p \ge 3,\ q \ge 3$
$\rho(H)$	H simple, $\rho \in \mathrm{Irr}_{\mathbb{C}}$, $\deg \rho = n$

Table 8: Maximal subgroups of $\mathrm{Sp}(n)$

$\mathrm{U}(n)$	
$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$	$1 \ge k \ge n-1$
$\mathrm{SO}(p) \otimes \mathrm{Sp}(q)$	$pq = n,\ p \ge 3,\ q \ge 1$
$\rho(H)$	H simple, $\rho \in \mathrm{Irr}_{\mathbb{H}}$, $\deg \rho = 2n$

Lucio Bedulli
Dipartimento di Matematica
Università di Firenze
viale Morgagni 67/A
50134 Firenze, Italy
bedulli@math.unifi.it

Anna Gori
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
gori@math.unifi.it