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A Nonsmooth Continuous Unitary Representation of a Banach-Lie Group

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Abstract. In this note we show that the representation of the additive group of the Hilbert space $L^2([0,1],\mathbb{R})$ on $L^2([0,1],\mathbb{C})$ given by the multiplication operators $\pi(f) := e^{if}$ is continuous but its space of smooth vectors is trivial. This example shows that a continuous unitary representation of an infinite dimensional Lie group need not be smooth.
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1. Introduction

Definition 1.1. Let G be a Lie group modeled on a locally convex space (cf. [4] for a survey on locally convex Lie theory).

Let $\mathcal{H}$ be a complex Hilbert space and $U(\mathcal{H})$ be its unitary group. A *unitary representation of* G *on* $\mathcal{H}$ is a pair $(\pi, \mathcal{H})$, where $\pi\colon G \to U(\mathcal{H})$ is a group homomorphism.

A unitary representation $(\pi, \mathcal{H})$ is said to be *continuous* if the group action $G \times \mathcal{H} \to \mathcal{H},\ (g, v) \mapsto \pi(g)v$ is continuous. Since G acts by isometries on $\mathcal{H}$, it is easy to see that this condition is equivalent to the continuity of all orbit maps $\pi^v\colon G \to \mathcal{H}, g \mapsto \pi(g)v$.

A unitary representation $(\pi, \mathcal{H})$ is said to be *smooth* if the space

$$\mathcal{H}^\infty := \{v \in \mathcal{H}\colon \pi^v \in C^\infty(G, \mathcal{H})\}$$

of *smooth vectors* is dense.

Clearly, every smooth representation is continuous (see the general observation used in the proof of Proposition 2.1 below), and it is a natural question to which extent the converse also holds.

Remark 1.2. If G is finite dimensional, then each continuous unitary representation is smooth. Even the subspace $\mathcal{H}^\omega \subseteq \mathcal{H}^\infty$ of analytic vectors is dense (cf. [2]).

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For the class of groups which are direct limits of finite dimensional Lie groups, Samoilenko's book [7] contains a variety of positive results on the existence of smooth vectors, in particular for abelian Lie groups, restricted direct products of $\mathrm{SU}_2(\mathbb{C})$ and the group of infinite upper triangular matrices. More general existence results on differentiable vectors for direct limits of finite dimensional Lie groups can be found in [1]. See also [8] for existence of smooth vectors for particular classes of representations of diffeomorphism groups.

However, the purpose of this note is to show that there is no automatic smoothness result for continuous unitary representations of infinite dimensional Lie groups. Even for the otherwise rather well-behaved class of abelian Banach–Lie groups. This will be shown by verifying that for the abelian Hilbert–Lie group $G = (L^2([0,1],\mathbb{R}),+)$, the unitary representation

$$\pi\colon G \to U(L^2([0,1],\mathbb{C})), \quad \pi(f)\xi := e^{if}\xi$$

is continuous, but its space $L^2([0,1],\mathbb{C})^\infty$ of smooth vectors is trivial.

Smoothness of a representation is a property that is crucial to make it accessible to Lie theoretic methods. In particular, for any smooth representation $(\pi,\mathcal{H})$ we obtain a representation of its Lie algebra $\mathfrak{g}$ on the space $\mathcal{H}^\infty$ of smooth vectors by skew-hermitian operators (cf. [5]). Our example shows that smoothness of a representation is an assumption that does not follow from continuity.

2. The exponential representation

Proposition 2.1. *The unitary representation $(\pi, L^2([0,1],\mathbb{C}))$ of the additive group $L^2([0,1],\mathbb{R})$, defined by $\pi(f)\xi = e^{if}\xi$, is continuous.*

Proof. First we observe that for any $t\in[0,1]$ and $f,g\in L^2([0,1],\mathbb{R})$ we have $|e^{if(t)} - e^{ig(t)}| \le |f(t)-g(t)|$. For any $\xi \in L^2([0,1],\mathbb{C})\cap L^\infty([0,1],\mathbb{C})$ we thus obtain

$$\begin{aligned}\|\pi(f)\xi - \pi(g)\xi\|_2^2 &= \int_0^1 |e^{if(t)} - e^{ig(t)}|^2\cdot|\xi(t)|^2\,dt \le \|\xi\|_\infty^2 \int_0^1 |f(t)-g(t)|^2\,dt \\ &= \|\xi\|_\infty^2\|f-g\|_2^2.\end{aligned}$$

This implies that the orbit map π^ξ is continuous if ξ is essentially bounded, and since the set of essentially bounded elements is dense in $L^2([0,1],\mathbb{C})$, the continuity of π follows by the following general observation:

> A unitary representation $\pi\colon G\to U(\mathcal{H})$ is continuous if and only if there exists a dense set of vectors $v\in\mathcal{H}$ for which the orbit map $\pi^v\colon G\to\mathcal{H}$ is continuous.

In fact, for arbitrary $g,h\in G$ and $v,w\in\mathcal{H}$ we have

$$\|\pi(g)v-\pi(h)w\| \le \|\pi(g)v-\pi(h)v\|+\|\pi(h)v-\pi(h)w\| = \|(\pi(g)-\pi(h))v\|+\|v-w\|$$

and then one can suitably approximate an arbitrary orbit map by a continuous one, in order to prove that the representation π is continuous if the aforementioned density condition is satisfied. ∎

To show that the space $L^2([0,1],\mathbb{C})^\infty$ of smooth vectors is trivial, we put $\mathcal{H} := L^2([0,1],\mathbb{C})$ and consider the functions

$$f_\lambda(t) := |t-\lambda|^{-\frac{1}{4}}, \quad \lambda \in [0,1],$$

in $L^2([0,1],\mathbb{R})$. For each λ, the continuous unitary representation π defines a continuous unitary one-parameter group

$$\pi_\lambda(t)\xi := e^{itf_\lambda}\xi,$$

whose infinitesimal generator is the multiplication operator

$$M_\lambda\colon \mathcal{D}_\lambda \to \mathcal{H}, \quad M_\lambda\xi := f_\lambda\xi, \quad \mathcal{D}_\lambda := \{\xi \in \mathcal{H}\colon \|f_\lambda\xi\|_2 < \infty\}.$$

Since the n-th derivative of $t \mapsto e^{itf_\lambda}\xi$ exists if and only if $f_\lambda^n \cdot e^{itf_\lambda}\xi$ is in $L^2([0,1],\mathbb{C})$, the set of smooth vectors for this one-parameter group is the dense subspace

$$\mathcal{D}_\lambda^\infty := \{\xi \in \mathcal{H}\colon (\forall n \in \mathbb{N})\ \|f_\lambda^n\xi\|_2 < \infty\}. \tag{1}$$

Therefore it remains to show that $\bigcap_{\lambda\in[0,1]} \mathcal{D}_\lambda^\infty = \{0\}$.

Proposition 2.2. *If $\xi \in L^2([0,1],\mathbb{C})$ has the property that $f_\lambda^4\xi \in L^2([0,1],\mathbb{C})$ holds for each $\lambda \in [0,1]$, then $\xi = 0$.*

Proof. Replacing ξ by $|\xi|$, we may w.l.o.g. assume that $\xi \geq 0$.

For $n \in \mathbb{N}$, let $M_n := \{t \in [0,1]\colon 1/n \leq \xi(t) \leq n\}$ and note that $\xi = \lim_{n\to\infty} \xi\chi_{M_n}$ holds in $L^2([0,1],\mathbb{C})$. If $f_\lambda^k\xi \in L^2([0,1],\mathbb{C})$, then we have $f_\lambda^k\xi\chi_{M_n} \in L^2([0,1],\mathbb{C})$ for any $k,n \in \mathbb{N}$ and hence $f_\lambda^k\chi_{M_n} \in L^2([0,1],\mathbb{C})$. We may therefore assume that $\xi = \chi_M$ is the characteristic function of some measurable subset $M \subseteq [0,1]$.

Suppose that M has positive measure and that $f_\lambda^4\xi \in L^2([0,1],\mathbb{C})$ holds for each $\lambda \in [0,1]$. We have to show that this assumption leads to a contradiction. Let $\lambda \in M\cap]0,1[$ be a Lebesgue point of ξ ([6, Thm. 7.11]), so that

$$1 = \lim_{h\to 0} \frac{|M \cap [\lambda-h,\lambda+h]|}{2h} = \lim_{h\to 0} \frac{1}{2h}\int_{\lambda-h}^{\lambda+h} \chi_M(t)\,dt.$$

We then find the two estimates

$$\int_{\lambda-h}^{\lambda+h} f_\lambda^4\chi_M(t)\,dt \leq \|f_\lambda^4\chi_M\|_2 \cdot \sqrt{2h} \to 0 \quad \text{as} \quad h \to 0,$$

and, likewise, for $h \to 0$,

$$\int_{\lambda-h}^{\lambda+h} f_\lambda^4\chi_M(t)\,dt = \int_{\lambda-h}^{\lambda+h} \frac{1}{|t-\lambda|}\chi_M(t)\,dt \geq \frac{1}{h}\int_{\lambda-h}^{\lambda+h} \chi_M(t)\,dt \to 2.$$

These two estimates are contradictory, which completes the proof. ∎

Theorem 2.3. *The unitary representation of $G = L^2([0,1],\mathbb{R})$ on the Hilbert space $\mathcal{H} = L^2([0,1],\mathbb{C})$ defined by $\pi(f)\xi = e^{if}\xi$ is continuous, but all its smooth vectors are trivial.*

Proof. The continuity has been verified in Proposition 2.1. If $\xi \in \mathcal{H}^\infty$ is a smooth vector, then (1) shows that $f_\lambda^4 \xi \in L^2([0,1],\mathbb{C})$ holds for each $\lambda \in [0,1]$, so that Proposition 2.2 leads to $\xi = 0$. ■

Remark 2.4. We note that the kernel K of the exponential representation π referred to in Theorem 2.3 is a rather pathological subgroup of the additive group G, in the sense that K is closed, arcwise connected, and yet is not a Lie subgroup of G; see Example II.11 in [3]. In particular, the faithfully represented group G/K is a topological abelian group which is not a Lie group although it is connected, simply connected, and completely metrizable.

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