Journal of Lie Theory
Volume **18** (2008) 301–333

Homotopes and Conformal Deformations of Symmetric Spaces

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Communicated by K.-H. Neeb

Abstract. Homotopy is an important feature of associative and Jordan algebraic structures: such structures always come in families whose members need not be isomorphic among each other, but still share many important properties. One may regard homotopy as a special kind of deformation of a given algebraic structure. In this work, we investigate the geometric counterpart of this phenomenon on the level of the associated symmetric spaces. On this level, homotopy gives rise to conformal deformations of symmetric spaces. These results are valid in arbitrary dimension and over general base fields and -rings.
Mathematics Subject Index 2000: 17C37, 32G99, 17C27, 14D99l.
Keywords and phrases: Homotope, isotope, Jordan algebras (-triple systems, -pairs), Lie triple system, symmetric space, generalized projective (and polar) geometries.

0. Introduction

0.1. Conformal deformations. *Deformations* of Lie and other algebras have been much studied since the fundamental work of Gerstenhaber [G64], and related constructions on the level of Lie groups ("contractions") have attracted the interest of physicists even earlier ([IW53]; see also [DR85]). Following Gerstenhaber, one usually considers a given Lie algebra structure μ on $\mathfrak{g}$ as a point of the variety $\mathcal{L}$ of *all* Lie algebra structures on $\mathfrak{g}$, and one studies the structure of $\mathcal{L}$ on a neighborhood of μ. In order to cover the case of base fields different from $\mathbb{R}$, one considers *formal deformations* $\mu_t = \sum_{m=0}^{\infty} t^m \mu_m$, and then relates conditions on the formal derivatives of such one-parameter families of deformations to Lie algebra cohomology.

In this work we consider a special kind of deformations which we call *conformal* in order to distinguish them from the general and more formal approaches mentioned above. These deformations have the advantage to lift always to the space level and thus to define geometric deformations of spaces (manifolds

ISSN 0949–5932 / $2.50

equipped with certain structures, such as Lie groups and symmetric spaces), given by explicit formulas which ensure a good control on the simultaneous deformation of geometric properties of the spaces in question, regardless of dimension and nature of the base field. For this reason we hope that our approach may turn out to be useful for harmonic analysis (in the spirit of [DR85]; cf. [FP04], [Pe02] for topics to which our methods may apply) and maybe for physics (the contraction of de Sitter or anti-de Sitter models of general relativity to Minkowski-space of special relativity is a special case of conformal deformation in our sense, but many other cases, both finite and infinite dimensional and potentially interesting for physics, are also included in our setting).

The possibility of conformal deformations of a geometric space (symmetric space or Lie group) relies on the existence of an additional structure which we call *generalized conformal structure*. Algebraically, this means that we consider symmetric spaces or Lie groups which lie in the image of the *Jordan-Lie functor* (see Section 1.6 for its definition): there is an "overlying" Jordan algebraic structure from which one can recover the Lie algebraic structure, and this overlying Jordan structure corresponds geometrically to a kind of conformal structure. Now, on the Jordan algebraic level there exists a notion of *homotopy* – Jordan algebraic structures J always come in natural families $(J_i)_{i\in I}$, where the index set I typically is a vector space or a certain algebraic variety; the Jordan structures J_i are then called *homotopes* of J, and if in turn J can be obtained as a homotope of J_i, then J_i is called an *isotope* of J. In our previous work [Be02], the relation between geometric structures belonging to isotopic Jordan structures could be fully understood, but it was not clear how to understand "deformations" or "contractions" to more singular values, where J_i is only homotopic, but no longer isotopic to J. In the present work we close this gap.

To our knowledge, there is no interesting notion of homotopy in a purely Lie-theoretic setting, and therefore it seems quite hard to recover the present results without using Jordan-theory (if one wishes to avoid case-by-case calculations). On the other hand, once the geometric framework is clarified, some results given in this paper could be used to derive in a geometric way many of the identities and properties of Jordan pairs from [Lo75]. We illustrate this remark by giving independent proofs of some of our results, one proof based on the algebraic results from [Lo75], and another based on the geometric framework of *generalized projective geometries* ([Be02]; cf. Appendix B). Finally, let us add that it would be an interesting topic of further research to relate our approach to formal deformations in the sense mentioned above.

0.2. Example: associative algebras and general linear groups. For readers who are not experts in Jordan theory, let us give a very simple example that is well-suited to illustrate what we mean by "conformal deformation", and by "homotopy": if G is any group with unit element $\mathbf{1}$, then we can define new group laws on G by choosing an arbitrary element $a \in G$ and defining

$$x \cdot_a y := xay.$$

The unit element of $(G, \cdot_a)$ is then a^{-1}. Of course, all these group laws are isomorphic among each other: left translation $l_a : G \to G$ is an isomorphism from $(G, \cdot_a)$ onto $(G, \cdot)$ (one may also use right translation), and hence this construction seems not to be very interesting. However, things would start to become more interesting, if one could take a "limit" for a converging to some element in the "boundary of G"; then $(G, \cdot_a)$ should converge to some deformed group which is no longer isomorphic to G (a "homotope" of G).

In order to make this idea more precise, we need some additional structure. For instance, assume that $G = A^\times$ is the unit group of an associative unital algebra A over a commutative ring $\mathbb{K}$. As explained above, $G := A^\times$ is a group with product $\cdot_a$, for any $a \in A^\times$. We want to allow a to "converge" to some non-invertible element of A – the problem being that, although the *homotope algebra* $(A, \cdot_a)$ with product $x \cdot_a y = xay$ is an associative algebra for all $a \in A$, it will not have a unit element as soon as a is non-invertible. In order to circumvent this problem, we translate, for invertible a, the unit element a^{-1} by the translation $\tau = \tau_{-a^{-1}}$ to the origin $0 \in A$, that is, for invertible a we consider the group law

$$\begin{aligned} x \diamond_a y &= \tau(\tau^{-1}(x) \cdot_a \tau^{-1}(y)) \\ &= (x + a^{-1})a(y + a^{-1}) - a^{-1} \\ &= xay + y + x \end{aligned} \tag{0.1}$$

which is defined on the set

$$\begin{aligned} G_a &= \{x \in A |\, a^{-1} + x \in A^\times\} \\ &= \{x \in A |\, \mathbf{1} + ax \in A^\times\}. \end{aligned} \tag{0.2}$$

Now it is easy to take the "limit as a becomes non-invertible": one checks that the last expressions of (0.1) and (0.2) define a group $(G_a, \diamond_a)$ for any element $a \in A$. (The inverse of $x \in G_a$ is $y = -x(\mathbf{1} + ax)^{-1}$, as is seen by direct calculation.) For $a \in A^\times$ these groups are, by construction, all isomorphic to G, but for non-invertible $a \in A$ they are not: they are "homotopes" , but not isotopes of G. In particular, for $a = 0$, we get the translation group $(A, +)$ as a deformation of $A^\times$. Also, it is clear that, if $a, b \in A$ belong to the same $A^\times$-orbit (under left, or under right action), then G_a and G_b are isomorphic. Therefore, if we can classify the orbits of the action of $A^\times \times A^\times$ on A, then we can also classify the various groups G_a obtained by the preceding construction. For instance, this is possible in the case where $A = M(n, n; \mathbb{K})$ is the algebra of square matrices over a field $\mathbb{K}$, and then a complete description of the groups G_a is not difficult (see Section 6.1). Finally, one remarks that, on the level of Lie algebras, the corresponding situation is very simple: whenever the notion of a "Lie algebra of G_a" makes sense, then this is simply A with the a-deformed Lie bracket

$$[x, y]_a = xay - yax. \tag{0.3}$$

0.3. Plan of this work. In the first chapter we recall the definition of the three basic Jordan theoretic categories and the notions of homotopy we are interested in:

UJA unital (linear) Jordan algebras $(J, \bullet)$ (Section 1.7), with the a-homotopes $\bullet_a$, $a \in J$;

JP (linear) Jordan pairs (V^+, V^-) (Section 1.4), with the a-homotopes V_a^+, $a \in V^-$;

JTS (linear) Jordan triple systems (V, T), with the α-homotopes (V, T_α), where α is a *structural transformation* (Lemma 1.13).

In Chapter 2 we describe the Jordan theoretic analog of the associative construction given above (Th. 2.2). Thus we can construct symmetric spaces also for non-unital Jordan algebras and can thus deform, for instance, a symmetric cone (see [FK94]) into flat euclidean space. In Chapter 3 we give an independent and geometric version of these results (Th. 3.1 and Th. 3.3). Finally, in Chapter 4 we construct the geometric version of deformations corresponding to the α-homotope T_α (Th. 4.3). In some sense, this is the most general (but also most abstract) result, since one may choose $\alpha = Q(a)$, the quadratic operator corresponding to $a \in V^-$, in order to recover the a-homotope of T. In view of applications in harmonic analysis or physics (see above), we describe in Chapter 5 the behavior of geometric structures (for instance, affine connections or duals of metrics) under conformal deformations – for brevety, whenever calculations are essentially the same as in [Be00] or [Be02], we refer to these works for further details. In Chapter 6 we make our abstract results more explicit for some cases, namely for a-deformations belonging to *idempotents* or *tripotents* (in the semisimple case of finite dimension over a field this makes a classification possible), and for the case of *Grassmannians.* In the appendices, we collect some basic facts on reflection spaces and symmetric spaces (A), on generalized projective geometries (B) and on structural transformations (C); the simple Lemma C.3 on globalizations of such transformations is a main ingredient to be used in the proof of Theorem 4.3.

Acknowledgement. I thank Ottmar Loos for helpful comments.

Notation. Throughout, we work over a commutative base ring $\mathbb{K}$ in which 2 and 3 are invertible and hence use the concepts *linear* Jordan pairs, algebras and triple systems.

1. Homotopes of Lie and Jordan algebraic structures

1.1. Scalar homotopes and c-duality. If V is an algebra over $\mathbb{K}$ (Lie or other), with product $m(x, y)$, then for any scalar $\lambda \in \mathbb{K}$, we may define a new product $m_\lambda(x, y) := \lambda m(x, y)$, called a *scalar homotope of*(V, m). Then $\phi := \lambda \, \mathrm{id}_V$ is a homomorphism from (V, m_λ) to (V, m). In particular, the algebras (V, m_λ) for invertible λ are all isomorphic among each other. Thus, if $\mathbb{K}$ is a field, this construction gives us nothing interesting.

Now let (V, R) be a triple system (i.e., $R : V^3 \to V$ is a trilinear map). Then $R_\lambda(X, Y, Z) := \lambda R(X, Y, Z)$ defines a new triple product on V, again called a *scalar homotope.* But now $\lambda \, \mathrm{id}_V$ is a homomorphism from (V, R_{λ^2}) to (V, R), and hence we can only conclude that all homotopes (V, R_μ), with μ a square of an invertible element in $\mathbb{K}$, are isomorphic to (V, R). For instance, if $\mathbb{K} = \mathbb{R}$, we get a new triple system (V, R_{-1}) which is called the *c-dual of R*.

For a general field, we say that the (V, R_λ) with invertible λ are *scalar isotopes of* (V,R); they need not be isomorphic to (V,R).

The aim of this chapter is to recall some basic results showing that for Jordan algebraic structures there are interesting generalizations of the notion of scalar homotopy.

1.2. Graded Lie algebras. Let $(\Gamma,+)$ be an abelian group. A Γ-*graded Lie algebra* is a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ with a decomposition $\mathfrak{g} = \bigoplus_{n\in\Gamma} \mathfrak{g}_n$ such that $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{n+m}$ for all $n, m \in \Gamma$. The $\mathbb{Z}/(2)$-graded Lie algebras correspond bijectively to *symmetric Lie algebras*, i.e., to Lie algebras together with an automorphism of order 2. A 3-*graded* Lie algebra is a $\mathbb{Z}$-graded Lie algebra such that $\mathfrak{g}_n = 0$ if $n \neq -1, 0, 1$. Note that then $\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \oplus \mathfrak{g}_{-1})$ gives rise to a $\mathbb{Z}/(2)$-grading, so $\mathfrak{g}$ is then also a symmetric Lie algebra (of a rather special type).

1.3. Lie triple systems. A Lie triple system (LTS) is a $\mathbb{K}$-module $\mathfrak{q}$ together with a trilinear map

$$\mathfrak{q}\times\mathfrak{q}\times\mathfrak{q}\to\mathfrak{q}, \quad (X,Y,Z)\mapsto [X,Y,Z]$$

such that (where we use also the notation $R(X,Y,Z) := R(X,Y)Z := -[X,Y,Z]$, alluding to the interpretation of this expression as a *curvature tensor* in differential geometry)

(LT1) $R(X,X) = 0$

(LT2) $R(X,Y)Z + R(Y,Z)X + R(Z,Z)Y = 0$ (the Jacobi identity)

(LT3) For $D := R(X,Y)$ we have $DR(U,V,W) = R(DU,V,W) + R(U,DV,W) + R(U,V,DW)$.

If $\mathfrak{g}$ is a $\mathbb{Z}/(2)$-graded Lie algebra, then $\mathfrak{g}_{-1}$ with $[X,Y,Z] := [[X,Y],Z]$ becomes an LTS, and every LTS arises in this way since one may reconstruct a Lie algebra from an LTS via the *standard imbedding* (due to Lister, cf. [Lo69]).

1.4. Linear Jordan pairs. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ is a 3-graded Lie algebra, then we let $V^\pm := \mathfrak{g}_{\pm 1}$ and define trilinear maps by

$$T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \quad (x,y,z)\mapsto [[x,y],z].$$

The maps $T^\pm$ satisfy the following identies:

(LJP1) $T^\pm(x,y,z) = T^\pm(z,y,x)$

(LJP2) $T^\pm(u,v,T^\pm(x,y,z)) =$
$T^\pm(T^\pm(u,v,x),y,z) - T^\pm(x,T^\mp(v,u,y),z) + T^\pm(x,y,T^\pm(u,v,z))$

In general, a pair of $\mathbb{K}$-modules (V_+, V_-) with trilinear maps $T^\pm : V_\pm \times V_\mp \times V_\pm \to V_\pm$ is called a *linear Jordan pair* if (LJP1) and (LJP2) hold. Every linear Jordan pair arises by the construction just described.

1.5. Jordan triple systems. An *involution of a* 3-*graded Lie algebra* is an automorphism θ of order 2 such that $\theta(\mathfrak{g}_\pm) = \mathfrak{g}_\mp$ and $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$. Then we let $V := \mathfrak{g}_+$ and define

$$T : V\times V\times V\to V, \quad (X,Y,Z)\mapsto [[X,\theta Y],Z].$$

Then T satisfies the identities (LJT1) and (LJT2) obtained from (LJP1) and (LJP2) by omitting the indices $\pm$. A $\mathbb{K}$-module together with a trilinear map satisfying these identities is called a *(linear) Jordan triple system* (JTS). Every linear Jordan triple system arises by the construction just described. One may also define the notion of an *involution of a Jordan pair*; then Jordan triple systems are the same as Jordan pairs with involution (see [Lo75]). For any JTS (V,T), the Jordan pair $(V^+,V^-)=(V,V)$ with $T^\pm = T$ is called the *underlying Jordan pair.*

1.6. The Jordan-Lie functor. If T is a JTS on V, then

$$[X,Y,Z] = T(X,Y,Z) - T(Y,X,Z)$$

defines a LTS on V. This defines a functor from the category of JTS to the category of LTS over $\mathbb{K}$, which we call the *Jordan-Lie functor* (cf. [Be00]). The LTS (V,R) belongs the θ-fixed subalgebra $\mathfrak{g}^\theta$ of $\mathfrak{g}$ whose involution is induced by the canonical involution of the 3-graded Lie algebra $\mathfrak{g}$.

1.7. Jordan algebras. A *Jordan algebra* is a $\mathbb{K}$-module J with a bilinear and commutative product $x \bullet y$ such that the identity

(J2) $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$

holds (cf. [McC04]; the reader not familiar with Jordan theory may throughout think of the example $J = A^\tau$, the set of fixed points of an anti-automorphism τ of an associative algebra A, with bilinear product $x \bullet y = \frac{xy+yx}{2}$.) Jordan algebras always come in families: for any Jordan algebra J and element $a \in J$, the *homotope* J_a is defined as follows. Writing $L(x)$ for the left multiplication operator $L(x)y = x \bullet y$, we can define a new product on J by

$$x \bullet_a y := (L(xa) + [L(x), L(a)])y = (x \bullet a) \bullet y + x \bullet (a \bullet y) - a \bullet (x \bullet y).$$

One proves that $J_a = (J, \bullet_a)$ is a again a Jordan algebra (see [McC04]), called the a*-homotope of* $(V, \bullet)$, and that $T(x,a,y) := 2x \bullet_a y$ defines a Jordan triple product on J (see [Be00, Ch. II] for a geometric proof of this fact). Thus (J,J) is a Jordan pair, but not all Jordan pairs are of this form.

1.8. Jordan pairs with invertible elements. For any Jordan pair (V^+,V^-) and every $a \in V^-$, we get a Jordan algebra structure on V^+ by defining the bilinear product

$$x \bullet_a y := \frac{1}{2} T^+(x,a,y),$$

again called the a*-homotope algebra.* This algebra is unital if, and only if, the element a is *invertible in* (V^+,V^-), which means that the operator

$$Q(a) := Q^-(a) := \frac{1}{2} T^-(a,\cdot,a) : V^+ \to V^-$$

is invertible; then $a^{-1} := (Q(a))^{-1}(a)$ is the unit element of J_a. Every unital Jordan algebra arises in this way from a Jordan pair having invertible elements, after a choice of such an element.

1.9. Invertibility. In a Jordan algebra $(J,\bullet)$, the quadratic operator U_x is defined by

$$U_x y = (2L_x^2 - L_{x^2})y = 2x \bullet (x \bullet y) - (x \bullet x) \bullet y.$$

Note that $U_x y = x \bullet_y x = Q(x)y$ in the Jordan pair (J,J). An element $x \in J$ is called *invertible* if it is invertible in (J,J), that is, if U_x is invertible; let

$$J^\times = \{x \in J |\, U_x \in \mathrm{GL}(J)\},$$

be the set of invertible elements. For such x, we define the *inverse* $x^{-1} = U_x^{-1}x$ and call the map $j : J^\times \to J$, $x \mapsto x^{-1}$ the *Jordan inverse.* Then the following identities hold (cf. [McC04, p. 200 and 201]):

(SB1) $U_{U_x y} = U_x U_y U_x$ (the "fundamental formula")
(SB2) $U_{x^{-1}} = (U_x)^{-1}$
(SB3) $x^{-1} = U_x^{-1}x$

Geometrically, this means that $(J^\times, e)$ is a reflection space with base point e (Proposition 2.1).

1.10. Quasi-invertibility. In general, a Jordan pair (V^+,V^-) does not have invertible elements. The important notion is the one of *quasi-inverse*: let

$$B(x,y) := \mathrm{id}_V - T(x,y) + Q(x)Q(y)$$
$$\tilde{\tau}_y(x) := x^y := B(x,y)^{-1}(x - Q(x)y)$$

the *Bergman operator*, resp. the *quasi-inverse* (the latter provided that (x,y) is *quasi-invertible*, which means that $B(x,y)$ is invertible). If x is invertible and (x,y) quasi-invertible, then we have $\tilde{\tau}_y(x) = (x^{-1}+y)^{-1}$.

1.11. Homotopes with respect to symmetric transformations of Jordan triple systems. For Jordan triple systems, there is a much richer supply of homotopes than for Lie triple systems: let (V,T) be a Jordan triple system. A *symmetric transformation of* (V,T) is an endomorphism α of V with the property

$$\forall x,y,z \in V: \quad T(\alpha x, y, \alpha z) = \alpha T(x, \alpha y, z). \tag{1.1}$$

Lemma 1.12. *If α is a symmetric transformation, then the formula*

$$T^{(\alpha)}(x,y,z) := T(x,\alpha y, z) \tag{1.2}$$

*defines a Jordan triple product T_α on V, called the α-*homotope *of T. The quadratic map and the Bergmann operator of T_α are given by*

$$Q_\alpha(x) = Q(x)\circ\alpha, \qquad B_\alpha(x,y) = B(x,\alpha y).$$

Proof. It is clear that T_α satisfies (LJT1). For the proof of (LJT2), take the identity (LJT2) for T and replace the middle elements v and y by αv and αy to see that $T(u,\alpha v, T(x,\alpha y,z))$ equals

$$T(T(u,\alpha v,x),\alpha y,z) - T^{\pm}(x, T(\alpha v,u,\alpha y),z) + T^{\pm}(x,\alpha y, T^{\pm}(u,\alpha v,z))$$

Then apply the defining relation of α to this term to get the identity (LJT2) for T_α. The quadratic operator for this JTS clearly is $Q_\alpha(x) = Q(x) \circ \alpha$, and it follows that

$$\begin{aligned} B_\alpha(x,y) &= \mathrm{id} - T_\alpha(x,y) + Q_\alpha(x)Q_\alpha(y) = \mathrm{id} - T(x,\alpha y) + Q(x) \circ \alpha Q(y) \circ \alpha \\ &= \mathrm{id} - T(x,\alpha y) + Q(x)Q(\alpha y) = B(x,\alpha y). \end{aligned}$$

■

The set of symmetric transformations of (V,T),

$$\mathrm{Svar}(T) := \{\alpha \in \mathrm{End}(V) \,|\, \forall x,y,z \in V: \quad T(\alpha x, y, \alpha z) = \alpha T(x, \alpha y, z)\}$$

is called the *structure variety of the Jordan triple system* T. Note that, by the fundamental formula $Q(Q(x)z) = Q(x)Q(z)Q(x)$, all quadratic maps $Q(x)$ belong to $\mathrm{Svar}(T)$, and so do the Bergmann operators of the form $B(x,x)$ for all $x \in V$ (by Bergmann structurality $Q(B(x,y)z) = B(x,y)Q(z)B(y,x)$, cf. [Lo75, JP26]). Also, $\mathrm{Svar}(T)$ contains all involutive automorphisms of T, together with their negatives. More generally, $\mathrm{Svar}(T)$ is a cone, i.e., invariant under taking scalar multiples.

2. Deformations of quadratic prehomogeneous symmetric spaces

The definition and elementary properties of *reflection spaces* and *symmetric spaces* are recalled in Appendix A.

Proposition 2.1. *Assume J is a Jordan algebra over a ring $\mathbb{K}$ with Jordan product $x \bullet y$ and unit element 1. Then the set $J^\times$ of invertible elements of J is a reflection space with product map*

$$\mu(x,y) = \sigma_x(y) = U_x(y^{-1}) = U_x U_y^{-1} y.$$

The symmetry σ_a with respect to $a \in J^\times$ is the Jordan inverse in the homotope algebra $(J, \bullet_{a^{-1}})$ (which has a as unit element), and the algebraic differential at a of this map, defined via scalar extension by dual numbers ($\mathbb{K}[\varepsilon] = \mathbb{K} \oplus \varepsilon\mathbb{K}$, $\varepsilon^2 = 0$) by the relation

$$\sigma_a(a + \varepsilon v) = \sigma_a(a) + \varepsilon D\sigma_a(a)v,$$

is the negative of the identity of J: $D\sigma_a(a) = -\mathrm{id}_J$. Hence $(J^\times, \mu)$ is an algebraic symmetric space.

Proof. Clearly $J^\times$ is stable under the maps $j(x) = x^{-1}$ (the Jordan inverse) and U_x for every $x \in J^\times$. Let $\mathcal{Q}(x) := U_x$. Then the identities (SB1) – (SB3) from Section 1.9 are exactly the identities (SB1) – (SB3) from Appendix A.1, and therefore Prop. A.2 implies that $J^\times$ carries a natural structure of a reflection space given by $\mu(x,y) = \sigma_x(y) = U_x(y^{-1}) = U_x U_y^{-1} y$, which is the Jordan inverse of y in the Jordan algebra $(J, \bullet_{x^{-1}})$.

Now, the algebraic tangent map of j is given by

$$j(1 + \varepsilon v) = (\mathbf{1} + \varepsilon v)^{-1} = \mathbf{1} - \varepsilon v,$$

whence $Dj(1) = -\mathrm{id}_J$, and the same argument holds in the homotope algebra $(J, \bullet_{a^{-1}})$. ■

We call the spaces obtained by the preceding construction *quadratic prehomogenoues symmetric spaces* (cf. [Be00, Chapter II], where a more intrinsic definition is proposed). Now we describe what happens when we translate the space $J^\times$ such that a base point a is translated into the origin $0 \in J$. Recall from Section 1.10 the definition of the Bergmann-operator and the quasi-inverse, and define, for any linear Jordan pair (V^+, V^-) and some fixed element $a \in V^-$,

$$\mathcal{U}_a := \{x \in V^+ \,|\, B(x,-a) \in \mathrm{GL}(V^+)\}$$

(the set of elements $x \in V^+$ such that $(x,-a)$ is quasi-invertible), and, for $x, y \in \mathcal{U}_a$,

$$\begin{aligned} \mathcal{Q}(x)(y) &:= \tau_{2x+Q(x)a} \circ B(x,-a)(y) = 2x + Q(x)a + B(x,-a)y, \\ \sigma_0(y) &:= -\tilde{\tau}_{-a}(y) = \tilde{\tau}_a(-y) = -B(-y,a)^{-1}\big(y + Q(y)a\big), \\ \mu(x,y) &:= (\mathcal{Q}(x) \circ \sigma_0)(y) = 2x + Q(x)a - B(x,-a)B(-y,a)^{-1}\big(y + Q(y)a\big). \end{aligned}$$

Theorem 2.2.

(i) *Assume J is a Jordan algebra and a is invertible in J. Let $\tau = \tau_{-a^{-1}} : J \to J$ be translation by $-a^{-1}$. Then $\mathcal{U}_a = \tau(J^\times)$, and τ is an isomorphism of the symmetric space structure from* Proposition 2.1 *onto the product map μ defined by the formula above.*

(ii) *Assume (V^+, V^-) is any Jordan pair and $a \in V^-$ an arbitrary element. Then $(\mathcal{U}_a, \mathcal{Q}, \sigma_0)$ is a reflection space with base point 0. Moreover, the algebraic differential of the symmetry $\sigma_x = \mu(x,\cdot)$ at x equals $-\mathrm{id}_{V^+}$, and thus $\mathcal{U}_a$ is a symmetric space.*

Proof. (i) First of all, $x \in \tau(J^\times)$ if and only if $a^{-1} + x$ is invertible, if and only if $U_{a^{-1}+x}$ is invertible; but since U_a is assumed to be invertible, we have the identity

$$B(x,a) = U_{x-a^{-1}} U_a \tag{2.1}$$

([Lo75, I.2.12]) and hence x belongs to $\tau(J^\times)$ if and only if $B(x,-a)$ is invertible, i.e., if and only if $(x,-a)$ is a quasi-invertible pair.

Now let us calculate the push forward of the product map from Proposition 2.1 via τ, or, which amounts to the same (cf. Prop. A.2), of the quadratic map and inversion with respect to a base point. With respect to the old base point $a^{-1} \in J^\times$, the quadratic map was given by $\mathcal{Q}(x)y = U_x U_a y$, and the symmetry was the Jordan inverse j_a in the Jordan algebra J_a with unit element a^{-1}. We have to show that transformation by τ leads to the new quadratic map and new inverse

$$\begin{aligned} \tau\Big(\mathcal{Q}\big(\tau^{-1}(x)\big)\tau^{-1}(y)\Big) &= 2x + U_x a + B(x,-a)y, \\ \tau(\sigma_a(\tau^{-1}(x)) &= \tilde{\tau}_a(-x). \end{aligned} \tag{2.2}$$

The first equality in (2.2) follows from (2.1):

$$\begin{aligned} U_{x+a^{-1}} U_a (y + a^{-1}) - a^{-1} &= B(x,-a)(y + a^{-1}) \\ &= B(x,-a)y + a^{-1} + T(x,a,a^{-1}) + U_x U_a a^{-1} \\ &= B(x,-a)y + a^{-1} + 2x + U_x a \end{aligned}$$

For the proof of the second equality from (2.2), note that the inverse in J_a is $j_a(x) = U_a^{-1}x^{-1}$, and hence

$$\tau \circ j_a \circ \tau^{-1}(x) = -a^{-1} + U_a^{-1}(x + a^{-1})^{-1} = -a^{-1} + U_a^{-1}(a^{-x}) = (-x)^a$$

by the symmetry principle for the quasi-inverse, see [Lo75, 3.3].

(ii) We use the homotope $J = V_a^+$ with product $x \bullet_a y = \frac{1}{2}T(x,a,y)$ and adjoin a unit element $\hat{J} = \mathbb{K}\mathbf{1} \oplus J$ (if J happens to be unital, we forget the unit element of J). For $x, y \in J$, we calculate in $\hat{J}$,

$$\begin{aligned}
U_x y &= (2L_x^2 - L_{x^2})y = 2x \bullet_a (x \bullet_a y) - (x \bullet_a x) \bullet_a y = Q^+(x)Q^-(a)y \\
U_{\mathbf{1}\oplus x}(y) &= y + 2x \bullet_a y + U_x y = (\mathrm{id} + T(x,a) + Q(x)Q(a))y = B(x,-a)y, \\
U_{\mathbf{1}\oplus x}(\mathbf{1} \oplus y) &= U_{\mathbf{1}\oplus x}(\mathbf{1}) + U_{\mathbf{1}\oplus x}(y) = (\mathbf{1} + x)^2 + U_{\mathbf{1}\oplus x} y \\
&= \mathbf{1} \oplus \big(2x + x^2 + U_{\mathbf{1}\oplus x} y\big) \\
&= \mathbf{1} \oplus \big(2x + Q(x)a + B(x,-a)y\big). \qquad (2.3)
\end{aligned}$$

The second equality shows that $B(x,-a)$ is the restriction of $U_{\mathbf{1}\oplus x}$ to J, and hence $(x,-a)$ is quasi-invertible in (V^+, V^-) if and only if $\mathbf{1} + x$ is invertible in $\hat{J}$, i.e.,

$$\mathbf{1} + \mathcal{U}_a = \hat{J}^\times \cap (\mathbf{1} + J).$$

This set is a symmetric subspace of $\hat{V}^\times$. In fact, the third equation of (2.3) shows that it is stable under the quadratic map $\mathcal{Q}$ (with respect to the origin 1), and it is also stable under inversion: if $(x,-a)$ is quasi-invertible, then the inverse of $\mathbf{1} + x$ in $\hat{J}$ is given by (see [Lo75, I.3])

$$(\mathbf{1} + x)^{-1} = \mathbf{1} - \tilde{\tau}_{-a}(x).$$

Transferring the symmetric space structure from $\mathbf{1} + \mathcal{U}_a$ to $\mathcal{U}_a$ via the map $\mathbf{1} \oplus x \mapsto x$, we obtain the claimed formula.

Finally, we calculate the differential $D\sigma_0(0)$ of σ_0 at the origin:

$$\sigma_0(0 + \varepsilon v) = -B(\varepsilon x, -a)^{-1}(\varepsilon x + Q(\varepsilon x)a) = -(\mathrm{id} - \varepsilon T(x,a))\varepsilon x = -\varepsilon x,$$

whence $D\sigma_0(0) = -\mathrm{id}_{V^+}$. Since $\sigma_x = \mathcal{Q}(x) \circ \sigma_0$, it follows that $D\sigma_x(x) = -\mathrm{id}_{V^+}$. ■

Remark. By the preceding calculations, the fundamental formula

$$U_{\mathbf{1}-x} U_{\mathbf{1}-y} U_{\mathbf{1}-x} = U_{U_{\mathbf{1}-x}(\mathbf{1}-y)}$$

in $\mathbf{1} - \mathcal{U}_a$ translates to the identity

$$B(x,a)B(y,a)B(x,a) = B\big(2x - Q(x)a + B(x,a)y, a\big),$$

which in turn can be interpreted by saying that, for every $a \in V^-$, the map $x \mapsto B(x,a)$ is a homomorphism of the reflection space $\mathcal{U}_a$ into the automorphism group of (V^+, V^-), seen as a reflection space.

2.3. The topological case. Assume $\mathbb{K}$ is a topological ring with dense unit group. A *topological Jordan pair* is a Jordan pair together with topologies on V^+ and V^- such that $V^\pm$ are Hausdorff topological $\mathbb{K}$-modules $V^\pm$ and the trilinear maps $T^\pm$ are continuous. Following [BN05], we say that $V^\pm$ *satisfies the condition* (C2) if the following holds:

(C2) for all $a \in V^\mp$, the sets $\mathcal{U}_a$ are open and the maps $\mathcal{U}_a \times V^\pm \to V^\pm$, $(x,z) \mapsto B(x,-a)^{-1}z$ are continuous.

Theorem 2.4. *Assume* (V^+, V^-) *is a topological Jordan pair satisfying the condition* (C2)*.*

(1) *The symmetric space* $(\mathcal{U}_a, \mu)$ *from* Theorem 2.2 *is smooth, i.e., it is a symmetric space in the category of manifolds over* $\mathbb{K}$*.*

(2) *With respect to the base point* 0*, the Lie triple system of the symmetric space* $(\mathcal{U}_a, \mathcal{Q}, \sigma_0)$ *is given by antisymmetrizing the Jordan triple product* $S(u,v,w) := T^+(u, Q(a)v, w)$ *in the first two variables:*

$$[x,y,z] = S(x,y,z) - S(y,x,z).$$

Proof. (1) Smoothness of μ follows from [BN05, Prop. 5.2], and the Property (S4) of a symmetric space follows from the calculation of $D\sigma_0(0)$ given above (the differential in the sense of differential calculus is the same as the algebraic differential, cf. [Be06]).

(2) Once again, the easiest proof is by reduction to unital Jordan algebras: for a unital Jordan algebra (J, e), the Lie triple system of the pointed symmetric space $(J^\times, e)$ is given by

$$[x,y,z] = T(x,y,z) - T(y,x,z) = [L_x, L_y]z = x \bullet (y \bullet z) - y \bullet (x \bullet z) \qquad (2.4)$$

(see [BN05, Theorem 3.4] and its proof which is algebraic in nature). For invertible a, the Lie triple system of $(J^\times, a)$ is then given by the same formula, but replacing $\bullet$ by $\bullet_a$ and T by $T_a(x,y,z) = T(x, Q(a)y, z) = S(x,y,z)$.

Starting now with a Jordan pair (V^+, V^-) and arbitrary $a \in V^-$, we apply the preceding remarks to the unital Jordan algebra $\hat{J} = \mathbb{K}\mathbf{1} \oplus J$ with $J = V_a^+$, as in the proof of Theorem 2.2(ii). Then the Lie triple system of the pointed symmetric space $(\mathbf{1} + \mathcal{U}_a, \mathbf{1}) = \hat{J}^\times \cap (\mathbf{1} + J)$ is given by restricting the triple product (2.4) (with $\bullet = \bullet_a$) to $J = V_a^+$. ■

The preceding result applies to all finite-dimensional and to all Banach Jordan pairs over $\mathbb{R}$ or $\mathbb{C}$, but also to many other not so classical cases. Note that the (C2)-condition assures in particular that the sets $\mathcal{U}_a$ are not reduced to $\{0\}$ (if $V^+ \neq \{0\}$). As to Part (2) of the theorem, it is a general fact that antisymmetrizing a Jordan triple system in the first two variables gives a Lie triple system (the "Jordan-Lie functor", see Section 1.6). The preceding construction works also in other contexts where some sort of differential calculus can be implemented (cf. Remark 10.1 in [BN05]), and it can also be applied if we start with any non-unital Jordan algebra instead of J_a; then $\mathcal{U}_a$ corresponds to the set of quasi-invertible elements in this algebra.

3. Deformations of generalized projective geometries

The geometric or "integrated" version of a Jordan pair (V^+, V^-) is a *generalized projective geometry* $(\mathcal{X}^+, \mathcal{X}^-)$, see Appendix B. We give an analog of Theorem 2.2 in this context; it generalizes parts of [Be03, Section 6].

Theorem 3.1. *Let $(\mathcal{X}^+,\mathcal{X}^-)$ be the generalized projective geometry associated to a Jordan pair (V^+,V^-), fix $a,b\in\mathcal{X}^-$ such that $\mathcal{U}_{a,b}:=\mathfrak{v}_a\cap\mathfrak{v}_b$ is non-empty, and let $p:\mathcal{U}_{a,b}\to\mathcal{X}^-$, $x\mapsto\mathbf{\Pi}_{\frac{1}{2}}(a,x,b)$ be the midpoint map assigning to x the midpoint of a and b in the affine space $\mathfrak{v}_x$. Then the set $\mathcal{U}_{a,b}$ becomes a symmetric space when equipped with the multiplication map*

$$\mu(x,y)=(-1)_{x,p(x)}(y)=\mathbf{\Pi}_{-1}\big(x,\mathbf{\Pi}_{\frac{1}{2}}(a,x,b),y\big).$$

In case $(V^+,V^-)=(J,J)$ comes from a unital Jordan algebra J, then p extends to a bijection denoted by $p:\mathcal{X}^+\to\mathcal{X}^-$; in this case the symmetric space $\mathcal{U}_{a,b}$ is the symmetric space associated to a polar geometry (see Section 4.1), and it is isomorphic to the quadratic prehomogeneous symmetric space $J^\times$.

Proof. Let us show first that the set $\mathcal{U}_{a,b}$ is indeed stable under the symmetries $\sigma_x=(-1)_{x,p(x)}$ for $x\in\mathcal{U}_{a,b}$. To this end, note that, since $(-1)=(-1)^{-1}$, the identity (PG1) implies that, for $x\in\mathcal{U}_{c,b}$, the pair

$$(g^+,g^-):=(\sigma_x^+,\sigma_x^-)=\big((-1)_{x,p(x)},\,(-1)_{p(x),x}\big)$$

acts as an automorphism on $(\mathcal{X}^+,\mathcal{X}^-)$. By definition, $\sigma_x=\sigma_x^+$ is the first component of this automorphism. Then, using that $\mathbf{\Pi}_r(u,c,v)=\mathbf{\Pi}_{1-r}(v,c,u)$ and that $r_{v,c}^{-1}=(r^{-1})_{v,c}$, we see that

$$\sigma_x^-(c)=\mathbf{\Pi}_{-1}\big(\mathbf{\Pi}_{\frac{1}{2}}(a,x,b),x,a\big)=\mathbf{\Pi}_2(a,x,\mathbf{\Pi}_{\frac{1}{2}}(a,x,b))=b,$$

and similarly $\sigma_x^-(b)=a$. Therefore $\sigma_x^+(\mathfrak{v}_b)=\mathfrak{v}_{\sigma_x^-(b)}=\mathfrak{v}_a$ and $\sigma_x(\mathfrak{v}_a)=\mathfrak{v}_b$ and hence σ_x preserves $\mathcal{U}_{a,b}$. Moreover, it follows that the group $G(\mathcal{U}_{a,b})$ generated by all products of two symmetries preserves both a and b and hence acts affinely both on $\mathfrak{v}_a$ and on $\mathfrak{v}_b$.

The defining identies (S1) – (S4) of a symmetric space are now all easily checked: x is fixed by σ_x since $(-1)_{x,c}(x)=x$ for all $c\top x$; $\sigma_x^2=\mathrm{id}$ since $(-1)_{x,c}^2=1_{x,c}=\mathrm{id}$; the tangent map of σ_x at x is $-\mathrm{id}_{T_xM}$ since in the chart $\mathfrak{v}_{p(x)}$, centered at x, this map is just the negative of the identity; finally, in order to prove (S3) note first that, because of the fundamental identity (PG2), the pair (p,p) is self-adjoint, and thus for all scalars r,

$$p\circ r_{x,p(x)}(y)=p\,\mathbf{\Pi}_r(x,p(x),y)=\mathbf{\Pi}_r(p(x),x,p(y))=r_{p(x),x}\circ p(y)$$

In particular, for $r=-1$, we have $p\circ(-1)_{x,p(x)}=(-1)_{p(x),x}\circ p$, i.e., $p\circ\sigma_x=\sigma_x'\circ p$. Using this, we obtain

$$\begin{aligned}\sigma_x\mu(y,z)&=\sigma_x\mathbf{\Pi}_{-1}(y,p(y),z)=\mathbf{\Pi}_{-1}(\sigma_xy,\sigma_x^-p(y),\sigma_xz)\\&=\mathbf{\Pi}_{-1}(\sigma_xy,p(\sigma_xy),\sigma_xz)=\mu(\sigma_xy,\sigma_xz),\end{aligned}$$

which is precisely the property (S3).

Finally, assume that (V^+,V^-) comes from a unital Jordan algebra. In this case, there is a canonical identification of $\mathcal{X}^+$ and $\mathcal{X}^-$, denoted by $n:\mathcal{X}^+\to\mathcal{X}^-$ (see [BN04], [Be03]), and then the property that $\mathfrak{v}_a\cap\mathfrak{v}_b$ be non-empty is equivalent to saying that the pair $(a,n(b))$ be quasi-invertible. If this is the case, the map p is, via n, identified with $(-1)_{a,n(b)}$ ([Be03, 2.1]), and hence extends canonically to a bijection $\mathcal{X}^-\to\mathcal{X}^+$ (called an *inner polarity*). In [Be03, Section 6] it is shown that the symmetric space associated to an inner polarity is, in a suitable chart, the same as the quadratic prehomogeneous space $J^\times$. ∎

3.2. The quasi-inverse revisited. In order to show that the symmetric spaces of Theorem 3.1 and of Theorem 2.2 coincide, we need the global interpretation of the quasi-inverse via translation groups: in any module $\mathfrak{m}$ over a ring $\mathbb{K}$ define for $r \in \mathbb{K}$ the proper dilations $r_x(y) = (1-r)x + ry$; then if 2 is invertible in $\mathbb{K}$, vector addition can be recovered from the dilations via $x+y = 2\frac{x+y}{2} = 2_0 2_x^{-1}(y)$, i.e., translations are given by $\tau_x = 2_0 2_x^{-1}$. In a generalized projective geometry, and with respect to a fixed base point (o,o'), the identity (PG1) then implies that the pair

$$(\tau_x, \tilde{\tau}_x) = (2_{o,o'} 2_{x,o'}^{-1},\ 2_{o',o}^{-1} 2_{o',x})$$

is an automorphism. Dually, $(\tilde{\tau}_a, \tau_a)$ for $a \in V^-$ is then defined. One can show that, in the affine chart (V^+, V^-) corresponding to the base point (o,o'), $\tilde{\tau}_a$ is indeed the quasi-inverse from Jordan theory. Similarly, we have

$$(\tau_{-x}, \tilde{\tau}_{-x}) = (2_{x,o'} 2_{o,o'}^{-1},\ 2_{o',x}^{-1} 2_{o',o}).$$

Theorem 3.3. *Let notation be as in* Theorem 3.1 *and fix* $o' := b$ *as base point in* $\mathcal{X}^-$ *and choose an arbitrary base point* $o \in \mathfrak{v}_a \cap \mathfrak{v}_b$. *We write* (V^+, V^-) *for the Jordan pair associated to the base point* (o,o'). *Then* $\mathcal{U}_{a,b} = \mathcal{U}_{a,o'}$ *is naturally identified with* $\mathcal{U}_a$, *and in the global chart* V^+ *of* $\mathcal{U}_{a,b}$, *the symmetric space structure from* Theorem 3.1 *is given by the explicit formula from* Theorem 2.2.

Proof. We calculate in the Jordan pair $(V^+, V^-) = (\mathfrak{v}_{o'}, \mathfrak{v}'_o)$. For $a \in V^-$ and $x \in V^+$, the following are equivalent:

$$\begin{aligned} x \top a \quad &\Longleftrightarrow \quad x \in \mathfrak{v}_a = \tilde{\tau}_a(\mathfrak{v}_{o'}) = \tilde{\tau}_a(V^+) \\ &\Longleftrightarrow \quad \tilde{\tau}_{-a}(x) \in V^+ \\ &\Longleftrightarrow \quad (x,-a) \text{ quasi-invertible.} \end{aligned}$$

Hence $\mathfrak{v}_a \cap \mathfrak{v}_{o'} = \tilde{\tau}_a(V^+) \cap V^+ = \mathcal{U}_a$.

Now let us show that the symmetric space structure from Theorem 3.1 is given by the explicit formulae

$$\begin{aligned} \sigma_o &= -\tilde{\tau}_{-a} \\ \sigma_x \sigma_o &= \mathcal{Q}(x) = \tau_{2x + Q(x)a} \circ B(x,-a). \end{aligned} \tag{3.1}$$

The proof is in several steps. Let $d := \frac{1}{2}a$, i.e., $a = 2d = 2_{o,o'}d$. Then

$$\begin{aligned} \sigma_x(y) = \mathbf{\Pi}_{-1}(x, \mathbf{\Pi}_{\frac{1}{2}}(o', x, 2d), y) &= \mathbf{\Pi}_{-1}(x, 2_{o',x}^{-1} 2_{o',o} d, y) \\ &= \mathbf{\Pi}_{-1}(x, \tilde{\tau}_{-x} d, y) \\ &= \tau_{-x} \mathbf{\Pi}_{-1}(2x, d, y + x) \\ &= \tau_{-x} \circ (-1)_{2x,d} \circ \tau_x(y), \end{aligned} \tag{3.2}$$

and for $x = o$ this gives

$$\sigma_o = (-1)_{o,d} = (-1)_{\tilde{\tau}_d.o, \tau_d.o'} = \tilde{\tau}_d (-1)_{o,o'} \tilde{\tau}_{-d} = -\tilde{\tau}_{-2d} = -\tilde{\tau}_{-a},$$

proving the first relation from (3.1). Now we are going to prove the second relation from (3.1). Recall from [Be02, Cor. 5.8] or [Be00, X.3.2] that one can express the dilations $r_{z,b}$ for a scalar $r \in \mathbb{K}$ and a quasi-invertible pair (z,b) via translations, quasi-inverses and the dilation $r\,\mathrm{id}_{V^+} = r_{o,o'}$. For the scalar $r=-1$, the formula reads:

$$(-1)_{z,b} = \tau_z \circ \tilde{\tau}_{2\tilde{\tau}_{-z}(b)} \circ \tau_z \circ (-1) = (-1) \circ \tau_{-z} \circ \tilde{\tau}_{-2\tilde{\tau}_{-z}(b)} \circ \tau_{-z} \tag{3.3}$$

where $(-1) = (-1)_{o,o'} = -\,\mathrm{id}$. Using this, we get from (3.2) along with $\sigma_o = (-1)_{o,d}$,

$$\begin{aligned}\sigma_x\sigma_o &= \tau_{-x} \circ (-1)_{2x,d} \circ \tau_x \circ (-1)_{o,d} \\ &= \tau_{-x} \circ (-1) \circ \tau_{-2x} \circ \tilde{\tau}_{-2\tilde{\tau}_{-2x}(d)} \circ \tau_{-2x} \circ \tau_x \circ (-1) \circ \tilde{\tau}_{-2d} \\ &= \tau_x \circ \tilde{\tau}_{2\tilde{\tau}_{-2x}(d)} \circ \tau_x \circ \tilde{\tau}_{-2d}.\end{aligned} \tag{3.4}$$

¿From the proof of Theorem 3.1, we know already that $\sigma_x\sigma_o$ acts as an affine transformation on $\mathcal{U}_a = V^+ \cap \mathfrak{v}_a$. Thus, in order to prove the second relation from (3.1), it suffices to determine its linear part and its value at o. The value at o of this transformation is

$$\begin{aligned}\sigma_x\sigma_o(o) &= x + \tilde{\tau}_{2\tilde{\tau}_{-2x}(d)}(x) = x + \frac{1}{2}\tilde{\tau}_{\tilde{\tau}_{-2x}(d)}(2x) \\ &= x + \frac{1}{2}(2x + Q(2x)d) = 2x + 2Q(x)d = 2x + Q(x)a\end{aligned}$$

where we passed to the last line by using the covariance property of the quasi-inverse map [Lo75, Th. I.3.7] which implies, in the notation of [Lo75],

$$\begin{aligned}\tilde{\tau}_{\tilde{\tau}_{-z}b}(z) = z^{(b^{-z})} &= B(-z,b)\big((z-z)^b - (-z)^b\big) \\ &= -B(-z,b)B(-z,b)^{-1}(-z - Q(-z)b) = z + Q(z)b.\end{aligned}$$

Let us determine the linear part of $\sigma_x\sigma_o$. For a general automorphism g such that $g(o) \in V^+$, the linear part in the triple decomposition $g \in \tau_{V^+}\,\mathrm{Aut}(V^+,V^-)\tilde{\tau}_{V^-}$ is the inverse of the *denominator* $d_g(o)$ (see [BN04, Theorem 2.10 and its proof]). Now, the denominator $d_g(o)$ of the transformation $g = \tilde{\tau}_w \circ \tau_v$ is $d_g(o) = B(v,w)$ (see [BN04]), and hence the linear part of $\sigma_x\sigma_o$ is

$$\begin{aligned}(d_{\sigma_x\sigma_o}(o))^{-1} = (d_{\tilde{\tau}_{2\tilde{\tau}_{-2x}(d)}\circ\tau_x}(o))^{-1} &= B\big(x, 2\tilde{\tau}_{-2x}(d)\big)^{-1} = B\big(2x, \tilde{\tau}_{-2x}(d)\big)^{-1} \\ &= B(2x,-d) = B(x,-2d) = B(x,-a)\end{aligned}$$

where we used JP35 in passing to the last line. Summing up, we have $\sigma_x\sigma_0 = \tau_{2x+Q(x)a} \circ B(x,-a)$, and (3.1) is proved. ∎

Note that combining Theorem 3.1 and Theorem 3.3 gives another (in fact, independent) proof of Theorem 2.2(ii). Such a proof does not involve the technique (which is artificial from a geometric point of view) of adjoining a unit element to the Jordan algebras V_a^+.

4. Conformal deformations of polar geometries

4.1. The symmetric space of a generalized polar geometry. Recall from Appendix B.3 that Jordan triple systems (V, T) correspond to *generalized polar geometries* $(\mathcal{X}^+, \mathcal{X}^-; p)$, i.e., to generalized projective geometries with polarity $p : \mathcal{X}^+ \to \mathcal{X}^-$. We denote by

$$M^{(p)} := \{x \in \mathcal{X}^+ \,|\, x \top p(x)\} \tag{4.1}$$

the set of non-isotropic points of the polarity p. This set becomes a symmetric space when equipped with the product

$$\mu(x, y) := \sigma_x(y) := (-1)_{x,p(x)}(y) = \mathbf{\Pi}_{-1}\big(x, p(x), y\big) \tag{4.2}$$

(see [Be02, th. 4.1]; the proof is similar to the one of Theorem 3.1). Fixing a point $o \in M^{(p)}$ and choosing the transversal pair $(o, o') = (o, p(o))$ as base point in $(\mathcal{X}^+, \mathcal{X}^-)$, we can describe the affine image of $M^{(p)}$ as follows: when we identify $\mathcal{X}^+$ and $\mathcal{X}^-$ and V^+ and V^- via p, then $(x, p(x))$ is a transversal pair if and only $(x, -x)$ is a quasi-invertible pair (see Chapter 3), i.e., if and only the Bergman operator $B(x, -x)$ is invertible, and hence[1]

$$M^{(p)} \cap V = \{x \in V \,|\, B(x, -x) \,\text{is invertible}\,\}. \tag{4.3}$$

4.2. The symmetric space associated to the α-homotope. Now consider the Jordan triple systems $V_\alpha = (V, T_\alpha)$, the α-homotopes of V with respect to symmetric transformations $\alpha : V \to V$ (cf. Lemma 1.12). Since V_α is a JTS in its own right, the preceding construction associates to V_α a generalized polar geometry $(\mathcal{X}_\alpha^+, \mathcal{X}_\alpha^-, p_\alpha)$ with symmetric space M_α. The explicit formula for the affine image is again given by (4.3), with the same underlying $\mathbb{K}$-module, but T replaced by T_α and B by $B_\alpha(u, v) = B(u, \alpha v)$ (Lemma 1.12). However, *a priori*, the global spaces $\mathcal{X}_\alpha^+$ for various α, are different spaces. In order to compare the geometries $\mathcal{X}_\alpha$ and the symmetric spaces M_α with each other and to realize them as deformations of $M^{(p)}$, we need a common realization. This is given by the following result:

[1] One should note that the sign in the correspondence between spaces and triple systems is a matter of convention. Geometers look at Hermitian symmetric spaces of non-compact type as having "negative" curvature, and on the other hand, one is used to associate them to "positive" Hermitian JTS, see [Lo77]. In order to check signs, one may use that for the JTS $\mathbb{R}$ with $T(x,y,z) = 2xyz$, the positive Bergmann polynomial $B(x, -x) = (1 + x^2)^2$ must belong to the compact space $\mathbb{RP}^1$ and the "opposite" Bergmann polynomial $B(x, x) = (1 - x^2)^2$ to its non-compact dual.

Theorem 4.3. *Let* $(\mathcal{X}^+, \mathcal{X}^-; p)$ *be the generalized polar geometry of a Jordan triple system* (V, T)*, and let* $\alpha : V \to V$ *be a symmetric transformation, i.e.,* $\alpha \in \mathrm{Svar}(T)$*. Let* $V_\alpha = (V, T_\alpha)$ *be the* α*-homotope of* (V, T) *and let* $(\mathcal{X}^+_\alpha, \mathcal{X}^-_\alpha; p_\alpha)$ *be the generalized polar geometry associated to* (V, T_α)*, with symmetric space denoted by* $M_\alpha \subset \mathcal{X}^+_\alpha$*. Then there exists a canonical embedding* $\Phi_\alpha : \mathcal{X}^+_\alpha \to \mathcal{X}^+$ *having the following properties:*

(1) *The affine image of* M_α *under this imbedding coincides with the description of* M_α *in the chart* V_α*, i.e.*
$$\Phi_\alpha(M_\alpha) \cap V = \{x \in V \,|\, B(x, -\alpha x) \in \mathrm{GL}(V)\}$$

(2) *The tangent map of* Φ_α *at the origin can be algebraically defined, and it is the identity map* $V_\alpha \to V$*.*

(3) *If* α *is invertible, then the image of this imbedding is precisely the symmetric space* $M^{(p\alpha)}$ *associated to the polarity* $p \circ \alpha : \mathcal{X}^+ \to \mathcal{X}^-$*.*

Proof. First we describe the construction of Φ_α. Saying that $\alpha : V \to V$ is a symmetric transformation of the JTS (V, T) is equivalent to saying that the map of Jordan pairs
$$(\alpha, \alpha) : (V, V) \to (V, V)$$
is a structural transformation of the Jordan pair (V, V) (see Appendix C). We apply Lemma C.3, with $(W^+, W^-) = (V^+, V^-)$ and $(f, g) = (\alpha, \alpha)$; then the Jordan pair (V^+, W^-) from Lemma C.3 is (V_α, V_α), with the JTS V_α defined by Lemma 1.12. The second point of Lemma C.3 says that
$$\phi = (\mathrm{id}, \alpha) : (V_\alpha, V_\alpha) \to (V, V)$$
is a homomorphism of Jordan pairs. As in Section C.4, the homomorphism (id, α) induces a homomorphism
$$(\Phi^+, \Phi^-) : (\mathcal{X}^+(V_\alpha), \mathcal{X}^-(V_\alpha)) \to (\mathcal{X}^+(V), \mathcal{X}^-(V))$$
with injective first component $\Phi^+ : \mathcal{X}^+(V_\alpha) \to \mathcal{X}^+(V)$.

We define $\Phi_\alpha := \Phi^+$ and show that this imbedding has the desired properties. Since (Φ^+, Φ^-) is a homomorphism of generalized projective geometries, its restriction to affine parts with respect to the base points is given by the pair of linear maps which induces it, namely by (id, α), and hence the algebraic tangent map can be defined and is equal to the pair (id, α) (cf. [Be02]). Thus the tangent map of Φ^+ at the base point o is the identity map $V_\alpha \to V$. This proves (2).

We prove (1). Let $x \in V^+ = V$. Recall that the element $p(x) \in \mathcal{X}^-$ is identified with $x \in V^- = V$. By definition, x belongs to the symmetric space $M^{(p)}$ if and only if x and $p(x)$ are transversal, if and only if the pair $(x, -p(x)) = (x, -x)$ is transversal (see proof of 3.1), if and only if $B(x, -x)$ is invertible. The same argument, applied to the JTS T_α, shows that x belongs to M_α if and only if $B_\alpha(x, -x)$ is invertible. But $B_\alpha(x, y) = B(x, \alpha y)$ (Lemma 1.12), and hence x belongs to M_α if and only if $B(x, -\alpha x)$ is invertible.

We prove (3). Assume α is symmetric for T and invertible. Then $(\mathrm{id}, \alpha) : (V_\alpha, V_\alpha) \to (V, V)$ is a Jordan pair isomorphism, and the induced map (Φ^+, Φ^-) is an isomorphism of geometries. Under this isomorphism, the polarity of $(\mathcal{X}^+(V_\alpha), \mathcal{X}^-(V_\alpha))$ defining the symmetric space M_α (the exchange automorphism) corresponds to the map $p \circ \alpha$ which is therefore the polarity defining the symmetric space $\Phi_\alpha(M_\alpha)$. This had to be shown. ∎

Remark 4.4. (Realisation as a graph.) Assume that $\alpha \in \mathrm{Svar}(T)$. Then it is easily checked that the graph of α, $\Gamma_\alpha = \{(v, \alpha v) | \, v \in V\}$, is a sub-JTS of the polarized JTS $(V \oplus V, \tilde{T})$ with

$$\tilde{T}\big((x,a),(y,b),(z,c)\big) = (T(x,b,z), T(a,y,c)),$$

and that Γ_α is isomorphic to V_α as JTS. By the preceding construction, this situation globalizes as follows: the polarized JTS $(V \oplus V, \tilde{T})$ corresponds to the polar geometry $(\mathcal{X}^+ \times \mathcal{X}^-, \mathcal{X}^- \times \mathcal{X}^+)$ with the exchange automorphism (see [Be02, Prop. 3.6]) and with associated symmetric space

$$M = \{(x,\alpha) \in \mathcal{X}^+ \times \mathcal{X}^- | \, x \top \alpha\}.$$

Then M_α is the intersection of M with the graph of the map

$$\Phi^+(\mathcal{X}^+) \to \mathcal{X}^-, \quad \Phi^+(x) \mapsto \Phi^-(x).$$

Remark 4.5. (Alternative proof in the real finite-dimensional case.) In the finite-dimensional real or complex case, the main statements of Theorem 4.3 can be proved in a different way: if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ with involution θ is the involutive 3-graded Lie algebra associated to the JTS T, then

$$\mathfrak{q}_\alpha := \{v + \theta(\alpha v) | \, v \in \mathfrak{g}_1\}$$

is a sub-Lie triple system of $\mathfrak{g}$ (a "defining plane" in the terminology of [Ri70]). Let $\mathfrak{h}_\alpha = [\mathfrak{q}_\alpha, \mathfrak{q}_\alpha]$ and define the subalgebra $\mathfrak{g}_\alpha := \mathfrak{h}_\alpha \oplus \mathfrak{q}_\alpha$ of $\mathfrak{g}$. Let G_α and H_α be the analytic subgroups of $G = \mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}_\alpha$, resp. $\mathfrak{h}_\alpha$. Then the orbit $M_\alpha := G_\alpha.o$ of the base point o in $\mathcal{X}^+$ is open, and it is a homogeneous symmetric space isomorphic to G_α / H_α coinciding (possibly up to topological connected components) with the space M_α from Theorem 4.3. This construction can already been found in the work [Ri70] by A.A. Rivillis. Since it uses the exponential map as essential ingredient, it gives rather results of local nature, and it does not work for general base fields.

Theorem 4.6. *Assume that $\mathbb{K}$ is a topological ring with dense unit group, and that (V,T) is a topological Jordan triple system over $\mathbb{K}$ such that*

(C1) *The set $(V \times V)^\times = \{(x,y) \in V \times V | \, B(x,y) \in \mathrm{GL}(V)\}$ of quasi-invertible pairs is open in $V \times V$, and the map*

$$(V \times V)^\times \times V \to V, \quad (x,y,z) \mapsto B(x,y)^{-1} z$$

is continuous.

Assume moreover that $\alpha \in \mathrm{Svar}(T)$ is a continuous map. Then M_α is a smooth symmetric space over $\mathbb{K}$, and it is an open submanifold of $\mathcal{X} = \mathcal{X}^+$. The Lie triple system of the pointed symmetric space (M_α, o) is V equipped with the Lie triple product

$$[X,Y,Z]_\alpha := T_\alpha(X,Y,Z) - T_\alpha(Y,X,Z) = T(X,\alpha Y, Z) - T(Y, \alpha X, Z).$$

Proof. This follows directly from [BN05, Theorem 6.3] since the JTS T_α again satisfies the property (C1). ■

In particular, the LTS $[\cdot,\cdot,\cdot]_\alpha$ and $[\cdot,\cdot,\cdot]_{-\alpha}$ are negatives of each other, which means that the symmetric spaces M_α and $M_{-\alpha}$ are *c-duals* of each other.

5. Behavior of geometric structures under deformation

For applications in geometry and harmonic analysis one not only needs to know how the affine image $M_\alpha \cap V$ of the symmetric space M_α as a set, but also how its geometric structure depends on the deformation parameter α. We give some answers to this question. Roughly, the dependence on the parameter α can be made explicit if we simply mind the rules: in the following formulae, replace M by M_α, T by T_α, $B(x,y)$ by $B(x,\alpha y)$ and $Q(x)$ by $Q(x)\circ\alpha$.

5.1. The product map. The product map of the symmetric space M is given by Formula (3.3), by letting $x=z$, $b=p(x)$:

$$\begin{aligned}\sigma_x(y) = (-1)_{x,p(x)}(y) &= \tau_x \tilde{\tau}_{2\tilde{\tau}_{-x}(x)} \tau_x (-1)_{o,o'}(y) \\ &= \tau_{2\tilde{\tau}_x(x)} B(2\tilde{\tau}_x(x), -x)^{-1} \tilde{\tau}_{2\tilde{\tau}_x(x)}(-y)\end{aligned} \tag{5.1}$$

In particular, the "square" $x^2 := \sigma_x(o)$ in the symmetric space M is

$$\sigma_x(o) = 2\tilde{\tau}_x(x) = 2(\mathrm{id}_V - Q(x))^{-1}x \tag{5.2}$$

(the last equality is proved as in [Be00, p. 195]).

5.2. Second derivatives and second order tangent bundles. The tangent bundle TM of M, which is again a symmetric space, is simply obtained by scalar extension by dual numbers, as already used in Section 2.1 (cf. [BN05, Th. 5.5]). Hence the geometrically important second tangent bundle TTM is given by scalar extension by the second order tangent ring $TT\mathbb{K} = \mathbb{K}[\varepsilon_1,\varepsilon_2]$. The same statement holds for the geometry $(\mathcal{X}^+,\mathcal{X}^-)$ itself. For instance, by a straightforward calculation using that $\varepsilon_1^2 = 0 = \varepsilon_2^2$, we get from the definition of the quasi-inverse and from (5.1)

$$\begin{aligned}\tilde{\tau}_a(\varepsilon_1 v_1 + \varepsilon_2 v_2) &= B(\varepsilon_1 v_1 + \varepsilon_2 v_2, a)^{-1}(\varepsilon_1 v_1 + \varepsilon_2 v_2 - Q(\varepsilon_1 v_1 + \varepsilon_2 v_2)a) \\ &= \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 T(v_1, a, v_2), \\ \sigma_x(x + \varepsilon_1 v_1 + \varepsilon_2 v_2) &= x - \varepsilon_1 v_1 - \varepsilon_2 v_2 + \varepsilon_1\varepsilon_2 T(v_1, \tilde{\tau}_{-x}(x), v_2).\end{aligned}$$

On the other hand, ordinary second differentials and second order tangent maps of twice continuously differentiable maps f are related via

$$f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2) = f(x) + \varepsilon_1 Df(x)v_1 + \varepsilon_2 Df(x)v_2 + \varepsilon_1\varepsilon_2 D^2 f(x)(v_1, v_2)$$

(see [Be06, Eqn. (7.18)]), so that from the preceding calculation we get the second differential of $\tilde{\tau}_a$ at the origin and the one of σ_x at x.

5.3. The canonical affine connection. It is possible to define the canonical affine connection of the symmetric space in a purely algebraic way. In the situation of Theorem 4.6 it coincides with any of the various differential geometric definitions of a connection (cf. [Be06, Chapter 26]). More precisely, in [Be06, Chapter 26] it is shown that the canonical connection ∇ of a symmetric space M can be defined in terms of the second order tangent bundle TTM, and we have the following chart description: let $x \in M$, V a chart around x inducing the flat connection ∇^0 on some neighborhood of x; then the difference $C := \nabla - \nabla^0$ (i.e., the Christoffel tensor of the canonical connection with respect to the chart) at the point x is simply the second derivative of the symmetry σ_x at x: $C_x = D^2\sigma_x(x)$. Since we have calculated this second differential above, we get

$$C_x(u,v) = T(v_1, \tilde{\tau}_{-x}(x), v_2) = T(v_1, (\mathrm{id}_V - Q(x))^{-1}x, v_2) \tag{5.3}$$

(for the last equality, see (5.2)). In the finite-dimensional real case this result is already given in [BH01, Theorem 2.3]; however, the proof given there does not carry over to the general situation considered here. As explained in [BH01], (5.3) implies that the canonical connections of the symmetric spaces M_α are *conformally equivalent* among each other. Note also that, for $\alpha = 0$, we get $C_x = 0$ for all x, and hence $\nabla = \nabla_0$, as expected.

5.4. Trace form, pseudo-metrics and their duals. It is clear that semi-simplicity is not preserved under deformations. Let us assume that $\mathbb{K}$ is a field, the JTS V is finite-dimensional over $\mathbb{K}$, and that the *trace form* $g_0(u,v) := \mathrm{tr}(T(u,v,\cdot))$ is non-degenerate. If this is the case, then g_0 extends to a pseudo-Riemannian metric tensor (non-degenerate symmetric two fold covariant tensor field) with affine picture given by

$$g_x(u,v) := g_o(u, B(x,-x)^{-1}v), \quad (u,v \in V,\, x \in M) \tag{5.4}$$

(see [Be00, X.6] for the case $\mathbb{K} = \mathbb{R}$). On the other hand, the dual two fold contraviant tensor field with affine picture $\gamma_x(\phi,\psi) = g_o(\phi, B(x,-x)\psi)$ can always be defined and exists also for the deformed spaces.

5.5. Invariant measures on symmetric spaces. In general, a (real, finite-dimensional) symmetric space M does not admit a $G(M)$-invariant measure (since, to our knowledge, no explicit counterexample can be found in the literature, we will give one below). A sufficient and necessary condition for the existence of such a measure (if M is connected) is that $R(u,v)$ be trace-free, for all $u,v \in T_oM$, which in turn is equivalent to the symmetry of the Ricci-form $\rho(u,v) = \mathrm{tr}[u,\cdot,v]$ (cf. [Be00, p. 99]). This holds, in particular, if $M = M_\alpha$ is the symmetric space associated to JTS (V, T_α) with non-degenerate trace form (see [Be00, X.6.2]). Since $R_\alpha(u,v) = T_\alpha(u,\cdot,v) - T_\alpha(\cdot,u,v)$ depends continuously on α, it will still be trace-free for all α belonging to the closure $\overline{S}$ in the structure variety of the "non-degenerate set" $S := \{\alpha \in \mathrm{Svar}(T) | \, T_\alpha \, \mathrm{non-deg.}\}$. For such α, the formula for the invariant integral from [Be00, X.6.2] will continue to hold:

$$I_\alpha(f) = \int_{M_\alpha \cap V} f(x) |\det B(x, -\alpha x)|^{-\frac{1}{2}} \, dx \tag{5.5}$$

(recall that $M_\alpha \cap V$ is open dense in M_α and in V). One might guess that this formula defines an invariant integral for any Jordan triple system (i.e., also for T such that the set S is empty), but the following counterexample shows that this is not true; thus homotopes of non-degenerate spaces enjoy properties that are not shared by arbitrary symmetric spaces.

Example 5.6. We construct a symmetric space M which admits no invariant $G(M)$-invariant measure. Contrary to our claim in [Be00, p. 211], it is not enough just to take the group type space of a solvable, non-unimodular group S (as a symmetric space, it does admit an invariant measure, essentially because $G(S)$ will be strictly smaller than the $S \times S$-action on S). Here is a more suitable example: consider $\mathfrak{g} = \{X \in \mathfrak{sl}(3,\mathbb{R}) | Xe_3 = 0\}$, the subalgebra of matrices killing the last vector of the canonical basis. This algebra is 3-graded, with 3-grading inherited by the one of $\mathfrak{sl}(3,\mathbb{R})$, i.e., by the derivation $\mathrm{ad}(I_{1,2})$:

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & e & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \right\},$$

where parameters run over $\mathbb{R}$. Hence $\mathfrak{q} := \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ is a Lie triple system. Let M be its corresponding symmetric space. We show that M has no $G(M)$-invariant measure: First of all, note that $\mathfrak{g}$ contains a copy of $\mathfrak{sl}(2,\mathbb{R})$ in the upper left corner, where the usual basis of $\mathfrak{sl}(2,\mathbb{R})$ corresponds to the matrices $P = E_{12}$, $Q = E_{21}$, $H = E_{11} - E_{22}$. Let $R := E_{31}$; then (Q, R) is a basis of $\mathfrak{g}_{-1}$ and P a basis of $\mathfrak{g}_1$. By a direct calculation we check that (P, Q, R) is a basis of eigenvectors of $\mathrm{ad}(H)$ in $\mathfrak{q}$, with corresponding eigenvalues $2, -2, -1$ (equivalently, note that $\mathfrak{g}$ is the direct sum of the 2-dimensional and the the 3-dimensional irreducible $\mathfrak{sl}_2$-representation.) Therefore

$$\mathrm{tr}(R(P,Q)) = \mathrm{tr}(\mathrm{ad}(H)|_{\mathfrak{q}}) = -1 \neq 0,$$

and thus M does not have an invariant measure. Geometrically, M is the space of pairs (A, B) of subspaces of $\mathbb{R}^3$ with $\dim A = 1$, $\dim B = 2$, $\mathbb{R}^3 = A \oplus B$ and $e_3 \in B$. This is the smallest example of the following situation: take a semisimple Jordan pair (V^+, V^-) and choose a proper inner ideal I in V^+. Then $\mathfrak{q} := I \oplus V^-$ is a (polarized) Lie triple system, and the associated symmetric space M never admits an invariant measure. Note, finally, that all such symmetric spaces are associated to a Jordan pair, namely to (I, V^-).

5.7. Further results and topics. Further results on geometric structures on the spaces M_α in the real, non-degenerate case can be found in the paper [Ma79] by B.O. Makarevic, and it would certainly be interesting to generalize some of these results. Finally, one would like to study the dependence of harmonic analysis and representation theory on the symmetric spaces M_α on the deformation parameter α. Although the situation here is different from the one considered, e.g., in [DR85], it seems possible to transfer ideas developed in the context of contractions of semisimple Lie groups to the set-up just described. Since all relevant geometric data of M_α depend algebraically or at least analytically on α, one may expect that something similar is true on the level of data of harmonic analysis. See, for instance [FP04] and [Pe02] for examples of harmonic analysis on symmetric spaces of the type M_α. The parameter α is fixed in these works, so that the dependence on α is not visible. When varying α, not only the aspect of deformations of α to singular values is of interest, but also the one of *analytic continuation* between isotopic, but non isomorphic values of α. For instance, the duality between *compactly causal* and *non-compactly causal symmetric spaces* is reflected by the isotopy between α and $-\alpha$ (cf. [Be00, XI.3]).

6. Some examples

6.1. Quadratic matrices and associative algebras. Let us return to the setting of Section 0.2 and consider the associative algebra $A = M(n,n;\mathbb{K})$ of square matrices over a field $\mathbb{K}$. Then the $A^\times \times A^\times$-orbits are represented by the idempotent matrices

$$e := e_r := \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & 0 \end{pmatrix}, \qquad r = 0, \dots, n.$$

The group $G_e = \{x \in A | \, \mathbf{1} + ex \in A^\times\}$ is explicitly given by

$$G_e = \Big\{ x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} | \det \begin{pmatrix} \mathbf{1}_r + \alpha & \beta \\ 0 & \mathbf{1}_{n-r} \end{pmatrix} \neq 0 \Big\}$$

(where β, γ, δ are arbitrary), with product $x \diamond_e y = xey + x + y$, i.e.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \diamond_e \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\beta' \\ \gamma\alpha' & \gamma\beta' \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

This group is isomorphic to a semidirect product of $\mathrm{GL}(r,\mathbb{K})$ and a Heisenberg type group: for β, γ, δ zero, we get a subgroup isomorphic to $\mathrm{GL}(r,\mathbb{K})$, and for $\alpha = 0$, we obtain a subgroup with product

$$\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \diamond_e \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta' \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} 0 & \beta' \\ \gamma' & \beta' \end{pmatrix}.$$

It is easily checked that G_e is the semidirect product of these two subgroups. The Lie algebra of G_e is given by the bracket $[x,y]_e = xey - yex$, which yields explicitly

$$\Big[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \Big]_e = \begin{pmatrix} \alpha\alpha' - \alpha'\alpha & \alpha\beta' - \alpha'\beta \\ \gamma\alpha' - \gamma'\alpha & \gamma\beta' - \gamma'\beta \end{pmatrix}$$

so that for $\alpha = 0 = \alpha'$ we get the 2-step nilpotent algebra

$$\Big[\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} \Big]_e = \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta' - \gamma'\beta \end{pmatrix}.$$

If A is a general associative algebra and e an idempotent in A, then the preceding calculations carry over, where matrices have to be understood with respect to the *Peirce-decomposition* of A, where $c := \mathbf{1} - e$:

$$A = cA \oplus eA = cAc \oplus cAe \oplus eAc \oplus eAe.$$

Then G_e is isomorphic to a semidirect product of $(eAe)^\times$ with a Heisenberg type group $cAe \oplus eAc \oplus cAc$. However, in general not every element of A is similar to an idempotent, and hence other types of deformations can appear.

6.2. Rectangular matrices. We consider the Jordan pair $(V^+, V^-) = (M(p,q;\mathbb{K}), M(q,p;\mathbb{K}))$ of rectangular matrices over a field $\mathbb{K}$, with trilinear maps $T^{\pm}(x,y,z) = xyz + zyx$. The group $\mathrm{GL}(p,\mathbb{K}) \times \mathrm{GL}(q,\mathbb{K})$ acts by automorphisms of this Jordan pair(in the usual way from the right and from the left), and every matrix $a \in V^-$ is conjugate under this action to a matrix of the form $e = e_r$ with $e_{ii} = 1$ for $i = 1, \ldots, r$ and 0 else. Thus we may assume $a = e$, and then the symmetric space structure on $\mathcal{U}_e$ from Theorem 2.3 is calculated in a similar way as above. Recall that a pair $(x,y) \in (M(p,q;\mathbb{K}), M(q,p;\mathbb{K}))$ is quasi-invertible if and only if the matrix $\mathbf{1}_p - xy$ is invertible in $M(p,p;\mathbb{K})$; therefore $\mathcal{U}_e$ is the set of matrices $x \in M(p,q;\mathbb{K})$ such that $\mathbf{1}_p + xe$ is invertible, i.e.

$$\mathcal{U}_e = \Big\{ \begin{pmatrix} \mathbf{1}_r + \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \det(\mathbf{1}_r + \alpha) \neq 0, \quad \beta, \gamma, \delta \,\text{arbitrary} \Big\}.$$

Using that $Q(x)e = xex$ and $B(x,-e)y = (\mathbf{1}+xe)y(\mathbf{1}+ex)$, we get the quadratic map of the symmetric space $\mathcal{U}_e$

$$\mathcal{Q}(x)y = 2x + Q(x)e + B(x,-e)y = 2x + y + xex + xey + yex + xeyex.$$

Again one finds that $\mathcal{U}_e$ has a bundle structure: the basis is given be letting β, γ, δ equal to zero; it is the space of $r \times r$-Matrices, with the symmetric space structure

$$\mathcal{Q}(\alpha)\alpha' = 2\alpha + \alpha' + \alpha^2 + \alpha\alpha' + \alpha'\alpha + \alpha\alpha'\alpha$$

which is isomorphic to the group $\mathrm{GL}(r,\mathbb{K})$, seen as a symmetric space (group case, cf. Appendix A.3). The fiber is given by letting $\alpha = 0$, and we get explicitly

$$\mathcal{Q}\Big(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \Big) \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} 0 & 2\beta + \beta' \\ 2\gamma + \gamma' & 2\delta + \delta' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta + \gamma\beta' + \gamma'\beta \end{pmatrix}.$$

Knowing that the quasi-inverse in the Jordan pair of rectangular matrices is given by $\tilde{\tau}_a(x) = (\mathbf{1} - xa)^{-1}x = x(\mathbf{1} - ax)^{-1}$, we find that the symmetry with respect to the origin in the fiber is

$$\sigma_0\Big(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \Big) = \begin{pmatrix} 0 & -\beta \\ -\gamma & -\delta + \gamma\beta \end{pmatrix},$$

so that the product map of the fiber is finally

$$\mu\Big(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} \Big) = \begin{pmatrix} 0 & 2\beta - \beta' \\ 2\gamma - \gamma' & 2\delta - \delta' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (\gamma - \gamma')(\beta - \beta') \end{pmatrix}.$$

In contrast to the preceding section, where the fiber was a two-step nilpotent group, for the symmetric space structure the fiber is now simply a flat a symmetric space. This is not surprising since the Lie triple bracket on a two-step nilpotent group is zero. Explicitly, one can check by direct calculation that the map

$$F : \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 0 & \beta \\ \gamma & \delta + \gamma\beta \end{pmatrix}$$

is an isomorphism from the fiber onto the same set, but equipped with the flat symmetric space structure $2x-y$. (The situation is completely similar to the one considered in [Be06], where the symmetric space structure on the second order tangent bundle TTM of a symmetric space M is calculated, see loc. cit., Chapters 26 and BA.)

6.3. Symmetric spaces associated to idempotents. The preceding calculations and remarks can be generalized for an arbitrary Jordan pair (V^+,V^-), as follows: we assume that $e:=e^- \in V^-$ is an element which can be completed to an idempotent, i.e., there existes $e^+\in V^-$ such that $Q^+(e^+)e^-=e^+$ and $Q^-(e^-)e^+=e^+$. (In other words, e is *von Neumann regular*, cf. [Lo75].) Associated to an idempotent (e^+,e^-), there is a *Peirce-decomposition* $V^+=V_2^+\oplus V_1^+\oplus V_0^+$ which is the eigenspace decomposition of the operator $T(e^+,e^-)$. The space $V_1^+\oplus V_0^+$ depends only on e^- since $V_1^+\oplus V_0^+=\ker Q(e^-)$, and V_2^+ depends only on e^+ since $V_2=Q(e^+)V^-$. Moreover, V_2^+ is a unital Jordan algebra with unit element e^+ (see [Lo75] for all this).

Theorem 6.4. (Bundle structure of $\mathcal{U}_e$.) *Assume $e=e^-\in V^-$ as a von Neumann regular element and choose $e^+\in V^+$ such that (e^+,e^-) is an idempotent.*

1. $\mathcal{U}_e=\{x=x_0+x_1+x_2 \mid x_0\in V_0^+, x_1\in V_1^+, x_2+e^+\in (V_2^+)^\times\}$
2. *The space $\mathcal{U}_e\cap V_2^+=e^+-V_2^\times$ is a symmetric subspace of $\mathcal{U}_e$ (called the* base*), isomorphic to the quadratic prehomogeneous symmetric space $(V_2^+)^\times$.*
3. *The space $V_1^+\oplus V_0^+$ is a symmetric subspace of $\mathcal{U}_e$ (called the* fiber*), and its symmetric space structure is isomorphic to the flat symmetric structure $2x-y$ on this set.*

Proof. (1) We use the identity JP24 of [Lo75] which gives

$$B(x,e)B(x,-e)=B(x,Q(e)x)=B(Q(x)e,e).$$

Assume $x\in V_0^+\oplus V_1^+$, i.e., $Q(e)x=0$. Hence $B(x,e)B(x,-e)=B(x,0)=\mathrm{id}$. In the same way, $B(x,-e)B(x,e)=\mathrm{id}$, and so $B(x,-e)$ is always invertible, with inverse $B(x,e)$. This shows that $V_0^+\oplus V_1^+\subset\mathcal{U}_e$. Moreover, then $\tilde\tau_x(e)=e$ since $B(e,x)^{-1}(e-Q(e)x)=B(e,-x)(e-Q(e)x)=B(e,-x)e=e$.

Assume $x\in V_2^+$. Then $B(x,-e)$ is invertible in $\mathrm{End}(V^+)$ if and only if its restriction to V_2^+ is invertible, and hence, by the results on the Jordan algebra case (Prop. 2.2), we see that this is equivalent to saying that e^++x is invertible in V_2^+.

Now let $z\in V^+$ and decompose $z=x+y$ with $y\in V_0^+\oplus V_1^+$ and $y\in V_2^+$. Then (y,e) is quasi-invertible with $\tilde\tau_y(e)=e$, as we have seen above, and hence we may apply the identity JP34 of [Lo75] in order to obtain $B(x+y,e)=B(x,\tilde\tau_y(e))B(y,e)=B(x,e)B(y,e)$. This operator is invertible in $\mathrm{End}(V^+)$ if and only if x and y satisfy the previously described conditions, and this proves the claim.

(2) This follows directly from (1) since the formula for the product $\mu(x,y)$ with $x,y\in(V_2^\times+e^+)$ uses only data which depend on the Jordan algebra V_2^+.

(3) We first calculate $\sigma_0(y)$ for an element $y \in \ker Q(e)$: using $e^{-y} = e$, we get from the symmetry formula ([Lo75, 3.3])

$$\sigma_0(y) = \tilde{\tau}_e(-y) = (-y)^e = -y + Q(y)\, e^{-y} = -y + Q(y)e.$$

Next we calculate the map $\mathcal{Q}(x)y = 2x + Q(x)e + B(x,-e)(y)$. Since

$$B(x,-e)y = y + T(x,e,y) + Q(x)Q(e)y = y + T(x,e,y),$$

we get

$$\mathcal{Q}(x)y = 2x + y + Q(x)e + T(x,e,y) = 2x + y + \frac{1}{2}T(x,e,x) + T(x,e,y).$$

As in the matrix case (Section 6.2), it is checked by a direct calculation that the map

$$F : V_0^+ \oplus V_1^+ \to V_0^+ \oplus V_1^+, \quad z \mapsto z + Q(z)e$$

is an isomorphism from the symmetric space $V_0^+ \oplus V_1^+$ with the usual flat structure $(x,y) \mapsto 2x - y$ onto the symmetric space $V_0^+ \oplus V_1^+$ equipped with the symmetric space structure that we have just described. ■

We conjecture that the symmetric space $\mathcal{U}_e$ from the preceding Theorem is a *symmetric bundle over the base* $(V_2^+)^\times$, in the sense of [BD07]. To prove this, it would be useful to state and prove Theorem 6.4 in a geometric way, i.e., in the context of Theorem 3.1. The starting point should be a suitable geometric interpretation of von Neumann regularity. This is indeed possible and will be taken up elsewhere.

6.5. Grassmannians. Next we give some examples illustrating Theorem 4.3. Assume $W = \mathbb{K}^{p+q}$ is a vector space over a field $\mathbb{K}$ of characteristic different from 2 and let $(\mathcal{X}^+, \mathcal{X}^-) = (\mathrm{Gras}_p(W), \mathrm{Gras}_q(W))$ the Grassmannian geometry of p- and q-dimensional subspaces of W, respectively. Then $(\mathcal{X}^+, \mathcal{X}^-)$ is a generalized projective geometry; with respect to the base point $(o^+, o^-) = (\mathbb{K}^p \oplus 0, 0 \oplus \mathbb{K}^q)$, it corresponds to the Jordan pair $(M(q,p;\mathbb{K}), M(p,q;\mathbb{K}))$. We define a family of symmetric space structures associated to $(\mathcal{X}^+, \mathcal{X}^-)$: assume $\beta : W \times W \to \mathbb{K}$ is a symmetric or skew-symmetric bilinear form on W such that β is non-degenerate on the first summand $o^+ = \mathbb{K}^p$. If β is non-degenerate on all of W, then $p : \mathcal{X}^+ \to \mathcal{X}^-$, $E \mapsto E^\beta$, where E^β is the orthogonal complement of E with respect to β, is a globally defined polarity, but if β is degenerate, then p is only defined on the set $\{E \in \mathcal{X}^+ |\ \dim(E^\beta) = q\}$. In any case, the space of non-degenerate linear subspaces of W,

$$M^\beta := \{E \subset W |\, W = E \oplus E^\beta\}$$

is a symmetric space with product map

$$\mu(E,F) = \sigma_E(F) = (-1)_{E,E^\beta}(F),$$

where, for any scalar $r \in \mathbb{K}$ and decomposition $W = U \oplus V$, we let $r_{U,V}$ be the linear map which is 1 on U and r on V. Let us study how the symmetric space structure and the Jordan triple system of M^β depends on β. In order to do this, we have to assume that $p(o^+) = o^-$, i.e., that $\beta(o^+, o^-) = 0$. In other words, β is of the form

$$\beta = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \text{ i.e. } \beta((x,y),(x',y')) = \beta_1(x,x') + \beta_2(y,y') = x^t B_1 x' + y^t B_2 y'$$

with $x, x' \in \mathbb{K}^p$, $y, y' \in \mathbb{K}^q$ and matrices B_1 (size $p \times p$) and B_2 (size $q \times q$) which are both either symmetric or skew-symmetric. The property that $p(o^+) = o^-$ assures that the affine picture of p is a linear map from $M(q,p;\mathbb{K})$ to $M(p,q;\mathbb{K})$. We calculate this map explicitly: let $X : \mathbb{K}^p \to \mathbb{K}^q$ be linear and consider its graph

$$\Gamma_X = \big\{ (x, Xx) |\, x \in \mathbb{K}^p \big\} \in \mathcal{X}^+.$$

Using that B_1 is invertible, we find that the orthogonal complement of Γ_X is given by

$$\begin{aligned}
(\Gamma_X)^\beta &= \{(u,v) \in \mathbb{K}^p \times \mathbb{K}^q | \forall x \in \mathbb{K}^p : \beta((u,v),(x,Xx)) = 0\} \\
&= \{(u,v) \in \mathbb{K}^p \times \mathbb{K}^q | \forall x \in \mathbb{K}^p : u^t B_1 x + v^t B_2 X x = 0\} \\
&= \{(u,v) \in \mathbb{K}^p \times \mathbb{K}^q | \forall x \in \mathbb{K}^p : u^t x + v^t B_2 X B_1^{-1} x = 0\} \\
&= \{(u,v) \in \mathbb{K}^p \times \mathbb{K}^q | \forall x \in \mathbb{K}^p : u = -B_1^{-1} X^t B_2 v\} \\
&= \Gamma_{-B_1^{-1} X^t B_2}.
\end{aligned}$$

The linear picture of the map p is therefore

$$M(q,p;\mathbb{K}) \to M(p,q;\mathbb{K}), \quad X \mapsto -B_1^{-1} X^t B_2 = -(B_2 X B_1^{-1})^t = -(\alpha X)^t$$

with $\alpha(X) = B_2 X B_1^{-1}$, and the JTS of the symmetric space M^β is

$$T_\alpha(X,Y,Z) = X(\alpha Y)^t Z + Z(\alpha Y)^t X = X(B_2 Y B_1^{-1})^t Z + Z(B_2 Y B_1^{-1})^t X,$$

which is the α-homotope of $T(X,Y,Z) = XY^tZ + ZY^tX$.

Let us consider some special choices of B_1 and B_2. Recall that B_1 has to be invertible, and B_1 and B_2 are either both symmetric or both skew-symmetric. In case $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, topological connected components of the symmetric spaces we obtain are homogeneous and of the type $M^\beta \cong \mathrm{O}(B_1 \oplus B_2)/(\mathrm{O}(B_1) \times \mathrm{O}(B_2))$. The isomorphism classes of symmetric spaces obtained for invertible matrices B_2 are listed in [Be00, Tables XII.4]:

- Symmetric case: by choosing for B_1 and B_2 invertible diagonal matrices having coefficients $b_{ii} \in \{\pm 1\}$, we get all symmetric spaces of the type $\mathrm{O}(l+j, k+i)/(\mathrm{O}(l,k) \times \mathrm{O}(i,j))$.
- Skew-symmetric case: by choosing the usual matrices of symplectic forms (the dimensions have to be even), we get all symmetric spaces of the type $\mathrm{Sp}(l+j,\mathbb{K})/(\mathrm{Sp}(l,\mathbb{K}) \times \mathrm{Sp}(j,\mathbb{K}))$.

Standard examples of deformations to degenerate cases are obtained by taking some of the diagonal coefficients of B_2 equal to zero. Since orthogonal groups of degenerate forms can be described as semidirect products of vector groups, general linear groups and orthogonal groups of non-degenerate forms, it is possible to give explicit descriptions of such symmetric spaces. Note that in case p even and q odd a new series of symmetric spaces arises which does not arise as deformation of a semisimple series: choose B_1 to be a symplectic form and B_2 skew-symmetric and non-zero.

6.6. Other examples, and some final comments. Similar examples of deformations of semi-simple symmetric spaces arise also in all other classical series of symmetric spaces (see the classification in [Be00, Chapter XII]) and also for about half of the simple exceptional symmetric spaces (cf. comments in [Be00, Section 0.6]). In order to keep this work in reasonable bounds, we cannot go here into further details. Let us just mention that all classical groups arise as group cases by our construction (the general linear groups are related to associative algebras, see Section 0.1, and all orthogonal, symplectic and unitary groups are related to certain Jordan triple systems); it is then an interesting problem to determine the deformations of these symmetric spaces that are again of groupe type. In other words, we raise the problem: *which deformations of group type symmetric spaces come from deformations of the overlying group structure* (as was the case for the general linear group in Section 0.1)? This question should not be confused with another, also very natural problem: *if* $(M_\alpha)_{\alpha\in\mathrm{Svar}(T)}$ *is a deformation of the symmetric space* $M = M_{\mathrm{id}}$*, in which sense are then the transvection groups* $G_\alpha := G(M_\alpha)$ *deformations of the transvection group* $G = G(M)$*?* They will in general not be "contractions of G" in the sense of [DR85] since the dimension of $G(M_\alpha)$ will not be constant as a function of α. Replacing transvection groups by automorphism groups leads to a variant of this problem. It seems that contractions of $\mathrm{Aut}(M)$ in the sense of [DR85] can only be obtained by using some additional parameter which in some sense memorizes the structure of the stabilizer group of a point.

Appendix A. Symmetric spaces

A.1. Reflection spaces with base point: a group-like concept. A *reflection space* is a set M together with a family $(\sigma_x)_{x\in M}$ of *symmetries* $\sigma_x : M \to M$ such that

(S1) $\sigma_x(x) = x$

(S2) $\sigma_x \circ \sigma_x = \mathrm{id}_M$

(S3) for all $x, y \in M$, $\sigma_x\sigma_y\sigma_x = \sigma_{\sigma_x(y)}$.

The map

$$\mu : M \times M \to M, \quad (x, y) \mapsto \mu(x, y) := \sigma_x(y)$$

is called the *multiplication map.* The concept of a reflection space is "affine", in the sense that it does not refer to a distinguished base point. On the other hand, for some purposes it is necessary to choose a base point $o \in M$. Following Loos [Lo69], we then define the *quadratic map* and the *square*, for $x \in M$,

$$\mathcal{Q}(x) := \mathcal{Q}_o(x) := \sigma_x \circ \sigma_o : M \to M, \qquad x^2 := \mathcal{Q}(x)o = \sigma_x(o).$$

The symmetry σ_o is called *inversion* and is sometimes denoted by $x^{-1} := \sigma_o(x)$. Note that $\mu(x,y)$ can be recovered from these data via

$$\mu(x,y) = \sigma_x(y) = \sigma_x\sigma_o\sigma_o(y) = Q(x)j(y) = Q(x)Q(y)^{-1}y.$$

The following relations are easily checked (cf. [Lo69] or [Be00, Chapter I]): for all $x,y \in M$,

(SB1) $\mathcal{Q}(\mathcal{Q}(x)y) = \mathcal{Q}(x)\mathcal{Q}(y)\mathcal{Q}(x)$
(SB2) $\mathcal{Q}(x^{-1}) = \mathcal{Q}(x)^{-1}$
(SB3) $x^{-1} = \mathcal{Q}(x)^{-1}x$

Proposition A.2. *There is an isomorphism of categories between the category of reflection spaces with base point and the category of pointed sets* (M,o) *equipped with two maps* $\mathcal{Q} : M \times M \to M$, $(x,y) \mapsto \mathcal{Q}(x)y$ *and* $\sigma_o : M \to M$, $x \mapsto x^{-1}$ *such that* (SB1) – (SB3) *hold.*

Proof. Given $(M,o;\mathcal{Q},\sigma_o)$, we define

$$\mu(x,y) := \sigma_x(y) := \mathcal{Q}(x)y^{-1} = \mathcal{Q}(x)\mathcal{Q}(y)^{-1}y,$$

and (S1) – (S3) are easily proved from (SB1) – (SB3). (For (S3), start by proving that $\sigma_o\mathcal{Q}(x)\sigma_o = \mathcal{Q}(x)^{-1}$.) This construction clearly is inverse to the one described above, and it is also clear that we get the same morphisms. ■

The various spaces $(M,a,\mathcal{Q}_a,\sigma_a)$ for $a \in M$ are in general not isomorphic among each other (the map $\mathcal{Q}(a)$ is an isomorphism from (M,o) onto (M,a^2) and not onto (M,a)). In order to change base points, we have the "isotopy formula":

$$\mathcal{Q}_a(x) = \sigma_x\sigma_a = \sigma_x\sigma_o(\sigma_a\sigma_o)^{-1} = \mathcal{Q}(x)\mathcal{Q}(a)^{-1}.$$

A.3. Symmetric spaces. Assume $\mathbb{K}$ is a topological ring with dense unit group. A *smooth symmetric space* is a smooth manifold over $\mathbb{K}$ together with a reflection space structure such that the multiplication map μ is smooth and

(S4) the tangent map of the symmetry σ_x at the fixed point x is equal to the negative of the identity of the tangent space T_xM.

(Cf. [Be06] for the notion of smpooth manifold over $\mathbb{K}$; if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, it coincides with the usual notion.) We will use the term *symmetric space*, somewhat informally, also in other, purely algebraic contexts, whenever the notion of tangent maps of the symmetries σ_x makes sense and (S4) is satisfied. For instance, this will be the case if there is some natural chart around x such that σ_x is, in this chart, represented by the negative of the identity.

If (M,o) is a smooth symmetric space with base point, one may define its *Lie triple system* in a similar way as the Lie algebra of a Lie group; this is done most naturally by using the quadratic maps $\mathcal{Q}(x)$, cf. [Lo69] for the real finite-dimensional and [Be06, Chapters 5 and 27] for the general case.

Important examples of symmetric spaces arise from the *group cases*: a group G with the new product $\mu(g,h) = \sigma_g(h) = gh^{-1}g$ satisfies (S1), (S2), (S3), and if G is, e.g., a Lie group, then also (S4) holds. Subspaces of (G,μ) are given by *spaces of symmetric elements*: if τ is an involution of a group G (anti-automorphism of order 2), then $M := \{g \in G | \tau(g) = g^{-1}\}$ is stable under μ, i.e., it is a subspace.

Appendix B: Generalized projective geometries

The geometric counterpart of a Jordan pair is a *generalized projective geometry*, and the one of a JTS a *generalized polar geometry*. In this appendix we recall the basic definitions and refer to [Be02] and [BN04] for further details.

B.1. Affine pair geometries. A *pair geometry* is given by a pair of sets, denoted by $(\mathcal{X}^+, \mathcal{X}^-)$, and a binary *transversality relation* $(\mathcal{X}^+ \times \mathcal{X}^-)^\top \subset \mathcal{X}^+ \times \mathcal{X}^-$; we write $x \top \alpha$ or $\alpha \top x$ if (x, α) belongs to $(\mathcal{X}^+ \times \mathcal{X}^-)^\top$, and we require that for all $\alpha \in \mathcal{X}^\mp$ there exists $x \in \mathcal{X}^\pm$ such that $x \top \alpha$. In other words, $\mathcal{X}^\pm$ is covered by the sets

$$\mathfrak{v}_\alpha := \mathfrak{v}_\alpha^\pm := \alpha^\top := \{x \in \mathcal{X}^\pm \,|\, x \top \alpha\} \tag{B.1}$$

as α runs through $\mathcal{X}^\mp$. An *affine pair geometry over a commutative ring* $\mathbb{K}$ is a pair geometry such that, for every $\alpha \in \mathcal{X}^\mp$, the set $\mathfrak{v}_\alpha$ is equipped with the structure of an affine space over $\mathbb{K}$; then the sets $\mathfrak{v}_\alpha$ will be called *affine parts* or *affine charts* of $\mathcal{X}^\pm$. Given an affine pair geometry, we let, for $(x, a), (y, a) \in (\mathcal{X}^+ \times \mathcal{X}^-)^\top$ and $r \in \mathbb{K}$, $r_{x,a}(y) := ry$ be the product $r \cdot y$ in the $\mathbb{K}$-module $\mathfrak{v}_a$ with zero vector x, and we define the *structure maps* by

$$\mathbf{\Pi}_r := \mathbf{\Pi}_r^+ : (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top \to \mathcal{X}^+, \quad (x, a, y) \mapsto \mathbf{\Pi}_r(x, a, y) := r_{x,a}(y), \tag{B.2}$$

where

$$(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top := \{(x, a, y) \in \mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \,|\, x \top a,\ y \top a\}, \tag{B.3}$$

and dually the maps $\mathbf{\Pi}_r^-$ are defined. Similarly, we get structure maps defined by vector addition:

$$\mathbf{\Sigma} := \mathbf{\Sigma}^+ : (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top \to \mathcal{X}^+, \quad (x, a, y, z) \mapsto \mathbf{\Sigma}(x, a, y, z) := y +_{x,a} z, \tag{B.4}$$

where the sum is taken in the $\mathbb{K}$-module $(\mathfrak{v}_a, x)$ and the set

$$(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top$$

is defined by a similar condition as in (B.3). Dually, $\mathbf{\Sigma}^-$ is defined.

B.2. Categorical notions. Assume $(\mathcal{X}^+, \mathcal{X}^-)$ and $(\mathcal{Y}^+, \mathcal{Y}^-)$ are affine pair geometries over $\mathbb{K}$.

1. Duality. All axioms of an affine pair geometry appear together with their dual version: thus $(\mathcal{X}^-, \mathcal{X}^+; \top; \mathbf{\Pi}^-, \mathbf{\Pi}^+, \mathbf{\Sigma}^-, \mathbf{\Sigma}^+)$ is again an affine pair geometry, called the *dual geometry* of $(\mathcal{X}^+, \mathcal{X}^-; \top; \mathbf{\Pi}^+, \mathbf{\Pi}^-, \mathbf{\Sigma}^+, \mathbf{\Sigma}^-)$.
2. Homomorphisms. These are pairs of maps

$$(g^+ : \mathcal{X}^+ \to \mathcal{Y}^+, g^- : \mathcal{X}^- \to \mathcal{Y}^-)$$

which are compatible with the transversality relations and with the structure maps, in the obvious sense.

3. Antiautomorphisms. These are isomorphisms $(g^+, g^-) : (\mathcal{X}^+, \mathcal{X}^-) \to (\mathcal{X}^-, \mathcal{X}^+)$ onto the dual geometry. If $g^- = (g^+)^{-1}$, then (g^+, g^-) is a *correlation*, and if, moreover, there exists a *non-isotropic point* (i.e., there is $x \in \mathcal{X}^+$ with $x \top g^+(x)$), then (g^+, g^-) is called a *polarity.*
4. Adjoint (or: structural) pairs of morphisms. These are pairs of maps $(f : \mathcal{X}^+ \to \mathcal{Y}^+, h : \mathcal{Y}^- \to \mathcal{X}^-)$ which are compatible with transversality and with structure maps in the sense that $x \top h(a)$ if and only if $f(x) \top a$ and

$$f.\mathbf{\Pi}_s(u, h.v, w) = \mathbf{\Pi}_s(f.u, v, f.w), \;\; f.\mathbf{\Sigma}(u, h.v, w, h.z) = \mathbf{\Sigma}(f.u, v, h.w, z).$$

It is necessary to allow here also maps that are not everywhere defined (see Appendix C). A map $f : \mathcal{X}^+ \to \mathcal{X}^-$ is called *selfadjoint* if (f, f) is an adjoint pair of morphisms into the dual geometry (compare with Section 1.12).
5. Stability. A pair geometry $(\mathcal{X}^+, \mathcal{X}^-, \top)$ is called *stable* if any two points $x, y \in \mathcal{X}^\pm$ are on a common affine chart, i.e., there exists $a \in \mathcal{X}^\mp$ such that $x, y \in \mathfrak{v}_a$.

B.3. The affine pair geometry associated to a Jordan pair. Assume that $V = (V^+, V^-)$ is a linear Jordan pair over $\mathbb{K}$, which we obtain in the form $(V^+, V^-) = (\mathfrak{g}_1, \mathfrak{g}_{-1})$ for a 3-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ with $T^\pm(x, y, z) = [[x, y], z]$ (see Section 1.4). Let $G := \mathrm{PE}(V^+, V^-) = \langle U^+, U^+ \rangle$ be the *projective elementary group* associated to this 3-graded Lie algebra, i.e., the group generated by the vector groups $U^\pm = \exp(\mathfrak{g}_{\pm 1})$, let H be the intersection of G with the automorphism group of the 3-grading (called the *(inner) structure group*) and let $P^\pm := U^\pm H$. Since $U^\pm$ are vector groups, it is easily seen that

$$(\mathcal{X}^+, \mathcal{X}^-) := (\mathcal{X}^+(V), \mathcal{X}^-(V)) := (G/P^-, G/P^+)$$

is an affine pair geometry. The base point (eP^-, eP^+) is often denoted by (o^+, o^-), and the affine parts $(\mathfrak{v}_{o^-}, \mathfrak{v}_{o^+})$ corresponding to this base point are identified with $(V^+, V^-) = (\exp(\mathfrak{g}^+).o^+, \exp(\mathfrak{g}^-).o^-)$. (In [BN04] a realization of this geometry and of its transversality relation in terms of *Lie algebra filtrations* is given.)

The construction is functorial: a Jordan pair homomorphism induces a homomorphism of affine pair geometries ([Be02, Th. 10.1]). In particular, an involution of a Jordan pair induces a polarity of $(\mathcal{X}^+, \mathcal{X}^-)$. Since Jordan triple systems are nothing but Jordan pairs with involution, we thus associate a polar geometry to any JTS.

The geometry associated to a Jordan pair (V^+, V^-) is stable if and only if the maps

$$V^\mp \times V^\pm \to \mathcal{X}^\pm, \quad (a, x) \mapsto \exp(a)\exp(x).o^\pm$$

are surjective; then $\mathcal{X}^\pm$ can be seen as $V^\pm \times V^\mp$ modulo the equivalence relation given by fibers of this map (called *projective equivalence*; cf. [Lo94]).

Another equivalent version of stability is that the projective group G admits the "Harish-Chandra decomposition" $G = \exp(V^+)\exp(V^-)H\exp(V^+)$ and $G = \exp(V^-)\exp(V^+)H\exp(V^-)$, where H is the structure group.

B.4. Generalized projective geometries are affine pair geometries $(\mathcal{X}^+, \mathcal{X}^-)$ such that the structure maps satisfy some algebraic identities (PG1) and (PG2) which should be seen as global analogs of certain Jordan algebraic identies. Recall from B.1 that $r_{x,a}(y) = \mathbf{\Pi}_r(x,a,y)$, $r_{a,x}(b) = \mathbf{\Pi}_r^-(a,x,b)$ ("left multiplication operators"), and denote by $M^{(r)}_{x,y}(a) := \mathbf{\Pi}_r(x,a,y)$, $M^{(r)}_{a,b}(x) := \mathbf{\Pi}_r(a,x,b)$ the "middle multiplication operators". Then the geometry $(\mathcal{X}^+, \mathcal{X}^-)$ associated to a Jordan pair (V^+, V^-) satisfies the following properties ([Be02, Th. 10.1]):

(PG1) For any invertible scalar $r \in \mathbb{K}$ and any transversal pair (x,a), the pair of maps $(g,g') = (r_{x,a}, r_{a,x}^{-1})$ is an automorphism of the affine pair geometry $(\mathcal{X}^+, \mathcal{X}^-)$.

(PG2) For any $r \in \mathbb{K}$ and $a,b \in \mathcal{X}^-$ with $\mathfrak{v}_a \cap \mathfrak{v}_b \neq \emptyset$, the pair of middle multiplications $(f,h) = (M^{(r)}_{a,b}, M^{(r)}_{b,a})$ is an adjoint pair of (locally defined) morphisms from $(\mathcal{X}^+, \mathcal{X}^-)$ to $(\mathcal{X}^-, \mathcal{X}^+)$, and dually.

Applied to elements of $\mathcal{X}^+$, resp. $\mathcal{X}^-$, (PG1) and (PG2) are indeed algebraic identities for the structure maps. One should think of (PG1) as a global analog of the defining identity (LJP2) (Section 1.4) of a Jordan pair, and of (PG2) as the global analog of the fundamental formula.

Appendix C: Globalization of structural transformations

C.1. Homomorphisms and structural transformations. There are two ways to turn Jordan pairs over $\mathbb{K}$ into a category: one defines *homomorphisms* in the usual way as pairs of linear maps $(h^+, h^-) : (V^+, V^-) \to (W^+, W^-)$ which are compatible with the maps $T^\pm$, and one defines a *structural transformation* to be a pair of linear maps $f : V^+ \to W^+$, $g : W^- \to V^-$ such that

$$T^+_W(f(x), y, f(z)) = f(T^+_V(x, g(y), z)), \qquad T^-_V(g(u), v, g(w)) = g(T^-_W(u, f(v), w))$$

(cf. [Lo94]). In both cases we get a category; these categories are different, but have essentially the same isomorphisms.

C.2. Globalization: example of projective spaces. A homomorphism of Jordan pairs $(\phi^+, \phi^-) : (V^+, V^-) \to (W^+, W^-)$ always lifts to a homomorphism of associated geometries ([Be02, Th. 10.1]), whereas for structural transformations the situation is more difficult. In order to show that structural transformations do not always extend globally, consider the example of ordinary projective geometry $\mathcal{X}^+ = \mathbb{KP}^n$ over a field. Its Jordan pair is (W, W^*), with $W = \mathbb{K}^n$, and

$$T^+(x, \lambda, y) = x\lambda(y) + y\lambda(x), \qquad T^-(\lambda, x, \mu) = \lambda(x)\mu + \mu(x)\lambda.$$

This Jordan pair is simple, and hence homomorphisms $(W, W^*) \to (V^+, V^-)$ are either injective or trivial. Clearly, every *injective* linear map $\mathbb{K}^{n-1} \to \mathbb{K}^{m-1}$ of affine parts lifts to a map $\mathbb{KP}^n \to \mathbb{KP}^m$ of associated projective spaces, which shows that homomorphisms globalize, as stated above. On the other hand, every linear map $f : V \to W$ together with its dual map defines a structural transformation $(f, g) = (f : V \to W, f^* : W^* \to V^*)$. If f is not injective, then f does not lift to a globally defined map of projective spaces. The best we can do is to extend f to a not everywhere defined map $\tilde{f}$ from $\mathbb{KP}^n$ to $\mathbb{KP}^m$. The "natural domain of definition" of $\tilde{f}$ is

$$D(\tilde{f}) := \Big\{ [\begin{pmatrix} v \\ r \end{pmatrix}] \,|\, v \in V, r \in \mathbb{K} : \begin{pmatrix} fv \\ r \end{pmatrix} \neq 0 \Big\},$$

and then $\tilde{f}([\begin{pmatrix} v \\ r \end{pmatrix}]) = [\begin{pmatrix} fv \\ r \end{pmatrix}]$. The set $D(\tilde{f})$ is strictly bigger than the affine part V (except for the trivial case $f = 0$). The purpose of this appendix is to define such a natural domain of definition for any structural transformation of Jordan pairs.

Lemma C.3. *If $(f, g) = (f : V^+ \to W^+, g : W^- \to V^-)$ is a structural transformation between Jordan pairs, then the pair (V^+, W^-) with*

$$\begin{aligned} S^+ &: V^+ \times W^- \times V^+ \to V^-, \quad (a, b, c) \mapsto T^+_V(a, g(b), c), \\ S^- &: W^- \times V^+ \times W^- \to W^-, \quad (u, v, w) \mapsto T^-_W(u, f(v), w) \end{aligned}$$

is a linear Jordan pair, and the maps

$$\begin{aligned} \phi &:= (\phi^+, \phi^-) := (f, \mathrm{id}) : (V^+, W^-) \to (W^+, W^-), \quad (x, y) \mapsto (f(x), y) \\ \psi &:= (\psi^+, \psi^-) := (\mathrm{id}, g) : (V^+, W^-) \to (V^+, V^-), \quad (x, y) \mapsto (x, g(y)) \end{aligned}$$

are homomorphisms of Jordan pairs.

Proof. The first claim can be checked by a direct computation which is completely analogous to the proof of Lemma 1.12. We show that ϕ is a homomorphism:

$$\begin{aligned} \phi^+ S^+(a, b, c) &= f T^+(a, gb, c) = T^+(fa, b, fc) = T^+(\phi^+ a, \phi^- b, \phi^+ c), \\ \phi^- S^-(x, y, z) &= T^-(x, fy, z) = T^-(\phi^- x, \phi^+ y, \phi^- z). \end{aligned}$$

Similarly, we see that ψ is a homomorphism. ∎

C.4. The globalization of a structural transformation. Assume (f, g) is a structural transformation as above, and let ϕ and ψ the homomorphisms defined in the lemma. By functoriality, we have induced homomorphisms on the level of geometries:

$$\begin{array}{ccccccc} & (V^+, W^-) & & & & \mathcal{X}(V^+, W^-) & \\ {\scriptstyle (f,\mathrm{id})} \swarrow & & \searrow {\scriptstyle (\mathrm{id},g)} & & {\scriptstyle (\Phi^+,\Phi^-)} \swarrow & & \searrow {\scriptstyle (\Psi^+,\Psi^-)} \\ (W^+, W^-) & & (V^+, V^-) & & \mathcal{X}(W) & & \mathcal{X}(V) \end{array}$$

In general, if the first (resp. second) component of a Jordan pair homomorphism is injective, then so is the first (resp. second) component of the induced map of geometries[1], and hence the first component Ψ^+ and the second component Φ^- are injective. The following maps together with their domains of definition are called the *globalization of* (f,g):

$$\tilde{f} : D(\tilde{f}) \to \mathcal{X}^+(W), \quad D(\tilde{f}) := \Psi^+(\mathcal{X}^+(V^+,W^-)), \quad \tilde{f}(\Psi^+(x)) := \Phi^+(x),$$
$$\tilde{g} : D(\tilde{g}) \to \mathcal{X}^-(V), \quad D(\tilde{g}) := \Phi^-(\mathcal{X}^-(V^+,W^-)), \quad \tilde{g}(\Phi^-(a)) := \Psi^-(a).$$

The following result will not be needed in this work and its proof is therefore omitted:

Theorem C.5. *The pair* $(\tilde{f},\tilde{g})$ *is an adjoint pair of morphisms between* $\mathcal{X}(V)$ *and* $\mathcal{X}(W)$ *whose domain of definition* $(D(\tilde{f}),D(\tilde{g}))$ *is complete in the following sense: whenever* $x \in D(\tilde{f})$*, then* $\tilde{f}(x)^\top \subset D(\tilde{g})$*, and whenever* $a \in D(\tilde{g})$*, then* $\tilde{g}(a)^\top \subset D(\tilde{f})$*. Moreover,* $(D(\tilde{f}),D(\tilde{g}))$ *is a generalized projective geometry, isomorphic to* $\mathcal{X}(V^+,W^-)$*, with structure maps given by*

$$\mathbf{\Pi}_r^+(x,a,y) = \mathbf{\Pi}_r(x,\tilde{g}(a),y), \quad \mathbf{\Pi}_r^-(a,y,b) = \mathbf{\Pi}_r^-(a,\tilde{f}(y),b). \qquad \blacksquare$$

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[1] cf. [Lo94, 1.3 (3)] for the case of a stable geometry; the proof of the corresponding fact for the general non-stable case is similar but rather lengthy and is therefore omitted here.

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Received January 15, 2007
and in final form February 19, 2008