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A Manifold Structure for the Group of Orbifold Diffeomorphisms of a Smooth Orbifold

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Abstract. For a compact, smooth C^r orbifold (without boundary), we show that the topological structure of the orbifold diffeomorphism group is a Banach manifold for $1 \le r < \infty$ and a Fréchet manifold if $r = \infty$. In each case, the local model is the separable Banach (Fréchet) space of $C^r(C^\infty,\text{resp.})$ orbisections of the tangent orbibundle.
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1. Introduction

Our goal in this paper is to determine the topological structure of the orbifold diffeomorphism group of a smooth compact orbifold. It is well known that in the case of a closed smooth C^r manifold, the group of C^r diffeomorphisms ($1 \le r \le \infty$) is a smooth manifold whose local model is $\mathcal{D}^r(M)$, the space of C^r tangent vector fields on M. See, for example [Ban97] or [Nit71]. $\mathcal{D}^r(M)$ is a separable Banach space for $1 \le r < \infty$ and a separable Fréchet space for $r = \infty$. One might naively think that the orbifold diffeomorphism group is itself an infinite dimensional orbifold, but one only need remember that the orbifold diffeomorphism group is a (topological) group and hence must be homogeneous. As such, it cannot be a non-trivial orbifold. In fact, in the case of a smooth compact orbifold, the structure of the orbifold diffeomorphism group holds no surprises, and we have the following

Theorem 1.1. *Let $r \ge 1$ and let $\mathcal{O}$ be a compact, smooth C^r orbifold (without boundary). Denote by $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ the group of C^r orbifold diffeomorphisms equipped with the C^r topology. Then $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is a manifold modeled on the topological vector space $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ of C^r orbisections of the tangent orbibundle equipped*

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with the C^r topology. This separable vector space is a Banach space if $1 \leq r < \infty$ and is a Fréchet space if $r = \infty$.

This particular result was first conjectured with a plausibility argument in [BB02]. Here, we provide a complete proof using techniques in the spirit of the classical result for the manifold case. There are many competing and useful notions of smooth orbifold map in the literature. In [BB02], the statement of theorem 1.1 referred to *unreduced* orbifold diffeomorphisms. The main result of [BB03] concerned the *reduced* orbifold diffeomorphism group $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$. It is possible to recover the topology of $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ as a quotient of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. In fact, we have the following structure theorem for $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ as a corollary of theorem 1.1.

Corollary 1.2. *Let $r \geq 1$ and let $\mathcal{O}$ be a compact, smooth C^r orbifold (without boundary). Let $\mathcal{ID} = \{f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \mid f(x) = x \text{ for all } x \in \mathcal{O}\}$. That is, $\mathcal{ID}$ is the set comprised of all lifts of the identity map. Then $|\mathcal{ID}| < \infty$ and there is a short exact sequence of groups*

$$1 \to \mathcal{ID} \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \to \mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O}) \to 1.$$

Thus, $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O}) \cong \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})/\mathcal{ID}$ is a Banach manifold if $r < \infty$ and a Fréchet manifold if $r = \infty$.

Remark 1.3. Using methods detailed in [KM97], it will follow that these diffeomorphism groups have the structure of smooth manifolds. Furthermore, composition and inversion in these groups will be continuous, and in the $r = \infty$ case, both $\mathrm{Diff}^\infty_{\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^\infty_{\mathrm{red}}(\mathcal{O})$ will be convenient Fréchet Lie groups. Details will appear in a future revision to the preprint [BB08] on the topological structure of the set of smooth mappings between orbifolds $\mathcal{O}$ and $\mathcal{P}$.

The next few sections of the paper will define and describe the notions that appear in the statement of theorem 1.1 and corollary 1.2. In particular, in section 2, we define the notion of smooth orbifold and its natural stratification. We also define the notion of product orbifold and suborbifold and give some examples. In section 3, we define the notion of orbifold map. Section 4 defines the (strong) C^r topology on maps between smooth orbifolds. In section 5, we define the tangent orbibundle and its orbisections. The space of orbisections provide the local model for the orbifold diffeomorphism group. In section 6, we look at smooth Riemannian structures and define a smooth Riemannian exponential map. Finally, we prove theorem 1.1 and corollary 1.2 in section 7.

It should be noted that we have chosen not to use the language of Lie groupoids and Morita equivalence in our description of orbifolds and their maps, but rather we have chosen a more "classical" approach. The reason for this choice is that a treatment using groupoids, in our opinion, would not add clarity to the exposition or enhance our results. In fact, we believe that much of the useful geometric and topological intuition becomes obscured. A reader interested in the groupoid approach to orbifolds and its utility should consult the recent monograph [ALR07] and the references therein, especially the article [Moe02].

We should also note that our definition of orbifold is modeled on the definition in Thurston [Thu78]. The orbifolds that concern us here are referred to as *classical effective orbifolds* in [ALR07]. While our notion of orbifold map is more general than that given in [ALR07, Ch. 1], our notion of *reduced* orbifold map and *reduced* orbifold diffeomorphism agrees with that book's definitions 1.3 and 1.4.

2. Orbifolds

In this section, we review the (classical) definition of smooth orbifold and related constructions.

Definition 2.1. An n-dimensional (topological) *orbifold* $\mathcal{O}$, consists of a paracompact, Hausdorff topological space $X_{\mathcal{O}}$ called the *underlying space*, with the following local structure. For each $x \in X_{\mathcal{O}}$ and neighborhood U of x, there is a neighborhood $U_x \subset U$, an open set $\tilde{U}_x \cong \mathbb{R}^n$, a finite group Γ_x acting continuously and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \to U_x$ with $\phi_x(0) = x$. These actions are subject to the condition that for a neighborhood $U_z \subset U_x$ with corresponding $\tilde{U}_z \cong \mathbb{R}^n$, group Γ_z and homeomorphism $\phi_z : \tilde{U}_z/\Gamma_z \to U_z$, there is an embedding $\tilde{\psi}_{zx} : \tilde{U}_z \to \tilde{U}_x$ and an injective homomorphism $\theta_{zx} : \Gamma_z \to \Gamma_x$ so that $\tilde{\psi}_{zx}$ is equivariant with respect to θ_{zx} (that is, for $\gamma \in \Gamma_z, \tilde{\psi}_{zx}(\gamma \cdot \tilde{y}) = \theta_{zx}(\gamma) \cdot \tilde{\psi}_{zx}(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_z$), such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_z \arrow[r, "\tilde{\psi}_{zx}"] \arrow[d] & \tilde{U}_x \arrow[d] \\
\tilde{U}_z/\Gamma_z \arrow[r, "\psi_{zx}=\tilde{\psi}_{zx}/\Gamma_z"] \arrow[dd, "\phi_z"] & \tilde{U}_x/\theta_{zx}(\Gamma_z) \arrow[d] \\
 & \tilde{U}_x/\Gamma_x \arrow[d, "\phi_x"] \\
U_z \arrow[r, "\subset"] & U_x
\end{tikzcd}$$

Remark 2.2. Note that if $\delta \in \Gamma_x$ then $\overline{\psi}_{zx} = \delta \cdot \tilde{\psi}_{zx}$ is also an embedding of $\tilde{U}_z$ into $\tilde{U}_x$. It is equivariant relative to the injective homomorphism $\overline{\theta}_{zx}(\gamma) = \delta \cdot \theta_{zx}(\gamma) \cdot \delta^{-1}$. Thus, we regard $\tilde{\psi}_{zx}$ as being defined only up to composition with elements of Γ_x, and θ_{zx} defined only up to conjugation by elements of Γ_x. In general, it is not true that $\tilde{\psi}_{zx} = \tilde{\psi}_{yx} \circ \tilde{\psi}_{zy}$ when $U_z \subset U_y \subset U_x$, but there should be an element $\delta \in \Gamma_x$ such that $\delta \cdot \tilde{\psi}_{zx} = \tilde{\psi}_{yx} \circ \tilde{\psi}_{zy}$ and $\delta \cdot \theta_{zx}(\gamma) \cdot \delta^{-1} = \theta_{yx} \circ \theta_{zy}(\gamma)$. Also, the covering $\{U_x\}$ of $X_{\mathcal{O}}$ is not an intrinsic part of the orbifold structure. We regard two coverings to give the same orbifold structure if they can be combined to give a larger covering still satisfying the definitions.

Definition 2.3. We say that an n-dimensional orbifold $\mathcal{O}$ is *locally smooth* if the action of Γ_x on $\tilde{U}_x \cong \mathbb{R}^n$ is topologically conjugate to an *orthogonal* action for all $x \in \mathcal{O}$. That is, for each $x \in \mathcal{O}$, there exists a (faithful) representation

$\rho_x : \Gamma_x \to O(n)$, the orthogonal group, such that if $\gamma \cdot y$ denotes the Γ_x action on $\tilde{U}_x$, there exists a homeomorphism h of $\tilde{U}_x$ such that $h \circ (\gamma \cdot y) = [\rho_x(\gamma)](h(y))$ for all $y \in \tilde{U}_x$. By standard results, [Wol84, lemma 4.7.1], the class of locally smooth orbifold remains unchanged if we replace $O(n)$ by the general linear group, $GL(n)$, in our definition.

Definition 2.4. Let $0 \le r \le \infty$. An orbifold $\mathcal{O}$ is a *smooth C^r orbifold* if each Γ_x acts by C^r diffeomorphisms on $\tilde{U}_x$ and each embedding $\tilde{\psi}_{zx}$ is C^r. When $r = 0$, a smooth C^0 orbifold is understood to be locally smooth.

Proposition 2.5. *If $\mathcal{O}$ is a smooth C^r orbifold with $r > 0$, then it is locally smooth. Moreover, the action of the local isotropy groups is smoothly C^r conjugate to an orthogonal action.*

Proof. Let Γ_x be the isotropy group of x, U_x a neighborhood of x with corresponding neighborhood $\tilde{U}_x$ of 0 in $\mathbb{R}^n$ and homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \to U_x$ with $\phi_x(0) = x$. By assumption, Γ_x acts by C^r diffeomorphisms on $\tilde{U}_x$. We denote the action of Γ_x by $(\gamma, \tilde{y}) \to \gamma \cdot \tilde{y}$ for all $\gamma \in \Gamma_x$ and $\tilde{y} \in \tilde{U}_x$. Note that $\Gamma_x \cdot 0 = 0$. Let $L_\gamma : T_0\tilde{U}_x \to T_0\tilde{U}_x$ be the linearization at 0 of $\tilde{y} \to \gamma \cdot \tilde{y}$. Note that L_γ, being the linearization *at* 0, is a *fixed* linear map, and is therefore C^∞. Define $F : \tilde{U}_x \to \mathbb{R}^n$ by

$$F(\tilde{y}) = \frac{1}{|\Gamma_x|} \sum_{\eta \in \Gamma_x} L_\eta(\eta^{-1} \cdot \tilde{y})$$

Then F is C^r since L_η is C^∞ and the action of Γ_x is by C^r diffeomorphisms. Also, $dF(0) = \mathrm{Id}$ and $F(\gamma \cdot \tilde{y}) = L_\gamma(F(\tilde{y}))$. To see the last statement, note that

$$\begin{aligned}
F(\gamma \cdot \tilde{y}) &= \frac{1}{|\Gamma_x|} \sum_{\eta \in \Gamma_x} L_\eta(\eta^{-1}\gamma \cdot \tilde{y}) \\
&= \frac{1}{|\Gamma_x|} \sum_{\eta \in \Gamma_x} L_\eta((\gamma^{-1}\eta)^{-1} \cdot \tilde{y}) \\
&= \frac{1}{|\Gamma_x|} \sum_{\mu \in \Gamma_x} L_{\gamma\mu}(\mu^{-1} \cdot \tilde{y}) && \text{where } \mu = \gamma^{-1}\eta \\
&= \frac{1}{|\Gamma_x|} \sum_{\mu \in \Gamma_x} L_\gamma(L_\mu(\mu^{-1} \cdot \tilde{y})) \\
&= L_\gamma\Big(\frac{1}{|\Gamma_x|} \sum_{\mu \in \Gamma_x} L_\mu(\mu^{-1} \cdot \tilde{y})\Big) = L_\gamma(F(\tilde{y}))
\end{aligned}$$

So by the inverse function theorem, there is a neighborhood $\tilde{V}_x$ of 0 in $\tilde{U}_x$ on which F is an equivariant C^r diffeomorphism. Thus, F conjugates the action of Γ_x to the linear action L_γ which in turn is linearly conjugate to an orthogonal action which we denote by $\rho_x(\gamma)$. ρ_x is the required representation making $\mathcal{O}$ locally smooth. ∎

Definition 2.6. An *orbifold chart* about x in a (locally) smooth orbifold $\mathcal{O}$ is a 4-tuple $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ where $\tilde{U}_x \cong \mathbb{R}^n$, Γ_x is a finite group, ρ_x is a (faithful) representation of $\Gamma_x : \rho_x \in \mathrm{Hom}(\Gamma_x, O(n))$, and ϕ_x is a homeomorphism: $\phi_x : \tilde{U}_x/\rho_x(\Gamma_x) \to U_x$, where $U_x \subset X_{\mathcal{O}}$ is a (sufficiently small) open relatively compact neighborhood of x, and $\phi_x(0) = x$.

For convenience we will often refer to the neighborhood U_x or $(\tilde{U}_x, \Gamma_x)$ as an orbifold chart, and ignore the representation ρ_x and write $U_x = \tilde{U}_x/\Gamma_x$. If necessary, we can assume that $\tilde{U}_x$ is an open metric ball in $\mathbb{R}^n$ centered at the origin and denote by $\pi_x : \tilde{U}_x \to \tilde{U}_x/\rho_x(\Gamma_x)$, the quotient map defined by the action of $\rho_x(\Gamma_x)$ on $\tilde{U}_x$.

Proposition 2.7. *Let $r \geq 0$. If $\mathcal{O}$ is a smooth C^r orbifold then in each orbifold chart $\tilde{U}_x$ the fixed point set $\tilde{S}_x = \{\tilde{y} \in \tilde{U}_x \mid \Gamma_x \cdot \tilde{y} = \tilde{y}\}$ is a connected C^r submanifold of $\tilde{U}_x$.*

Proof. Let $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ be an orbifold chart about x. Since $\mathcal{O}$ is C^r smooth, the proof of proposition 2.5 gives the existence of Γ_x-equivariant C^r diffeomorphism $F : \tilde{U}_x \to \mathbb{R}^n_{\rho_x}$, where $\mathbb{R}^n_{\rho_x}$ denotes $\mathbb{R}^n$ with the orthogonal Γ_x-action induced by the representation ρ_x. Thus, we have $F(\gamma \cdot \tilde{y}) = [\rho_x(\gamma)](F(\tilde{y}))$. If $\tilde{y} \in \tilde{S}_x$, and $\tilde{z} = F(\tilde{y})$ then we have that $\tilde{z} = [\rho_x(\gamma)](\tilde{z})$, hence $F(\tilde{S}_x) \subset \bigcap_{\gamma \in \Gamma_x} \ker(\rho_x(\gamma) - I)$. Let $\tilde{W} = \bigcap_{\gamma \in \Gamma_x} \ker(\rho_x(\gamma) - I)$ and let $\tilde{w} \in \tilde{W}$, with $F(\tilde{v}) = \tilde{w}$ for some $\tilde{v} \in \tilde{U}_x$. Then

$$\tilde{v} = F^{-1}(\tilde{w}) = F^{-1}[\rho_x(\gamma)](\tilde{w}) = F^{-1}[\rho_x(\gamma)]F(\tilde{v}) = F^{-1}F(\gamma \cdot \tilde{v}) = \gamma \cdot \tilde{v}$$

for all $\gamma \in \Gamma_x$. Hence $\tilde{v} \in \tilde{S}_x$. We have shown $F(\tilde{S}_x) = \tilde{W}$. Since $\tilde{W}$ is a subspace, we have that $\tilde{S}_x = F^{-1}(\tilde{W})$ is a connected C^r submanifold of $\tilde{U}_x$. ■

Stratification of an Orbifold.

Definition 2.8. Let $\mathcal{O}$ be a connected n-dimensional locally smooth orbifold. Given a point $x \in \mathcal{O}$, there is a neighborhood U_x of x which is homeomorphic to a quotient $\tilde{U}_x/\Gamma_x$ where $\tilde{U}_x$ is homeomorphic to $\mathbb{R}^n$ and Γ_x is a finite group acting orthogonally on $\mathbb{R}^n$. The definition of orbifold implies that the germ of this action in a neighborhood of the origin of $\mathbb{R}^n$ is unique. We define the *isotropy group of* x to be the group Γ_x. The *singular set*, Σ_1, of $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ with $\Gamma_x \neq \{e\}$.

We wish to define the notion of a *stratum* $\mathcal{S}$ of $\mathcal{O}$. Roughly speaking, a stratum of $\mathcal{O}$ is a maximal connected subset $\mathcal{S}$ of $\mathcal{O}$ for which the Γ_x action is constant for $x \in \mathcal{S}$. The formal definition is:

Definition 2.9. Two points x, y belong to the same stratum $\mathcal{S} \subset \mathcal{O}$ if there exists a chain of orbifold charts $\{U_x = U_0, U_1, \ldots, U_m = U_y\}$ so that for $0 \leq i \leq m-1$ we have

1. $U_i \cap U_{i+1} \neq \emptyset$
2. $\mathrm{Im}(\rho_i) = \mathrm{Im}(\rho_{i+1})$, and

3. Γ_i acts on $\tilde{U}_i \cap \tilde{U}_{i+1}$; that is, $\tilde{U}_i \cap \tilde{U}_{i+1}$ is Γ_i invariant

Here, $\rho_i \in \mathrm{Hom}(\Gamma_i, O(n))$ is the faithful representation of Γ_i corresponding to the chart U_i. By construction, the diagram below commutes (horizontal maps are simply inclusions):

$$\begin{tikzcd}
\tilde{U}_i \cap \tilde{U}_{i+1} \arrow[r, "\subset"] \arrow[d] & \tilde{U}_{i+1} \arrow[d] \\
(\tilde{U}_i \cap \tilde{U}_{i+1})/\Gamma_i \arrow[r, "\subset"] \arrow[d] & \tilde{U}_{i+1}/\Gamma_{i+1} \arrow[d] \\
U_i \cap U_{i+1} \arrow[r, "\subset"] & U_{i+1}
\end{tikzcd}$$

It is easy to see that belonging to the same stratum is an equivalence relation on $\mathcal{O}$. Also, there can only be a finite number of distinct strata on a compact orbifold. We have the following structure result for strata:

Proposition 2.10. *Let $\mathcal{S}$ be a stratum of a smooth C^r orbifold $\mathcal{O}$. Then $\mathcal{S}$ is connected and there exists a connected smooth C^r manifold $\tilde{U}$ and a C^r action by a finite group Γ on $\tilde{U}$ such that $\tilde{U}/\Gamma$ is a neighborhood of $\mathcal{S}$ in $\mathcal{O}$.*

Proof. From the definition of smooth orbifold we see that $\tilde{U} = \bigcup_{i=0}^m \tilde{U}_i$ inherits the structure of a connected smooth C^r manifold. Let $\Gamma = \Gamma_0$ and $\rho = \rho_0$. By construction, we have an orthogonal action given by $\rho_0(\Gamma)$ of Γ on $\tilde{U}$ and it is clear that $\tilde{U}/\Gamma$ is a neighborhood of $\mathcal{S}$ in $\mathcal{O}$. That $\mathcal{S}$ is connected follows from proposition 2.7 and the fact that $\mathcal{S}$ is the (continuous) projection of the fixed point subset $\tilde{S} = \{\tilde{u} \in \tilde{U} \mid \Gamma \cdot \tilde{u} = \tilde{u}\}$. ∎

Definition 2.11. Let $\mathcal{O}$ be a smooth C^r orbifold. For $x \in \mathcal{O}$, the stratum containing x will be denoted by $\mathcal{S}_x$. It is a suborbifold of $\mathcal{O}$ (see definition 2.13). The corresponding C^r manifold covering and finite group given in proposition 2.10 will be denoted by $\tilde{U}_{\mathcal{S}_x}$ and $\Gamma_{\mathcal{S}_x}$, respectively. The neighborhood $\tilde{U}_{\mathcal{S}_x}/\Gamma_{\mathcal{S}_x}$ of $\mathcal{S}_x$ will be denoted by $U_{\mathcal{S}_x}$ and the inverse image of $\mathcal{S}_x$ in $\tilde{U}_{\mathcal{S}_x}$ will be denoted by $\tilde{S}_x$.

Products of Orbifolds. Cartesian products of (locally) smooth orbifolds inherit a natural (locally) smooth orbifold structure:

Definition 2.12. Let $\mathcal{O}_i$ for $i = 1, 2$ be orbifolds. The *orbifold product* $\mathcal{O}_1 \times \mathcal{O}_2$ is the orbifold having the following structure:

1. $X_{\mathcal{O}_1 \times \mathcal{O}_2} = X_{\mathcal{O}_1} \times X_{\mathcal{O}_2}$.
2. For each $(x_1, x_2) \in X_{\mathcal{O}_1 \times \mathcal{O}_2}$ and orbifold charts U_i of x_i, $U_1 \times U_2$ is an orbifold chart around (x_1, x_2). Explicitly,

$$(\tilde{U}_1 \times \tilde{U}_2, \Gamma_{x_1} \times \Gamma_{x_2}, \rho_{x_1} \times \rho_{x_2}, \phi_{x_1} \times \phi_{x_2})$$

is an orbifold chart around (x_1, x_2).

Note that the isotropy group $\Gamma_{(x_1,x_2)} = \Gamma_{x_1} \times \Gamma_{x_2}$.

Suborbifolds. The definition of a suborbifold is somewhat more delicate than the corresponding notion for a manifold.

Definition 2.13. A *suborbifold* $\mathcal{P}$ of an orbifold $\mathcal{O}$ consists of the following.

1. A subspace $X_{\mathcal{P}} \subset X_{\mathcal{O}}$ equipped with the subspace topology
2. For each $x \in X_{\mathcal{P}}$ and neighborhood W of x in $X_{\mathcal{P}}$ there is an orbifold chart $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ about x in $\mathcal{O}$ with $U_x \subset W$, a subgroup $\Lambda_x \subset \Gamma_x$ of the isotropy group of x in $\mathcal{O}$ and a $\rho_x(\Lambda_x)$ invariant vector subspace $\tilde{V}_x \subset \tilde{U}_x = \mathbb{R}^n$, so that $(\tilde{V}_x, \Lambda_x, \rho_x|_{\Lambda_x}, \psi_x)$ is an orbifold chart for $\mathcal{P}$ and
3. $V_x = \psi_x(\tilde{V}_x/\rho_x(\Lambda_x)) = U_x \cap X_{\mathcal{P}} = \phi_x(\pi_x(\tilde{V}_x))$ is an orbifold chart for x in $\mathcal{P}$ where $\pi_x : \tilde{U}_x \to \tilde{U}_x/\rho_x(\Gamma_x)$ is the quotient map.

Remark 2.14. It is tempting to define the notion of an m–suborbifold $\mathcal{P}$ of an n–orbifold $\mathcal{O}$ simply by requiring $\mathcal{P}$ to be locally modeled on $\mathbb{R}^m \subset \mathbb{R}^n$ modulo finite groups. That is, the local action on $\mathbb{R}^m$ is induced by the local action on $\mathbb{R}^n$. This is the definition adopted in [Thu78]. It is equivalent to the added condition in our definition that $\Lambda_x = \Gamma_x$ at all x in the underlying topological space of $\mathcal{P}$. This more restrictive definition is not adequate for our needs as the following example shows.

Example 2.15. Let $\mathcal{O}$ be a smooth C^r orbifold. Let $\operatorname{diag}(\mathcal{O}) = \{(x,x) \mid x \in \mathcal{O}\} \subset \mathcal{O} \times \mathcal{O}$ be the *diagonal*. Then $\operatorname{diag}(\mathcal{O})$ is a suborbifold of $\mathcal{O} \times \mathcal{O}$ with isotropy group $\Gamma_{(x,x)} \cong \Gamma_x$ via the diagonal action $\gamma \cdot (\tilde{x}, \tilde{x}) = (\gamma \cdot \tilde{x}, \gamma \cdot \tilde{x})$. See proposition 3.8. If we had chosen the more restrictive definition of suborbifold given in the last remark, then $\operatorname{diag}(\mathcal{O})$ would *not* have been a suborbifold. For example, consider the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $\gamma \cdot x = -x$. The underlying topological space $X_{\mathcal{O}}$ of $\mathcal{O}$ is $[0,\infty)$ and the isotropy subgroups are $\{1\}$ for $x \in (0,\infty)$ and $\mathbb{Z}_2$ for $x = 0$. The isotropy subgroup of $(0,0) \in \mathbb{R}/\mathbb{Z}_2 \times \mathbb{R}/\mathbb{Z}_2$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, whereas the isotropy subgroup of $(0,0)$ in the diagonal suborbifold $\operatorname{diag}(\mathbb{R}/\mathbb{Z}_2) \subset \mathbb{R}/\mathbb{Z}_2 \times \mathbb{R}/\mathbb{Z}_2$ must be isomorphic to $\mathbb{Z}_2$, as $\operatorname{diag}(\mathbb{R}/\mathbb{Z}_2)$ is a 1-dimensional suborbifold.

Remark 2.16. Let $\mathcal{P} \subset \mathcal{O}$ be a suborbifold. Note that even though a point $p \in X_{\mathcal{P}}$ may be in the singular set of $\mathcal{O}$, it need not be in the singular set of $\mathcal{P}$.

3. Orbifold Maps

Intuitively, an orbifold map should be a map between underlying topological spaces that has local lifts, but unfortunately axiomatizing such a simple idea has proven difficult if one wants to provide a definition that is very flexible. We now discuss one such natural definition of maps between orbifolds. This definition will elaborate on the definition that was given in the paper [BB02]. In that paper, these maps were referred to as unreduced orbifold maps because we distinguished among different liftings of the same map of underlying topological spaces. From now on, we will refer to such maps simply as orbifold maps. In [BB03], our definition of (reduced) orbifold map did not distinguish among different liftings. We will retain the term *reduced* for orbifold maps for which the particular choice of local lifts is ignored. Thus, a reduced orbifold map agrees with the notion of orbifold map given in [ALR07, Def. 1.3]. In what follows, we use the notation given in definitions 2.1, 2.6 and 2.11.

Definition 3.1. A C^0 *orbifold map* $(f, \{\tilde{f}_x\})$ between locally smooth orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ consists of the following:

1. A continuous map $f : X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$ of the underlying topological spaces.
2. For each $y \in \mathcal{S}_x$, a group homomorphism $\Theta_{f,y} : \Gamma_{\mathcal{S}_x} \to \Gamma_{f(y)}$.
3. A $\Theta_{f,y}$-equivariant lift $\tilde{f}_y : \tilde{U}_y \subset \tilde{U}_{\mathcal{S}_x} \to \tilde{V}_{f(y)}$ where $(\tilde{U}_y, \Gamma_{\mathcal{S}_x}, \rho_y, \phi_y)$ is an orbifold chart at y and $(\tilde{V}_{f(y)}, \Gamma_{f(y)}, \rho_{f(y)}, \phi_{f(y)})$ is an orbifold chart at $f(y)$. That is, the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_y \arrow[r, "\tilde{f}_y"] \arrow[d] & \tilde{V}_{f(y)} \arrow[d] \\
\tilde{U}_y/\Gamma_{\mathcal{S}_x} \arrow[r, "\tilde{f}_y/\Theta_{f,y}(\Gamma_{\mathcal{S}_x})"] \arrow[dd] & \tilde{V}_{f(y)}/\Theta_{f,y}(\Gamma_{\mathcal{S}_x}) \arrow[d] \\
 & \tilde{V}_{f(y)}/\Gamma_{f(y)} \arrow[d] \\
U_y \subset U_{\mathcal{S}_x} \arrow[r, "f"] & V_{f(y)}
\end{tikzcd}$$

4. (Equivalence) Two orbifold maps $(f, \{\tilde{f}_x\})$ and $(g, \{\tilde{g}_x\})$ are considered equivalent if for each $x \in \mathcal{O}_1$, $\tilde{f}_x = \tilde{g}_x$ as germs. That is, there exists an orbifold chart $(\tilde{U}_x, \Gamma_x)$ at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$. Note that this implies that $f = g$.

Remark 3.2. Note that equivalence of two orbifold maps does not require that $\Theta_{f,x} = \Theta_{g,x}$. To see that this is justifiable, consider the example where $\mathcal{O}$ is the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $x \to -x$ and f is the constant map $f \equiv 0$. The underlying topological space $X_{\mathcal{O}}$ of $\mathcal{O}$ is $[0, \infty)$ and the isotropy subgoups are trivial for $x \in (0, \infty)$ and $\mathbb{Z}_2$ for $x = 0$. The map $\tilde{f}_0 \equiv 0$ is a local equivariant lift of f at $x = 0$ using either of the homomorphisms $\Theta_{f,0} = \mathrm{Id}$ or $\Theta'_{f,0} = \{e\}$. We do not wish to consider these as distinct orbifold maps.

We will often denote an orbifold map $(f, \{\tilde{f}_x\})$ simply by f for convenience.

Definition 3.3. An orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$ of C^r smooth orbifolds is C^r *smooth* if each of the local lifts $\tilde{f}_x$ may be chosen to be C^r.

The next lemma is a technical result that states that a local lift $\tilde{f}_x$ chosen on a particular orbifold chart about x uniquely specifies a local lift on any other orbifold chart about x. Hence, in definition 3.1, the $\tilde{f}_x$'s, once chosen, are independent of the choice of local charts.

Lemma 3.4. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a C^r orbifold map, $x \in \mathcal{O}_1$, $U_x \subset W_x$ connected orbifold charts around x and $V_{f(x)} \subset Z_{f(x)}$ connected orbifold charts around $f(x)$ in $\mathcal{O}_2$ with $f(U_x) \subset V_{f(x)}$ and $f(W_x) \subset Z_{f(x)}$. If $\tilde{f}_{U_x}$ is a lift of f to $\tilde{U}_x$, then there is a unique lift $\tilde{f}_{W_x}$ of f to $\tilde{W}_x$ extending $\tilde{f}_{U_x}$.*

Proof. Let $\tilde{D}_x \subset \tilde{W}_x$ and $\tilde{D}_{f(x)} \subset \tilde{Z}_{f(x)}$ be Dirichlet fundamental domains for the actions of the isotropy groups Γ_x and $\Gamma_{f(x)}$ on $\tilde{W}_x$ and $\tilde{Z}_{f(x)}$ respectively. Then, $\tilde{D}_x \cap \tilde{U}_x$ and $\tilde{D}_{f(x)} \cap \tilde{V}_{f(x)}$ are also Dirichlet fundamental domains for the actions of the respective isotropy groups on $\tilde{U}_x$ and $\tilde{V}_{f(x)}$ respectively. Let $\tilde{y} \in \tilde{U}_x \cap \tilde{D}_x$ be a point in the non-singular set of $\mathcal{O}_1$. Without loss of generality, we may take $\tilde{D}_{f(x)}$ to be the Dirichlet fundamental domain containing $\tilde{f}_{\tilde{U}_x}(\tilde{y})$ and so for any $\tilde{z} \in \tilde{D}_x$, there is a unique $\tilde{w} \in \tilde{D}_{f(x)}$ with $\pi_{f(x)}(\tilde{w}) = f(\pi_x(\tilde{z}))$. Now define the extension $\tilde{f}_{\tilde{W}_x} : \tilde{W}_x \to \tilde{Z}_{f(x)}$ via:

$$\tilde{f}_{\tilde{W}_x}(\gamma \cdot \tilde{z}) = \Theta_{f,x}(\gamma) \cdot \tilde{w}$$

Uniqueness and continuity of the extension follow from the properties of Dirichlet domains. ■

Given two orbifolds $\mathcal{O}_i$, $i = 1, 2$, the class of C^r orbifold maps from $\mathcal{O}_1$ to $\mathcal{O}_2$ will be denoted by $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$. If $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$, we use the notation $C^r_{\mathrm{Orb}}(\mathcal{O})$ instead. The following was stated as a proposition without proof in [BB02].

Example 3.5. (Lifts of the Identity Map) Consider the identity map $\mathrm{Id} : \mathcal{O} \to \mathcal{O}$. Let $x \in \mathcal{O}$ and $(\tilde{U}_x, \Gamma_x)$ be an orbifold chart at x. From the definition of orbifold map, it follows (since Γ_x is finite) that there exists $\gamma \in \Gamma_x$ such that a lift $\widetilde{\mathrm{Id}}_x : \tilde{U}_x \to \tilde{U}_x$ is given by $\widetilde{\mathrm{Id}}_x(\tilde{y}) = \gamma \cdot \tilde{y}$ for all $\tilde{y} \in \tilde{U}_x$. Since $\widetilde{\mathrm{Id}}_x$ is $\Theta_{\mathrm{Id},x}$ equivariant we have for $\delta \in \Gamma_x$:

$$\begin{aligned}
\widetilde{\mathrm{Id}}_x(\delta \cdot \tilde{y}) &= \Theta_{\mathrm{Id},x}(\delta) \cdot \widetilde{\mathrm{Id}}_x(\tilde{y}) && \text{hence} \\
\gamma\delta \cdot \tilde{y} &= \Theta_{\mathrm{Id},x}(\delta)\gamma \cdot \tilde{y} && \text{which implies} \\
& && \text{since } \Gamma_x \text{ acts effectively that} \\
\gamma\delta &= \Theta_{\mathrm{Id},x}(\delta)\gamma && \text{or, equivalently,} \\
\Theta_{\mathrm{Id},x}(\delta) &= \gamma\delta\gamma^{-1}
\end{aligned}$$

Thus, $\Theta_{\mathrm{Id},x}$ is an isomorphism of Γ_x, in fact, an inner automorphism. Since two inner automorphisms, $I_{\gamma_i}(\delta) = \gamma_i\delta\gamma_i^{-1}$, give rise to the same automorphism of Γ_x precisely when $\gamma_1 = \zeta\gamma_2$ where $\zeta \in \mathrm{Center}(\Gamma_x)$, the number of possible distinct choices for the homomorphism $\Theta_{\mathrm{Id},x}$ is $\dfrac{|\Gamma_x|}{|\mathrm{Center}(\Gamma_x)|}$. In particular, if x is non–singular, or more generally, if Γ_x is abelian, $\Theta_{\mathrm{Id},x}$ is the identity isomorphism on

Γ_x, and the identity map has exactly $|\Gamma_x|$ local lifts over x. Moreover, we see that the identity map between C^r orbifolds is C^r. In fact, it is an example of a C^r orbifold diffeomorphism (definition 3.9).

Example 3.6. Let $\mathcal{O}$ be an orbifold and $X_{\mathcal{O}}$ its underlying topological space. Let N be a manifold or manifold with boundary (with trivial orbifold structure). Let $f : X_{\mathcal{O}} \to N$ be a (topologically) continuous map; that is $f \in C^0(X_{\mathcal{O}}, N)$. Then f is naturally an orbifold continuous map; that is $f \in C^0_{\mathrm{Orb}}(\mathcal{O}, N)$. To see this, note that since N is a trivial orbifold, $\Gamma_{f(x)} = \{e\}$ for all $x \in \mathcal{O}$. Thus, $\Theta_{f,x}$ is the constant homomorphism $\gamma \mapsto e$. Therefore, equivariant local lifts $\tilde{f}_x : \tilde{U}_x \to \tilde{V}_{f(x)} = V_{f(x)}$ may be defined via $\tilde{f}_x(\tilde{y}) = f \circ \pi_x(\tilde{y})$ for $\tilde{y} \in \tilde{U}_x$. By construction $\tilde{f}$ is well-defined, continuous and unique, and thus $f \in C^0_{\mathrm{Orb}}(\mathcal{O}, N)$.

Example 3.7. Let $\mathcal{O}$ be a smooth orbifold and let N be a smooth manifold or manifold with boundary (with trivial orbifold structure). If $f \in C^r_{\mathrm{Orb}}(N, \mathcal{O})$, then since $\Gamma_x = \{e\}$ for all $x \in N$ the homomorphism $\Theta_{f,x} : \Gamma_x \to \Gamma_{f(x)}$ is just $e \mapsto e$. Thus f is merely a map from N to $\mathcal{O}$ with choice of local C^r lifts. In the case where $\partial N \neq \emptyset$, this means that a local lift is C^r over $N - \partial N$ with continuous extension to ∂N.

Proposition 3.8. *Let $f \in C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$, then the graph of f, graph(f), defined by*

$$\mathrm{graph}(f) = \{(x, f(x)) \in \mathcal{O}_1 \times \mathcal{O}_2\} \subset \mathcal{O}_1 \times \mathcal{O}_2$$

is a C^r suborbifold. Note the isotropy group $\Gamma_{(x,y)} \cong \Gamma_x$ is acting on $\tilde{U}_x \times \tilde{V}_y$, a chart in $\mathcal{O}_1 \times \mathcal{O}_2$, via the twisted diagonal action $\gamma \cdot (\tilde{x}, \tilde{y}) = (\gamma \cdot \tilde{x}, \Theta_{f,x}(\gamma) \cdot \tilde{y})$.

Proof. Let $x \in \mathcal{O}_1$, $(\tilde{U}_x, \Gamma_x)$ a chart at x, $\Theta_{f,x} \in \mathrm{Hom}(\Gamma_x, \Gamma_{f(x)})$, $(\tilde{V}_{f(x)}, \Gamma_{f(x)})$ a chart at $f(x)$ and equivariant lift $\tilde{f}_x : \tilde{U}_x \to \tilde{V}_{f(x)}$ of f. That is, $\Theta_{f,x}(\gamma) \cdot \tilde{f}(\tilde{x}') = \tilde{f}(\gamma \cdot \tilde{x}')$ for all $\gamma \in \Gamma_x$ and $\tilde{x}' \in \tilde{U}_x$. For $(x, f(x)) \in \mathrm{graph}(f) \subset \mathcal{O}_1 \times \mathcal{O}_2$ we have $\Gamma_{(x,f(x))} = \Gamma_x \times \Gamma_{f(x)}$. We need to give a suborbifold structure for graph(f). Define the subgroup

$\Gamma_\Theta = \{(\gamma, \Theta_{f,x}(\gamma)) \mid \gamma \in \Gamma_x\} \subset \Gamma_x \times \Gamma_{f(x)}$ and let $\tilde{W}_x = \{(\tilde{x}', \tilde{f}(\tilde{x}')) \mid \tilde{x}' \in \tilde{U}_x\} \subset \tilde{U}_x \times \tilde{V}_{f(x)}$. Note that $\tilde{W}_x$ is Γ_Θ invariant: Suppose $(\tilde{x}', \tilde{f}(\tilde{x}')) \in \tilde{W}_x$ and $\delta = (\gamma, \Theta_{f,x}(\gamma)) \in \Gamma_\Theta$. Then

$$\delta \cdot \left(\tilde{x}', \tilde{f}(\tilde{x}')\right) = \left(\gamma \cdot \tilde{x}', \Theta_{f,x}(\gamma) \cdot \tilde{f}(\tilde{x}')\right) = \left(\gamma \cdot \tilde{x}', \tilde{f}(\gamma \cdot \tilde{x}')\right) \in \tilde{W}_x$$

Thus, $\left(\tilde{U}_x \times \tilde{V}_{f(x)}, \Gamma_x \times \Gamma_{f(x)}, \rho_x \times \rho_{f(x)}, \phi_x \times \phi_{f(x)}\right)$ is an orbifold chart around $(x, f(x))$ with $\left(\tilde{W}_x, \Gamma_\Theta, \rho_x \times \rho_{f(x)}\big|_{\Gamma_\Theta}, \psi_x = \phi_x \times \phi_{f(x)}\big|_{\mathrm{graph}(f)}\right)$ the required suborbifold chart around $(x, f(x)) \in \mathrm{graph}(f)$. ∎

Definition 3.9. For any topological space, let Homeo(X) denote its group of homeomorphisms. For a C^0 orbifold $\mathcal{O}$, denote by $\mathrm{Homeo}_{\mathrm{Orb}}(\mathcal{O})$ the subgroup of $\mathrm{Homeo}(X_{\mathcal{O}})$ with $f, f^{-1} \in C^0_{\mathrm{Orb}}(\mathcal{O})$. If $\mathcal{O}$ is a C^r orbifold, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$, the C^r *orbifold diffeomorphism group*, is the subgroup of $\mathrm{Homeo}_{\mathrm{Orb}}(\mathcal{O})$ with $f, f^{-1} \in C^r_{\mathrm{Orb}}(\mathcal{O})$.

Example 3.10. Consider the case of a so-called $\mathbb{Z}_p$-football $\mathcal{O} = S^2/\mathbb{Z}_p$ where $\mathbb{Z}_p$ acts on $S^2 \subset \mathbb{R}^3$ by rotation about the z-axis by an angle $2\pi/p$. It is an example, in the language of Thurston, of a *good* orbifold $\mathcal{O} = M/\Gamma$ where M is a smooth manifold and Γ acts effectively on M as a proper discontinuous group of diffeomorphisms on M. This type of orbifold is referred to as an *effective global quotient* in [ALR07]. There are two singular points corresponding to the north and south poles. Let $\mathcal{ID}$ denote the subgroup of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ comprised of all lifts of the identity map. Then $\mathcal{ID} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. If we let $\mathrm{Diff}^r_{\mathbb{Z}_p}(M) \subset \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ denote the (global) $\mathbb{Z}_p$-equivariant diffeomorphisms of M and let $\mathcal{ID}_{\mathbb{Z}_p} \subset \mathrm{Diff}^r_{\mathbb{Z}_p}(M)$ denote the $\mathbb{Z}_p$-equivariant lifts of the identity, then $\mathcal{ID}_{\mathbb{Z}_p} \cong \mathbb{Z}_p$. This example shows that, in general, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ will be strictly larger than $\mathrm{Diff}^r_{\Gamma}(M)$ for a good orbifold $\mathcal{O} = M/\Gamma$.

Recall the following terminology [Hir76]: Let $\mathcal{R}$ be a C^r smooth structure on an orbifold $\mathcal{O}$. A C^s smooth structure $\mathcal{S}$ on $\mathcal{O}$, $s > r$, is *compatible with* $\mathcal{R}$ if $\mathcal{S} \subset \mathcal{R}$. This means that orbifold charts in $(\mathcal{O}, \mathcal{S})$ are orbifold charts in $(\mathcal{O}, \mathcal{R})$ in the sense that the identity map of $\mathcal{O}$ is a element of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. As in the classical case of smooth manifolds [Whi36], we have the following result on raising the differentiablity of smooth orbifold structures.

Proposition 3.11. *Let $\mathcal{R}$ be a C^r smooth structure on an orbifold $\mathcal{O}$, $r \geq 1$. For every s, $r < s \leq \infty$, there exists a compatible C^s smooth structure $\mathcal{S} \subset \mathcal{R}$, and $\mathcal{S}$ is unique up to C^s orbifold diffeomorphism.*

Proof. In light of definition 2.4 and example 3.5, one merely need use the results of Palais [Pal70]. ∎

4. Function Space Topologies

In this section, we assume that $\mathcal{O}_i$ are smooth C^r orbifolds and define the (strong/fine/Whitney) C^r topology on $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$. For $f \in C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$, we first define a C^0 neighborhood of f and corresponding C^0 topology on $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$. Although we will introduce a Riemannian structure later, for our purposes now we make the observation that orbifolds are metrizable: Just let $U = \tilde{U}/\Gamma = \pi(U)$ be any orbifold chart of $\mathcal{O}$. Since Γ is finite, we may define a metric on U by $d_U(x, y) = d_{\tilde{U}}\left(\pi^{-1}(x), \pi^{-1}(y)\right)$ where $d_{\tilde{U}}$ is the usual Euclidean metric on $\tilde{U}$. This makes $\mathcal{O}$ locally metrizable. Since all orbifolds are assumed paracompact and Hausdorff, the Smirnov metrization theorem [Mun75] implies $\mathcal{O}$ is metrizable and second countable.

Definition 4.1. Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a C^r orbifold map. Let $\mathcal{C} = \{C_i\}$ be a locally finite covering of $\mathcal{O}_1$ by relatively compact, open sets such that $\overline{C}_i \subset U_i$ and $f(\overline{C}_i) \subset V_i$ where U_i and V_i are (open) relatively compact orbifold charts. Let $\{\varepsilon_i\}$ be a collection of positive constants. Let $\mathcal{N}^0(f, \varepsilon_i; \mathcal{C})$ consist of all $g \in C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ such that for all i, $g(C_i) \subset V_i$ and $\|\tilde{f}_x(\tilde{y}) - \tilde{g}_x(\tilde{y})\|_{\tilde{V}_i} < \varepsilon_i$ for all $x \in C_i$ and $\tilde{y} \in \pi_x^{-1}(C_i \cap U_x)$. The sets $\mathcal{N}^0(f, \varepsilon_i; \mathcal{C})$ form a neighborhood base for a topology on $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$, which we call the (orbifold) C^0 *topology* relative to $\mathcal{C}$ and we refer to $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ with this topology as $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C})$.

To define the (strong/fine/Whitney) C^s topology on $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ for $1\le s\le r$, we simply require, in addition, that local lifts are C^s close in the usual C^s topology. In particular we have,

Definition 4.2. Let $f:\mathcal{O}_1\to\mathcal{O}_2$ be a C^r orbifold map. Define $\mathcal{N}^s(f,\varepsilon_i;\mathcal{C})$ to be those maps $g\in\mathcal{N}^0(f,\varepsilon_i;\mathcal{C})$ such that for all $1\le k\le s$, $\|\partial^k\tilde f_x(\tilde y)-\partial^k\tilde g_x(\tilde y)\|<\varepsilon_i$ for all $x\in C_i$ and $\tilde y\in\pi_x^{-1}(C_i\cap U_x)$. This means that the local lifts of f and g have all partial derivatives of order $\le s$ within ε_i at each point of $\tilde y\in\pi_x^{-1}(C_i\cap U_x)$. Sets of this type form a neighborhood base for the (orbifold) C^s *topology* on $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ relative to the atlas $\mathcal{C}$. The C^∞ *topology* relative to $\mathcal{C}$ on $C^\infty_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ is defined to be the union of the topologies induced by the inclusion maps $C^\infty_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C})\hookrightarrow C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C})$ for finite r and as above, and $C^\infty_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ with this topology will be denoted by $C^\infty_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C})$ as above.

Remark 4.3. If both $\mathcal{O}_1$ and $\mathcal{O}_2$ are compact, then the coverings $\{C_i\}$ are finite and ε_i may be chosen to be a constant ε for all i. The resulting topologies induced by the neighborhood base $\mathcal{N}^s(f,\varepsilon)$ on $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ are equivalent to the topologies in definitions 4.1 and 4.2 given above.

Proposition 4.4. *The topology on $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ is independent of the cover $\mathcal{C}$. That is, the spaces $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C})$ and $C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2;\mathcal{C}')$ are homeomorphic for any two covers $\mathcal{C}$ and $\mathcal{C}'$ as in definition 4.2 and any value of r where $0\le r\le\infty$.*

The proof depends on the following lemma. To aid both the statement and proof of the following lemma, the following notation will be useful. For $f\in C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$, U a chart about $x\in\mathcal{O}_1$, V a chart about $f(x)\in\mathcal{O}_2$ and relatively compact connected open sets $x\in C'\subset\overline{C}'\subset C\subset\overline{C}\subset U$, define

$$\begin{aligned}\mathcal{N}^s(f,\varepsilon;C)=\{g\in C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)&\text{ such that}\\ &\|\partial^k\tilde f(\tilde y)-\partial^k\tilde g(\tilde y)\|<\varepsilon\text{ for all }\tilde y\in\tilde C\text{ and all }k\le s\}\end{aligned}$$

$$\begin{aligned}\mathcal{N}^s(f,\varepsilon;C,C')=\{g\in\mathcal{N}^s(f,\varepsilon;C')&\text{ such that}\\ &\|\partial^k f(y)-\partial^k g(y)\|<\varepsilon\text{ for all }y\in C-\Sigma_1\text{ and}\\ &\|f(y)-g(y)\|<\varepsilon\text{ for all }y\in C\}\end{aligned}$$

Lemma 4.5. *Let f, x, U, C and $C'\subset C$ be as above, then for each $\varepsilon>0$ there is a $\delta>0$ so that*

$$\mathcal{N}^r(f,\delta;C,C')\subset\mathcal{N}^r(f,\varepsilon;C)$$

Proof. The proof is by contradiction. Assuming the contrary implies that there is an $\varepsilon>0$ and a sequence $\{g_n\}\subset C^r_{\mathrm{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ so that

$$g_n\in\mathcal{N}^r(f,2^{-n};C,C')\text{ and }g_n\notin\mathcal{N}^r(f,\varepsilon;C)$$

For each $y \in C$, let $\Gamma_{f(y)}$ to be the isotropy group of $f(y)$ and $\theta_{f(y)f(x)} : \Gamma_{f(y)} \to \Gamma_{f(x)}$ the injective homomorphism of definition 2.1. Let $N(x, y)$ denote the index of $\theta_{f(y)f(x)}(\Gamma_{f(y)})$ in $\Gamma_{f(x)}$, $\left|\Gamma_{f(x)} : \theta_{f(y)f(x)}(\Gamma_{f(y)})\right|$ and let γ_i, $i = 1, \ldots, N(x, y)$ the corresponding coset representatives. Then there is a neighborhood, $\tilde{V}_{\tilde{f}(\tilde{y})}$ of $\tilde{f}(\tilde{y})$ in $\tilde{V}$ so that $\gamma_i \cdot \tilde{V}_{\tilde{f}(y)} \cap \gamma_j \cdot \tilde{V}_{\tilde{f}(\tilde{y})} = \emptyset$ if $i \neq j$. Thus, the projection $\pi : \tilde{V}/\theta_{f(x)f(y)}(\Gamma_{f(y)}) \to V$ is a local isometry over $\tilde{V}_{f(y)}$ by our choice of metric. For any $\tilde{y} \in \tilde{C}$ let $\tilde{W}_{\tilde{y}} = \tilde{f}^{-1}\left(\tilde{V}_{\tilde{f}(\tilde{y})}\right)$. $\left\{\tilde{W}_{\tilde{y}}\right\}$ is an open cover of $\tilde{C}$. Compactness of $\overline{\tilde{C}}$ yields a finite subcover $\tilde{W}_{\tilde{y}_1}, \ldots, \tilde{W}_{\tilde{y}_M}$. Without loss of generality, we may also uniformly bound the radii of the neighborhoods $V_{f(y)}$ in the range so that this cover is non-trivial.

Now let $\tilde{D} \subset \tilde{C}$ be the maximal domain defined by

$$\tilde{D} = \{\tilde{z} \in \tilde{C} \mid \tilde{g}_n(\tilde{z}) \to \tilde{f}(\tilde{z}) \text{ pointwise}\}$$

A Cantor diagonal argument shows that the limit point of any sequence $\tilde{z}_n \to \tilde{z}$ is also in $\tilde{D}$ and so $\tilde{D}$ is closed and therefore a compact set containing $\tilde{C}'$. Thus, there are points $\tilde{y}_{\alpha_1}, \ldots, \tilde{y}_{\alpha_k} \subset \{\tilde{y}_1, \ldots, \tilde{y}_M\}$ so that $\tilde{W}_{\tilde{y}_{\alpha_1}}, \ldots, \tilde{W}_{\tilde{y}_{\alpha_k}}$ cover $\tilde{D}$ and $\tilde{D} \cap \tilde{W}_{\tilde{y}_{\alpha_i}} \neq \emptyset$ for $i = 1, \ldots, k$. By shrinking the $\tilde{W}_{\tilde{y}_{\alpha_i}}$'s we may assume that they still cover $\tilde{D}$ and they also satisfy $\tilde{g}_n(\tilde{W}_{\tilde{y}_{\alpha_i}}) \subset \tilde{V}_{\tilde{f}(\tilde{y}_{\alpha_i})}$ for n sufficiently large and all i. Picking $\tilde{z}_i \in \tilde{D} \cap \tilde{W}_{\tilde{y}_{\alpha_i}}$ for each i we have by definition of the $\tilde{W}$'s that

$$\|\tilde{g}_n(\tilde{z}) - \gamma_i \cdot \tilde{f}(\tilde{z})\| = \|g_n(z) - f(z)\|$$

for all $\tilde{z} \in \tilde{W}_{\tilde{y}_{\alpha_i}}$ and some coset representative γ_i of $\theta_{f(x)f(y)}(\Gamma_{f(y)})$ in $\Gamma_{f(x)}$. By evaluating at some $\tilde{z}_i \in \tilde{D} \cap \tilde{W}_{\tilde{y}_{\alpha_i}}$, the definition of $\tilde{D}$ implies we must have $\gamma_i = e$ and thus, $\tilde{g}_n(\tilde{z}) \to \tilde{f}(\tilde{z})$ for all $\tilde{z} \in \tilde{W}_{\tilde{y}_{\alpha_i}}$. Since this holds for each $i = 1, \ldots, k$, $\tilde{g}_n(\tilde{z}) \to \tilde{f}(\tilde{z})$ for all $\tilde{z} \in \bigcup_{i=1}^{k} \tilde{W}_{\tilde{y}_{\alpha_i}}$ of which $\tilde{D}$ is a proper subset. This contradicts the maximality of $\tilde{D}$. ■

Proof. (Proof of proposition 4.4) Given two open covers $\mathcal{C}$ and $\mathcal{C}'$, take an open cover $\mathcal{C}''$ that refines them both. Clearly the inclusion maps

$$C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}) \hookrightarrow C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'') \text{ and } C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}') \hookrightarrow C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$$

induced by restriction to the common refinement $\mathcal{C}''$ in each of the covers $\mathcal{C}$ and $\mathcal{C}'$ show that the topology on $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ is coarser than either of the topologies induced by $\mathcal{C}$ or $\mathcal{C}'$. We now show that $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ is, in fact, homeomorphic to $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C})$.

Since sets of the form $\mathcal{N}^r(f, \varepsilon; C)$ for $C \in \mathcal{C}$ form a subbase for the topology of $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C})$, it suffices to find a neighborhood of f in $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ contained in $\mathcal{N}^r(f, \varepsilon; C)$. Let $C''_1, \ldots, C''_k \in \mathcal{C}''$ be a cover of $C \in \mathcal{C}$. For any $\delta > 0$

$$\bigcap_{i=1}^{k} \mathcal{N}^r(f, \delta; C''_k) \subset \mathcal{N}^r(f, \delta; C, C''_i)$$

Therefore, by lemma 4.5, $\mathcal{N}^r(f, \varepsilon; C)$ is open in $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ and thus we may conclude that $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ and $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C})$ are homeomorphic. Similarly,

$C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}'')$ and $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}')$ are homeomorphic. Thus, $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C})$ and $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C}')$ are homeomorphic as claimed. ■

From now on, we drop the dependence of topology on $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2; \mathcal{C})$ on the cover $\mathcal{C}$, and will simply use the notation $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ for the set of orbifold functions with the C^r topology as in definition 4.2. For the remainder of the paper, whenever function spaces between orbifolds are mentioned, we will assume that the source orbifolds are compact.

Definition 4.6. For a fixed cover $\mathcal{C}$ by orbifold charts and any $\varepsilon > 0$, put

$$\mathcal{N}^s(f, \varepsilon) = \{g \in C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2) \mid g \in \mathcal{N}^s(f, \varepsilon; C) \text{ for all } C \in \mathcal{C}\}$$

As in the case for compact manifolds, for a compact orbifold $\mathcal{O}_1$, we define for f and $g \in C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ a distance

$$d_s(f, g) = \inf\{\varepsilon > 0 \mid f \in \mathcal{N}^s(g, \varepsilon) \text{ and } g \in \mathcal{N}^s(f, \varepsilon)\}$$

where the dependence on the orbifold atlas used has been supressed.

Remark 4.7. Compactness of $\mathcal{O}_1$ implies (as in the usual manifold case) that the metric topology induced by the metric d_s as above is equivalent to the C^s topology on $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ given by the orbifold atlas $\mathcal{C}$ (and hence to the topology induced by any other atlas by proposition 4.4).

Proposition 4.8. *Let $\mathcal{O}_i$ be compact C^r orbifolds, $1 \le r \le \infty$. For $1 \le s \le r$, $C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ with the C^s topology relative to $\mathcal{C}$ is a separable metric space. If $s = r$, then this metric space is complete.*

Proof. Let $\{f_n\} \subset C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ be a Cauchy sequence in the C^r topology. For any $x \in \mathcal{O}_1$, orbifold charts U_x about x and $V \subset \mathcal{O}_2$ containing $\bigcup_n f_n(U_x)$, the lifts $\{\tilde{f}_n : \tilde{U}_x \to \tilde{V}\}$ are a sequence of Γ_x-equivariant functions converging uniformly in the C^r topology on compact subsets of $\tilde{U}_x$. Therefore they converge to a C^r, Γ_x-equivariant function $\tilde{f} : U_x \to \tilde{V}$ which is a lift of the function $f(x) = \lim f_n(x)$. Thus, the limit function $f \in C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ which proves completeness. For separability, note that for any $f \in C^r_{\text{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$, each lift $\tilde{f}_x : \tilde{U}_x \to \tilde{V}_{f(x)}$ may be approximated by a polynomial $\tilde{g}_x : \tilde{U}_x \to \tilde{V}_{f(x)}$. To get a Γ_x-equivariant approximation by a polynomial we average $\tilde{g}_x$ over Γ_x. That is, we define $\tilde{G}_x : \tilde{U}_x \to \tilde{V}_{f(x)}$ by

$$\tilde{G}_x(\tilde{z}) = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \Theta_{f,x}(\gamma) \cdot \tilde{g}_x(\gamma^{-1} \cdot \tilde{z})$$

Since

$$\begin{aligned}
\tilde{G}_x(\delta\cdot\tilde{z}) &= \frac{1}{|\Gamma_x|}\sum_{\gamma\in\Gamma_x}\Theta_{f,x}(\gamma)\cdot\tilde{g}_x(\gamma^{-1}\delta\cdot\tilde{z})\\
&= \frac{1}{|\Gamma_x|}\sum_{\gamma\in\Gamma_x}\Theta_{f,x}(\delta)\Theta_{f,x}(\delta^{-1}\gamma)\cdot\tilde{g}_x((\delta^{-1}\gamma)^{-1}\cdot\tilde{z})\\
&= \Theta_{f,x}(\delta)\cdot\frac{1}{|\Gamma_x|}\sum_{\mu\in\Gamma_x}\Theta_{f,x}(\mu)\cdot\tilde{g}_x(\mu^{-1}\cdot\tilde{z}) \qquad \text{where } \mu=\delta^{-1}\gamma\\
&= \Theta_{f,x}(\delta)\cdot\tilde{G}_x(\tilde{z})
\end{aligned}$$

we see that $\tilde{G}_x$ satisfies the same equivariance relation as $\tilde{f}_x$ and thus $\tilde{G}_x \in C^r_{\text{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$. Since averaging is distance nonincreasing, we have produced an approximation of $\tilde{f}_x$ by Γ_x-equivariant polynomials. Furthermore, because there can be only finitely many lifts of f over any orbifold chart, compactness of $\mathcal{O}_1$ implies that the space $C^r_{\text{Orb}}(\mathcal{O}_1,\mathcal{O}_2)$ is separable as the equivariant polynomials form a countable dense set. ■

5. The Tangent Orbibundle and its Sections

We now define the tangent orbibundle of a smooth C^{r+1} orbifold. It is a special case of the more general notion of a linear orbibundle given in [BB02].

Definition 5.1. Let $\mathcal{O}$ be an n–dimensional C^{r+1} orbifold. The *tangent orbibundle* of $\mathcal{O}$, $p: T\mathcal{O}\to\mathcal{O}$, is the C^r orbibundle defined as follows. If $(\tilde{U}_x,\Gamma_x)$ is an orbifold chart around $x\in\mathcal{O}$ then $p^{-1}(U_x)\cong(\tilde{U}_x\times\mathbb{R}^n)/\Gamma_x$ where Γ_x acts on $\tilde{U}_x\times\mathbb{R}^n$ via $\gamma\cdot(\tilde{y},\tilde{v})=(\gamma\cdot\tilde{y},d\gamma_{\tilde{y}}(\tilde{v}))$. In keeping with tradition, we denote the fiber $p^{-1}(x)$ over $x\in U_x$ by $T_x\mathcal{O}\cong\mathbb{R}^n/\Gamma_x$. Note that, in general, if Γ_x is non-trivial then $T_x\mathcal{O}$ will be a convex cone rather than a vector space. Locally we have the diagram:

$$\begin{tikzcd}
\tilde{U}_x\times\mathbb{R}^n \arrow[r, "\Pi_x"] \arrow[d, "\mathrm{pr}_1"'] & (\tilde{U}_x\times\mathbb{R}^n)/\Gamma_x \arrow[d, "p"] \\
\tilde{U}_x \arrow[r, "\pi_x"] & U_x
\end{tikzcd}$$

where $\mathrm{pr}_1:\tilde{U}_x\times\mathbb{R}^n\to\tilde{U}_x$ denotes the projection onto the first factor $(\tilde{y},\tilde{v})\mapsto\tilde{y}$ (which is a specific choice of lift of p).

Definition 5.2. A C^r *orbisection* of the tangent orbibundle $T\mathcal{O}$ is a C^r orbifold map $\sigma:\mathcal{O}\to T\mathcal{O}$ such that $p\circ\sigma=\mathrm{Id}_{\mathcal{O}}$ and for any $x\in\mathcal{O}$ and chart U_x about x, we have $\mathrm{pr}_1\circ\tilde{\sigma}_x=\mathrm{Id}_{\tilde{U}_x}$. In particular, it follows that $\Theta_{\sigma,x}=\mathrm{Id}:\Gamma_x\to\Gamma_x$ and thus orbisections have unique equivariant lifts over orbifold charts.

We have the following structure result which was first stated in [BB02].

Proposition 5.3. *The set $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ of C^r orbisections of the tangent orbibundle $T\mathcal{O}$ is naturally a real vector space with the vector space operations being defined pointwise.*

Proof. Let $\sigma \in \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$. Locally we have the diagram:

$$\begin{tikzcd}
\tilde{U}_x \arrow[r, "\tilde{\sigma}_x"] \arrow[d, "\pi_x"'] & \tilde{U}_x \times \mathbb{R}^n \arrow[d, "\Pi_x"] \\
U_x \arrow[r, "\sigma_x"] \arrow[dr, "\mathrm{Id}"'] & p^{-1}(U_x) = (\tilde{U}_x \times \mathbb{R}^n)/\Gamma_x \arrow[d, "p"] \\
& U_x
\end{tikzcd}$$

and we can write for $y \in U_x$, $\sigma(y) = (y, s(y))$ where $s(y) \in T_y\mathcal{O} \cong \mathbb{R}^n/\theta_y(\Gamma_y)$ (θ_y is the injective homomorphism which appears in definition 2.1). Let $\tilde{\sigma}_x$ be the lift of σ. Then $\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, \tilde{s}(\tilde{y}))$, where $\tilde{s} : \tilde{U}_x \to \mathbb{R}^n$ is such that $\tilde{s}(\delta \cdot \tilde{y}) = d\delta_{\tilde{y}}(\tilde{s}(\tilde{y}))$. In particular, since $\tilde{x}$ is a fixed point of the Γ_x action on $\tilde{U}_x$, we have $\tilde{s}(\tilde{x}) = \tilde{s}(\delta \cdot \tilde{x}) = d\delta_{\tilde{x}}(\tilde{s}(\tilde{x}))$. Thus $\tilde{s}(\tilde{x})$ is a fixed point of the (linear) action of Γ_x on $\mathbb{R}^n$. Note that the set of such fixed points forms a *vector subspace* of $\mathbb{R}^n$. As a result we may define a real vector space structure on $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ as follows: For $\sigma_i \in \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$, let $\tilde{\sigma}_{i,x}$ be local lifts at x as above. Define

$$(\sigma_1 + \sigma_2)(y) = \Pi_x\big((\tilde{\sigma}_{1,x} + \tilde{\sigma}_{2,x})(\tilde{y})\big) = \Pi_x\big((\tilde{y}, \tilde{s}_1(\tilde{y}) + \tilde{s}_2(\tilde{y}))\big) = \sigma_1(y) + \sigma_2(y)$$
$$(\lambda\sigma)(y) = \Pi_x\big((\lambda\tilde{\sigma}_x)(\tilde{y})\big) = \Pi_x\big((\tilde{y}, \lambda\tilde{s}(\tilde{y}))\big) = \lambda(\sigma(y)) \qquad \blacksquare$$

In light of the previous proposition, we make the following

Definition 5.4. Let $\mathcal{O}$ be a smooth orbifold. Let $x \in \mathcal{O}$. Denote by $A_x\mathcal{O}$ the set of *admissible tangent vectors at* x

$$A_x\mathcal{O} = \big\{v \in T_x\mathcal{O} \mid (x, v) = \sigma(x) \text{ for some } \sigma \in \mathcal{D}^0_{\mathrm{Orb}}(\mathcal{O})\big\} \subset T_x\mathcal{O}$$

By proposition 5.3, $A_x\mathcal{O}$ is a vector space for each x, and a suborbifold of $T_x\mathcal{O}$. The *admissible tangent bundle of* $\mathcal{O}$ is the subset $A\mathcal{O} = \bigcup_{x \in \mathcal{O}} A_x\mathcal{O} \subset T\mathcal{O}$ with the subspace topology. It is not hard to see that, in general, $A\mathcal{O}$ is not an orbifold. See example 5.5.

Example 5.5. Let $\mathcal{O}$ be the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $x \to -x$. The underlying topological space $X_{\mathcal{O}}$ of $\mathcal{O}$ is $[0, \infty)$ and the isotropy subgoups are trivial for $x \in (0, \infty)$ and $\mathbb{Z}_2$ for $x = 0$. The tangent orbibundle $T\mathcal{O}$ is given by $(\mathbb{R} \times \mathbb{R})/\mathbb{Z}_2$ with the $\mathbb{Z}_2$ action being given by $(x, y) \to (-x, -y)$, with underlying topological space the quotient of $[0, \infty) \times \mathbb{R}$ by the equivalence relation $(0, y) \sim (0, -y)$. Note that $T_x\mathcal{O} = \mathbb{R}$ if $x \neq 0$ but that $T_0\mathcal{O} = [0, \infty)$. It also follows from proposition 5.3 that the set of admissible vectors at $x = 0$ consists only of the zero vector. Thus, all orbisections $\sigma \in \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ must vanish at 0. In particular, $A\mathcal{O} \cong \{(0, 0)\} \cup \{(0, \infty) \times \mathbb{R}\}$ and a neighborhood of $(0, 0)$ is not covered by an orbifold chart, and thus $A\mathcal{O}$ is not an orbifold. See Figure 1.

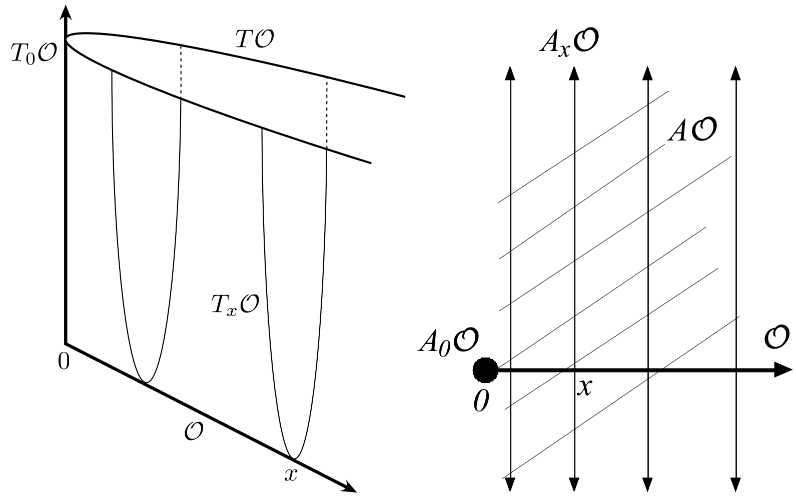

Figure 1: The tangent and admissible tangent bundles of example 5.5

Proposition 5.6. *For a compact orbifold $\mathcal{O}$, the inclusion $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}) \hookrightarrow C^r_{\mathrm{Orb}}(\mathcal{O}, T\mathcal{O})$ induces a separable Banach space structure on $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ for $1 \leq r < \infty$ and a separable Fréchet space structure if $r = \infty$.*

Proof. Let $\mathcal{C} = \{C_i\}_{i=1}^N$ be a cover of $\mathcal{O}$ by a finite number of compact charts (obtained by passing to a finite subcover of a covering by orbifold charts and then shrinking if necessary), equipped with trivializations $\Psi_i : T_{C_i}\mathcal{O} \to (\tilde{C}_i \times \mathbb{R}^n)/\Gamma_i$ of the tangent orbibundle over C_i where the lifts $\tilde{\Psi}_i$ are linear in the fiber. Let $V_{i,r} = C^r(\tilde{C}_i, \mathbb{R}^n)$ for $i = 1, \ldots, N$ and $0 \leq r \leq \infty$ with topology of uniform convergence of derivatives of order $\leq r$. This is a Banach space for finite r and a Fréchet space for $r = \infty$. For finite r, let $\| \|_{i,r}$ be a C^r norm on $V_{i,r}$. Define a linear map $T : \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}) \to \bigoplus_{i=1}^N V_{i,r}$ by

$$T(\sigma) = \Big(\mathrm{pr}_2(\tilde{\Psi}_1(\tilde{\chi}_1\tilde{\sigma})), \ldots, \mathrm{pr}_2(\tilde{\Psi}_N(\tilde{\chi}_N\tilde{\sigma}))\Big)$$

where $\chi_i \in C^r_{\mathrm{Orb}}(\mathcal{O}, [0,1])$, $i = 1, \ldots, N$, is a partition of unity subordinate to the cover $\mathcal{C}$ (see proposition 6.1 for a proof of the existence of such partitions of unity) and $\mathrm{pr}_2 : \tilde{C}_i \times \mathbb{R}^n \to \mathbb{R}^n$ is bundle projection onto the second factor. Continuity of T is immediate from the definitions of the C^r topology on $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ and the topology on $\bigoplus_{i=1}^N V_{i,r}$. Moreover, given a neighborhood of the zero section $\mathbf{0} \in \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ of the form $\mathcal{N}^r(\mathbf{0}, \varepsilon_i; \mathcal{C})$, it is apparent that there is a neighborhood of the zero section $\mathbf{0}$ in $\bigoplus_{i=1}^N V_{i,r}$ of the form $\max\{\|s_1\|_{1,r}, \ldots, \|s_N\|_{N,r}\} < \delta$ where $\delta \leq \min\{\varepsilon_1, \ldots, \varepsilon_N\}$ contained in $T\left(\mathcal{N}^r(\mathbf{0}, \varepsilon_i; \mathcal{C})\right)$. Thus, with the subspace topology on $T(\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}))$, $T : \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}) \to T(\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}))$ is a linear homeomorphism. Since $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}) \subset C^r_{\mathrm{Orb}}(\mathcal{O}, T\mathcal{O})$ is a closed subset, we see that $T(\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O}))$ is a closed subspace of the direct sum and thus $\mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$ inherits a Banach space structure if $r < \infty$ and a Fréchet space structure if $r = \infty$. ■

Curves in Orbifolds. In this paragraph we study the notion of curves in orbifolds. As a special case of example 3.7 we make the following

Definition 5.7. Let I be an interval (finite or infinite, closed, open or half-open) with trivial orbifold structure and $\mathcal{O}$ a smooth orbifold. Then elements of $C^r_{\mathrm{Orb}}(I, \mathcal{O})$ are the C^r *orbifold curves in* $\mathcal{O}$.

Definition 5.8. Let $\mathcal{O}$ be a smooth C^{r+1} orbifold, and let $c \in C^r_{\mathrm{Orb}}(I, \mathcal{O})$ be an orbifold curve. Suppose $\tilde{c}_{\tilde{x}}$ is a C^r lift of c to a chart $\tilde{U}_x$. Let $\tilde{c}'_{\tilde{x}}(t)$ be the tangent vector at t. If $\Pi_x\left(\tilde{c}_{\tilde{x}}(t), \tilde{c}'_{\tilde{x}}(t)\right) = (c(t), v) \in TU_x$, then $v \in T_{c(t)}U_x$ is called the *tangent vector to* c *at* t and we denote it by $c'(t)$.

Proposition 5.9. *If* $c \in C^r_{\mathrm{Orb}}(I, \mathcal{O})$, *then the tangent vector* $c'(t)$ *is well-defined.*

Proof. Let $x_0 = c(t_0)$ and consider an orbifold chart $(\tilde{U}_{x_0}, \Gamma_{x_0})$ at x_0. Let $t_0 \in J \subset I$ be an interval such that $c(t) \in U_{x_0}$ for all $t \in J$. Let $\hat{c}(t)$ be a C^r lift of $c(t)$ to $\tilde{U}_{x_0}$. If x_0 is non-singular, then Γ_{x_0} is trivial and $\hat{c}(t)$ is unique. Thus, $c'(t_0)$ is well defined when x_0 is non-singular.

Now suppose that x_0 is singular. If $t_0 \in \partial I$, it is not hard to see (since Γ_{x_0} is finite, acts discretely, and lifts are continuous) that there is a subinterval $t_0 \in J' \subset J$ such that any other lift of $c(t)$ is of the form $\tilde{c}(t) = \gamma \cdot \hat{c}(t)$. This is a C^r lift of c for any $\gamma \in \Gamma_{x_0}$. The tangent vector $\tilde{c}'(t_0) = d\gamma_{\hat{c}(t_0)}\hat{c}'(t_0)$. Thus, $\tilde{c}'(t_0)$ is in the same orbit as $\hat{c}'(t_0)$ of the Γ_{x_0} action on $T_{\tilde{x}_0}\tilde{U}_{x_0}$ and so their projections to $T_{x_0}U_{x_0}$ are equal and thus $c'(t_0)$ is well-defined. If t_0 is an interior point of I, then it is possible to build a C^0 lift of c by concatenation:

$$\tilde{c}(t) = \begin{cases} \hat{c}(t) & \text{for } t \le t_0 \\ \gamma \cdot \hat{c}(t) & \text{for } t \ge t_0 \end{cases}$$

Note that by our previous observations this is the only way to produce another lift around t_0. The condition that $\tilde{c}$ be at least C^1 implies that $\hat{c}'(t_0) = d\gamma_{\hat{c}(t_0)}\hat{c}'(t_0)$. Thus, like above, we see that $c'(t_0)$ is well defined and furthermore that $\hat{c}'(t_0)$ is fixed by the action of γ on $T_{\tilde{x}_0}\tilde{U}_{x_0}$. Note that $c'(t_0)$ is not necessarily an admissible tangent vector, as $\hat{c}'(t_0)$ is not necessarily fixed by all elements of Γ_{x_0}. ■

Example 5.10. Let $\mathcal{O}$ be the orbifold $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ via $(x, y) \to (x, -y)$. The underlying topological space $X_{\mathcal{O}}$ of $\mathcal{O}$ is the closed upper half-plane and the isotropy subgoups at (x, y) are $\mathbb{Z}_2$ if $y = 0$ and trivial otherwise. Let $I = [-1, 1]$ and consider the curves $b(t) = (t, |t|)$ and $c(t) = (t, t^2)$. It's easy to see that b and c have four C^0 lifts. They are of the form:

$$\tilde{b}^{\pm}_{\pm}(t) = \begin{cases} (t, \pm t) & \text{for } t \le 0 \\ (t, \pm t) & \text{for } t \ge 0 \end{cases} \qquad \tilde{c}^{\pm}_{\pm}(t) = \begin{cases} (t, \pm t^2) & \text{for } t \le 0 \\ (t, \pm t^2) & \text{for } t \ge 0 \end{cases}$$

b has two C^r lifts, b^+_+ and b^-_- for $r \ge 1$. However, all four lifts of c are C^1 while only two, c^+_+ and c^-_-, are C^r for $r \ge 2$. One sees that in the case of b the C^1 lifts do not arise from a non-trivial concatenation. Note that the tangent vectors of these lifts at $t = 0$ are not fixed by the action of $\mathbb{Z}_2$. On the other hand, two of the four C^1 lifts of c do arise as non-trivial concatenations. Their tangent vectors at $t=0$ are fixed by the $\mathbb{Z}_2$ action.

6. Smooth Riemannian Orbifold Structures

In this section we show that any smooth orbifold admits a smooth Riemannian orbifold structure. Although orbifolds are metrizable, this is not sufficient for our needs as we will need to make use of a smooth orbifold Riemannian exponential map: $\exp : T\mathcal{O} \to \mathcal{O}$. In order to do this, we proceed as in the classical situation of Riemannian manifolds.

Proposition 6.1. *Let $\mathcal{O}$ be a smooth orbifold and let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be a locally finite open covering of $\mathcal{O}$ by orbifold charts. Then there exists a C^∞ partition of unity subordinate to $\mathfrak{U}$.*

Proof. Paracompactness of $\mathcal{O}$ implies the existence of the covering $\mathfrak{U}$. Without loss of generality, by proposition 3.11, we may assume that $\mathcal{O}$ is a C^∞ orbifold. Let $\tilde{\mathfrak{U}} = \{(\tilde{U}_\alpha, \Gamma_\alpha)\}_{\alpha\in I}$ be the corresponding covering charts and let $\pi_\alpha : \tilde{U}_\alpha \to U_\alpha$ be the quotient map. Since $\mathcal{O}$ is paracompact and Hausdorff, we let $\{\chi'_\alpha\} : \mathcal{O} \to [0,1]$ be a C^0 partition of unity subordinate to the cover $\{U_\alpha\}$. If we give $[0,1]$ the trivial orbifold structure, we may regard each χ'_α as an element of $C^0_{\mathrm{Orb}}(\mathcal{O},[0,1])$ (See example 3.6). That is, each local lift of χ_α, $\tilde{\chi}'_{\alpha,\beta} : \tilde{U}_\beta \to [0,1]$, is C^0 equivariant and $\tilde{\chi}'_{\alpha,\beta}(\tilde{x}) = \chi'_\alpha \circ \pi_\beta(\tilde{x})$ for all $\tilde{x} \in \tilde{U}_\beta$. Note that for fixed $x \in \mathcal{O}$, $\pi_\beta^{-1}(x) \neq \emptyset$ for only finitely many β and furthermore, $\tilde{\chi}'_{\alpha,\beta}\left(\pi_\beta^{-1}(x)\right) \neq 0$ for all but a finite number of α. In order to produce a C^∞ partition of unity we choose, for each pair (α,β), a nonnegative C^∞ map $\tilde{\chi}''_{\alpha,\beta} : \tilde{U}_\beta \to [0,1]$ which is sufficiently C^0 close to $\tilde{\chi}'_{\alpha,\beta}$. For $\tilde{x} \in \tilde{U}_\beta$ define

$$\hat{\chi}_{\alpha,\beta}(\tilde{x}) = \frac{1}{|\Gamma_\beta|} \sum_{\gamma\in\Gamma_\beta} \tilde{\chi}''_{\alpha,\beta}(\gamma\cdot\tilde{x})$$

By defining $\tilde{\chi}_{\alpha,\beta} = \dfrac{\hat{\chi}_{\alpha,\beta}}{\sum_{\mu,\nu\in I}\hat{\chi}_{\mu,\nu}}$ we get a C^∞ Γ_β-equivariant map on $\tilde{U}_\beta$ that is C^0 close to $\tilde{\chi}'_{\alpha,\beta}$ for each pair (α,β). Thus the map

$$\chi_\alpha(x) = \begin{cases} \sum_\beta \tilde{\chi}_{\alpha,\beta}\left(\pi_\beta^{-1}(x)\right) & \text{for } x \in U_\alpha \\ 0 & \text{for } x \in \mathcal{O} - U_\alpha \end{cases}$$

is well-defined, each $\chi_\alpha \in C^\infty_{\mathrm{Orb}}(\mathcal{O},[0,1])$ and the collection $\{\chi_\alpha\}$ is a smooth partition of unity subordinate to the cover $\{U_\alpha\}$. ∎

We now prove the existence of a smooth orbifold Riemannian metric. We could, of course, do this by defining appropriate notions of tensor bundles over orbifolds and their sections. However, since our needs are limited, we choose to do this in an elementary way following the classical development. Since the tangent space $T_x\mathcal{O} \cong \mathbb{R}^n/\Gamma_x$ is, in general, a convex cone rather than a vector space, we make the following

Definition 6.2. A function $g_x : T_x\mathcal{O} \times T_x\mathcal{O} \to \mathbb{R}$ is a *positive definite, real, orbifold inner product* if it has a $\Gamma_x \times \Gamma_x$ equivariant lift $\tilde{g}_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is a positive definite real inner product on $\mathbb{R}^n$. Note that we gave the natural product orbifold structure to $T_x\mathcal{O} \times T_x\mathcal{O}$.

Definition 6.3. Let $\mathcal{O}$ be a smooth C^{r+1} orbifold. A *smooth C^r orbifold Riemannian metric on $\mathcal{O}$* is a collection $g = \{g_x\}_{x\in\mathcal{O}}$ of positive definite real orbifold inner products so that the functions $g(\sigma, \tau) : x \mapsto g_x(\sigma(x), \tau(x))$ are elements of $C^r_{\mathrm{Orb}}(\mathcal{O}, \mathbb{R})$ for all orbisections $\sigma, \tau \in \mathcal{D}^r_{\mathrm{Orb}}(\mathcal{O})$. An orbifold equipped with a C^r Riemannian metric will be called a C^r *Riemannian orbifold.*

Proposition 6.4. *Let $\mathcal{O}$ be a smooth orbifold. Then there exists on $\mathcal{O}$ a smooth C^∞ orbifold Riemannian metric.*

Proof. Without loss of generality, by proposition 3.11, we may assume that $\mathcal{O}$ is a C^∞ orbifold. Using the notation from proposition 6.1, let $\{\chi_\alpha\}$ be a C^∞ partition of unity and let $\tilde{g}'_\alpha$ be a C^∞ Riemannian metric on $\tilde{U}_\alpha$. Define

$$\tilde{g}_\alpha(\tilde{v}, \tilde{w}) = \frac{1}{|\Gamma_\alpha|^2} \sum_{(\gamma,\mu)\in\Gamma_\alpha\times\Gamma_\alpha} \tilde{g}'_\alpha(d\gamma_{\tilde{x}}(\tilde{v}), d\mu_{\tilde{x}}(\tilde{w}))$$

for $\tilde{v}, \tilde{w} \in T_{\tilde{x}}\tilde{U}_\alpha$. Then $\tilde{g}_\alpha$ is a C^∞, $\Gamma_\alpha \times \Gamma_\alpha$ equivariant positive definite, real inner product on each $T_{\tilde{x}}\tilde{U}_\alpha$ which descends to a smooth orbifold Riemannian metric g_α on U_α. Thus, $g = \sum_\alpha \chi_\alpha g_\alpha$ is the required C^∞ orbifold Riemannian metric on $\mathcal{O}$. ∎

Remark 6.5. Note that the proof of proposition 6.4 shows that the action of Γ_α on $\tilde{U}_\alpha$ is by isometries relative to $\tilde{g}_\alpha$, and that the equivariant transition maps $\tilde{\psi}$ that appear in definition 2.1 are isometric embeddings. By shrinking the cover $\{U_\alpha\}$ if necessary, we may assume that each orbifold covering chart $\tilde{U}_\alpha$ is convex making $\mathcal{O}$ a Riemannian orbifold as defined in [Bor93] and [Bor92]. Recall that for a Riemannian manifold to be convex means there exists a unique minimal geodesic joining any two points.

If $\mathcal{O}$ is a smooth C^r Riemannian orbifold, then we may give $\mathcal{O}$ the structure of a length space. A general reference is [Gro99]. In particular, given two points $x, y \in \mathcal{O}$ we may define the distance between x and y to be

$$d(x, y) = \inf\{\mathrm{Length}(c) \mid c \in C^0_{\mathrm{Orb}}(I, \mathcal{O}) \text{ and } c \text{ joins } x \text{ to } y\}$$

The length of a curve c is defined by adding up the lengths of local lifts in each orbifold chart $\tilde{U}_\alpha$. This can be shown to be well-defined and independent of the choice of lift [Bor92]. This length metric structure generates a topology that is the same as the as the topology of the underlying space of $\mathcal{O}$. If $(\mathcal{O}, d)$ is complete any two points can be joined by a *minimal geodesic* realizing the distance $d(x, y)$ [Gro99], since $\mathcal{O}$ is locally compact. Moreover, the local lifts of any such minimal geodesic must be a smooth C^r minimal geodesic in each $\tilde{U}_\alpha$, justifying the use of the terminology. Additionally, if $c \in C^r_{\mathrm{Orb}}(I, \mathcal{O})$ is a minimal geodesic it can be shown that $\Gamma_{c(t)} = \Gamma_{c(t')}$ for all $t, t' \in I - \partial I$ [Bor93].

We now proceed to define the exponential map for a Riemannian orbifold. For a general reference for standard results of Riemannian geometry that we need see [Pet98]. As in the proof of proposition 6.4, assume the collection $\{U_\alpha\}$ is a locally finite open covering of $\mathcal{O}$ by orbifold charts that are relatively compact. Let $TU_\alpha \cong (\tilde{U}_\alpha \times \mathbb{R}^n)/\Gamma_\alpha$ be a local trivialization of the tangent bundle over U_α. Denote the Riemannian exponential map on $T\tilde{U}_\alpha$ by $\widetilde{\exp}_{\tilde{U}_\alpha} : T\tilde{U}_\alpha \to \tilde{U}_\alpha$. Thus, for $\tilde{x} \in \tilde{U}_\alpha$ and $\tilde{v} \in T_{\tilde{x}}\tilde{U}_\alpha$ we have $\widetilde{\exp}_{\tilde{U}_\alpha}(\tilde{x}, t\tilde{v}) = \tilde{c}_{\tilde{x},\tilde{v}}(t)$ where $\tilde{c}_{\tilde{x},\tilde{v}}$ is the unit speed geodesic in $\tilde{U}_\alpha$ which starts at $\tilde{x}$ and has initial velocity $\tilde{v}$. Recall that there is an open neighborhood $\tilde{\Omega}_{\tilde{U}_\alpha} \subset T\tilde{U}_\alpha$ of the 0-section of $T\tilde{U}_\alpha$ such that $\tilde{c}_{\tilde{x},\tilde{v}}(1)$ is defined for $\tilde{v} \in T_{\tilde{x}}\tilde{U}_\alpha \cap \tilde{\Omega}_{\tilde{U}_\alpha}$. Furthermore, by shrinking $\tilde{\Omega}_{\tilde{U}_\alpha}$ if necessary, we may assume that on $T_{\tilde{x}}\tilde{U}_\alpha \cap \tilde{\Omega}_{\tilde{U}_\alpha}$, $\widetilde{\exp}_{\tilde{U}_\alpha}(\tilde{x}, \cdot)$ is a local diffeomorphism onto a neighborhood of $\tilde{x} \in \tilde{U}_\alpha$ for each $\tilde{x} \in \tilde{U}_\alpha$. Let $\Omega_\alpha = \Pi_\alpha(\tilde{\Omega}_{\tilde{U}_\alpha})$, an open subset of $T\mathcal{O}$, and define $\Omega = \bigcup_\alpha \Omega_\alpha$. Ω is an open neighborhood of the 0-orbisection of $T\mathcal{O}$.

Definition 6.6. Let $x \in U_\alpha$, and $(x, v) \in \Omega_\alpha$. Choose $(\tilde{x}, \tilde{v}) \in \Pi_\alpha^{-1}(x, v)$. Then the *Riemannian exponential map* $\exp : \Omega \subset T\mathcal{O} \to \mathcal{O}$ is defined by $\exp(x, v) = \pi_\alpha \circ \widetilde{\exp}_{\tilde{U}_\alpha}(\tilde{x}, \tilde{v})$.

Proposition 6.7. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold. Then the exponential map $\exp(x, v) = \pi_\alpha \circ \widetilde{\exp}_{\tilde{U}_\alpha} \circ \Pi_\alpha^{-1}(x, v)$ is well–defined.*

Proof. Since the metric $\tilde{g}_\alpha$ is equivariant relative to the action of Γ_α by isometries on $\tilde{U}_\alpha$ we see that (since isometries map geodesics to geodesics) $\widetilde{\exp}_{\tilde{U}_\alpha}[\gamma \cdot (\tilde{x}, \tilde{v})] = \gamma \cdot \widetilde{\exp}_{\tilde{U}_\alpha}(\tilde{x}, \tilde{v})$. Thus, $\widetilde{\exp}_{\tilde{U}_\alpha} : \tilde{\Omega}_\alpha \subset T\tilde{U}_\alpha \to \tilde{U}_\alpha$ is equivariant and hence exp is well-defined for each $\tilde{U}_\alpha$. If $x \in U_\alpha \cap U_\beta$, then there is an orbifold chart $U_{\alpha\beta} \subset U_\alpha \cap U_\beta$ of x, and equivariant isometric embeddings $\tilde{\psi}_\alpha : \tilde{U}_{\alpha\beta} \to \tilde{U}_\alpha$ and $\tilde{\psi}_\beta : \tilde{U}_{\alpha\beta} \to \tilde{U}_\beta$. This observation is enough to show that exp is independent of local chart. ∎

As usual we denote by $\exp_x$ the restriction of exp to a single tangent cone $T_x\mathcal{O}$. We let $B(x, r)$ denote the metric r-ball centered at x and use tildes to denote corresponding points in local coverings.

Proposition 6.8. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold. Then $\exp_x$ is a local (topological) homeomorphism. That is, there exists $\varepsilon > 0$ such that $\exp_x : B(0, \varepsilon) \subset T_x\mathcal{O} \to B(x, \varepsilon) \subset \mathcal{O}$ is a (topological) homeomorphism with C^r local lifts for each $x \in \mathcal{O}$.*

Proof. First note that a lift of $\exp_x$ to $\tilde{U}_x$ is of the form $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, \cdot)$. Since the classical Riemannian exponential map is as smooth as its tangent bundle, we see that $\exp_x$ has local C^r lifts.

Choose $\varepsilon > 0$ so that $\overline{B(\tilde{0}, \varepsilon)} \subset \tilde{\Omega}_{\tilde{U}_x} \cap T_{\tilde{x}}\tilde{U}_x$. Then $\widetilde{\exp}_{\tilde{U}_\alpha}(\tilde{x}, \cdot)$ is a local C^r diffeomorphism from $B(\tilde{0}, \varepsilon) \subset T_{\tilde{x}}\tilde{U}_x$ onto $B(\tilde{x}, \varepsilon) \subset \tilde{U}_x$. By construction of the length metric on $\mathcal{O}$, it is easy to see that $\pi_x\left(\overline{B(\tilde{x},\varepsilon)}\right) = \overline{B(x,\varepsilon)}$, thus $\exp_x$ maps $\overline{B(0,\varepsilon)} \subset T_x\mathcal{O}$ onto $\overline{B(x,\varepsilon)} \subset \mathcal{O}$.

To see that $\exp_x$ is injective, suppose that $\exp_x(v) = \exp_x(w)$ for $v, w \in \overline{B(0,\varepsilon)}$. Then there is $\gamma \in \Gamma_x$ such that $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x},\tilde{v}) = \gamma \cdot \widetilde{\exp}_{\tilde{U}_x}(\tilde{x},\tilde{w}) = \widetilde{\exp}_{\tilde{U}_x}(\gamma \cdot \tilde{x}, d\gamma_{\tilde{x}}\tilde{w}) = \widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, d\gamma_{\tilde{x}}\tilde{w})$. Thus, $\tilde{v} = d\gamma_{\tilde{x}}\tilde{w}$, since $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x},\cdot)$ is a local diffeomorphism and therefore $v = w$.

Finally since $\exp_x$ is continuous, bijective and $\overline{B(0,\varepsilon)}$ is compact, we see that $\exp_x$ is a local homeomorphism. ■

If we restrict the exponential map $\exp_x$ to admissible vectors at x, we can say a little more.

Proposition 6.9. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold. Let $\varepsilon > 0$ be as in proposition 6.8. Then the restriction of $\exp_x$ to $B(0,\varepsilon) \cap A_x\mathcal{O}$ is a C^r local diffeomorphism of $A_x\mathcal{O}$ (with trivial suborbifold structure) onto a neighborhood of x in the stratum $\mathcal{S}_x$ (with trivial suborbifold structure).*

Proof. Let $v \in B(0,\varepsilon) \cap A_x\mathcal{O}$, and choose $(\tilde{x},\tilde{v}) \in \Pi_x^{-1}(x,v) \cap B(\tilde{0},\varepsilon)$. Then, by the proof of proposition 5.3, $d\gamma_{\tilde{x}}\tilde{v} = \tilde{v}$ for all $\gamma \in \Gamma_x$. Thus, by equivariance of $\widetilde{\exp}_{\tilde{U}_x}$, we have for $t \in [0,1]$,

$$\widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, t\tilde{v}) = \widetilde{\exp}_{\tilde{U}_x}[\gamma \cdot (\tilde{x}, t\tilde{v})] = \gamma \cdot \widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, t\tilde{v})$$

Hence, $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, t\tilde{v})$ is fixed by the action of Γ_x for all $t \in [0,1]$. This implies that for all $t \in [0,1]$ we have, $\exp_x(tv) = \exp(x, tv) = \pi_x \circ \widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, t\tilde{v}) \in B(x,\varepsilon) \cap \mathcal{S}_x$. Thus, $\exp_x$ maps onto $B(x,\varepsilon) \cap \mathcal{S}_x$. In fact, since the restriction of the action of Γ_x to $\tilde{\mathcal{S}}_x$ is trivial ($\Gamma_x \cdot \tilde{s} = \tilde{s}$ for all $\tilde{s} \in \tilde{\mathcal{S}}_x$), we may identify $\mathcal{S}_x \subset \mathcal{O}$ with $\tilde{\mathcal{S}}_x \subset \tilde{U}_x$ and under this identification our restriction of $\exp_x$ to $A_x\mathcal{O}$ is nothing more than the map $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x},\cdot)$ restricted to $T_{\tilde{x}}\tilde{\mathcal{S}}_x \cap T_{\tilde{x}}\tilde{U}_x$. Hence $\exp_x$ is a local C^r (manifold) diffeomorphism. ■

The composition of the exponential map with an orbisection turns out to be a smooth orbifold map.

Proposition 6.10. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold. Let σ be a C^r orbisection of $T\mathcal{O}$. Then the map $E^\sigma(x) = (\exp \circ \sigma)(x) : \mathcal{O} \to \mathcal{O}$ is a smooth C^r orbifold map, provided $\sigma(x) \in \Omega$. That is, $E^\sigma \in C^r_{\mathrm{Orb}}(\mathcal{O})$.*

Proof. Let $(\tilde{U}_x, \Gamma_x)$ be an orbifold chart at $x \in \mathcal{O}$. For $y \in U_x$, $\sigma(y) = (y, s(y))$ where $s(y) \in A_y\mathcal{O}$. Then as in the proof of proposition 5.3, if $\tilde{\sigma}_x$ is a lift of σ, then $\Theta_{\sigma,x}(\delta) = \delta$ for all $\delta \in \Gamma_x$ and $\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, \tilde{s}(\tilde{y}))$, where $\tilde{s} : \tilde{U}_x \to \mathbb{R}^n$ satisfies $\tilde{s}(\delta \cdot \tilde{y}) = (d\delta)_{\tilde{y}}\tilde{s}(\tilde{y})$.

The map $\tilde{E}^\sigma_x = \widetilde{\exp}_{\tilde{U}_x} \circ \tilde{\sigma}_x$ is a C^r lift of E^σ and thus we need to check equivariance:

$$\begin{aligned}
\tilde{E}^{\sigma}_x(\delta\cdot\tilde{y}) &= \widetilde{\exp}_{\tilde{U}_x}\big(\delta\cdot\tilde{y},\tilde{s}(\delta\cdot\tilde{y})\big)\\
&= \widetilde{\exp}_{\tilde{U}_x}\big(\delta\cdot\tilde{y},(d\delta)_{\tilde{y}}\tilde{s}(\tilde{y})\big)\\
&= \widetilde{\exp}_{\tilde{U}_x}\big[\delta\cdot\big(\tilde{y},\tilde{s}(\tilde{y})\big)\big]\\
&= \delta\cdot\widetilde{\exp}_{\tilde{U}_x}\big(\tilde{y},\tilde{s}(\tilde{y})\big)\\
&= \delta\cdot\tilde{E}^{\sigma}_x(\tilde{y})
\end{aligned}$$

Thus, $\tilde{E}^{\sigma}_x$ is $\Theta_{E^{\sigma},x}$ equivariant if we define $\Theta_{E^{\sigma},x}(\delta)=\delta$. Hence $E^{\sigma}\in C^r_{\mathrm{Orb}}(\mathcal{O})$. ■

Denote by $\mathbf{0}:\mathcal{O}\to T\mathcal{O}$, $\mathbf{0}(x)=0_x\in T_x\mathcal{O}$, the 0-orbisection of $T\mathcal{O}$. The next proposition shows that if σ is sufficiently C^1 close to the 0-orbisection $\mathbf{0}$, then E^{σ} is a local orbifold diffeomorphism.

Proposition 6.11. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold and $U_\alpha\subset\mathcal{O}$, where U_α is a relatively compact orbifold chart. Then there is a open neighboorhood $\Lambda_\alpha\subset\Omega_\alpha\subset TU_\alpha$ of $U_\alpha\times\{0\}\subset TU_\alpha$, such that if σ is a C^r orbisection with $\sigma(x)\in\Lambda_\alpha$ and σ is sufficiently C^1 close to $\mathbf{0}$ on U_α, then $E^{\sigma}|_{U_\alpha}$ is a C^r orbifold diffeomorphism onto its image. That is, $E^{\sigma}|_{U_\alpha}$ is an orbifold embedding.*

Proof. Without loss of generality, we may assume by shrinking U_α and Λ_α if necessary that U_α and $E^{\sigma}(U_\alpha)$ are contained in a single relatively compact orbifold chart $(\tilde{U},\Gamma)$. Let $\Lambda_\alpha=\Pi_\alpha\left(\tilde{\Omega}_{\tilde{U}}\cap\tilde{\Omega}_{\tilde{U}_\alpha}\right)$. By proposition 6.10, we know already that $E^{\sigma}(x)$ is a C^r orbifold map. We need to show that E^{σ} has an inverse that is also a C^r orbifold map. We first show that $E^{\sigma}(x)$ is injective.

There exists $\gamma\in\Gamma$ such that $\left(\widetilde{\exp}_{\tilde{U}_u}\circ\tilde{\mathbf{0}}_u\right)(\tilde{x})=\gamma\cdot\tilde{x}$ since this map is a lift of the identity map. If σ is C^1 close enough to $\mathbf{0}$ with lift $\tilde{\mathbf{0}}_u=(\tilde{x},0)$, then $\tilde{\sigma}_u=(\tilde{x},\tilde{s}(\tilde{x}))$ for $u\in U$. Suppose that $E^{\sigma}(x)=E^{\sigma}(y)=u$ for $x,y,u\in U$. (This implies that the isotropy groups of x,y,u are equal, by proposition 6.9). Then there exists $\delta\in\Gamma$ such that $\tilde{E}^{\sigma}_u(\tilde{x})=\delta\cdot\tilde{E}^{\sigma}_u(\tilde{y})$. Thus,

$$\begin{aligned}
\left(\widetilde{\exp}_{\tilde{U}_u}\circ\tilde{\sigma}_u\right)(\tilde{x}) &= \delta\cdot\left[\left(\widetilde{\exp}_{\tilde{U}_u}\circ\tilde{\sigma}_u\right)(\tilde{y})\right]\\
&= \delta\cdot\left(\widetilde{\exp}_{\tilde{U}_u}\big(\tilde{y},\tilde{s}(\tilde{y})\big)\right)\\
&= \widetilde{\exp}_{\tilde{U}_u}\big(\delta\cdot\tilde{y},(d\delta)_{\tilde{y}}\tilde{s}(\tilde{y})\big)\\
&= \widetilde{\exp}_{\tilde{U}_u}\big(\tilde{w},\tilde{s}(\tilde{w})\big) \qquad \text{where } \tilde{w}=\delta\cdot\tilde{y}\\
&= \left(\widetilde{\exp}_{\tilde{U}_u}\circ\tilde{\sigma}_u\right)(\tilde{w})
\end{aligned}$$

Since a sufficiently small C^1 neighborhood of an embedding is an embedding [Mun66], by choosing σ sufficiently C^1 close to $\mathbf{0}$, we may conclude that $\tilde{x}=\tilde{w}$ which in turn implies that $\tilde{x}$ and $\tilde{y}$ are in the same orbit of the Γ action on $\tilde{U}$. Thus $x=y$.

We now show that $(E^{\sigma})^{-1}$ is a C^r orbifold map. Denote by $\widetilde{\exp}^{-1}_{\tilde{U}_u,\tilde{x}}$ the C^r map $\left[\widetilde{\exp}_{\tilde{U}_u}(\tilde{x},\cdot)\right]^{-1}:\tilde{U}\to T_{\tilde{x}}\tilde{U}$. Also, let $\mathrm{pr}_1:T\tilde{U}\to\tilde{U}$ be the bundle projection

$(\tilde{x},\tilde{v}) \mapsto \tilde{x}$. Suppose $\tilde{y} = \tilde{E}^\sigma_u(\tilde{x})$. We claim that $\left(\tilde{E}^\sigma_u\right)^{-1}(\tilde{y}) = \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\gamma\tilde{x}}(\tilde{y})\right)$, a composition of C^r maps. To see the formula is correct we compute:

$$\begin{aligned}
\operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\gamma\tilde{x}}(\tilde{y})\right) &= \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\gamma\tilde{x}}\left(\tilde{E}^\sigma_u(\tilde{x})\right)\right)\\
&= \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\gamma\tilde{x}}\left(\widetilde{\exp}_{\tilde{U}_u}\circ\tilde{\sigma}_u(\tilde{x})\right)\right)\\
&= \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\gamma\tilde{x}}\left(\widetilde{\exp}_{\tilde{U}_u}(\tilde{x},\tilde{s}(\tilde{x})\right)\right)\\
&= \operatorname{pr}_1(\tilde{x},\tilde{s}(\tilde{x}))\\
&= \tilde{x}
\end{aligned}$$

Now we need to check equivariance. From the computation in proposition 6.10, for any $\delta\in\Gamma$, we have $\tilde{E}^\sigma_u(\delta\cdot\tilde{x}) = \delta\cdot\tilde{y}$. Thus,

$$\begin{aligned}
\left(\tilde{E}^\sigma_u\right)^{-1}(\delta\cdot\tilde{y}) &= \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\delta\gamma\tilde{x}}\left(\tilde{E}^\sigma_u(\delta\cdot\tilde{x})\right)\right)\\
&= \operatorname{pr}_1\left(\widetilde{\exp}^{-1}_{\tilde{U}_u,\delta\gamma\tilde{x}}\left[\widetilde{\exp}_{\tilde{U}_u}(\delta\cdot\tilde{x},\tilde{s}(\delta\cdot\tilde{x}))\right]\right)\\
&= \operatorname{pr}_1((\delta\cdot\tilde{x},\tilde{s}(\delta\cdot\tilde{x})))\\
&= \delta\cdot\tilde{x}\\
&= \delta\cdot\left(\tilde{E}^\sigma_u\right)^{-1}(\tilde{y})
\end{aligned}$$

Thus, $\left(\tilde{E}^\sigma_u\right)^{-1}$ is $\Theta_{(E^\sigma)^{-1},u}$ equivariant if we define $\Theta_{(E^\sigma)^{-1},u}(\delta)=\delta$. Note that $\Theta_{(E^\sigma)^{-1},u} = (\Theta_{E^\sigma,u})^{-1}$ as to be expected. ∎

The next lemma is a standard result of differential topology adapted to orbifolds:

Lemma 6.12. *Let* Id : $\mathcal{O}\to\mathcal{O}$ *be the identity map. Then there is a* C^0 *neighborhood of* Id *such that if* f *lies in this neighborhood, then* f *is surjective.*

Proof. The proof is essentially a minor modification of the argument in [Mun66, lemma 3.11]. For completeness, we give it here. Let $\{C_i\}$ be a locally finite covering of $\mathcal{O}$ by compact sets whose interiors also cover $\mathcal{O}$. Assume further that the corresponding orbifold charts $(\tilde{C}_i,\Gamma_i)$ have $\tilde{C}_i=$ unit ball $B^n\subset\mathbb{R}^n$, and let $(\tilde{V}_i,\Gamma_i)$ be an orbifold chart with $\tilde{C}_i\subset\operatorname{int}(\tilde{V}_i)$. Let $\widetilde{\operatorname{Id}}_i$ be the corresponding lift of the identity map Id to $\tilde{V}_i$ and let $B^n(r)$ denote the metric r-ball centered at 0 in $\mathbb{R}^n$. Choose ε_i small enough so that if $\tilde{D}_i = \widetilde{\operatorname{Id}}_i^{-1}(B(1-\varepsilon_i))$ then the collection $\{D_i=\pi_i(\tilde{D}_i)\}$ covers $\mathcal{O}$ and also that $B(1+\varepsilon_i)\subset\tilde{V}_i$.

Let $f:\mathcal{O}\to\mathcal{O}$ be a map such that $\|\tilde{f}_i(\tilde{x})-\widetilde{\operatorname{Id}}_i(\tilde{x})\|_{\tilde{V}_i}<\varepsilon_i$ for $\tilde{x}\in\tilde{C}_i$ and all i. We want to show that f is surjective.

Define $\tilde{g}_i=\tilde{f}_i\circ\widetilde{\operatorname{Id}}_i^{-1}$. Then $\tilde{g}_i$ is a map from $B^n=\tilde{C}_i$ into $\mathbb{R}^n$ and the image of the unit sphere $S^{n-1}=\partial B^n$ under $\tilde{g}_i$ lies outside $B(1-\varepsilon_i)$. We will show that $\tilde{D}_i\subset\tilde{g}_i(B^n)$. Since $\{D_i\}$ cover $\mathcal{O}$ and $D_i=\pi_i(\tilde{D}_i)\subset\pi_i\circ\tilde{g}_i(B^n)=\pi_i\circ\tilde{f}_i(\tilde{C}_i)=f(C_i)$, this will imply that f is surjective.

Suppose to the contrary that $\tilde{y} \in B(1-\varepsilon_i)$, but $\tilde{y} \notin \tilde{g}_i(B^n)$. Let $\lambda : \mathbb{R}^n - \{\tilde{y}\} \to S^{n-1}$ be the radial projection from $\tilde{y}$. Then $\lambda \circ \tilde{g}_i$ maps B^n into S^{n-1}. On the other hand, the restriction $\tilde{g}_i|_{S^{n-1}} : S^{n-1} \to \mathbb{R}^n$ is homotopic to the identity map via $F_t(\tilde{x}) = t\tilde{g}_i(\tilde{x}) + (1-t)\tilde{x}$ for $\tilde{x} \in S^{n-1}$. This homotopy carries $\tilde{g}_i(\tilde{x})$ along the straight line between $\tilde{g}_i(\tilde{x})$ and $\tilde{x}$ so $F_t(\tilde{x})$ lies outside $B(1-\varepsilon_i)$. Thus, $\lambda \circ F_t$ is a well-defined homotopy between $(\lambda \circ \tilde{g}_i)|_{S^{n-1}} : S^{n-1} \to S^{n-1}$ and the identity map. It is not necessary that F_t and λ be equivariant. Now consider the homology sequence of the pair (B^n, S^{n-1}):

$$\begin{tikzcd}
0 \arrow[r] & H_n(B^n, S^{n-1}) \arrow[r] \arrow[d, "(\lambda\circ\tilde{g}_i)_*"'] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\left((\lambda\circ\tilde{g}_i)|_{S^{n-1}}\right)_*"] & 0 \\
0 \arrow[r] & H_n(B^n, S^{n-1}) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

$(\lambda \circ \tilde{g}_i)_*$ is the zero homomorphism since $(\lambda \circ \tilde{g}_i)$ sends B^n into S^{n-1}. However, $\left((\lambda \circ \tilde{g}_i)|_{S^{n-1}}\right)_*$ is the identity homomorphism since $(\lambda \circ \tilde{g}_i)|_{S^{n-1}}$ is homotopic to the identity map. Since $H_n(B^n, S^{n-1}) \cong \mathbb{Z}$ and the diagram commutes we have a contradiction. Thus, f is surjective. ■

The following is a culmination of the results of this section.

Theorem 6.13. *Let $\mathcal{O}$ be a C^{r+1} Riemannian orbifold. If σ is a C^r orbisection sufficiently C^1 close to the 0-orbisection $\mathbf{0}$ of $T\mathcal{O}$ then E^σ is a C^r orbifold diffeomorphism. That is, $E^\sigma \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$.*

Proof. Let $\{C_i\}$ be a locally finite covering of $\mathcal{O}$ by compact sets. By proposition 6.11, there exist positive constants ε_i such that if σ is C^1 ε_i-close to $\mathbf{0}$ on C_i, then $E^\sigma|_{C_i}$ is a C^r orbifold embedding. Since $\mathrm{Id} = E^{\mathbf{0}} = (\exp \circ \mathbf{0})$, by choosing ε_i smaller if necessary, we may conclude that E^σ is surjective by lemma 6.12. We need only to show that E^σ is globally injective. To do this, we modify the argument in [Mun66, theorem 3.10].

Let $\{D_i\}$ be a covering of $\mathcal{O}$ by compact sets with $D_i \subset \mathrm{int}(C_i)$. Let $\delta_i = d\left(D_i, \mathcal{O} - \mathrm{int}(C_i)\right) > 0$. By choosing ε_i smaller if necessary, we may assume that E^σ is C^1 $\frac{1}{2}\delta_i$-close to Id for $x \in D_i$ and that $E^\sigma(D_i) \subset C_i$. Suppose that $E^\sigma(x) = E^\sigma(y)$, where $x \in D_i$ and $y \in D_j$ and $\delta_i \le \delta_j$. Then

$$d(x,y) \le d(x, E^\sigma(x)) + d(E^\sigma(x), E^\sigma(y)) + d(E^\sigma(y), y) < \frac{1}{2}\delta_i + \frac{1}{2}\delta_j \le \delta_j$$

However, since E^σ is injective on C_j, $x \notin C_j$. Thus, $d(x,y) \ge \delta_j$, a contradiction. Hence E^σ is injective and thus a C^r orbifold diffeomorphism. ■

7. Proof of Theorem 1.1 and Corollary 1.2

Throughout this section, we assume that $\mathcal{O}$ is a smooth compact orbifold (without boundary). Without loss of generality, we may assume, by propositions 3.11 and 6.4, that $\mathcal{O}$ is a C^∞ orbifold with C^∞ Riemannian metric. We let $\mathcal{B}^r(\sigma, \varepsilon) = \mathcal{N}^r(\sigma, \varepsilon) \cap \mathcal{D}^r(\mathcal{O})$. That is, $\mathcal{B}^r(\sigma, \varepsilon)$ is the set of C^r orbisections ε-close to σ in the C^r topology on $C^r_{\mathrm{Orb}}(\mathcal{O}, T\mathcal{O})$. We prove theorem 1.1 in a series of propositions.

Proposition 7.1. *There exists $\varepsilon > 0$ such that $E^\sigma = \exp \circ \sigma \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ for $\sigma \in \mathcal{B}^r(\mathbf{0}, \varepsilon)$. That is, there exists a map $E : \mathcal{B}^r(\mathbf{0}, \varepsilon) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ defined by $E(\sigma) = E^\sigma$.*

Proof. This follows from compactness of $\mathcal{O}$ and theorem 6.13. ∎

Proposition 7.2. *The map $E : \mathcal{B}^r(\mathbf{0}, \varepsilon) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is injective.*

Proof. Suppose $E(\sigma) = E(\tau)$ for $\sigma, \tau \in \mathcal{B}^r(\mathbf{0}, \varepsilon)$. Then $(\exp \circ \sigma)(x) = (\exp \circ \tau)(x)$ for all $x \in \mathcal{O}$. Thus, in each orbifold chart $(\tilde{U}_x, \Gamma_x)$, we have $\pi_x \circ \widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, \tilde{v}) = \pi_x \circ \widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, \tilde{w})$. Since $\widetilde{\exp}_{\tilde{U}_x}(\tilde{x}, \cdot)$ is a local C^r diffeomorphism we must have $\tilde{v} = (d\gamma)_{\tilde{x}}(\tilde{w})$ for some $\gamma \in \Gamma_x$. Thus, $v = w \in A_x\mathcal{O}$. Hence $\sigma = \tau$ and E is injective. ∎

Proposition 7.3. *The map $E : \mathcal{B}^r(\mathbf{0}, \varepsilon) \to \mathcal{N}^0(\mathrm{Id}, \varepsilon) \cap \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is surjective.*

Proof. Let $f \in \mathcal{N}^0(\mathrm{Id}, \varepsilon) \cap \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. Let $\{C_i\}$ be a finite covering of $\mathcal{O}$ by compact sets such that C_i is an orbifold chart and $f(C_i) \subset V_i$ where V_i is a relatively compact orbifold chart. Let $x \in C_i$, and $\tilde{U}_x \subset \operatorname{int} \tilde{C}_i$ an orbifold chart at x where the local lift $\tilde{f}_x$ to $\tilde{U}_x$ is C^0 ε-close to the lift $\widetilde{\mathrm{Id}}_x = \mathrm{Id}_{\tilde{U}_x}$ of the identity map and not ε-close to any other lift of the identity over $\tilde{U}_x$. For ε small enough it follows that $\Theta_{f,x}(\delta) = \Theta_{\mathrm{Id},x}(\delta) = \delta$ for all $\delta \in \Gamma_x$. This is because for each $\delta \in \Gamma_x$ we have

$$
\begin{aligned}
\|\tilde{f}_x(\delta \cdot \tilde{y}) - \widetilde{\mathrm{Id}}_x(\delta \cdot \tilde{y})\|_{\tilde{V}_i} < \varepsilon &\iff \\
\|\Theta_{f,x}(\delta) \cdot \tilde{f}_x(\tilde{y}) - \delta \cdot \tilde{y}\|_{\tilde{V}_i} < \varepsilon &\iff \\
\|\delta^{-1}\Theta_{f,x}(\delta) \cdot \tilde{f}_x(\tilde{y}) - \tilde{y}\|_{\tilde{V}_i} < \varepsilon &\iff \text{(since } \Gamma_x \text{ acts by isometries)} \\
\|\delta^{-1}\Theta_{f,x}(\delta) \cdot \tilde{f}_x(\tilde{y}) - \widetilde{\mathrm{Id}}_x(\tilde{y})\|_{\tilde{V}_i} < \varepsilon &
\end{aligned}
$$

Thus, by our choice of local lift of the identity map over $\tilde{U}_x$, it follows that $\delta^{-1}\Theta_{f,x}(\delta) = e$ which implies that $\Theta_{f,x}(\delta) = \delta$.

We wish to define a C^r orbisection σ so that $E(\sigma) = f$. We do this by defining appropriate local lifts $\tilde{\sigma}_x$. In particular, let

$$\tilde{\sigma}_x(\tilde{y}) = \left(\tilde{y}, \widetilde{\exp}^{-1}_{\tilde{U}_x,\tilde{y}}\left(\tilde{f}_x(\tilde{y})\right)\right) \in T\tilde{U}_x$$

Before we show that $\tilde{\sigma}_x$ satisfies the correct equivariance relation observe that, in general, $\widetilde{\exp}^{-1}_{\tilde{U}_x,\tilde{y}}(\gamma \cdot \tilde{z}) = (d\gamma)_{\gamma^{-1}\tilde{y}} \circ \widetilde{\exp}^{-1}_{\tilde{U}_x,\gamma^{-1}\tilde{y}}(\tilde{z}) = \gamma \cdot \widetilde{\exp}^{-1}_{\tilde{U}_x,\gamma^{-1}\tilde{y}}(\tilde{z})$. Thus,

$$\begin{aligned}\tilde{\sigma}_x(\delta\cdot\tilde{y}) &= \left(\delta\cdot\tilde{y}, \widetilde{\exp}^{-1}_{\tilde{U}_x,\delta\tilde{y}}\left(\tilde{f}_x(\delta\cdot\tilde{y})\right)\right)\\ &= \left(\delta\cdot\tilde{y}, \widetilde{\exp}^{-1}_{\tilde{U}_x,\delta\tilde{y}}\left(\delta\cdot\tilde{f}_x(\tilde{y})\right)\right)\\ &= \left(\delta\cdot\tilde{y}, \delta\cdot\widetilde{\exp}^{-1}_{\tilde{U}_x,\delta^{-1}\delta\tilde{y}}\left(\tilde{f}_x(\tilde{y})\right)\right)\\ &= \left(\delta\cdot\tilde{y}, \delta\cdot\widetilde{\exp}^{-1}_{\tilde{U}_x,\tilde{y}}\left(\tilde{f}_x(\tilde{y})\right)\right)\\ &= \delta\cdot\tilde{\sigma}_x(\tilde{y})\end{aligned}$$

which is the correct equivariance relation for an orbisection. As a result we see that the map $\sigma(x) = \Pi_x \circ \tilde{\sigma}_x(\tilde{x})$ defines a C^r orbisection of $T\mathcal{O}$ and that $E(\sigma) = f$ since $\tilde{\sigma}_x(\tilde{x}) = \left(\tilde{x}, \widetilde{\exp}^{-1}_{\tilde{U}_x,\tilde{x}}\left(\tilde{f}_x(\tilde{x})\right)\right)$. ■

The following proposition is the last ingredient needed to complete the proof of theorem 1.1.

Proposition 7.4. *The map* $E : \mathcal{B}^r(\mathbf{0},\varepsilon) \to \mathcal{N}^0(\mathrm{Id},\varepsilon) \cap \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ *is a homeomorphism.*

Proof. Propositions 7.2 and 7.3 show that E is bijective. Continuity of E follows from the formula for a local lift of E given in Propositon 6.10 and continuity of E^{-1} follows from the formula for $\tilde{\sigma}_x$ given in the last line of proposition 7.3. ■

Proof of Theorem 1.1.

Proof. Let $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(O)$. By proposition 7.4, the map

$$f \circ E : \mathcal{B}^r(\mathbf{0},\varepsilon) \to \mathcal{N}^0(f,\varepsilon) \cap \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$$

is a homeomorphism giving a local chart about f. Let $\mathcal{N}_{fg} = \mathcal{N}^0(f,\varepsilon) \cap \mathcal{N}^0(g,\varepsilon) \cap \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ denote a chart overlap, and let $\mathcal{B}_{fg} = (f \circ E)^{-1}(\mathcal{N}_{fg}) \subset \mathcal{B}^r(\mathbf{0},\varepsilon)$. Then the corresponding transition map

$$(g \circ E)^{-1} \circ (f \circ E)\big|_{\mathcal{B}_{fg}} : \mathcal{B}_{fg} \subset \mathcal{B}^r(\mathbf{0},\varepsilon) \to (g \circ E)^{-1}(\mathcal{N}_{fg}) \subset \mathcal{B}^r(\mathbf{0},\varepsilon)$$

is a homeomorphism. This gives the desired C^0 manifold structure to $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ where the model space is the topological vector space of C^r orbisections of the tangent orbibundle with the C^r topology. ■

Proof of Corollary 1.2.

Proof. It follows from the arguments in example 3.5 that for a given $f \in \mathcal{ID}$ and any $x \in \mathcal{O}$ with orbifold chart U_x of x there is a $\gamma_x \in \Gamma_x$ so that $\tilde{f}(\tilde{y}) = \gamma_x \cdot \tilde{y}$ for all $\tilde{y} \in \tilde{U}_x$. A finite cover of $\mathcal{O}$ by charts $\{U_{x_1}, \ldots, U_{x_M}\}$ shows that $\mathcal{ID}$ is a subgroup of $\prod_{i=1}^{M} \Gamma_{x_i}$ and is therefore finite. Clearly $\mathcal{ID}$ is a normal subgroup of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ as $\tilde{g} \circ \tilde{f} \circ \tilde{g}^{-1}$ covers the identity for any $g \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ and $f \in \mathcal{ID}$. Also, any two lifts $\tilde{h}_0$ and $\tilde{h}_1$ of $h \in \mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ by definition must satisfy $\tilde{h}_0 \circ \tilde{h}_1^{-1} \in \mathcal{ID}$ from which follows the existence of the short exact sequence. Moreover, the finiteness of $\mathcal{ID}$ shows that the quotient topology on $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ is again that of a Banach manifold if $r < \infty$ and of a Fréchet manifold if $r = \infty$. ■

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