Journal of Lie Theory
Volume **18** (2008) 83–91

A Simple Proof of the Algebraic Version of a Conjecture by Vogan

Tim Bratten and Sergio Corti

Communicated by B. Kostant

Abstract. In a recent manuscript, D. Vogan conjectures that four canonical globalizations of Harish-Chandra modules commute with certain $\mathfrak{n}-$cohomology groups. In this article we prove that Vogan's conjecture holds for one of the globalizations if and only if it holds for the dual. Using a previously published result of one of the authors, which establishes the conjecture for the minimal globalization, we can therefore deduce Vogan's conjecture for the maximal globalization.
Mathematics Subject Index 2000: 22E46.
Keywords and phrases: Representations of reductive Lie groups, $\mathfrak{n}$-homology groups, globalizations of Harish-Chandra modules.

1. Introduction

In a recent manuscript [6, Conjecture 10.3], D.Vogan conjectures that four canonical globalizations of Harish-Chandra modules commute with certain $\mathfrak{n}$-cohomology groups. In this article we consider an algebraic version of Vogan's conjecture which entails that the conjecture holds for one of the globalizations if and only if it holds for the dual. We prove this result for a reductive Lie group of Harish-Chandra class. Using the result that Vogan's conjecture is known for the minimal globalization [1] we can therefore conclude the conjecture is true for the maximal globalization.

This article is organized as follows. In the second section we define the $\mathfrak{n}$-homology and cohomology groups and recall some formulas that relate them. The third section treats Harish-Chandra modules and globalizations. The fourth section introduces Vogan's conjecture and establishes the algebraic version. In the fifth section we review the Hochschild-Serre spectral sequence and use it to prove the main result.

The authors would like to take this opportunity to thank David Vogan for contacting us about his work relating to this conjecture. We would also like to thank the referee for pointing out the possibility of a formal proof based on the Hochschild-Serre spectral sequence. Our original proof, which applied to the case of a connected, complex reductive group, relied on the Beilinson-Bernstein classification and characterized the Harish-Chandra dual of a standard module.

ISSN 0949–5932 / $2.50

2. $\mathfrak{n}$-homology and cohomology

Suppose G_0 is a reductive group of Harish-Chandra class with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. By definition, a *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra and a *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra that contains a Borel subalgebra. If $\mathfrak{p} \subseteq \mathfrak{g}$ is a parabolic subalgebra then *the nilradical* $\mathfrak{n}$ of $\mathfrak{p}$ is the largest solvable ideal in $[\mathfrak{p}, \mathfrak{p}]$. A *Levi factor* is a complementary subalgebra to $\mathfrak{n}$ in $\mathfrak{p}$. One knows that Levi factors exist and that they are exactly the subalgebras which are maximal with respect to being reductive in $\mathfrak{p}$.

Fix a parabolic subalgebra $\mathfrak{p}$ with nilradical $\mathfrak{n}$ and Levi factor $\mathfrak{l}$. Let $U(\mathfrak{n})$ denote the enveloping algebra of $\mathfrak{n}$ and let $\mathbb{C}$ be the irreducible trivial module. If M is a $\mathfrak{g}$-module then *the zero* $\mathfrak{n}$*-homology of* M is the $\mathfrak{l}$-module

$$H_0(\mathfrak{n}, M) = \mathbb{C} \otimes_{U(\mathfrak{n})} M.$$

This definition determines a right exact functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{l}$-modules. *The* $\mathfrak{n}$*-homology groups of* M are the $\mathfrak{l}$-modules obtained as the corresponding derived functors.

The right standard resolution of $\mathbb{C}$ is the complex of free right $U(\mathfrak{n})$-modules given by

$$0 \cdots \to \Lambda^{p+1}\mathfrak{n} \otimes U(\mathfrak{n}) \to \Lambda^{p}\mathfrak{n} \otimes U(\mathfrak{n}) \to \cdots \to \mathfrak{n} \otimes U(\mathfrak{n}) \to U(\mathfrak{n}) \to 0.$$

Applying the functor

$$- \otimes_{U(\mathfrak{n})} M$$

to the standard resolution we obtain a complex

$$\cdots \to \Lambda^{p+1}\mathfrak{n} \otimes M \to \Lambda^{p}\mathfrak{n} \otimes M \to \cdots \to \mathfrak{n} \otimes M \to M \to 0$$

of left $\mathfrak{l}$-modules called *the standard* $\mathfrak{n}$*-homology complex.* Here $\mathfrak{l}$ acts via the tensor product of the adjoint action on $\Lambda^{p}\mathfrak{n}$ with the given action on M. Since $U(\mathfrak{g})$ is free as $U(\mathfrak{n})$-module, a routine homological argument identifies the pth homology of the standard complex with the pth $\mathfrak{n}$-homology group

$$H_{\mathrm{p}}(\mathfrak{n}, M).$$

One knows that the induced $\mathfrak{l}$-action on the homology groups of the standard complex is the correct one.

The zero $\mathfrak{n}$*-cohomology* of a $\mathfrak{g}$-module M is the $\mathfrak{l}$-module

$$H^0(\mathfrak{n}, M) = \mathrm{Hom}_{U(\mathfrak{n})}(\mathbb{C}, M).$$

This determines a left exact functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{l}$-modules. By definition, *the* $\mathfrak{n}$*-cohomology groups of* M are the $\mathfrak{l}$-modules obtained as the corresponding derived functors. These $\mathfrak{l}$-modules can be calculated by applying the functor

$$\mathrm{Hom}_{U(\mathfrak{n})}(-, M)$$

to the standard resolution of $\mathbb{C}$, this time by free left $U(\mathfrak{n})$-modules. In a natural way, one obtains a complex of $\mathfrak{l}$-modules and pth cohomology of this complex realizes the pth $\mathfrak{n}$-cohomology group

$$H^p(\mathfrak{n}, M).$$

Let $\mathfrak{n}^*$ denote the $\mathfrak{l}$-module dual to $\mathfrak{n}$. Then, using the standard complexes and the natural isomorphism of $\mathfrak{l}$-modules

$$\Lambda^p\mathfrak{n}^* \otimes M \cong \mathrm{Hom}(\Lambda^p\mathfrak{n}, M)$$

one can deduce the following well known fact [3, Section 2]:

Proposition 2.1. *Suppose M is a $\mathfrak{g}$-module. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra with nilradical $\mathfrak{n}$ and Levi factor $\mathfrak{l}$.*
(a) Let M^* denote the $\mathfrak{g}$-module dual to M. Then there is a natural isomorphism

$$H_p(\mathfrak{n}, M^*) \cong H^p(\mathfrak{n}, M)^*$$

where $H^p(\mathfrak{n}, M)^*$ denotes the $\mathfrak{l}$-module dual to $H^p(\mathfrak{n}, M)$.
(b) Let d denote the dimension of $\mathfrak{n}$. Then there is a natural isomorphism

$$H_p(\mathfrak{n}, M) \cong H^{d\text{-}p}(\mathfrak{n}, M) \otimes \Lambda^d\mathfrak{n}$$

3. Harish-Chandra modules and globalizations

Fix a maximal compact subgroup K_0 of G_0. Suppose we have a linear action of K_0 on a complex vector space M. A vector $m \in M$ is called *K_0-finite* if the span of the K_0-orbit of m is finite-dimensional and if the action of K_0 in this subspace is continuous. The linear action of K_0 in M is called *K_0-finite* when every vector is K_0-finite. By definition, a *Harish-Chandra module* for G_0 is a finite length $\mathfrak{g}$-module M equipped with a compatible, K_0-finite, linear action. One knows that an irreducible K_0-module has finite multiplicity in a Harish-Chandra module. For our purposes, it will also be useful to refer to a category of *good* K_0-modules. A *good* K_0-module will mean a locally finite module such that each irreducible K_0-module has finite multiplicity therein.

A representation of G_0 in a complete locally convex topological vector space V is called *admissible* if V has finite length (with respect to closed invariant subspaces) and if each irreducible K_0-module has finite multiplicity in V. When V is admissible, then each K_0-finite vector in V is differentiable and the subspace of K_0-finite vectors is a Harish-Chandra module. The representation is called *smooth* if every vector in V is differentiable. In this case, V is a $\mathfrak{g}$-module. For example, one knows that an admissible representation in a Banach space is smooth if and only if the representation is finite-dimensional.

Given a Harish-Chandra module M, a *globalization M_{glob} of M* is an admissible representation of G_0 whose underlying $(\mathfrak{g}, K_0)$-module of K_0-finite vectors is isomorphic to M. By now, four canonical globalizations of Harish-Chandra modules are known to exist. These are: the smooth globalization of Casselman and Wallach [2], its dual (called: the distribution globalization), Schmid's minimal globalization [5] and its dual (the maximal globalization). All four globalizations are smooth and functorial. In this article we focus on the minimal and maximal globalizations of Schmid.

The minimal globalization $M_{\min}$ of a Harish-Chandra module M is uniquely characterized by the property that any $(\mathfrak{g}, K_0)$-equivariant linear of M onto the K_0-finite vectors of an admissible representation V lifts to a unique, continuous G_0-equivariant linear map of $M_{\min}$ into V. In particular, $M_{\min}$ embeds G_0-equivariantly and continuously into any globalization of M. The construction of the minimal globalization shows that it's realized on a *DNF space*. This means that its continuous dual, in the strong topology, is a nuclear Frechét space. One knows that $M_{\min}$ consists of analytic vectors and that it surjects onto the analytic vectors in a Banach space globalization. Like each of the canonical globalizations, the minimal globalization is functorially exact. In particular, a closed G_0-invariant subspace of a minimal globalization is the minimal globalization of its underlying Harish-Chandra module and a continuous G_0-equivariant linear map between minimal globalizations has closed range.

To characterize the maximal globalization, we introduce the K_0-finite dual on the category of Harish-Chandra modules. In particular, let M be a Harish-Chandra module. Then the algebraic dual M^* of M is a $\mathfrak{g}$-module and a K_0-module, but in general not K_0-finite. We define $M^\vee$, *the* K_0*-finite (or Harish-Chandra) dual to* M, to be the subspace of K_0-finite vectors in M^*. Thus $M^\vee$ is a Harish-Chandra module. In fact, the functor $M \mapsto M^\vee$ is exact on the category of good K_0-modules. We also have the formula

$$(M^\vee)^\vee \cong M.$$

The maximal globalization $M_{\max}$ of M can be defined by the equation

$$M_{\max} = ((M^\vee)_{\min})'$$

where the last prime denotes the continuous dual equipped with the strong topology. In particular, $M_{\max}$ is a globalization of M. Observe that the maximal globalization is an exact functor, since all functors used in the definition are exact. Because of the minimal property of $M_{\min}$, it follows that any globalization of M embeds continuously and equivariantly into $M_{\max}$. Note that the continuous dual of a maximal globalization is the minimal globalization of the dual Harish-Chandra module.

4. A conjecture by Vogan

In order to introduce Vogan's conjecture, we need to be more specific about the parabolic subalgebras we consider. Suppose $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. We say that $\mathfrak{p}$ is *nice* if $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{l}_0$ is the real form of a Levi factor $\mathfrak{l}$ of $\mathfrak{p}$. In this case $\mathfrak{l}$ is called *the stable Levi factor.* When $\mathfrak{p}$ is nice, then every G_0-conjugate of $\mathfrak{p}$ is nice.

Suppose $\mathfrak{p}$ is nice and $\mathfrak{l}$ is the stable Levi factor. Then we define *the associated Levi subgroup* L_0 to be the normalizer of $\mathfrak{p}$ in G_0. One knows that L_0 is a real reductive group of Harish-Chandra class with Lie algebra $\mathfrak{l}_0$. Let

$$\theta : G_0 \to G_0$$

be a Cartan involution with fixed point set K_0. The parabolic subalgebra will be called *very nice* if $\theta(L_0) = L_0$. In this case $K_0 \cap L_0$ is a maximal compact subgroup of L_0. One knows that a nice parabolic subalgebra is G_0-conjugate to a very nice parabolic subalgebra and that two very nice parabolic subalgebras are conjugate under K_0 if and only if they are conjugate under G_0.

Throughout the remainder of this discussion, when $\mathfrak{p}$ is a very nice parabolic subalgebra, then $\mathfrak{n}$ will denote the nilradical of $\mathfrak{p}$, $\mathfrak{l}$ will denote the stable Levi factor and L_0 will denote the associated Levi subgroup. We fix the maximal subgroup $K_0 \cap L_0$ of L_0 and speak of Harish-Chandra modules for L_0 accordingly. We have the following result [3, Proposition 2.24].

Proposition 4.1. *Suppose $\mathfrak{p}$ is a very nice parabolic subalgebra and let M be a Harish-Chandra module for G_0. Then the $\mathfrak{n}$-homology groups and $\mathfrak{n}$-cohomology groups of M are Harish-Chandra modules for L_0.*

Vogan's conjecture is the following.

Conjecture 4.2. Suppose $\mathfrak{p}$ is a very nice parabolic subalgebra and let M be a Harish-Chandra module for G_0. Suppose M_{glob} indicates one of the four canonical globalizations of M. Then the induced topologies of the $\mathfrak{n}$-cohomology groups $H^{\text{p}}(\mathfrak{n}, M_{\text{glob}})$ are Hausdorff and there are natural isomorphisms of L_0-representations

$$H^{\text{p}}(\mathfrak{n}, M_{\text{glob}}) \cong H^{\text{p}}(\mathfrak{n}, M)_{\text{glob}}$$

where $H^{\text{p}}(\mathfrak{n}, M)_{\text{glob}}$ denotes the corresponding canonical globalization to L_0 of the Harish-Chandra module $H^{\text{p}}(\mathfrak{n}, M)$. .

Proposition 2.1 (b) shows that Vogan's conjecture holds for the $\mathfrak{n}$-cohomology groups if and only if it holds for the $\mathfrak{n}$-homology groups.

The conjecture is known to be true for the case of the minimal globalization [1]. We will now observe that Vogan's conjecture for the dual representation is in fact equivalent to a certain purely algebraic statement about $\mathfrak{n}$-homology groups and the Harish-Chandra dual of a Harish-Chandra module.

Proposition 4.3. *Suppose that $\mathfrak{p}$ is a very nice parabolic subalgebra and suppose that Vogan's conjecture holds for the $\mathfrak{n}$-cohomology groups of one of the four canonical globalizations. In particular, when M is a Harish-Chandra module, let M_{glob} denote the globalization for which Vogan's conjecture holds and let M^{glob} denote the dual globalization. Thus*

$$M^{glob} \cong \left((M^{\vee})_{glob}\right)'.$$

Then, in a natural way, the $\mathfrak{n}-$homology group $H_p(\mathfrak{n}, M^{glob})$ is isomorphic to the dual globalization of $H^p(\mathfrak{n}, M^{\vee})^{\vee}$. That is:

$$H_p(\mathfrak{n}, M^{glob}) \cong (H^p(\mathfrak{n}, M^{\vee})^{\vee})^{glob}$$

In particular, Vogan's conjecture holds for the dual globalization if and only if there are natural isomorphisms

$$H_p(\mathfrak{n}, M^{\vee}) \cong H^p(\mathfrak{n}, M)^{\vee}$$

for each p.

Proof. We assume the conjecture holds for M_{glob}. Since the continuous dual is exact on the category obtained by applying the canonical globalization to Harish-Chandra modules, it follows, as in the proof of Proposition 2.1, that

$$H_{\mathrm{p}}(\mathfrak{n},(M_{\mathrm{glob}})') \cong H^{\mathrm{p}}(\mathfrak{n},M_{\mathrm{glob}})'.$$

Since M^{glob} is given by $((M^\vee)_{\mathrm{glob}})'$ it follows that

$$\begin{aligned} H_{\mathrm{p}}(\mathfrak{n},M^{\mathrm{glob}}) &\cong H_{\mathrm{p}}\left(\mathfrak{n},\left((M^\vee)_{\mathrm{glob}}\right)'\right) \cong H^{\mathrm{p}}\left(\mathfrak{n},(M^\vee)_{\mathrm{glob}}\right)' \\ &\cong \left((H^{\mathrm{p}}(\mathfrak{n},(M^\vee))_{\mathrm{glob}}\right)' \cong \left(H^{\mathrm{p}}(\mathfrak{n},M^\vee)^\vee\right)^{\mathrm{glob}} \end{aligned}$$

■

In this article we will show there are natural isomorphisms

$$H_{\mathrm{p}}(\mathfrak{n},M^\vee) \cong H^{\mathrm{p}}(\mathfrak{n},M)^\vee$$

for $\mathfrak{p}$ a very nice parabolic subalgebra. We call this isomorphism *the algebraic version of Vogan's conjecture.*

5. The Natural Map and the Hochschild-Serre Spectral Sequence

Through out the remainder of the discussion we fix a very nice parabolic subalgebra $\mathfrak{p}$. Suppose M is a Harish-Chandra module for G_0. Then then the natural inclusion

$$M^\vee \to M^*$$

induces a map

$$H_{\mathrm{p}}(\mathfrak{n},M^\vee) \to H_{\mathrm{p}}(\mathfrak{n},M^*) \cong H^{\mathrm{p}}(\mathfrak{n},M)^*.$$

Since $H_{\mathrm{p}}(\mathfrak{n},M^\vee)$ is a Harish-Chandra module for L_0, it follows that the image of this map lies inside $H^{\mathrm{p}}(\mathfrak{n},M)^\vee$. Our point is to show the resulting natural map

$$H_{\mathrm{p}}(\mathfrak{n},M^\vee) \to H^{\mathrm{p}}(\mathfrak{n},M)^\vee$$

is an isomorphism.

Before giving a general argument, we first make the following observation. Suppose that $K_0 \subseteq L_0$ (this is what happens, for example when $G_0 = SL(2,\mathbb{R})$). Then the standard resolution is a complex of good K_0-modules (Section 3). Since the K_0-finite dual is an exact functor on the category of good K_0-modules, one can argue directly, as in the case for the ordinary dual that

$$H_{\mathrm{p}}(\mathfrak{n},M^\vee) \cong H^{\mathrm{p}}(\mathfrak{n},M)^\vee.$$

Our general proof builds on this observation by introducing the Hochschild-Serre spectral sequence [4]. In particular, let $\mathfrak{t} \subseteq \mathfrak{g}$ be the complexified Lie algebra of K_0 and let $\mathfrak{s} \subseteq \mathfrak{g}$ be the complexification of the negative one eigenspace space for the Cartan involution θ. Thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$$

complexifies the Cartan decomposition for $\mathfrak{g}_0$. Since $\mathfrak{p}$ is θ-stable

$$\mathfrak{n} = \mathfrak{n} \cap \mathfrak{k} \oplus \mathfrak{n} \cap \mathfrak{s}.$$

Observe that $\mathfrak{n}\cap\mathfrak{k}$ acts on $\mathfrak{n}\cap\mathfrak{s}$ by the adjoint representation. Roughly speaking, the Hochschild-Serre spectral sequence gives us canonical ways to relate $H_{\mathrm{q}}(\mathfrak{n}\cap\mathfrak{k}, M^\vee)\otimes\wedge(\mathfrak{n}\cap\mathfrak{s})$ to $H_{\mathrm{p}}(\mathfrak{n}, M^\vee)$ and $H^{\mathrm{q}}(\mathfrak{n}\cap\mathfrak{k}, M)\otimes\wedge(\mathfrak{n}\cap\mathfrak{s})^*$ to $H^{\mathrm{p}}(\mathfrak{n}, M)$. We will use this to deduce the desired result.

We follow the development in [4, Chapter V, Section 10] and begin by selecting in $\wedge(\mathfrak{n}\cap\mathfrak{s})$ a sequence $(V_{\mathrm{p}})_{\mathrm{p}=0}^{\mathrm{N}}$ of $K_0\cap L_0$-invariant subspaces such that:
(a) $\wedge(\mathfrak{n}\cap\mathfrak{s}) = \oplus_{\mathrm{p}=0}^{\mathrm{N}} V_{\mathrm{p}}$
(b) $V_0 = \wedge^0(\mathfrak{n}\cap\mathfrak{s}) = \mathbb{C}$ and $V_{\mathrm{N}} = \wedge^{\mathrm{R}}(\mathfrak{n}\cap\mathfrak{s})$ where R$=$dim$(\mathfrak{n}\cap\mathfrak{s})$.
(c) There is a monotone increasing function $r(\mathrm{p}) \leq$p such that $V_{\mathrm{p}} \subseteq \wedge^{r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{s})$
(d) $(\mathfrak{n}\cap\mathfrak{k})\cdot V_{\mathrm{p}} \subseteq \oplus_{\mathrm{k}=0}^{\mathrm{p-1}} V_{\mathrm{k}}$.
The spectral sequences we need are as follows and can be phrased in terms of a $\mathfrak{g}$-module M:

(i) There is a convergent spectral sequence

$$E_{\mathrm{r}}^{\mathrm{p,q}} \implies H_{\mathrm{p+q}}(\mathfrak{n}, M)$$

with E_1 term

$$E_1^{\mathrm{p,q}} = H_{\mathrm{p+q}-r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{k}, M)\otimes V_{\mathrm{p}}.$$

The differential d_{r} has bidegree $(-\mathrm{r},\mathrm{r}-1)$ and is a $K_0\cap L_0$ map when M is a $K_0\cap L_0$-module.

(ii) There is a convergent spectral sequence

$$E_{\mathrm{r}}^{\mathrm{p,q}} \implies H^{\mathrm{p+q}}(\mathfrak{n}, M)$$

with E_1 term

$$E_1^{\mathrm{p,q}} = H^{\mathrm{p+q}-r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{k}, M)\otimes (V_{\mathrm{p}})^*.$$

The differential d_{r} has bidegree $(\mathrm{r},1.-\mathrm{r})$ and is a $K_0\cap L_0$ map when M is a $K_0\cap L_0$-module.

We are now ready to prove the algebraic version of Vogan's conjecture.

Theorem 5.1. *Suppose $\mathfrak{n}$ is the nilradical of a very nice parabolic subalgebra and let M be a Harish-Chandra module. Then the natural map*

$$H_p(\mathfrak{n}, M^\vee) \to H^p(\mathfrak{n}, M)^\vee$$

is an isomorphism.

Proof. The proof is reminiscent of the proof given for [4, Corollary 5.141]. First consider the spectral sequence associated to the object $H_{\mathrm{p}}(\mathfrak{n}, M^*) \cong H^{\mathrm{p}}(\mathfrak{n}, M)^*$. In particular, by dualizing everything in sight in the spectral sequence for $H^{\mathrm{p}}(\mathfrak{n}, M)$ we obtain a spectral sequence naturally isomorphic to the spectral sequence for $H_{\mathrm{p}}(\mathfrak{n}, M^*)$ and with E_1 term

$$E_1^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^*)) = H^{\mathrm{p+q}-r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{k}, M)^*\otimes V_{\mathrm{p}}.$$

Thus the space of $K_0\cap L_0$-finite vectors in this term is given by

$$H^{\mathrm{p+q}-r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{k}, M)^\vee\otimes V_{\mathrm{p}}.$$

Next we show that this object is naturally isomorphic to the E_1 term for the spectral sequence associated to $H_{\mathrm{p}}(\mathfrak{n}, M^\vee)$. Letting $\widehat{K_0}$ denote the unitary dual of the group K_0, we write

$$M = \oplus_{\pi\in\widehat{K_0}} m(\pi)V_\pi$$

where V_π is a copy of the irreducible representation corresponding to $\pi \in \widehat{K_0}$ and $m(\pi)$ is the multiplicity of π in M. Thus we have

$$H_{\mathrm{p+q}-r(\mathrm{p})}(\mathfrak{n}\cap\mathfrak{k}, M^\vee) = H_{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, \left(\oplus_{\pi\in\widehat{K_0}} m(\pi)V_\pi\right)^\vee\right) =$$

$$H_{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, \oplus_{\pi\in\widehat{K_0}} m(\pi)V_\pi^*\right) = \oplus_{\pi\in\widehat{K_0}} m(\pi)H_{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, V_\pi^*\right) =$$

$$\oplus_{\pi\in\widehat{K_0}} m(\pi)H^{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, V_\pi\right)^* = \left(\oplus_{\pi\in\widehat{K_0}} m(\pi)H^{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, V_\pi\right)\right)^\vee =$$

$$H^{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, \oplus_{\pi\in\widehat{K_0}} m(\pi)V_\pi\right)^\vee = H^{\mathrm{p+q}-r(\mathrm{p})}\left(\mathfrak{n}\cap\mathfrak{k}, M\right)^\vee.$$

We now show how this leads to the desired result. Notably, let $E_r^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^\vee))$, $E_r^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^*))$ and $E_r^{\mathrm{p,q}}(H^\bullet(\mathfrak{n}, M))$denote the E_{r} terms of the corresponding spectral sequences. Using induction, we want to see that

$$E_r^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^\vee)) \cong E_r^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^*))_{K_0\cap L_0}$$

where $E_r^{\mathrm{p,q}}(H_\bullet(\mathfrak{n}, M^*))_{K_0\cap L_0}$ indicates the corresponding space of $K_0 \cap L_0$-finite vectors. Indeed, using the fact that the $E_{\mathrm{r+1}}$ terms are given by the homology of a complex

$$d_{\mathrm{r}} : E_{\mathrm{r}} \to E_{\mathrm{r}}$$

we can determine the result from the fact the complex associated to $H_\bullet(\mathfrak{n}, M^*)$ is obtained by dualizing the complex associated to $H^\bullet(\mathfrak{n}, M)$ and the fact that terms $E_r^{\mathrm{p,q}}(H^\bullet(\mathfrak{n}, M))$ are good $K_0 \cap L_0$-modules. Specifically

$$E_r^{\mathrm{p,q}}(H\left(_\bullet(\mathfrak{n}, M^*)\right)_{K_0\cap L_0} \cong E_r^{\mathrm{p,q}}\left(H^\bullet(\mathfrak{n}, M)\right)^*_{K_0\cap L_0} = E_r^{\mathrm{p,q}}\left(H^\bullet(\mathfrak{n}, M)\right)^\vee$$

therefore the result follows by induction since the $K_0 \cap L_0$- finite dual is an exact functor on the category of good $K_0 \cap L_0$-modules.

Finally, to deduce the main result, we use the filtrations of $H_{\mathrm{p}}(\mathfrak{n}, M^\vee)$ and $H_{\mathrm{p}}(\mathfrak{n}, M^*)$ given by the corresponding spectral sequences. Then we can conclude the final result from the following analog to [4, Lemma 5.142].

Lemma 5.2. *Let A be a good $(\mathfrak{l}, K_0 \cap L_0)$-module and suppose B is an $(\mathfrak{l}, K_0 \cap L_0)$-module (we do not assume B is necessarily $K_0 \cap L_0$-finite). Suppose each module is endowed with $(\mathfrak{l}, K_0 \cap L_0)$-filtrations*

$$\begin{aligned} A^N \supseteq A^{N-1} \supseteq \cdots \supseteq A^0 \supseteq A^{-1} = 0 \\ B^N \supseteq B^{N-1} \supseteq \cdots \supseteq B^0 \supseteq B^{-1} = 0 \end{aligned}$$

and let $j : A \to B$ be an $(\mathfrak{l}, K_0 \cap L_0)$-map such that $j(A^p) \subseteq B^p$ for each p. Suppose the induced maps

$$j^p : A^p/A^{p\text{-}1} \to \left(B^p/B^{p\text{-}1}\right)_{K_0\cap L_0}$$

are isomorphisms for each p. Then the map

$$j : A \to B_{k_0 \cap L_0}$$

is an isomorphism.

Proof of Lemma. Since the functor taking the space of $K_0 \cap L_0$-finite vectors is left exact on an appropriately defined category of $(\mathfrak{l}, K_0 \cap L_0)$-modules, the lemma follows as in the proof of [4, Lemma 5.142] after applying the functor of $K_0 \cap L_0$-finite vectors ■

We can therefore conclude our main result:

Theorem 5.3. *Let G_0 be a reductive Lie group of Harish-Chandra class, $K_0 \subseteq G_0$ a maximal compact subgroup and $\mathfrak{g}$ the complexified Lie algebra of G_0. Suppose $\mathfrak{n}$ is the nilradical of a very nice parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Let $L_0 \subseteq G_0$ denote the associated Levi subgroup and let M_{max} denote the maximal globalization of a Harish-Chandra module M. Then, in a natural way, the $\mathfrak{n}$-cohomology groups $H_p(\mathfrak{n}, M_{max})$ are representations of L_0 and for each p, there are canonical isomorphisms*

$$H^p(\mathfrak{n}, M_{max}) \cong H^p(\mathfrak{n}, M)_{max}.$$

References

[1] Bratten, T., *A comparison theorem for Lie algebra homology groups*, Pacific J. Math. **182** (1998), 23–36.

[2] Casselman, W., *Canonical extensions of Harish-Chandra modules to representations of G*, Can. J. Math. **41** (1989), 385–438.

[3] Hecht, H., and W. Schmid, *Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules*, Acta Math. **151** (1983), 49–151.

[4] Knapp, A., and D. Vogan, "Cohomological Induction and Unitary Representations," Princeton University Press, Princeton, New Jersey, 1995.

[5] Schmid, W., *Boundary value problems for group invariant differential equations*, The mathematical heritage of Elie Cartan, Astérique 1985, 311–321.

[6] Vogan, D., *Unitary representations and complex analysis*, Notes from the Cime summer school, Venice, Italy 2004.

Tim Bratten
Facultad de Ciencias Exactas
UNICEN
Tandil, Argentina
bratten@exa.unicen.edu.ar

Sergio Corti
Facultad de Ciencias Exactas
UNICEN
Tandil, Argentina
csdaniel2003@yahoo.com.ar

Received June 2, 2006
and in final form November 13, 2007