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Compact Symmetric Spaces, Triangular Factorization, and Poisson Geometry

Arlo Caine*

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Abstract. Let X be a simply connected compact Riemannian symmetric space, let U be the universal covering group of the identity component of the isometry group of X, and let $\mathfrak{g}$ denote the complexification of the Lie algebra of U, $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$. Each $\mathfrak{u}$-compatible triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ determines a Poisson Lie group structure π_U on U. The Evens-Lu construction produces a (U, π_U)-homogeneous Poisson structure on X. By choosing the basepoint in X appropriately, X is presented as U/K where K is the fixed point set of an involution which stabilizes the triangular decomposition of $\mathfrak{g}$. With this presentation, a connection is established between the symplectic foliation of the Evens-Lu Poisson structure and the Birkhoff decomposition of U/K. This is done through reinterpretation of results of Pickrell. Each symplectic leaf admits a natural torus action. It is shown that the action is Hamiltonian and the momentum map is computed using triangular factorization. Finally, local formulas for the Evens-Lu Poisson structure are displayed in several examples.
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1. Introduction

Let (U, π_U) be a Poisson Lie group and $\mathfrak{d}$ be the double of its Lie bialgebra. Let $\mathcal{L}(\mathfrak{d})$ denote the variety of Lagrangian subalgebras of $\mathfrak{d}$. Drinfeld showed in [2] that the U-equivariant isomorphism classes of (U, π_U)-homogeneous Poisson spaces with connected stability subgroups are in one-to-one correspondence with the U-orbits in a certain subset of $\mathcal{L}(\mathfrak{d})$. Evens and Lu gave a general construction in [3] which produces a (U, π_U)-homogeneous Poisson structure on each U-orbit in $\mathcal{L}(\mathfrak{d})$.

Let X be a connected, and simply connected, compact Riemannian symmetric space. Let U be the universal covering group of the identity component of the isometry group of X and denote by π_U a Poisson Lie group structure on U.

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The complexification of U will be denoted by G with corresponding Lie algebra $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$. This paper concerns (U, π_U)-homogeneous Poisson structures on X, i.e., structures for which the action map $U \times X \to X$ is a Poisson map. By selecting a basepoint, which determines a stability subgroup K in U and an involution θ of U fixing K, the symmetric space X is presented as a coset space U/K. This choice determines a model point $\mathfrak{g}_0$ in $\mathcal{L}(\mathfrak{d})$ for X. The Evens-Lu construction generates a (U, π_U)-homogeneous Poisson structure on X.

The difference of any two (U, π_U)-homogeneous Poisson structures on a U-homogeneous space M is a U-invariant bivector field on M. If M is an irreducible compact Hermitian symmetric space, then M admits a one parameter family of U-invariant bivector fields. The elements of this family are the scalar multiples of the non-degenerate U-invariant Poisson structure π_{KKS} that a Hermitian symmetric space carries because it is a coadjoint orbit. The structure π_{KKS} is the contravariant version of the symplectic structure on coadjoint orbits discovered by Kostant, Kirillov, and Souriau. If M is a non-Hermitian irreducible symmetric space, then only the trivial bivector field is U-invariant. Thus, there is exactly one (U, π_U)-homogeneous Poisson structure on an irreducible non-Hermitian symmetric space. For standard Poisson Lie group structures π_U, the (U, π_U)-homogeneous Poisson structures on irreducible Hermitian symmetric spaces were classified in [8].

Each choice of a $\mathfrak{u}$-compatible triangular decomposition of $\mathfrak{g}$ determines a standard Poisson Lie group structure π_U on U. In this paper the (U, π_U)-homogeneous Poisson structure on X resulting from the Evens-Lu construction is studied. The key point of this paper is that one can choose the basepoint of X in such a way that the triangular decomposition of $\mathfrak{g}$ defining π_U is stable with respect to the involution selecting the stability subgroup K. With such a presentation of X as U/K there is a beautiful connection between the symplectic foliation of the Evens-Lu Poisson structure on X and the corresponding Birkhoff decomposition of U/K. This link enables the explicit description of the symplectic foliation. Each symplectic leaf has a natural Hamiltonian torus action. The connection with the Birkhoff decomposition enables the computation of the momentum map. It should also be emphasized that this presentation is very useful for doing explicit calculations to produce examples.

This Poisson structure on compact symmetric spaces was also studied by Foth and Lu in [5]. They gave an alternate construction which one may interpret as follows. It is possible to choose the basepoint of X, determining a different stability subgroup K' and corresponding model point $\mathfrak{g}_0'$, in such a way that the Borel subalgebra $\mathfrak{h} + \mathfrak{n}_+$ is Iwasawa relative to the noncompact real form $\mathfrak{g}_0'$ of $\mathfrak{g}$. This means that the intersection $(\mathfrak{h} + \mathfrak{n}_+) \cap \mathfrak{g}_0'$ contains $\mathfrak{a}_0' + \mathfrak{n}_0'$ for some Iwasawa decomposition $\mathfrak{g}_0' = \mathfrak{k}' + \mathfrak{a}_0' + \mathfrak{n}_0'$ of $\mathfrak{g}_0'$. In this presentation, the push-forward of π_U under the projection map $U \to U/K'$ gives the Evens-Lu Poisson structure. This method of construction has the advantage that the natural quotient map is Poisson but the drawback that it is difficult to explicitly calculate examples. Also, it is not clear that there are torus actions on the symplectic leaves.

It is important to note that the stability subgroup K' is not necessarily a Poisson Lie subgroup of (U, π_U), even though the projection of π_U defines a (U, π_U)-homogeneous Poisson structure on U/K'. Using results of Lu and Weintstein ([10]), the authors Khoroshkin, Radul, and Rubtsov proved the existence in the Hermitian symmetric case of a parabolic subgroup P of G such that

$U \cap P$ is a Poisson Lie subgroup of (U, π_U). With X presented as $U/(U \cap P)$, the natural projection of π_U to $U/(U \cap P)$ defines another (U, π_U)-homogeneous Poisson structure on X. This structure will be denoted π_{PL} as in [8]. Khoroshkin, Radul, and Rubtsov proved that the Schouten bracket $[\pi_{PL}, \pi_{KKS}]$ vanishes. Thus, $\pi_{PL} + \lambda \pi_{KKS}$ is the one parameter family of (U, π_U)-homogeneous Poisson structures on the Hermitian symmetric space X.

This paper is organized as follows. In section 2 it is shown that each $\mathfrak{u}$-compatible triangular decomposition of $\mathfrak{g}$ determines a Poisson Lie group structure π_U on U. The Evens-Lu construction is reviewed producing a (U, π_U)-homogeneous Poisson structure on X. In section 3, the Birkhoff decomposition of X is reviewed and it is proven that one can choose a basepoint in X in such a way that the corresponding Birkhoff decomposition aligns with the symplectic foliation of X. By reinterpreting the results on the Birkhoff decomposition of X in [11], the leaves of the symplectic foliation are characterized and tori acting on the symplectic leaves are determined. In section 4 it is shown that the tori act in a Hamiltonian fashion and the momentum maps are computed. Section 5 contains a finer discussion for the cases when X is a compact Lie group or the involution θ is an inner automorphism. Finally, in section 6, local formulas for the Evens-Lu Poisson structure are displayed. This is done for the complex Grassmannian, complex projective space, and the compact Lie group SU(2).

2. Poisson Geometry

Let X be a simply connected compact symmetric space. For simplicity, further assume that X is irreducible. Let U be the universal covering group of the identity component of the isometry group of X. The complexification of U will be denoted G with corresponding Lie algebra $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$.

Each choice of a $\mathfrak{u}$-compatible triangular decomposition of $\mathfrak{g}$ determines a Poisson Lie group structure on U as follows. Fix a $\mathfrak{u}$-compatible triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$. This means that: $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; a set of positive roots for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ has been chosen; $\mathfrak{n}_\pm$ is the direct sum of the positive (respectively negative) root spaces; and $-(\mathfrak{n}_\pm)^* = \mathfrak{n}_\mp$ where $-(\cdot)^*$ denotes the Cartan involution selecting $\mathfrak{u}$ in $\mathfrak{g}$. The sum $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ is a direct sum of vector spaces. As every sum of vector spaces that will need to be written down in this paper will be direct, the notation $+$ will be used instead of the more cumbersome $\oplus$ to denote direct sum. Set $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{u}$ and $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$. Define a $\mathbb{C}$-linear transformation $\mathcal{H} \colon \mathfrak{g} \to \mathfrak{g}$ relative to the given triangular decomposition by

$$\mathcal{H}(Z_- + Z_{\mathfrak{h}} + Z_+) = -iZ_- + iZ_+ \tag{1}$$

for each $Z = Z_- + Z_{\mathfrak{h}} + Z_+ \in \mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$. The real subspace $\mathfrak{u}$ of $\mathfrak{g}$ is stable under $\mathcal{H}$. Thus $\mathcal{H}$ is the complexification of a $\mathbb{R}$-linear transformation $\mathcal{H}_{\mathbb{R}} \colon \mathfrak{u} \to \mathfrak{u}$ which is skew-symmetric relative to the Killing form on $\mathfrak{u}$. Using the Killing form, denoted $\langle \cdot, \cdot \rangle$, identify the dual of $\mathfrak{u}$ with $\mathfrak{u}$ itself and then view $\mathcal{H}_{\mathbb{R}}$ as an element of $\mathfrak{u} \wedge \mathfrak{u}$. The bivector field

$$\pi_U\big|_g = r_{g*}\mathcal{H}_{\mathbb{R}} - \ell_{g*}\mathcal{H}_{\mathbb{R}} \tag{2}$$

defines a Poisson Lie group structure on U. (Here ℓ_g and r_g denote left and right translation by $g \in U$). The Lie algebra structure induced by π_U on the dual of $\mathfrak{u}$ is isomorphic to the real Lie algebra $\mathfrak{n}_- + \mathfrak{h}_\mathbb{R}$. The identification is given by the imaginary part of the Killing form. The double of this Lie bialgebra can then be identified with $\mathfrak{g}$ (regarded as a real Lie algebra) via the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h}_\mathbb{R} + \mathfrak{u}$. The Poisson Lie group structure π_U corresponds to the Manin triple $(\mathfrak{g}, \mathfrak{u}, \mathfrak{n}_- + \mathfrak{h}_\mathbb{R})$. Following terminology common in the literature, the Poisson structure in (2) will be referred to as a *standard Poisson Lie group structure* on U.

More often, this Poisson structure is presented in the literature in terms of a basis. For each positive root α of the action of $\mathfrak{h}$ on $\mathfrak{g}$ choose a root vector E_α such that $\langle E_\alpha, -(E_\alpha)^* \rangle = -1$. The $\mathfrak{u}$-compatibility of the triangular decomposition implies that E_α^* is a root vector for $-\alpha$. Set $E_{-\alpha} = E_\alpha^*$. Then $X_\alpha = E_\alpha - E_{-\alpha}$ and $Y_\alpha = i(E_\alpha + E_{-\alpha})$ are in $\mathfrak{u}$ for each positive root α and $\mathcal{H}_\mathbb{R} = \frac{1}{2}\sum_{\alpha>0} X_\alpha \wedge Y_\alpha$. It is known that the Schouten bracket $[\mathcal{H}_\mathbb{R}, \mathcal{H}_\mathbb{R}] \in \bigwedge^3 \mathfrak{u}$ is $\mathrm{Ad}(U)$-invariant (cf. [10]) and thus, by a theorem of Drinfeld (cf. Proposition 10.13 in [12]), the bivector field π_U in (2) defines a Poisson Lie group structure on the compact group U.

For the remainder of this paper, fix a triangular decomposition of $\mathfrak{g}$ and thus a standard Poisson Lie group structure π_U on U.

Write $\mathfrak{d}$ for $\mathfrak{g}$ regarded as a real Lie algebra. The imaginary part of the Killing form for $\mathfrak{g}$, denoted $\mathrm{Im}\langle \cdot, \cdot \rangle$, gives a real symmetric bilinear form on $\mathfrak{d}$. A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ is said to be *Lagrangian* with respect to $\mathrm{Im}\langle \cdot, \cdot \rangle$, if $2\dim_\mathbb{R} \mathfrak{l} = \dim_\mathbb{R} \mathfrak{g}$, and $\mathrm{Im}\langle a, b \rangle = 0$ for all $a, b \in \mathfrak{l}$. The set of all Lie subalgebras of $\mathfrak{d}$ which are Lagrangian with respect to $\mathrm{Im}\langle \cdot, \cdot \rangle$ will be denoted $\mathcal{L}(\mathfrak{d})$. This is naturally a subvariety of the real Grassmannian $Gr(d, \mathfrak{d})$ of d-dimensional subspaces of $\mathfrak{d}$, where $2d = \dim_\mathbb{R} \mathfrak{d}$. The adjoint action of G on $\mathfrak{g}$ induces an action of G (and therefore any subgroup of G) on $\mathcal{L}(\mathfrak{d})$. Each U-orbit in $\mathcal{L}(\mathfrak{d})$ is smooth (Theorem 1.1 part 2) in [3]).

Using the data defining π_U, Evens and Lu ([3]) construct a smooth bivector field Π on $Gr(d, \mathfrak{d})$ with the property that the Schouten bracket $[\Pi, \Pi]$ vanishes at each point of the subvariety $\mathcal{L}(\mathfrak{d})$. Furthermore, they show that Π is tangent to each U-orbit in $\mathcal{L}(\mathfrak{d})$, so that each U-orbit is a Poisson manifold. The construction is carried out in such a way that each U-orbit in $\mathcal{L}(\mathfrak{d})$ becomes a (U, π_U)-homogenous Poisson space.

Each choice of basepoint in X determines a model point in $\mathcal{L}(\mathfrak{d})$ as follows. Let K be the stability subgroup in U of a basepoint in X. The subgroup K is closed and is the fixed point set of an involution θ of U. Let θ also denote the complex extension of the involution from U to all of G and write g^θ for $\theta(g)$. The Cartan involution of G fixing U will be denoted $g \mapsto g^{-*}$. Since $(\cdot)^{-1}$, $(\cdot)^{-*}$, and θ all commute, the practice of writing θ as a superscript will not cause confusion. Write G_0 for the connected subgroup of G which is fixed by the involution $g \mapsto g^\sigma = g^{-*\theta}$. The intersection of U and G_0 in G is K and the Lie algebra of G_0, denoted $\mathfrak{g}_0$, is a real form of $\mathfrak{g}$ and thus a Lagrangian subalgebra of $\mathfrak{d}$. The coset space U/K is a finite sheeted covering of the U orbit through $\mathfrak{g}_0$ in $\mathcal{L}(\mathfrak{d})$. This is why $\mathfrak{g}_0$ is called a model point for $X \simeq U/K$ in $\mathcal{L}(\mathfrak{d})$.

The diagram of groups and Lie algebras shown in figure 1 lists this information for reference. The upward arrows are inclusions in both diagrams. In the group diagram the quotients are listed for each leg. Each quotient is also a

$$\begin{tikzcd}
 & \mathfrak{g} & \\
\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p} \arrow[ur] & & \mathfrak{u} = \mathfrak{k} + i\mathfrak{p} \arrow[ul] \\
 & \mathfrak{k} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G & \\
G_0 \arrow[ur, "G/G_0"] & & U \arrow[ul, "G/U"'] \\
 & K \arrow[ul, "G_0/K"] \arrow[ur, "U/K"'] &
\end{tikzcd}$$

Figure 1: Algebras, groups, and quotients.

symmetric space. The quotient G_0/K is a model for the non-compact symmetric space dual to X presented as U/K. Also shown in the diagram is the decomposition of the Lie algebras $\mathfrak{u}$ and $\mathfrak{g}_0$ into the eigenspaces of θ as $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. At times in this paper it will be convenient to write X^g for $\mathrm{Ad}(g)(X)$ to compactify notation. Unfortunately, the adjoint action of g does not necessarily commute with the involutions which are also being written as superscripts. Thus one notation or the other will be used at different points in the paper depending on the situation.

The Poisson structure Π on the orbit $U \cdot \mathfrak{g}_0$ may be lifted to a Poisson structure π on X and it is this structure, making X into a (U, π_U)-homogeneous Poisson space, that is of interest in this paper. It will be referred to as the *Evens-Lu Poisson structure* on X. The Evens-Lu construction will now be reviewed and an expression will be derived for π amenable to the discussion in later sections.

Let $(\cdot,\cdot)$ denote the non-degenerate $\mathbb{R}$-bilinear form on $\mathfrak{d} = (\mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}}) + \mathfrak{u}$ defined by

$$(\xi_1 + x_1, \xi_2 + x_2) = \mathrm{Im}\langle \xi_1, x_2\rangle + \mathrm{Im}\langle \xi_2, x_1\rangle$$

for each $\xi_k \in (\mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}})$ and $x_k \in \mathfrak{u}$, $k = 1, 2$. Identify the dual and double dual of $\mathfrak{d}$ with $\mathfrak{d}$ itself using this non-degenerate pairing. This allows one to define an element of $\bigwedge^2 \mathfrak{d}$ by its action on a pair of elements of $\mathfrak{d}$. Evens and Lu define $R \in \bigwedge^2 \mathfrak{d}$ by

$$R(\xi_1 + x_1, \xi_2 + x_2) = \mathrm{Im}\langle \xi_2, x_1\rangle - \mathrm{Im}\langle \xi_1, x_2\rangle \tag{3}$$

and use it to generate a bivector field on the Grassmannian of d-dimensional subspaces of $\mathfrak{d}$. The adjoint action of G on $\mathfrak{g}$ induces a $\mathbb{R}$-Lie algebra anti-homomorphism

$$\kappa\colon \mathfrak{d} \to \Gamma(TGr(d, \mathfrak{d}))$$

whose multi-linear extension $\bigwedge^{\cdot} \mathfrak{d} \to \Gamma(\bigwedge^{\cdot} TGr(d, \mathfrak{d}))$ will also be denoted by κ. The bivector field Π is defined by $\Pi = \frac{1}{2}\kappa(R)$.

To do calculations, identify $T(U/K)$ with $U \times_K i\mathfrak{p}$ using right translation. Further, identify the dual of $i\mathfrak{p}$ with $i\mathfrak{p}$ itself using the Killing form so that $T^*(U/K)$ may also be represented by $U \times_K i\mathfrak{p}$. In this setting, the action of π on a pair of cotangent vectors represented by classes $[u, X]$ and $[u, Y]$ in $U \times_K i\mathfrak{p}$ may be computed by

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y\rangle$$

where $\Omega_u\colon i\mathfrak{p} \to i\mathfrak{p}$ is a skew-symmetric $\mathbb{R}$-linear transformation which is K-equivariant in its dependence on u.

Theorem 2.1. *The Evens-Lu Poisson bivector* π *can be expressed as*

$$\pi([u,X],[u,Y]) = \langle \Omega_u X, Y\rangle$$

where $X, Y \in i\mathfrak{p}$*, the transformation* $\Omega_u\colon i\mathfrak{p} \to i\mathfrak{p}$ *is given by*

$$\Omega_u(X) = \{\mathrm{Ad}(u^{-1}) \circ \mathcal{H} \circ \mathrm{Ad}(u)(X)\}_{i\mathfrak{p}} \tag{4}$$

and $\{\cdot\}_{i\mathfrak{p}}$ *denotes the projection to* $i\mathfrak{p}$ *along the decomposition* $\mathfrak{g} = i\mathfrak{u} + \mathfrak{k} + i\mathfrak{p}$.

Proof. Let $\mathrm{pr}_{\mathfrak{u}}\colon \mathfrak{g} \to \mathfrak{u}$ denote the projection onto $\mathfrak{u}$ along the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}} + \mathfrak{u}$. In terms of the triangular decomposition $Z = Z_- + Z_{\mathfrak{h}} + Z_+ \in \mathfrak{g}$, this projection is given by the formula

$$\mathrm{pr}_{\mathfrak{u}}(Z) = -(Z_+)^* + Z_{\mathfrak{t}} + Z_+.$$

The composition $\mathfrak{u} \xrightarrow{i} i\mathfrak{u} \xrightarrow{\mathrm{pr}_{\mathfrak{u}}} \mathfrak{u}$ agrees with the transformation $\mathcal{H}$ when restricted to $\mathfrak{u}$ (recall the definition of $\mathcal{H}$ in (1)). Indeed, each $Z \in \mathfrak{u}$ satisfies $Z_- = -(Z_+)^*$ and $Z_{\mathfrak{h}} = Z_{\mathfrak{t}}$; hence $(iZ)_{\mathfrak{t}} = 0$, and $-((iZ)_+)^* = i(Z_+)^* = iZ_-$. It follows that

$$\mathrm{pr}_{\mathfrak{u}}(iZ) = -iZ_- + iZ_+ = \mathcal{H}(Z). \tag{5}$$

In equation (4) the equivalence class $[u,X]$ represents a linear functional on the tangent space to U/K at uK. Using the covering map to identify this space with the tangent space to the U-orbit through $\mathfrak{g}_0$ at $u\cdot\mathfrak{g}_0$, each such tangent vector corresponds to an element $\chi \in \mathfrak{u}/\mathfrak{k}^u$ via the map $\chi \mapsto \frac{d}{dt}\big|_{t=0} e^{t\chi}u\cdot\mathfrak{g}_0$. The corresponding class in $U\times_K i\mathfrak{p}$ is $[u,\chi^{u^{-1}}]$. Moreover, the action of $[u,X]$ as a linear functional on $[u,\chi^{u^{-1}}]$ is given by

$$[u,X]\left([u,\chi^{u^{-1}}]\right) = \langle X, \chi^{u^{-1}}\rangle = \mathrm{Im}\langle iX^u, \chi\rangle.$$

By the Evens-Lu construction

$$\pi([u,X],[u,Y]) = \frac{1}{2}\,\kappa(R)|_{uK}\,([u,X],[u,Y]) = \frac{1}{2}R(iX^u, iY^u).$$

Combining the definition of R in (3), the result in (5), and skew-symmetry of $\mathcal{H}$, it follows that

$$\begin{aligned} R(iX^u, iY^u) &= \mathrm{Im}\langle \mathrm{pr}_{\mathfrak{u}}(iX^u), iY^u\rangle - \mathrm{Im}\langle iX^u, \mathrm{pr}_{\mathfrak{u}}(iY^u)\rangle \\ &= \langle \mathcal{H}(X^u), Y^u\rangle - \langle X^u, \mathcal{H}(Y^u)\rangle \\ &= 2\langle (\mathcal{H}(X^u))^{u^{-1}}, Y\rangle. \end{aligned}$$

This completes the proof of the theorem. ■

¿From this formula, the connection with triangular factorization is evident and a group theoretic interpretation of the symplectic foliation can be given. At the group level write the corresponding Iwasawa decomposition for G as

$$\begin{array}{rcl} G & \simeq & N^- \times A \times U. \\ g & \mapsto & (\mathbf{l}(g), \mathbf{a}(g), \mathbf{u}(g)) \end{array} \tag{6}$$

where $A = \exp(\mathfrak{h}_{\mathbb{R}})$. There is a natural action of G (and therefore any subgroup of G) on U coming from the identification of U with the right coset space $N^-A\backslash G$.

$$\begin{array}{rcl} U \times G & \to & U \\ u\cdot g & \mapsto & \mathbf{u}(ug) \end{array}$$

As observed in [10], the symplectic leaves of the Poisson Lie group structure π_U on U are precisely the N^-A orbits in U.

Proposition 2.2. *For the Evens-Lu Poisson structure π the leaves of the symplectic foliation are the projections of the G_0-orbits in U to U/K.*

Proof. The symplectic foliation of U/K is generated by the distribution in $T(U/K)$ which is image of the map $\pi^{\#}\colon T^*(U/K) \to T(U/K)$ given by $\pi^{\#}(\alpha) = \pi(\alpha, \cdot)$. In terms of the identifications $T^*(U/K) \simeq U \times_K i\mathfrak{p} \simeq T(U/K)$,

$$\pi^{\sharp}([u, X]) = [u, \{(\mathcal{H}(X^u))^{u^{-1}}\}_{i\mathfrak{p}}] = [u, \{(\mathrm{pr}_{\mathfrak{u}}(iX^u))^{u^{-1}}\}_{i\mathfrak{p}}].$$

Fix $u \in U$. The map $G_0 \to U$ given by $g_0 \mapsto \mathbf{u}(ug_0)$ is equivariant for the right K-action on both G_0 and U and thus descends to a map $G_0/K \to U/K$. Now, suppose that $iX \in \mathfrak{p}$, so that $X \in i\mathfrak{p}$. Consider the curve

$$t \mapsto u \cdot e^{tiX} = \mathbf{u}(ue^{tiX}) = \mathbf{u}(e^{tiX^u}u).$$

Differentiating at $t = 0$, one obtains the tangent vector $(r_u)_*(\mathrm{pr}_{\mathfrak{u}}(iX^u))$. It follows that the distribution tangent to the projection of the G_0-orbit at uK and is spanned by vectors of the form

$$\kappa(\mathrm{pr}_{\mathfrak{u}}(iX^u))|_{uK} = [u, \{(\mathrm{pr}_{\mathfrak{u}}(iX^u))^{u^{-1}}\}_{i\mathfrak{p}}] \tag{7}$$

which is exactly the image of $\pi^{\#}$. ■

This proposition has been established in several contexts for this Poisson structure. Using their construction, Foth and Lu give an alternate proof using general principles (cf. Proposition 1.1 in [5]).

3. Symplectic Leaves and Triangular Factorization

Let G be a simply connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$, and a set of positive roots for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Let $\mathfrak{n}_{\pm}$ denote the sum of the positive (resp. negative) root spaces. This data gives a triangular decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$$

Set $H = \exp(\mathfrak{h})$, and $N^{\pm} = \exp(\mathfrak{n}_{\pm})$. Corresponding to this decomposition is a Birkhoff decomposition (a.k.a. triangular decomposition a.k.a LDU decomposition) of the group

$$G = \coprod_{w \in W} \Sigma^G_{\mathbf{w}}, \text{ where } \Sigma^G_w = N^- wHN^+$$

and w is an element of the Weyl group $W = N_G(H)/H$. Each component Σ^G_w is a manifold diffeomorphic to $N^- \cap wN^-w^{-1} \times H \times N^+$. The codimension of Σ^G_w increases with the length of w in the Weyl group and Σ^G_1 is a Zariski open subset of G. Each element in Σ^G_w can be factored as a product of an element of N^-, an element of $w \subset N_G(H)/H$, an element of H, and an element of N^+. For $\mathrm{SL}(n, \mathbb{C})$ in an appropriate representation this would correspond to the factorization of an $n \times n$ complex matrix of determinant one into a product of a lower triangular unipotent matrix, a unitary permutation matrix, a diagonal matrix of determinant one, and an upper triangular unipotent matrix. Each element of Σ_1^G admits a unique factorization of this form. If w is a non-trivial element of the Weyl group then the elements in Σ_w^G admit several such factorizations. Further conditions are required to guarantee uniqueness in those cases.

This section concerns a generalization of this decomposition for symmetric spaces and, in particular, its relationship to the Poisson geometry of (X,π). Recall the setup considered in this paper: X is a compact, connected, and simply connected Riemannian symmetric space; U is the universal covering group of the identity component of the isometry group; G is the complexification of U. The Poisson Lie group structure π_U on U was defined by a triangular decomposition of $\mathfrak{g}$ which had the additional property of being $\mathfrak{u}$-compatible, i.e. $-(\mathfrak{n}_\pm)^* = \mathfrak{n}_\mp$.

For each choice of basepoint in X, Cartan defined an embedding of X into $U \subset G$.

$$\begin{array}{rcccccl} \phi\colon & U/K & \to & U & \hookrightarrow & G \\ & uK & \mapsto & uu^{-\theta} & & \end{array} \tag{8}$$

In (8), θ is the involution fixing the stability subgroup of the basepoint in X. If U is viewed as a Riemannian manifold with the metric induced from the Killing form, then this map gives a totally geodesic embedding of X into $U \subset G$. The intersection of this image with the decomposition of G induces a decomposition of X which will be called a *Birkhoff decomposition of* X. Such a decomposition depends on the choice of basepoint in X and the triangular decomposition of $\mathfrak{g}$. In this paper, the triangular decomposition of $\mathfrak{g}$ is regarded as fixed, having defined π_U.

Lemma 3.1. *There exists a basepoint in X such that the given triangular decomposition of $\mathfrak{g}$ is stable with respect to the involution selecting the stability subgroup of x in U.*

Proof. Fix a point $x \in X$. This determines the data: K, the stability subgroup of x; θ, the involution of U fixing K; G_0, a non-compact real form of G; $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, a decomposition of $\mathfrak{u}$ into the eigenspaces of θ. The Lie algebra of G_0 is $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$. First, it will be shown that a triangular decomposition of $\mathfrak{g}$, which is both $\mathfrak{u}$-compatible and θ-stable, exists.

Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{h}_0$ denote the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. Theorem 6.60 of [9] shows that $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, where $\mathfrak{a}_0 \subset \mathfrak{p}$. Thus, the subalgebras

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0,\ \mathfrak{t} = \mathfrak{t}_0 + i\mathfrak{a}_0, \text{ and } \mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$$

are Cartan subalgebras of $\mathfrak{g}_0$, $\mathfrak{u}$, and $\mathfrak{g}$ respectively. Selecting a Weyl chamber in $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$, chooses a set of positive roots for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, and thus determines a triangular decomposition for $\mathfrak{g}$. For each choice the resulting decomposition will be $\mathfrak{u}$-compatible. The Cartan subalgebra $\mathfrak{h}$ is θ-stable by construction. However, θ will stabilize the positive root spaces only if the chosen Weyl chamber contains an element of $i\mathfrak{t}_0$. The fact that $i\mathfrak{t}_0$ contains a regular element is equivalent to the fact that $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Thus, a $\mathfrak{u}$-compatible θ-stable triangular decomposition of $\mathfrak{g}$ exists.

To finish the proof, note that all $\mathfrak{u}$-compatible triangular decompositions of $\mathfrak{g}$ are U-conjugate. Therefore, one may conjugate the constructed triangular decomposition to the given $\mathfrak{u}$-compatible triangular decomposition of $\mathfrak{g}$ by an element $u \in U$. Conjugating the stability subgroup K by the same element selects a new basepoint in X with corresponding involution $\theta' = \mathrm{Ad}(u) \circ \theta \circ \mathrm{Ad}(u^{-1})$. The given $\mathfrak{u}$-compatible decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ is stable with respect to the involution θ'. ∎

For the remainder of this section, fix a presentation of the symmetric space X as U/K such that the triangular decomposition of $\mathfrak{g}$ defining π_U is θ-stable. Under the assumption of θ-stability, Pickrell was able to characterize the connected components of the resulting Birkhoff decomposition of X (cf. [11]). It is through reinterpretation of these results that an explicit description of the geometry of the Evens-Lu Poisson structure on X can be given. For the convenience of the reader, the same notation as that in [11] has been adopted in this paper.

The first observation is that $\phi(U/K)$ does not necessarily intersect each component of the Birkhoff decomposition of G. In fact, $\phi(U/K) \cap \Sigma_w^G$ is non-empty if and only if $\phi(U/K) \cap w$ is non-empty where $w \subset N_U(T)$ represents the Weyl group element $w \in W = N_G(H) \simeq N_U(T)$ (cf. Theorem 2 (a) combined with Theorem 1 (a) in [11]).

One should think of a Birkhoff decomposition of X as consisting of a number of layers indexed by the possible elements w. In this paper, *the layer corresponding to w*, will refer to the set $\phi(U/K) \cap \Sigma_w^G$, or its equivalent in U/K or X when the context is clear. The reader should understand that when this terminology is used they are to implicitly assume that w is such that $\phi(U/K) \cap w$ is non-empty.

Each layer consists of a number of connected components. For a given $w \in N(T)/T$ the connected components of the layer corresponding to w are indexed by the set

$$\{\mathbf{w} \in \phi(U/K) \cap w\}/T \tag{9}$$

where T acts on the right by $\mathbf{w} \cdot t = t^{-1}\mathbf{w}t^{\theta}$ (cf. Theorem 2 (c) in [11]). This characterization uses θ-stability. The elements $\mathbf{w} \in \phi(U/K) \cap T$ are the images under the Cartan embedding of the preferred basepoints in X in the sense of Lemma 3.1. Given an element $\mathbf{w}$ in the layer corresponding to w, write $\Sigma_{\mathbf{w}}^{\phi(U/K)}$ for the connected component of $\phi(U/K) \cap \Sigma_w^G$ containing $\mathbf{w}$.

In Proposition 2.2 it was shown that the symplectic leaves of the Evens-Lu Poisson structure on $X \simeq U/K$ are the projections modulo K of the G_0-orbits in U. The right action of G_0 on U was that induced by the identification $U \simeq N^-A\backslash G$ coming from the Iwasawa decomposition $G \simeq N^-AU$. By combining this with an action of the torus $T = \exp(\mathfrak{h} \cap \mathfrak{u})$ as

$$\begin{array}{ccc} U \times (T \times G_0) & \to & U \\ u \cdot (t, g_0) & \mapsto & t^{-1}\mathbf{u}(ug_0) \end{array}$$

Pickrell was able to characterize each component $\Sigma_{\mathbf{w}}^{\phi}(U/K)$. The following proposition is Theorem 4 (a) in [11].

Proposition 3.2. *Consider the layer of the Birkhoff decomposition of X corresponding to $w \in N_U(T)/T$. Let $\mathbf{w} \in \phi(U/K) \cap w$ and fix a choice of $\mathbf{w}_1 \in U$ such that $\phi(\mathbf{w}_1 K) = \mathbf{w}$. The map*

$$\begin{array}{ccc} T \times G_0 & \to & \phi(U/K) \cap \Sigma_{\mathbf{w}}^{\phi(U/K)} \\ (t, g_0) & \mapsto & \phi(t^{-1}\mathbf{u}(\mathbf{w}_1 g_0)) \end{array}$$

is surjective and induces a diffeomorphism

$$T \times_{\exp(\ker\{\mathrm{Ad}(w)\theta|_{\mathfrak{t}}-1\})} R\backslash G_0/K \to \Sigma_{\mathbf{w}}^{\phi(U/K)}$$

where $R = (N^- A)^{\mathbf{w}_1^{-1}} \cap G_0$ *is a contractible subgroup of* G_0 *and*

$$\lambda \in \exp(\ker\{\mathrm{Ad}(w)\theta|_{\mathfrak{t}} - 1\})$$

is identified with the pair $(\lambda, \lambda^{\mathbf{w}_1^{-1}})$.

The proof is fairly involved and will not be reproduced here. The key point is that the theorem can be reinterpreted as characterizing the symplectic leaves of the Evens-Lu Poisson structure. The projection modulo K of the G_0-orbit through $\mathbf{w}_1$ maps to a sub-manifold of $\Sigma_{\mathbf{w}}^{\phi(U/K)}$ passing through $\mathbf{w}$. This shows that with the appropriate choice of basepoint in X, the symplectic foliation of the Evens-Lu Poisson structure aligns with the Birkhoff decomposition of X.

Corollary 3.3. *Each connected component of the layer of the Birkhoff decomposition corresponding to* w *is foliated by contractible symplectic leaves each diffeomorphic to the double coset space* $R\backslash G_0/K$.

More can be seen from Proposition 3.2. Each leaf admits a natural torus action. The acting torus is a subgroup of the torus T determined by the layer of the Birkhoff decomposition in which the leaf is contained.

Corollary 3.4. *Each symplectic leaf foliating the layer corresponding to* w *is acted on by the torus*

$$T_w = \exp(\ker\{\mathrm{Ad}(w)\theta|_{\mathfrak{t}} - 1\}) \subset T.$$

Moreover, the coset $\mathbf{w}_1 K$ *represents the unique fixed point for the* T_w*-action in the symplectic leaf through* $\mathbf{w}_1 K$.

In the next section, it will be shown that the action of T_w on each symplectic leaf foliating the layer indexed by w is Hamiltonian. The following proposition and theorem will be needed in order to compute the momentum map. First, notice that an element g in $\phi(U/K) \cap G$ first satisfies the equation $g^{-1} = g^\theta$ and the further condition that $g \in U$ so that $g^{-1} = g^*$. In fact $\phi(U/K)$ can be realized as the identity component of the set $\{g^* = g^\theta\} \cap U$ (cf. Theorem 1 (a) in [11]). The following proposition is Theorem 2 (d) in [11].

Proposition 3.5. *Fix* $w \in W$. *Suppose* $\mathbf{w} \in w \subset N_G(H)$ *with* $\mathbf{w}^* = \mathbf{w}^\theta$. *For brevity, write* $D_{\mathbf{w}}$ *for the set*

$$\{(h, L) \in H \times (N^- \cap (N^+)^w) \colon \theta(h^{w^{-1}}) = h^*, (\theta(L^{-*}))^{\mathbf{w}h} = L^{-1}\}.$$

The map

$$\begin{array}{rcl} (N^- \cap (N^-)^w) \times D_{\mathbf{w}} & \to & \{g^* = g^\theta\} \cap \Sigma_w^G \\ (l, (h, L)) & \mapsto & lL^{-1}\mathbf{w}h(lL^{-1})^{*\theta} \end{array}$$

is a diffeomorphism onto the connected component of $\{g^* = g^\theta\} \cap \Sigma_w^G$ *containing* $\mathbf{w}$.

Applied to $\phi(U/K) \cap \Sigma_w^G$, this proposition provides further conditions to guarantee the uniqueness of the triangular factorization of elements in the layers indexed by non-trivial elements w. The following corollary reinterprets this result.

Corollary 3.6. *Let* $\mathbf{w} \in \phi(U/K) \cap w$ *be an element in the layer corresponding to* w*. Let* $S_{\mathbf{w}}$ *denote the symplectic leaf of* (X, π) *for which* $\phi(S_{\mathbf{w}})$ *passes through* $\mathbf{w}$ *in* $\phi(U/K) \cap w$*. Each element in* $\phi(S_{\mathbf{w}})$ *can be factored as* $\ell \mathbf{w} h \ell^{*\theta}$ *where* $\ell \in N^-$*, and* $h \in \exp(\ker\{Ad(w)\sigma|_{\mathfrak{h}} + 1\})$*. Furthermore, the magnitude* $|h| = \sqrt{h^* h}$ *is a smooth function on the leaf.*

Proof. Existence of the factorization is guaranteed by Proposition 3.5 with the observation that h satisfies the condition $\theta(h^{w^{-1}}) = h^*$ if and only if $h \in \exp(\ker\{\mathrm{Ad}(w)\sigma|_{\mathfrak{h}} + 1\})$. Note that Proposition 3.5 also guarantees that h is a smooth function on the leaf and thus $|h| = \sqrt{h^* h} \in \exp(\mathfrak{h}_{\mathbb{R}})$ is smooth as well. ∎

4. The Momentum Map

Given the amount of notation introduced and the number of objects and parameters involved it seems apropos to include a brief summary of the setup and results of the paper thus far. With a compact, connected, and simply connected symmetric space X comes the data: U, the universal cover of the identity component of the isometry group of X; G, the complexification of U; the Cartan involution $g \mapsto g^{-*}$ selecting U as a compact real form of G. In section 2 a $\mathfrak{u}$-compatible (i.e., $-(\mathfrak{n}_{\pm})^* = \mathfrak{n}_{\mp}$) triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ was fixed, which determined a standard Poisson Lie group structure π_U on U. The Evens-Lu construction produced a (U, π_U)-homogeneous Poisson structure π on X. In section 3 a basepoint was chosen in X whose corresponding involution θ stabilized the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ (i.e., $\theta(\mathfrak{n}_{\pm}) = \mathfrak{n}_{\pm}$, and $\theta(\mathfrak{h}) = \mathfrak{h}$). This choice determined: a presentation of X as a coset space U/K; an embedding $\phi\colon U/K \to U$ of X into $U \subset G$. The intersection of the image of this map with the Birkhoff decomposition of G corresponding to the fixed triangular decomposition of $\mathfrak{g}$ induced a Birkhoff decomposition of X.

In section 3 it was shown that the layers of the Birkhoff decomposition of X are foliated by symplectic leaves, each of which is contractible. There is a bijection between $\cup_{w \in W}(\phi(U/K) \cap w)$ and the symplectic leaves in X assigning a leaf to the element its ϕ-image passes through. Given $\mathbf{w} \in \phi(U/K) \cap \Sigma_w^G$ the corresponding leaf was denoted $S_{\mathbf{w}}$. The torus $T = \exp(\mathfrak{h} \cap \mathfrak{u})$ acts from the right on U/K via $uK \cdot t = t^{-1} uK$. The sub-torus $T_w = \exp(\ker\{\mathrm{Ad}(w)\theta|_{\mathfrak{t}} - 1\})$ preserves the symplectic leaf $S_{\mathbf{w}}$ with a unique fixed point. Each element of $\phi(S_{\mathbf{w}})$ can be factored as $\ell \mathbf{w} h \ell^{*\theta}$ where $\ell \in N^-$ and h is uniquely determined. The magnitude of h, denoted $|h| = \sqrt{h^* h}$ is a smooth function on $S_{\mathbf{w}}$.

Theorem 4.1. *The action of the torus* T_w *on the symplectic leaf* $S_{\mathbf{w}}$ *is Hamiltonian with momentum map*

$$\begin{array}{rccc} \mu\colon & S_{\mathbf{w}} & \to & \mathfrak{t}_w^* \\ & uK & \mapsto & \langle \frac{1}{2} i\theta(\log|h|), \cdot \rangle \end{array}$$

where h *is the diagonal part of* $\phi(uK) = uu^{-\theta} \in \phi(S_{\mathbf{w}})$*.*

Proof. Let $X \in \mathfrak{t}_w$. Practically, this means that X satisfies the equation $\mathbf{w} X^{\theta} \mathbf{w}^{-1} = X$ for each $\mathbf{w} \in w$. The vector field $\tilde{X}$ induced by the action of T_w

on U/K is represented by the class $[u, \{-X^{u^{-1}}\}_{i\mathfrak{p}}]$ in $U \times_K i\mathfrak{p}$. It needs to be shown that $\mu_X \colon S_{\mathbf{w}} \to \mathbb{R}$ given by $\mu_X(uK) = \langle \frac{1}{2} i\theta(\log|h|), X\rangle$ is a Hamiltonian function for the vector field $\tilde{X}$. This will follow if $\pi^{\#}(d\mu_X) = \tilde{X}$.

Let $[u, Y] \in U \times_K i\mathfrak{p}$ be a tangent vector to U/K at uK. This tangent vector is represented by the curve $t \mapsto ue^{tY}K$ which passes through uK at $t = 0$. Decompose $\phi(uK) = uu^{-\theta}$ as $\ell\mathbf{w}h\ell^{*\theta}$ using Corollary 3.6. Then

$$ue^{tY}(ue^{tY})^{-\theta} \;=\; e^{2tY^u}uu^{-\theta} = e^{2tY^u}\ell\mathbf{w}h\ell^{*\theta} = \ell\mathbf{w}e^{2t(\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w}}h\ell^{*\theta}$$

and thus $\frac{d}{dt}\big|_{t=0} \log|h(ue^{tY}K)| = 2\operatorname{pr}_{\mathfrak{h}_{\mathbb{R}}}((\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w})$. With this calculation it follows that

$$\begin{aligned} d\mu_X([u,Y]) &= \langle i\theta(\operatorname{pr}_{\mathfrak{h}_{\mathbb{R}}}((\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w})), X\rangle = \langle Y, \operatorname{Ad}(u^{-1})((i\mathbf{w}X^\theta\mathbf{w}^{-1})^\ell)\rangle \\ &= \langle Y, \operatorname{Ad}(u^{-1})(iX^\ell)\rangle. \end{aligned}$$

Hence, $d\mu_X$ corresponds to the class $[u, \{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{i\mathfrak{p}}]$.

Assertion 1: $\{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{i\mathfrak{p}} = \frac{1}{2}\{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{\mathfrak{u}}$ where $\{\cdot\}_{\mathfrak{u}}$ denotes the projection to $\mathfrak{u}$ along the decomposition $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$.

Given $Z \in \mathfrak{g}$, the projection to $i\mathfrak{p}$ is given by

$$\{Z\}_{i\mathfrak{p}} = \frac{1}{4}\left(Z + Z^{*\theta} - (Z + Z^{*\theta})^*\right) = \frac{1}{2}\{Z + Z^{*\theta}\}_{\mathfrak{u}}. \tag{10}$$

The equation $uu^{-\theta} = \ell\mathbf{w}h\ell^{*\theta}$ implies that $u^{-\theta}\ell^\sigma = u^{-1}\ell\mathbf{w}h$. This equality gives

$$\begin{aligned} (\operatorname{Ad}(u^{-1})(iX^\ell))^{*\theta} &= u^{-\theta}\ell^\sigma iX^\theta\ell^{-\sigma}u^\theta & (11)\\ &= u^{-1}\ell\mathbf{w}iX^\theta\mathbf{w}^{-1}\ell^{-1}u & (12)\\ &= \operatorname{Ad}(u^{-1})(iX^\ell) & (13) \end{aligned}$$

where (12) follows from (11) after substituting for $u^{-\theta}\ell^\sigma$ and noting the fact that $h \in H$ trivially on iX^θ. The assertion follows.

In terms of the identifications of $T(U/K)$ and $T^*(U/K)$ with $U \times_K i\mathfrak{p}$ the map $\pi^{\#}$ is given by

$$[u, Y] \mapsto [u, \{(\mathcal{H}(Y^u))^{u^{-1}}\}_{i\mathfrak{p}}] \tag{14}$$

where $\mathcal{H}(Y) = -i(Y)_- + i(Y)_+$ is the transformation from (1). Using Assertion 1 and the fact that $\operatorname{Ad}(u)$ commutes with the projection to $\mathfrak{u}$ one can see that $\pi^{\#}(d\mu_X) = [u, \{\operatorname{Ad}(u^{-1}) \circ \mathcal{H}(\{iX^\ell\}_{\mathfrak{u}})\}_{i\mathfrak{p}}]$.

Assertion 2: $\mathcal{H}(\{iX^\ell\}_{\mathfrak{u}}) = \frac{1}{2}\{X^\ell\}_{\mathfrak{u}} - X$.

The key point is that $iX^\ell = (iX^\ell)_- + iX$ since $\ell \in N^-$ and $X \in H$. Hence,

$$2\{iX^\ell\}_{\mathfrak{u}} = \left((iX^\ell)_- - ((iX^\ell)_-)^*\right) = \left((\ell iX\ell^{-1})_- - (\ell^{-*}iX\ell^*)_+\right).$$

Applying $\mathcal{H}$ to this expression gives

$$\begin{aligned} 2\mathcal{H}(\{iX^\ell\}_{\mathfrak{u}}) &= (\ell X\ell^{-1})_- + (\ell^{-*}X\ell^*)_+ = \ell X\ell^{-1} - X + \ell^{-*}X\ell^* - X \\ &= \{X^\ell\}_{\mathfrak{u}} - 2X. \end{aligned}$$

This proves Assertion 2.

To finish the calculation of $\pi^{\#}(d\mu_X)$, observe that $\mathrm{Ad}(u^{-1})$ preserves $\mathfrak{u}$, and use Assertion 2 to conclude that

$$\pi^{\#}(d\mu_X) = [u, \frac{1}{2}\{\mathrm{Ad}(u^{-1})(X^{\ell})\}_{i\mathfrak{p}} - \{X^{u^{-1}}\}_{i\mathfrak{p}}].$$

The same calculation in (11) through (13) shows that

$$(\mathrm{Ad}(u^{-1})(X^{\ell}))^{*\theta} = -\mathrm{Ad}(u^{-1})(X^{\ell}).$$

Combining this observation with the formula in (10) yields that the $i\mathfrak{p}$ part of $\mathrm{Ad}(u^{-1})(X^{\ell})$ is zero. Thus $\pi^{\#}(d\mu_X) = [u, -\{X^{u^{-1}}\}_{i\mathfrak{p}}]$, which completes the proof of the theorem. ∎

5. Comments

This section addresses several special cases and sets the stage for the explicit examples in section 6.

Let K be a connected and simply connected compact Lie group. Fix a triangular decomposition

$$\mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{n}}_{-} + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_{+}. \tag{15}$$

Write $\mathcal{H}_{\mathfrak{k}}$ for the linear transformation in (1) relative to the decomposition (15). Trivialize the tangent bundle to K using right translation. This identifies TK with $K \times \mathfrak{k}$. Further identify $\mathfrak{k}$ will its dual using the Killing form so that T^*K is also identified with $K \times \mathfrak{k}$. Using the definition of π_K from (2), a short calculation yields the following formula.

Proposition 5.1. *The Lu-Weinstein Poisson Lie group structure π_K can be expressed by*

$$\pi_K((k,P),(k,Q)) = \langle(\mathcal{H}_{\mathfrak{k}} - \mathrm{Ad}(k) \circ \mathcal{H}_{\mathfrak{k}} \circ \mathrm{Ad}(k^{-1}))(P), Q\rangle. \tag{16}$$

for each $(k,P),(k,Q) \in K \times \mathfrak{k}$.

With this formula, one can see that the maximal torus $T = \exp(\mathfrak{t})$, $\mathfrak{t} = \tilde{\mathfrak{h}} \cap \mathfrak{k}$, is a Poisson Lie subgroup of K, as π_K vanishes identically there. Thus, the push-forward of π_K under the natural projection map $K \to K/T$ defines a (K, π_K)-homogeneous Poisson structure on the flag manifold K/T. The symplectic leaves of this induced structure are precisely the corresponding Bruhat cells in K/T because the symplectic leaves of π_K foliate the components of the Bruhat decomposition of K. This was established in [10].

When equipped with the invariant metric induced by the Killing form, K is also a Riemannian symmetric space. The isometries of K are given by either left or right translation by elements of K. In this case, $U = K \times K$, $G = K^{\mathbb{C}} \times K^{\mathbb{C}}$, and the Cartan involution selecting U in G is $(g_1, g_2) \mapsto (g_1^{-*}, g_2^{-*})$ where $g \mapsto g^{-*}$ is the Cartan involution selecting K inside of $K^{\mathbb{C}}$. The left action of $u = (k_1, k_2)$ on $k \in K$ is given by $(k_1, k_2) \cdot k = k_1 k k_2^{-1}$. The decomposition

$$\mathfrak{g} = \underbrace{(\tilde{\mathfrak{n}}_{-} \times \tilde{\mathfrak{n}}_{-})}_{\mathfrak{n}_{-}} + \underbrace{(\tilde{\mathfrak{h}} \times \tilde{\mathfrak{h}})}_{\mathfrak{h}} + \underbrace{(\tilde{\mathfrak{n}}_{+} \times \tilde{\mathfrak{n}}_{+})}_{\mathfrak{n}_{+}} \tag{17}$$

built using (15) is a $\mathfrak{u}$-compatible triangular decomposition of $\mathfrak{g}$.

Using the identity in K as a basepoint gives a presentation of K as the coset space U/Δ where the stability subgroup $\Delta = \{(k,k)\colon k \in K\}$ is the diagonal image of K in U. The involution θ fixing Δ in U is the automorphism which interchanges the two factors of U. The triangular decomposition in (17) is stable with respect to this outer automorphism.

The image of the Cartan embedding of U/Δ into U is the anti-diagonal image of K in U, namely $\phi(U/\Delta) = \{(k,k^{-1})\colon k \in K\} \subset U$. The corresponding Birkhoff decomposition of K as a symmetric space is identical to the decomposition of K induced by the Birkhoff decomposition of $K^{\mathbb{C}}$ with respect to $\mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$.

Theorem 5.2. *The Evens-Lu Poisson structure π on K can be expressed as*

$$\pi((k,P),(k,Q)) = \langle (\mathcal{H}_{\mathfrak{k}} + \mathrm{Ad}(k)\circ \mathcal{H}_{\mathfrak{k}} \circ \mathrm{Ad}(k^{-1}))(P), Q\rangle. \tag{18}$$

Proof. The isometry $\psi\colon U/\Delta \to K$ given by $(k_1,k_2)\Delta \mapsto k_1k_2^{-1}$ identifies U/Δ with K. Use right translation to identify the tangent bundle to U/Δ with $U \times_\Delta i\mathfrak{p}$ and TK with $K \times \mathfrak{k}$. Given $(k_1,k_2) \in U$ write $k = \psi(k_1,k_2) = k_1k_2^{-1}$. In this setting $i\mathfrak{p} = \{(X,-X)\colon X \in \mathfrak{k}\}$. A curve representing $(X,-X)$ in $i\mathfrak{p}$ passing through $(k_1,k_2)\Delta$ is given by $(k_1e^{tX}, k_2e^{-tX})\Delta$ and its image under ψ is $k_1e^{tX}e^{tX}k_2^{-1} = e^{2tX^{k_1}}k$. Thus, $\psi_*[(k_1,k_2),(X,-X)]$ is represented by $(k, 2X^{k_1}) \in K\times\mathfrak{k}$. Using the Killing form to identify the dual of $\mathfrak{k}$ with $\mathfrak{k}$, the cotangent bundle can also be represented by $K\times\mathfrak{k}$ using right translation. Then, the pull-back of a cotangent vector at k represented by (k,P) is given by the class $[(k_1,k_2),(P^{k_1^{-1}}, -P^{k_1^{-1}})]$. Let π_{EL} denote the Evens-Lu Poisson structure U/Δ. From Theorem 2.2 it follows that

$$(\psi_*\pi_{EL})((k,P),(k,Q)) = \langle \Omega_{(k_1,k_2)}(P^{k_1^{-1}}, -P^{k_1^{-1}}), (Q^{k_1^{-1}}, -Q^{k_1^{-1}})\rangle_{\mathfrak{u}} \tag{19}$$

where $\Omega_{(k_1,k_2)}(X,-X) = \{\mathrm{Ad}(k_1^{-1},k_2^{-1})\circ\mathcal{H}_{\mathfrak{k}\times\mathfrak{k}}\circ\mathrm{Ad}(k_1,k_2)(X,-X)\}_{i\mathfrak{p}}$. The result of applying the transformation $\Omega_{(k_1,k_2)}$ to $(P^{k_1^{-1}}, -P^{k_1^{-1}})$ is equivalent to the $i\mathfrak{p}$ part of

$$\mathrm{Ad}(k_1^{-1},k_2^{-1})\circ\mathcal{H}_{\mathfrak{k}\times\mathfrak{k}}(P,-P^{k^{-1}}) = ((\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}}, -(\mathcal{H}_{\mathfrak{k}}(P^{k^{-1}}))^{k_2^{-1}}). \tag{20}$$

Note that the left and right hand side of (20) are in $\mathfrak{u} = \mathfrak{k}\times\mathfrak{k}$. Temporarily denote the right hand side of (20) by Z. Since $Z\in\mathfrak{u}$, the projection to $i\mathfrak{p}$ is given by $\frac{1}{2}(Z - Z^\theta)$ or

$$\frac{1}{2}\left((\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k^{-1}}))^{k_2^{-1}}, -((\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k^{-1}}))^{k_2^{-1}})\right). \tag{21}$$

Substituting (21) into the right hand side of (19) yields the expression

$$\langle (\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k^{-1}}))^{k_2^{-1}}, Q^{k_1^{-1}}\rangle$$

from which it follows that

$$(\psi_*\pi_{EL})((k,P),(k,Q)) = \langle (\mathcal{H}_{\mathfrak{k}} + \mathrm{Ad}(k)\circ\mathcal{H}_{\mathfrak{k}}\circ\mathrm{Ad}(k^{-1}))(P), Q\rangle.$$

completing the proof of the theorem. ∎

For a compact group K there are essentially two Poisson structures intimately related to the Lie theory of K. On the one hand there is the standard Poisson Lie group structure whose symplectic foliation respects the Bruhat decomposition of K. This is given by the difference of the right and left invariant bivector fields generated by $\mathcal{H}_{\mathbb{R}}$. On the other hand there is the Evens-Lu $K \times K$-homogeneous Poisson structure on K whose symplectic foliation respects the Birkhoff decomposition of K. This is given by the sum of the left and right invariant bivector fields generated by $\mathcal{H}_{\mathbb{R}}$.

Returning to the general case where X is not necessarily a group, note that all stability subgroups of U corresponding to points in X are conjugate in U. Thus, the class of θ in the outer automorphism group of U is an invariant of X. When this class is trivial, the Evens-Lu Poisson structure is non-degenerate on an dense open subset of points of X as will be shown below. This stems from the fact that when θ is an inner automorphism, each Cartan subalgebra of $\mathfrak{k}$ is, in fact, a Cartan subalgebra of $\mathfrak{u}$. Thus $\mathfrak{h} \cap \mathfrak{u} = \mathfrak{t}$ and $\mathfrak{h} \cap \mathfrak{k} = \mathfrak{t}_0$ are equal. Many statements simplify dramatically in the inner case. For example, the torus $T_w = \exp(\ker\{\mathrm{Ad}(w)\theta|_t - 1\})$ which acts on the layer of the Birkhoff decomposition of X corresponding to w admits a much simpler description in the inner case as $T_0 \cap (T_0)^w$.

Theorem 5.3. *In symmetric spaces for which θ is an inner automorphism, each connected component of the layer of the Birkhoff decomposition of X corresponding to the trivial element of the Weyl group is an open symplectic leaf. The components are indexed by the elements of order two in $T_0 \cap \phi(U/K)$.*

Proof. ¿From Corollary 3.3: the symplectic leaves of maximal dimension foliate the connected components of the layer corresponding to $w = T_0 \in W = N_U(T_0)/T_0$. For brevity, this layer will be referred to as the *top layer.* The leaves are indexed by the elements $\mathbf{w} \in \phi(U/K) \cap T_0$. Such an element satisfies the equation $\mathbf{w}^{-1} = \mathbf{w}^\theta$, but T_0 is fixed by θ, so $\mathbf{w}$ must be an element of order 2. Theorem 3 in [11] shows that for each such $\mathbf{w}$ there exists an element $\mathbf{w}_1 \in N_U(T_0)$ such that $\phi(\mathbf{w}_1 K) = \mathbf{w}$.

In Theorem 2.2 it was shown that with the presentation of $T^*(U/K) \simeq U \times_K i\mathfrak{p}$ the Evens-Lu Poisson bivector can be expressed relative to the Killing form as

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y \rangle$$

where $X, Y \in i\mathfrak{p}$, and $\Omega_u \colon i\mathfrak{p} \to i\mathfrak{p}$ is given by

$$\Omega_u(X) = \{\mathrm{Ad}(u^{-1}) \circ \mathcal{H} \circ \mathrm{Ad}(u)(X)\}_{i\mathfrak{p}}.$$

The kernel of $\Omega_{\mathbf{w}_1}(X)$ is equal to $\mathfrak{t}_0^{\mathbf{w}_1^{-1}} \cap i\mathfrak{p} = \mathfrak{t}_0 \cap i\mathfrak{p} = 0$. Thus, the leaves in X whose ϕ-images lie in the top layer of the Birkhoff decomposition are open. ∎

The elements of order two in $T_0 \cap \phi(U/K)$ correspond precisely to the preferred basepoints in X whose existence was established in Lemma 3.1.

¿From the classification of symmetric spaces in [7], the list of irreducible compact symmetric spaces for which the involution is an inner automorphism includes, but is not limited to, the compact Hermitian symmetric spaces. Complex variables will be used in the following section to exhibit locally the Evens-Lu Poisson structure on some spaces of this type.

6. Examples

In this section, local expressions are recorded for the Evens-Lu Poisson structure in a number of explicit examples. The first example is the complex Grassmannian, i.e., the space of m-planes in in $\mathbb{C}^{m+n}$. As a symmetric space, this may be presented as the quotient of the compact group $U = \mathrm{SU}(m+n)$ by the closed subgroup $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$. This presentation arises from the natural action of $\mathrm{SU}(m+n)$ on $\mathbb{C}^{m+n}$ by linear isometries of the standard Hermitian inner product, which descends to a transitive action on the set of complex m-planes through the origin.

The complexification of $\mathfrak{u} = \mathfrak{su}(m+n)$ is $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{C})$. Let $\mathfrak{h}$ be the diagonal matrices in $\mathfrak{sl}(m+n,\mathbb{C})$ and the triangular decomposition be the usual one where $\mathfrak{n}_+$ consists of the strictly upper triangular matrices and $\mathfrak{n}_-$ the strictly lower triangular matrices. This $\mathfrak{u}$-compatible triangular decomposition generates a standard Poisson Lie group structure on U. The corresponding Birkhoff decomposition of $G = \mathrm{SL}(m+n,\mathbb{C})$ corresponds to the factorization produced in linear algebra through Gaussian elimination.

The point in the Grassmannian corresponding to the plane spanned by the first m standard basis vectors is an example of a preferred basepoint. Denote this plane by $\mathbb{C}^m$. The stability subgroup of this basepoint consists of the special unitary transformations preserving this plane (and by necessity its Hermitian orthogonal complement), i.e. $\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$. ¿From this presentation one can readily see that θ is the automorphism which negates the off-diagonal blocks. This involution is an inner automorphism, given by conjugation by a scalar multiple of the block diagonal matrix with an $m \times m$ identity matrix in the upper diagonal block and an $n \times n$ diagonal matrix with negative ones on the diagonal in the lower block. When n is even, the scalar multiple is one. When n is odd, the multiple is a primitive $2(m+n)$-th root of unity.

The complex Grassmannian is, additionally, a Hermitian symmetric space. It is diffeomorphic to the quotient of $\mathrm{SL}(m+n,\mathbb{C})$ by the parabolic subgroup of the upper block triangular matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is $m \times m$ and D is $n \times n$. It is through an identification such as this that the Grassmannian inherits a complex structure. Holomorphic coordinates can thus be used to present local formulas.

The graph of a $\mathbb{C}$-linear transformation $Z \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ in $\mathbb{C}^{m+n}$ written

$$\{(X, ZX) \colon X \in \mathbb{C}^m\}$$

uniquely determines a point in the Grassmannian. In fact, every complex m-dimensional subspace of $\mathbb{C}^{m+n}$ which is transverse to $(\mathbb{C}^m)^\perp = \mathbb{C}^n$ can be realized in this way. In this fashion $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ provides an affine coordinate chart for the Grassmannian with each point in a Zariski open subset described by an $n \times m$ matrix Z of complex numbers.

Each coset of $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(m))$ corresponding the graph of a linear transformation contains a unique element u which has positive definite diagonal blocks. This can be seen as follows. Apply polar decomposition to the diagonal blocks A

and D of a special unitary matrix u, writing $A = |A|P_A$ and $D = |D|P_D$ and then factor u to find

$$u = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} |A| & BP_D^* \\ CP_A^* & |D| \end{pmatrix} \begin{pmatrix} P_A & 0 \\ 0 & P_D \end{pmatrix}. \tag{22}$$

The diagonal blocks of a special unitary matrix corresponding to the graph of a linear transformation are invertible, so their polar decomposition is unique. Furthermore, the diagonal blocks of a special unitary matrix have conjugate determinants (see Proposition 1.3 of [4]), thus P_A and P_D have conjugate determinants. Therefore, the factorization in (22) produces a unique representative for u modulo $\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ with the desired properties.

This preferred coset representative can then be determined as a function of Z. It is given by

$$u(Z) = \begin{pmatrix} (1+Z^*Z)^{-1/2} & -(1+Z^*Z)^{-1/2}Z^* \\ Z(1+Z^*Z)^{-1/2} & (1+ZZ^*)^{-1/2} \end{pmatrix}.$$

This formula for $u(Z)$ will be referred to as the *canonical representative* for the coset corresponding to uK depending on Z.

Cotangent vectors at Z will be represented by $m \times n$ complex matrices using the identification of the real cotangent space to the Grassmannian with the holomorphic cotangent space which, at a point, is further identified with $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.

6.1. The Grassmannian. With respect to these coordinates the action of the bivector π on two cotangent vectors represented by complex $m \times n$ matrices V and W may be computed by

$$\pi(V, W) = i[\mathrm{tr}\,(L_Z V)^* W) - \mathrm{tr}\,((L_Z V) W^*)]$$

where L_Z is the $\mathbb{R}$-linear transformation $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n) \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ given by

$$\begin{aligned} L_Z V &= V - Z^*ZVZZ^* + Z^*((ZV - V^*Z^*)_+ + c.t.) \\ &\quad -((Z^*V^* - VZ)_+ + c.t)Z^*. \end{aligned} \tag{23}$$

In the above expression, $(\cdot)_+$ denotes the upper triangular part as before, V^* denotes the conjugate transpose of the matrix V, and *c.t.* denotes the conjugate transpose of the preceding term. This local formula is obtained by direct calculation from the equivariant formula for π in Theorem 2.2. The complete derivation of this formula can be found in [1].

6.2. Complex Projective Space. Complex projective space of dimension n, denoted $\mathbb{C}\mathrm{P}^n$, is the space of complex lines through the origin in $\mathbb{C}^{n+1}$ and is the Grassmannian with $m = 1$. In this case, the coordinate Z is a column vector and the matrices representing cotangent vectors are row vectors. The quantity Z^*Z is a scalar which, for brevity, we write as

$$Z^*Z = \|Z\|^2 = |z_1|^2 + \cdots + |z_n|^2.$$

Furthermore, the third term of $L_Z V$ in (23) vanishes as the matrix $Z^*V^* - VZ$ is one by one and thus has no upper triangular part. In this case the bivector can

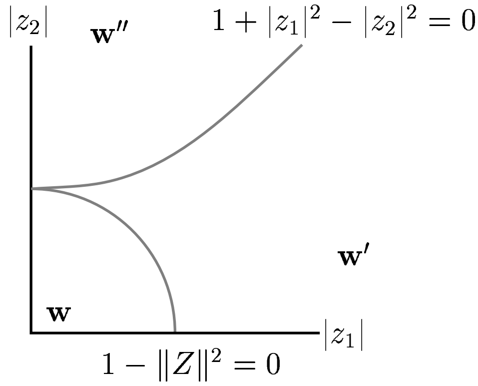

Figure 2: The degeneracy locus for $\mathbb{CP}^2$. The three elements $\mathbf{w} = \mathrm{diag}(1,1,1)$, $\mathbf{w}' = \mathrm{diag}(-1,-1,1)$, $\mathbf{w}'' = \mathrm{diag}(-1,1,-1)$ in $T_0^{(2)} \cap \phi(U/K)$ are depicted with each component of the top stratum they determine.

be written more explicitly as

$$\begin{aligned}\pi &= -i\left\{\sum_{j=1}^{n} S_j \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \overline{z}_j}\right.\\ &\quad \left. + \left(\sum_{j<k} z_j z_k \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k} - \sum_{j<k} z_j \overline{z}_k \|Z\|^2 \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \overline{z}_k}\right) - c.c.\right\}\end{aligned}$$

where

$$S_j = 1 + \sum_{k=1}^{j-1} |z_k|^2 - |z_j|^2 \|Z\|^2 - \sum_{k=j+1}^{n} |z_k|^2.$$

This expression is obtained from the more abstract formula for for the Evens-Lu bivector on the Grassmannian. The complete calculation can be found in [1].

It is interesting to consider $\mathbb{CP}^2$ and $\mathbb{CP}^1$ in further detail. It appears that the coefficients S_j are reducible polynomials in the variables $|z_j|^2$ only in these cases. It is not clear what the significance of this is (if any). For $\mathbb{CP}^2$ the bivector can be expressed locally as

$$\begin{aligned}\pi &= -i\left\{S_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \overline{z}_1} + S_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \overline{z}_2}\right.\\ &\quad \left. + \left(z_1 z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} - z_1 \overline{z}_2 \|Z\|^2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \overline{z}_2}\right) - c.c.\right\}\end{aligned}$$

where $S_1 = (1+|z_1|^2)(1-\|Z\|^2)$ and $S_2 = (1-|z_2|^2)(1+\|Z\|^2)$. Since the symplectic leaves are open, one can write down the induced symplectic structure in these coordinates by inverting the Poisson tensor. Explicitly,

$$\begin{aligned}\omega &= \frac{i}{p(Z,Z^*)}\left\{-S_2 dz_1 \wedge d\overline{z}_1 - S_1 dz_2 \wedge d\overline{z}_2\right.\\ &\quad \left. + \left(z_1 z_2 d\overline{z}_1 \wedge d\overline{z}_2 + z_1 \overline{z}_2 \|Z\|^2 d\overline{z}_1 \wedge dz_2\right) - c.c\right\}\end{aligned}$$

where $p(Z,Z^*) = (1+|z_1|^2-|z_2|^2)(1-\|Z\|^2)(1+\|Z\|^2)$. The degeneracy locus of π is given by the variety $p(Z,Z^*) = 0$.

To witness the connection with triangular factorization the canonical representatives can be used to compute the complement of the top layer of the Birkhoff decomposition in these coordinates. By introducing the real analytic function $\varphi = (\sqrt{1+\|Z\|^2}-1)/\|Z\|^2$, one can compute the matrix $(1+ZZ^*)^{-1/2}$. Then

$$u = \frac{1}{\sqrt{1+\|Z\|^2}} \begin{pmatrix} 1 & -\overline{z}_1 & -\overline{z}_2 \\ z_1 & 1+|z_2|^2\varphi & -z_1\overline{z}_2\varphi \\ z_2 & -\overline{z}_1 z_2\varphi & 1+|z_1|^2\varphi \end{pmatrix}$$

and

$$uu^{-\theta} = \frac{1}{1+\|Z\|^2} \begin{pmatrix} 1-\|Z\|^2 & -2\overline{z}_1 & -2\overline{z}_2 \\ 2z_1 & 1-|z_1|^2+|z_2|^2 & -2z_1\overline{z}_2 \\ 2z_2 & -2\overline{z}_1 z_2 & 1+|z_1|^2-|z_2|^2 \end{pmatrix}. \tag{24}$$

The matrix $uu^{-\theta}$ in (24) is in $\phi(U/K)\cap\Sigma_1^G$ provided the principal minors are non-vanishing. Thus, the complement of the top layer is given in coordinates by the vanishing locus of the product of the principal minors. One can check that, in this case, this product is given by the smooth rational function $p(Z,Z^*)/(1+\|Z\|^2)^3$.

In general, the degeneracy locus for π on the complex Grassmannian is given by the vanishing locus of a reducible polynomial whose factors are given by the explicit formulas of Habermas ([6]).

6.3. SU(2) **and** $\mathbb{C}\mathrm{P}^1$**.**

For $X = \mathbb{C}\mathrm{P}^1$ the group U is SU(2) and $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$. As is typical with this example, everything can be computed. The Lu-Weinstein Poisson Lie group structure, the Evens-Lu Poisson structure that SU(2) inherits as a symmetric space, and the Evens-Lu Poisson structure on $\mathbb{C}\mathrm{P}^1$ will all be displayed. The end of this subsection returns to a topic discussed in the introduction, relating the structure produced by the Evens-Lu construction to the one produced by the Foth-Lu construction ([5]). To conclude, the Evens-Lu Poisson structure is exhibited as an element of the one parameter family from [8].

Consider the standard triangular decomposition of $\mathfrak{g}$ as for the Grassmannian: $\mathfrak{h}$ is the set of traceless diagonal matrices and $\mathfrak{n}_\pm$ are the strictly upper triangular (resp. lower triangular) matrices. Temporarily, set

$$E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \text{ and } E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{H\}$, $\mathfrak{n}_\pm = \mathrm{span}_{\mathbb{C}}\{E_\pm\}$, and $\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{H,X,Y\}$ where $X = E_+ - E_-$, and $Y = i(E_+ + E_-)$. The triangular decomposition is stable with respect to the involution selecting the stability subgroup $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1))$.

Using right translation, the tangent bundle to U can be identified with $U\times\mathfrak{u}$. Thus, a bivector field on U can be identified with a smooth map $U \to \bigwedge^2\mathfrak{u}$. In this presentation the value of the Lu-Weinstein Poisson Lie group structure π_U at an element

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}(2)$$

is given by

$$\pi_U = (1-|a|^2+|b|^2)X\wedge Y + 2\mathrm{Im}(\overline{a}b)Y\wedge H - 2\mathrm{Re}(a\overline{b})H\wedge X \tag{25}$$

whereas the Evens-Lu Poisson structure that SU(2) inherits as a symmetric space is given by

$$\pi^{\mathrm{EL}} = (1 + |a|^2 - |b|^2) X \wedge Y + 2\mathrm{Im}(ab) Y \wedge H - 2\mathrm{Re}(ab) H \wedge X.$$

One can see that the Evens-Lu Poisson structure π^{EL} vanishes precisely when the principal minor, i.e., a, is zero. This shows that the symplectic foliation of π^{EL} respects the Birkhoff decomposition of $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$.

With the presentation of $\mathbb{C}\mathrm{P}^1$ as $\mathrm{SU}(2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ the Evens-Lu construction gives the Poisson structure

$$\pi = -i(1 - |z|^4) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}. \tag{26}$$

The top layer of the Birkhoff decomposition has two connected components, the upper and lower hemispheres. The degeneracy locus of π is the equator.

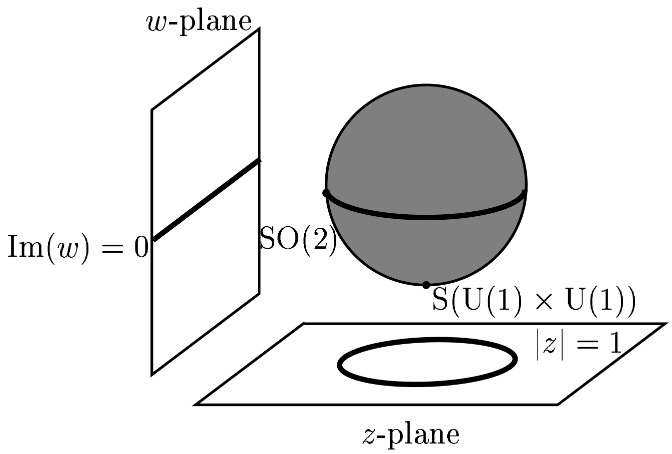

Figure 3: The symplectic foliation for the (U, π_U)-homogeneous Poisson structure on the $\mathbb{C}\mathrm{P}^1$. The degeneracy locus is pictured in the z-coordinate when the stability subgroup of the basepoint is $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ and the w-coordinate when the stability subgroup is $\mathrm{SO}(2)$.

Alternatively, Foth and Lu choose the basepoint of this symmetric space so that the stability subgroup is $\mathrm{SO}(2) \subset \mathrm{SU}(2)$. In this case, the corresponding non-compact real form $\mathfrak{g}_0$ is $\mathfrak{sl}(2, \mathbb{R})$ and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}_+$ is Iwasawa relative to $\mathfrak{g}_0$. With this choice of basepoint the quotient map $U \to U/K$ projects π_U to a (U, π_U)-homogeneous Poisson structure on U/K. The map

$$w \mapsto \frac{1}{\sqrt{1 + |w|^2}} \begin{pmatrix} 1 + i\mathrm{Re}(w) & i\mathrm{Im}(w) \\ i\mathrm{Im}(w) & 1 - i\mathrm{Re}(w) \end{pmatrix}$$

gives a local cross section of the projection $\mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{SO}(2)$. In this coordinate, the projection of π_U (25) is given by

$$\pi = -2i\mathrm{Im}(w)(1 + |w|^2) \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \overline{w}}. \tag{27}$$

These two points of view are illustrated in figure 3. There, one can see that the Foth-Lu construction and the Evens-Lu construction produce the same (U, π_U)-homogeneous Poisson structure on $\mathbb{C}\mathrm{P}^1$.

In [8] the existence of a parabolic subgroup of G such that $U \cap P$ is a Poisson Lie subgroup was established under the assumption that X was a Hermitian symmetric space. The projection of π_U under the quotient map $U \mapsto U/(U \cap P)$ then defines a (U, π_U)-homogeneous Poisson structure on X which is compatible with the invariant Poisson structure π_{KKS}.

In this example, the parabolic subgroup P is actually the Borel subgroup $HN^+ \subset \mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SU}(2) \cap P$ is the diagonal torus $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$. Pushing forward π_U from (25) under the quotient map gives the Poisson structure

$$\pi_{PL} = 2i|z|^2(1 + |z|^2)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}} \tag{28}$$

which is degenerate only at the basepoint. Furthermore, the invariant Poisson structure of Kostant-Kirillov-Souriau is given in the z-coordinate by

$$\pi_{KKS} = i(1 + |z|^2)^2 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}. \tag{29}$$

The reader is invited to check that the Evens-Lu Poisson structure π in (26) is equal to $\pi_{PL} + \lambda \pi_{KKS}$ with $\lambda = -1$.

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A. Caine
Max Planck Institut für Mathematik,
Bonn
Box 7280
53111, Bonn, Germany
caine@mpim-bonn.mpg.de

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