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Borel Subalgebras of Root-reductive Lie Algebras

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Abstract. This paper generalizes the classification in [Dimitrov, I., and I. Penkov, *Borel subalgebras of* $\mathfrak{gl}(\infty)$, Resenhas **6** (2004) 153–163] of Borel subalgebras of $\mathfrak{gl}_\infty$. Root-reductive Lie algebras are direct limits of finite-dimensional reductive Lie algebras along inclusions preserving the root spaces with respect to nested Cartan subalgebras. A Borel subalgebra of a root-reductive Lie algebra is by definition a maximal locally solvable subalgebra. The main general result of this paper is that a Borel subalgebra of an infinite-dimensional indecomposable root-reductive Lie algebra is the simultaneous stabilizer of a certain type of generalized flag in each of the standard representations.

For the three infinite-dimensional simple root-reductive Lie algebras more precise results are obtained. The map sending a maximal closed (isotropic) generalized flag in the standard representation to its stabilizer hits Borel subalgebras, yielding a bijection in the cases of $\mathfrak{sl}_\infty$ and $\mathfrak{sp}_\infty$; in the case of $\mathfrak{so}_\infty$ the fibers are of size one and two. A description is given of a nice class of toral subalgebras contained in any Borel subalgebra. Finally, certain Borel subalgebras of a general root-reductive Lie algebra are seen to correspond bijectively with Borel subalgebras of the commutator subalgebra, which are understood in terms of the special cases.
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1. Introduction

The representation theory of root-reductive Lie algebras is currently being approached through a structure theory program. Root-reductive Lie algebras are direct limits of finite-dimensional reductive Lie algebras along inclusions preserving the root spaces with respect to nested Cartan subalgebras. The appropriate generalization in this context of the notion of a Borel subalgebra of a finite-dimensional Lie algebra is that of a *maximal locally solvable* subalgebra. This paper describes the Borel subalgebras of root-reductive Lie algebras, generalizing the results of [3] in the case of $\mathfrak{gl}_\infty$.

The most general result of this paper, Theorem 4.1, states that a Borel subalgebra of an infinite-dimensional indecomposable root-reductive Lie algebra

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is the simultaneous stabilizer of a certain type of generalized flag in each of the standard representations. Any root-reductive Lie algebra is the direct sum of finite-dimensional Lie algebras and infinite-dimensional indecomposable root-reductive Lie algebras. Since Borel subalgebras of a direct sum are precisely direct sums of Borel subalgebras, the theorem can be used to understand Borel subalgebras of any root-reductive Lie algebra.

Theorems 8.3, 9.7, and 10.6 address the infinite-dimensional simple root-reductive Lie algebras. As in the case of $\mathfrak{gl}_\infty$ treated in [3], Borel subalgebras of $\mathfrak{sl}_\infty$ (or $\mathfrak{so}_\infty$, $\mathfrak{sp}_\infty$) are stabilizers of maximal closed (isotropic) generalized flags in the standard representation. The correspondence between Borel subalgebras and maximal closed (isotropic) generalized flags is bijective in the cases of $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, and $\mathfrak{sp}_\infty$; whereas a Borel subalgebra of $\mathfrak{so}_\infty$ corresponds to one or two maximal closed isotropic generalized flags. This phenomenon should not be surprising, since every Borel subalgebra of $\mathfrak{so}_{2n}$ is the stabilizer of a pair of maximal isotropic flags in the standard representation. We refer to any pair of maximal isotropic generalized flags corresponding to a single Borel subalgebra of $\mathfrak{so}_\infty$ as *twins*.

A nice class of toral subalgebras contained in a Borel subalgebra of $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, or $\mathfrak{sp}_\infty$ is described in Section 11. In these cases any Borel subalgebra is the span of such a toral subalgebra and the ad hoc nilradical. Thus irreducible representations of the Borel subalgebra are given by characters of the toral subalgebra.

Analysis of the general situation continues in Section 13. In Theorem 13.2 certain Borel subalgebras of a root-reductive Lie algebra $\mathfrak{g}$ are seen to correspond bijectively to the Borel subalgebras of $[\mathfrak{g},\mathfrak{g}]$. It remains unknown whether every Borel subalgebra of $\mathfrak{g}$ yields a Borel subalgebra of $[\mathfrak{g},\mathfrak{g}]$ when intersected with $[\mathfrak{g},\mathfrak{g}]$.

The argument which leads to the classification of Borel subalgebras of $\mathfrak{sl}_\infty$, Theorem 8.3, begins with Theorem 4.1 and Proposition 7.1, and continues with Lemmas 8.1 and 8.2. Many elements of the proofs are straightforward applications to $\mathfrak{sl}_\infty$ of work on $\mathfrak{gl}_\infty$ seen in [4]. The proof of Theorem 4.1, by contrast, is quite different from their proof in the case of $\mathfrak{gl}_\infty$; the modified proof allows for generalization to the isotropic cases.

I wish to acknowledge Ivan Dimitrov and Ivan Penkov for explaining their work in [3], and for sharing with me the proofs of the results announced there in the form of a manuscript [4]. The debt I owe Ivan Penkov goes further, for he introduced me to root-reductive Lie algebras and helped me greatly as I was first learning about them. I wish to express my gratitude to Joseph Wolf, for his warm guidance and frequent attention throughout the writing of this paper. I would also like to thank Vera Serganova and the referee for their helpful comments.

2. Preliminaries

Throughout the paper we fix the ground field to be the field of complex numbers $\mathbb{C}$. Let V and V_* be countable-dimensional vector spaces over $\mathbb{C}$, and let $\langle\cdot,\cdot\rangle : V \times V_* \to \mathbb{C}$ be a nondegenerate pairing. We denote by $\mathfrak{gl}(V,V_*)$ the Lie algebra associated to the associative algebra $V \otimes V_*$. Note that V is a faithful representation of $\mathfrak{gl}(V,V_*)$ under the action defined by $(x \otimes y) \cdot v := \langle v, y\rangle x$ for any $x, v \in V$ and $y \in V_*$, and hence $\mathfrak{gl}(V,V_*)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. By $\mathfrak{sl}(V,V_*)$ we denote the traceless part of $\mathfrak{gl}(V,V_*)$, i.e. $[\mathfrak{gl}(V,V_*),\mathfrak{gl}(V,V_*)]$. If

$\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ is a symmetric nondegenerate form, then for any $x,y,v,w\in V$ one may check that $[x\otimes y-y\otimes x, v\otimes w-w\otimes v]\in\bigwedge^2 V$, and thus $\bigwedge^2 V$ is a Lie subalgebra of $\mathfrak{gl}(V,V)$, denoted $\mathfrak{so}(V)$. If $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ is an antisymmetric nondegenerate form, then for any $x,y,v,w\in V$ one may check that $[x\otimes y+y\otimes x, v\otimes w+w\otimes v]\in\mathrm{Sym}^2(V)$, and thus $\mathrm{Sym}^2(V)$ is a Lie subalgebra of $\mathfrak{gl}(V,V)$, denoted $\mathfrak{sp}(V)$.

By a result of Mackey [5, p. 171], as long as the pairing $\langle\cdot,\cdot\rangle$ is nondegenerate, the above algebras do not depend on the pairing, up to isomorphism. The usual representatives of these isomorphism classes are called $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$, respectively.

We will need a notion of the closure of a subspace of a vector space, with respect to a pairing. Let X and Y be vector spaces, and let $\langle\cdot,\cdot\rangle : X\times Y\to\mathbb{C}$ be any pairing. Given a subspace $F\subseteq X$, we consider the subspace $F^{\perp\perp}$, denoted $\overline{F}$, to be its *closure* in X. A subspace $F\subseteq X$ is said to be *closed* if $F=\overline{F}$. One important identity is that $F^\perp = F^{\perp\perp\perp}$ for any $F\subset X$. As a result, for any $F\subset X$, the subspace $F^\perp\subset Y$ is closed. Furthermore, the closure of any subspace is closed. One may also check that the arbitrary intersection of closed subspaces is closed.

If $F\subset X$ is a closed subspace and $F\subset G\subset X$ with $\dim G/F<\infty$, then G is closed. This follows from the fact that $\dim F^\perp/G^\perp\le\dim G/F$ for arbitrary subspaces $F\subset G\subset X$. Explicitly, consider that

$$\begin{aligned}\dim G/F\le\dim\overline{G}/F=\dim\overline{G}/\overline{F} &= \dim(G^\perp)^\perp/(F^\perp)^\perp\\ &\le \dim F^\perp/G^\perp\le\dim G/F.\end{aligned}$$

Hence $\dim\overline{G}/F=\dim G/F<\infty$, and since $G\subset\overline{G}$, we know $G=\overline{G}$.

Now suppose $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ is a nondegenerate pairing. A subspace $F\subset V$ is said to be *isotropic* if $F\subset F^\perp$, and *coisotropic* if $F^\perp\subset F$. If $F\subset V$ is an isotropic subspace, then its closure $\overline{F}$ is also isotropic. That is, $F\subset F^\perp$ implies $\overline{F}\subset\overline{F^\perp}$, where $\overline{F^\perp}=F^\perp=\overline{F}^\perp$.

If $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ is a symmetric nondegenerate form, an isotropic subspace $M\subset V$ is maximal isotropic if and only if $\dim M^\perp/M\le 1$ and M is closed. If $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ is an antisymmetric nondegenerate form, a subspace $M\subset V$ is maximal isotropic if and only if $M=M^\perp$.

A Lie algebra $\mathfrak{g}$ is *locally finite* if every finite subset of $\mathfrak{g}$ is contained in a finite-dimensional subalgebra, i.e. if $\mathfrak{g}$ is a union of finite-dimensional subalgebras. One interesting class of locally finite Lie algebras is the root-reductive Lie algebras.

Definition 2.1.

1. An inclusion of finite-dimensional reductive Lie algebras $\mathfrak{l}\subseteq\mathfrak{m}$ is called a *root inclusion* if, for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{m}$, the subalgebra $\mathfrak{l}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}$ and each $\mathfrak{l}\cap\mathfrak{h}$-root space $\mathfrak{l}^\alpha$ is also an $\mathfrak{h}$-root space of $\mathfrak{m}$.

2. A Lie algebra $\mathfrak{g}$ is called *root-reductive* if it is isomorphic to a union $\bigcup_{i\in\mathbb{Z}_{>0}}\mathfrak{g}_i$ of nested reductive Lie algebras with respect to root inclusions for a fixed choice of nested Cartan subalgebras $\mathfrak{h}_i\subseteq\mathfrak{g}_i$ with $\mathfrak{h}_{i-1}=\mathfrak{h}_i\cap\mathfrak{g}_{i-1}$.

To understand the structure of root-reductive Lie algebras one uses the following theorem from [2]. The part of the classification in which it is shown that there are, up to isomorphism, three infinite-dimensional simple objects in this category is also given in [6].

Theorem 2.2. *Let $\mathfrak{g}$ be a root-reductive Lie algebra.*

1. *There is a split exact sequence of Lie algebras $0 \to \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] =: \mathfrak{a} \to 0$, i.e. $\mathfrak{g} \simeq [\mathfrak{g},\mathfrak{g}] \inplus \mathfrak{a}$, with $\mathfrak{a}$ abelian.*
2. *The Lie algebra $[\mathfrak{g},\mathfrak{g}]$ is isomorphic to a direct sum of finite-dimensional simple Lie algebras and copies of $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$.*

Since there are no nontrivial extensions of of an abelian Lie algebra by a finite-dimensional simple Lie algebra, any root-reductive Lie algebra is isomorphic to a direct sum of simple finite-dimensional Lie algebras and a root-reductive Lie algebra $\mathfrak{g}$ in which $[\mathfrak{g},\mathfrak{g}]$ is isomorphic to a direct sum of copies of $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$.

Let $\mathfrak{g}$ be an infinite-dimensional indecomposable root-reductive Lie algebra. Then $[\mathfrak{g},\mathfrak{g}] \cong \bigoplus_m \mathfrak{s}_m$ as Lie algebras, where for each m the component $\mathfrak{s}_m$ is isomorphic to $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, or $\mathfrak{sp}_\infty$. Let V_m denote the standard representation of $\mathfrak{s}_m$, and let $(V_m)_*$ denote the relevant dual representation. Consider V_m as a $[\mathfrak{g},\mathfrak{g}]$-module on which $\bigoplus_{n\neq m} \mathfrak{s}_n$ acts trivially. By Proposition 4.2 of [1], there exists a $\mathfrak{g}$-module structure on V_m extending the $[\mathfrak{g},\mathfrak{g}]$-module structure. Likewise, there exists a $\mathfrak{g}$-module structure on $(V_m)_*$ in which $\bigoplus_{n\neq m} \mathfrak{s}_n$ acts trivially. One may check that under this construction, the pairing $\langle\cdot,\cdot\rangle : V_m \times (V_m)_* \to \mathbb{C}$ is $\mathfrak{g}$-invariant. By *the standard representations* of $\mathfrak{g}$, we mean the representations V_m together with a choice of $\mathfrak{g}$-module structure on each.

If $\mathfrak{l} \subseteq \mathfrak{m}$ is a root inclusion, then $\mathfrak{m}$ is completely reducible as an $\mathfrak{l}$-module. Therefore the Jordan decomposition of any element of $\mathfrak{l}$ into a sum of commuting semisimple and nilpotent parts agrees with its Jordan decomposition as an element of $\mathfrak{m}$, and one obtains as a result a notion of Jordan decomposition of elements of a root-reductive Lie algebra [1]. A subalgebra of a root-reductive Lie algebra is called a *toral subalgebra* if it consists of elements which are semisimple in the sense of Jordan decomposition. We denote the normalizer in $\mathfrak{g}$ of a subalgebra $\mathfrak{k}$ by $\mathfrak{n}_\mathfrak{g}(\mathfrak{k})$.

A locally finite Lie algebra $\mathfrak{g}$ is *locally solvable* (respectively *locally nilpotent*) if every finite subset of $\mathfrak{g}$ is contained in a solvable (resp. nilpotent) subalgebra, i.e. if $\mathfrak{g}$ is a union of finite-dimensional solvable (resp. nilpotent) subalgebras. The *ad hoc nilradical* of a locally solvable subalgebra $\mathfrak{s}$ of a root-reductive Lie algebra is defined to be the set of all elements of $\mathfrak{s}$ which are nilpotent in the sense of Jordan decomposition. The ad hoc nilradical of a locally solvable subalgebra $\mathfrak{s}$ of a root-reductive Lie algebra contains $[\mathfrak{s},\mathfrak{s}]$, and thus the ad hoc nilradical is a subalgebra. Note that the ad hoc nilradical of a locally solvable subalgebra $\mathfrak{s}$ of a root-reductive Lie algebra is a locally nilpotent subalgebra of $\mathfrak{s}$. The following lemma is an essential result about representations of locally solvable Lie algebras, and its proof is reproduced from [4] with the permission of the authors.

Lemma 2.3. *Let $\mathfrak{s}$ be a locally finite locally solvable Lie algebra, i.e. $\mathfrak{s} = \bigcup_i \mathfrak{s}_i$ with $\mathfrak{s}_i$ finite-dimensional and solvable. If W is an irreducible $\mathfrak{s}$-module which is a union of finite-dimensional $\mathfrak{s}_i$-modules W_i, then $\dim W = 1$.*

Proof. Assume for the sake of a contradiction that $\dim W > 1$. Thus there exist linearly independent vectors $w_1, w_2 \in W$. Since W is an irreducible module over the universal enveloping algebra $U(\mathfrak{s})$ of $\mathfrak{s}$, the Jacobson density theorem implies that there exist elements $X_{ij} \in U(\mathfrak{s})$ such that $X_{ij} \cdot w_k = \delta_{jk} w_i$ for $i, j \in \{1, 2\}$. There is a finite-dimensional subalgebra $\mathfrak{s}_0 \subset \mathfrak{s}$ for which $X_{ij} \in U(\mathfrak{s}_0)$ for $i, j \in \{1, 2\}$. Denote by W_0 the $\mathfrak{s}_0$-module generated by w_1. Since W_0 is finite dimensional, Lie's Theorem implies that $\mathfrak{s}_0$ stabilizes a maximal flag

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = W_0$$

in W_0. By construction, $w_1 \notin F_{n-1}$. The vector w_2 is an element of W_0 since $X_{21} \cdot w_1 = w_2$. Thus w_2 can be written uniquely in the form $w_2 = cw_1 + w_2'$ with $w_2' \in F_{n-1}$. Clearly one has $X_{12} \cdot w_2' = w_1$, which contradicts the fact the F_{n-1} is stable under $\mathfrak{s}_0$. ■

Finally, a subalgebra of a root-reductive Lie algebra is called a *Borel subalgebra* if it is a maximal locally solvable subalgebra.

3. Generalized flags and isotropic generalized flags

Generalized flags and closed generalized flags were defined in [3] to study Borel subalgebras of $\mathfrak{gl}_\infty$. Let X be a complex vector space. A *chain* in X is a set of subspaces of X totally ordered by inclusion. A *generalized flag* $\mathfrak{F}$ *in* X is a chain in X such that each subspace $F \in \mathfrak{F}$ has an immediate predecessor or an immediate successor in the inclusion ordering, and for every nonzero $x \in X$ there exists an immediate predecessor-successor pair $F' \subset F'' \in \mathfrak{F}$ with $x \in F'' \setminus F'$. Let A be the set of immediate predecessor-successor pairs of $\mathfrak{F}$, and denote by F'_α the predecessor and by F''_α the successor of each pair $\alpha \in A$. The inclusion ordering on $\mathfrak{F}$ induces the following ordering on A: for any $\alpha, \beta \in A$, we define $\alpha \leq \beta$ if $F'_\alpha \subset F'_\beta$. Since every subspace in $\mathfrak{F}$ is either the immediate predecessor or the immediate successor of another subspace, a generalized flag $\mathfrak{F}$ may be considered as $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$.

Let $x \in X$ be nonzero. Then we denote by F'_x and F''_x the predecessor and successor, respectively, of the immediate predecessor-successor pair such that $x \in F''_x \setminus F'_x$, obtained from the definition of a generalized flag. A generalized flag $\mathfrak{G}$ is considered to be a *refinement* of $\mathfrak{F}$ if $F'_x \subset G'_x \subset G''_x \subset F''_x$ for every nonzero $x \in X$. A generalized flag $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ is maximal (with respect to refinements) if $\dim F''_\alpha / F'_\alpha = 1$ for all $\alpha \in A$ [3].

Suppose $\mathcal{C}$ is a chain of subspaces in X satisfying the property that for each $x \in X$, there exists a subspace $C \in \mathcal{C}$ containing x, as well as a subspace $C \in \mathcal{C}$ not containing x. (This is not terribly restrictive, as one sufficient condition is that 0 and X be elements of $\mathcal{C}$.) Then one may obtain a generalized flag $\mathrm{fl}(\mathcal{C})$ by defining:

$$\mathrm{fl}(\mathcal{C}) := \{ \bigcup_{x \notin C \in \mathcal{C}} C, \bigcap_{x \in C \in \mathcal{C}} C \}_{0 \neq x \in X}.$$

If $\mathfrak{F} = \mathrm{fl}(\mathcal{C})$, then for each nonzero $x \in X$, one has $F'_x = \bigcup_{x \notin C \in \mathcal{C}} C$ and $F''_x = \bigcap_{x \in C \in \mathcal{C}} C$. The generalized flag obtained from a chain is not necessarily a subset of that chain, nor must it contain every subspace in the chain. Take as an example a chain of the form

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots \subsetneq \bigcup V_i \subsetneq \cdots \subsetneq W_3 \subsetneq W_2 \subsetneq W_1 \subsetneq X.$$

If on the one hand $\bigcup_i V_i \neq \bigcap_j W_j$, then the generalized flag obtained from this chain is

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots \subsetneq \bigcup V_i \subsetneq \bigcap W_j \subsetneq \cdots \subsetneq W_3 \subsetneq W_2 \subsetneq W_1 \subsetneq X.$$

If on the other hand $\bigcup_i V_i = \bigcap_j W_j$, then the generalized flag obtained from this chain is

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots \subsetneq \cdots \subsetneq W_3 \subsetneq W_2 \subsetneq W_1 \subsetneq X.$$

Now suppose that there is a bilinear form $X \times Y \to \mathbb{C}$. For any chain $\mathcal{C}$ of subspaces of X, one may consider the set of subspaces given by $\mathcal{C}^\perp := \{C^\perp\}_{C \in \mathcal{C}}$, which is a chain in Y. A generalized flag $\mathfrak{F}$ is said to be *closed* if $\mathfrak{F} = \mathrm{fl}(\mathfrak{F}^{\perp\perp})$. A generalized flag is closed if and only if every immediate succesor is closed while every immediate predecessor has as its closure either itself or its immediate successor [3]. In the context of closed generalized flags, we use the term *good pair* to refer to any immediate predecessor-successor pair of which the predecessor is closed. A closed generalized flag is a maximal closed generalized flag if and only if every good pair has codimension 1 [3].

We say that a closed generalized flag $\mathfrak{F}$ is *bivalent* if every good pair has codimension 1 or ∞. Let $\mathfrak{F}$ be a bivalent closed generalized flag in X. A generalized flag $\mathfrak{G}$ refining $\mathfrak{F}$ is called a *Borel generalized flag*[2] if whenever a nonzero $x \in X$ yields a good pair with infinite codimension $F'_x \subset F''_x$ in $\mathfrak{F}$, it holds that $\dim G''_x/G'_x = 1$ and $\overline{G'_x} = F''_x$; and otherwise $F'_x = G'_x \subset G''_x = F''_x$. Note that maximal closed generalized flags are a subset of the bivalent closed generalized flags, and that any maximal closed generalized flag may be considered as a Borel generalized flag refining itself.

The following statement appears in [3], and its proof is replicated from [4].

Lemma 3.1. *If $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}$ is a generalized flag in V, then the stabilizer of $\mathfrak{F}$ in $\mathfrak{gl}(V, V_*)$ is $\mathrm{St}_{\mathfrak{F}} = \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$.*

Proof. If $v \in F''_\alpha$ and $w \in (F'_\alpha)^\perp$, then $(v \otimes w) \cdot V \subset \mathbb{C}v \subset F''_\alpha$ and $(v \otimes w) \cdot F'_\alpha = 0$. Thus the generalized flag $\mathfrak{F}$ is stable under $v \otimes w$. As a result $\sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp \subset \mathrm{St}_{\mathfrak{F}}$.

Conversely, let $X \in \mathrm{St}_{\mathfrak{F}}$. We have $X = \sum_{i=1}^n v_i \otimes w_i$, and we may assume $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, where $v_i \in F''_{\alpha_i} \setminus F'_{\alpha_i}$. We may also assume that if k is such that $\alpha_{k-1} < \alpha_k = \alpha_{k+1} = \cdots = \alpha_n$, then the images of the vectors $v_k, v_{k+1}, \ldots, v_n$ are linearly independent in $F''_{\alpha_n}/F'_{\alpha_n}$. For any $v \in F'_{\alpha_n}$, one has $X \cdot v = (\sum_{i=1}^n v_i \otimes w_i) \cdot v = \sum_{i=1}^n \langle v, w_i \rangle v_i \in F'_{\alpha_n}$ Since $v_i \in F'_{\alpha_n}$ for $i = 1, \ldots, k-1$ and the vectors v_k, v_{k+1}, $\ldots$, v_n are linearly independent modulo F'_{α_n}, one obtains that $\langle v, w_k \rangle = \langle v, w_{k+1} \rangle = \cdots = \langle v, w_n \rangle = 0$. This show that $w_i \in (F'_{\alpha_i})^\perp$ for $i = k, k+1, \ldots, n$. Since $X - \sum_{i=k}^n v_i \otimes w_i$ is also an element of $\mathrm{St}_{\mathfrak{F}}$, a simple induction argument shows that $X \in \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$. ■

[2]It may turn out that the only Borel generalized flags which are of interest are the maximal closed generalized flags.

Also, the span of the nilpotent elements of $\mathrm{St}_{\mathfrak{F}}$ (that is to say its ad hoc nilradical, since $\mathrm{St}_{\mathfrak{F}}$ is locally solvable as seen below) is given by the formula $\sum_\alpha F''_\alpha \otimes (F''_\alpha)^\perp$ [3].

The following proposition is a consequence of a more complicated statement in [4], and I present an alternative proof.

Proposition 3.2. *Let $\mathfrak{F}$ be a maximal generalized flag in V. Then the stabilizer in $\mathfrak{gl}(V, V_*)$ of $\mathfrak{F}$ is a locally solvable subalgebra.*

Proof. Let $X \subset V$ and $Y \subset V_*$ be finite-dimensional subspaces such that the restriction of $\langle \cdot, \cdot \rangle$ to $X \times Y$ is nondegenerate. Let d denote the dimension of X, and of course $X \otimes Y \cong \mathfrak{gl}_d$. Observe that $\mathfrak{gl}(V, V_*)$ may be exhausted by such subalgebras.

Let A be the set of immediate predecessor-successor pairs of $\mathfrak{F}$, and recall the notation $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$. We claim that for $i = 1, \ldots, d$ there exists $\alpha_i \in A$ such that $\dim(X \cap F''_{\alpha_i}) = i$. To see this, consider that since $d < \infty$, there exists $\alpha_d \in A$ such that

$$\dim(X \cap F''_{\alpha_d}) = \dim(X \cap \bigcup_{\alpha \in A} F''_\alpha) = \dim(X \cap V) = d.$$

Similarly, since $d < \infty$, there exists $\alpha_{min} \in A$ such that

$$\dim(X \cap F''_{\alpha_{min}}) = \dim(X \cap \bigcap_{\alpha \in A} F''_\alpha) \leq \dim(\bigcap_{\alpha \in A} F''_\alpha) \leq 1.$$

Now suppose that $\alpha, \beta \in A$ are such that $\dim(X \cap F''_\alpha) < \dim(X \cap F''_\beta)$. It suffices to show that there exist $\alpha', \beta' \in A$ such that

$$\dim(X \cap F''_\alpha) \leq \dim(X \cap F''_{\alpha'}) < \dim(X \cap F''_{\beta'}) \leq \dim(X \cap F''_\beta),$$

with $\dim(X \cap F''_{\beta'}) = \dim(X \cap F''_{\alpha'}) + 1$. Let $x \in (X \cap F''_\beta) \setminus (X \cap F''_\alpha)$. For some $\beta' \in A$ one has $x \in F''_{\beta'} \setminus F'_{\beta'}$. It follows that

$$\dim(X \cap F''_\alpha) \leq \dim(X \cap F'_{\beta'}) < \dim(X \cap F''_{\beta'}) \leq \dim(X \cap F''_\beta),$$

and that $\dim(X \cap F''_{\beta'}) = \dim(X \cap F'_{\beta'}) + 1$. Since $F'_{\beta'} = \bigcup_{\gamma < \beta'} F''_\gamma$, there exists $\alpha' < \beta'$ such that $\dim(X \cap F''_{\alpha'}) = \dim(X \cap F'_{\beta'})$. Thus we have proved the existence of $\alpha_i \in A$ such that $\dim(X \cap F''_{\alpha_i}) = i$ for $i = 1, \ldots, d$.

Let $X_i := X \cap F''_{\alpha_i}$. Then

$$0 \subset X_1 \subset \cdots \subset X_{d-1} \subset X_d = X$$

is a maximal flag in X. Choose for each i an element $x_i \in X_i \setminus X_{i-1}$. For each i there exists $\beta_i \in A$ such that $x_i \in F''_{\beta_i} \setminus F'_{\beta_i}$. Then one has

$$\mathrm{St}_{\mathfrak{F}} \cap (X \otimes Y) = \sum_{i=1}^{d} X_i \otimes ((F'_{\beta_i})^\perp \cap Y).$$

One may check that $(F'_{\beta_i})^\perp \cap Y \subset X_{i-1}^\perp$, where the perpendicular complement of F'_{β_i} is taken in V_* and the perpendicular complement of X_{i-1} is taken in Y. This follows immediately from the fact that $X_{i-1} \subset F'_{\beta_i}$. Therefore $\mathrm{St}_{\mathfrak{F}} \cap (X \otimes Y) \subset \sum_{i=1}^d X_i \otimes X_{i-1}^\perp$. The latter expression is the stabilizer of the maximal flag $0 \subset X_1 \subset \cdots \subset X_{d-1} \subset X$ in $X \otimes Y$, which is solvable since it is a Borel subalgebra. Therefore $\mathrm{St}_{\mathfrak{F}} \cap (X \otimes Y)$ is solvable. It follows that $\mathrm{St}_{\mathfrak{F}}$ is locally solvable. ■

Now let $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ be a bilinear form. We will say that $\mathfrak{F}$ is an *isotropic generalized flag in* V if every $F \in \mathfrak{F}$ is an isotropic subspace of V, and $\mathfrak{F}$ is a generalized flag in $\bigcup_{F\in\mathfrak{F}} F$. As before, we say an isotropic generalized flag $\mathfrak{F}$ is *closed* if $\mathfrak{F} = \mathrm{fl}(\mathfrak{F}^{\perp\perp})$. Again, an isotropic generalized flag is closed if and only if every immediate succesor is closed while every immediate predecessor has as its closure either itself or its immediate successor. A closed isotropic generalized flag $\mathfrak{F}$ in V is a maximal closed isotropic generalized flag if and only if the subspace $\bigcup_{F\in\mathfrak{F}} F$ is a maximal isotropic subspace of V, and every good pair has codimension 1.

If $\mathfrak{F}$ is a generalized flag in V, let $\mathfrak{F}_{iso}$ denote the pairs of $\mathfrak{F}$ which are isotropic, i.e. $\mathfrak{F}_{iso} := \{F'_\alpha, F''_\alpha : F''_\alpha \subset (F''_\alpha)^\perp\}$.

4. Stable generalized flags in the standard representations

The following result motivates the definition given in Section 3 of a Borel generalized flag.

Theorem 4.1. *Any Borel subalgebra of an infinite-dimensional indecomposable root-reductive Lie algebra is the simultaneous stabilizer of a Borel generalized flag in each of the standard representations.*

Proof. Let $\mathfrak{g}$ be an infinite-dimensional indecomposable root-reductive Lie algebra, and let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. Let $[\mathfrak{g},\mathfrak{g}] \cong \bigoplus_m \mathfrak{s}_m$ be the decomposition into simple root-reductive Lie algebras, where $\mathfrak{s}_m$ is isomorphic to one of $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$ for each m. Let V_m denote the standard representations of $\mathfrak{g}$, as defined in Section 2. For each m, let $\mathcal{C}_m$ be a maximal chain of closed $\mathfrak{b}$-stable subspaces in V_m. Take $\mathfrak{F}_m := \mathrm{fl}(\mathcal{C}_m)$.

Let $F' \subset F''$ be any immediate predeccessor-successor pair in $\mathfrak{F}_m$. One can see immediately that there are no closed subspaces properly between F' and F''. Observe that F'' is closed, since it is obtained as the intersection of closed subspaces of V_m. If F' is not closed, then $\overline{F'} = F''$ because there are no closed subspaces properly between F' and F''. This implies that $\mathfrak{F}_m$ is a closed generalized flag.

If F' is closed, then $\dim F''/F'$ is either 1 or infinite. In detail, let G be any $\mathfrak{b}$-stable subspace $F' \subset G \subset F''$. If $\dim F''/F' < \infty$, then $\dim G/F' < \infty$, and hence G is closed, which implies that G is equal to either F' or F''. That is, if $\dim F''/F' < \infty$, then F''/F' is an irreducible $\mathfrak{b}$-module, and because F''/F' satisfies the hypotheses of Lemma 2.3, it is necessarily 1-dimensional. Thus $\mathfrak{F}_m$ is a bivalent closed generalized flag.

Let $\mathcal{D}_m$ be obtained from $\mathfrak{F}_m$ by adding a maximal chain of $\mathfrak{b}$-stable subspaces between every pair good $F' \subset F''$ with $\dim F''/F' = \infty$. Define $\mathfrak{G}_m := \mathrm{fl}(\mathcal{D}_m)$. Clearly $\mathfrak{G}_m$ is a refinement of $\mathfrak{F}_m$. Let $0 \neq x \in V_m$ be such that F'_x closed and $\dim F''_x/F'_x = \infty$. The maximality of $\mathcal{D}_m$ implies $\dim G''_x/G'_x = 1$. Moreover, $F'_x \subsetneq G'_x$, otherwise G''_x would be a closed $\mathfrak{b}$-stable subspace. Similarly, $\overline{G'_x}$ is a closed $\mathfrak{b}$-stable subspace with $F'_x \subsetneq \overline{G'_x} \subset F''_x$, and therefore $\overline{G'_x} = F''_x$. Thus $\mathfrak{G}_m$ is a Borel generalized flag refining $\mathfrak{F}_m$.

Consider $\mathfrak{s}_m \subset \mathfrak{gl}(V_m,(V_m)_*)$. Observe that the stabilizer in $\mathfrak{gl}(V_m,(V_m)_*)$ of $\mathfrak{G}_m$ is equal to the stabilizer in $\mathfrak{gl}(V_m,(V_m)_*)$ of any maximal generalized flag refining $\mathfrak{G}_m$, which by Proposition 3.2 is locally solvable. As a result, $\mathrm{St}_{\mathfrak{G}_m} \cap \mathfrak{s}_m$ is locally solvable.

Since each flag $\mathfrak{G}_m$ is stable under $\mathfrak{b}$, indeed $\mathfrak{b} \subset \bigcap_m \mathrm{St}_{\mathfrak{G}_m}$. Calculate

$$\begin{aligned} [\bigcap_m \mathrm{St}_{\mathfrak{G}_m}, \bigcap_m \mathrm{St}_{\mathfrak{G}_m}] &\subset (\bigcap_m \mathrm{St}_{\mathfrak{G}_m}) \cap [\mathfrak{g}, \mathfrak{g}] \\ &= \bigoplus_m (\mathrm{St}_{\mathfrak{G}_m} \cap \mathfrak{s}_m). \end{aligned}$$

Since each $\mathrm{St}_{\mathfrak{G}_m} \cap \mathfrak{s}_m$ is locally solvable, it follows that $\bigoplus_m (\mathrm{St}_{\mathfrak{G}_m} \cap \mathfrak{s}_m)$ is locally solvable. Therefore $\bigcap_m \mathrm{St}_{\mathfrak{G}_m}$ is a locally solvable subalgebra of $\mathfrak{g}$. Since $\mathfrak{b}$ is maximal locally solvable, finally $\mathfrak{b} = \bigcap_m \mathrm{St}_{\mathfrak{G}_m}$. ∎

The general case is resumed in Section 13.

5. Isotropic subspaces in the standard representation of $\mathfrak{so}_\infty$

In this section we assume $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{so}(V)$.

Lemma 5.1. *Suppose $M \subset V$ is a maximal $\mathfrak{b}$-stable isotropic subspace. If G is a $\mathfrak{b}$-stable subspace with $M \subset G \subset M^\perp$, then $G \cap G^\perp = M$.*

Proof. Observe that $M \subset G^\perp \subset M^\perp$. Since $G \cap G^\perp$ is $\mathfrak{b}$-stable and isotropic, and moreover $M \subset G \cap G^\perp$, the maximality of M implies $G \cap G^\perp = M$. ∎

Proposition 5.2. *A maximal $\mathfrak{b}$-stable isotropic subspace of V is maximal isotropic.*

Proof. Since M is isotropic, its closure $\overline{M}$ is also isotropic. Moreover, $\overline{M}$ is stable under $\mathfrak{b}$ because M is stable under $\mathfrak{b}$, by the $\mathfrak{g}$-invariance of $\langle \cdot, \cdot \rangle$. By the maximality of M, indeed M is closed.

Let $\mathcal{C}$ be a maximal chain of $\mathfrak{b}$-stable subspaces of V between M and $M^\perp$. Let $\mathfrak{F} := \mathrm{fl}(\mathcal{C})$, so that $0 \subset M \subset \mathfrak{F} \subset M^\perp \subset V$ is a generalized flag in V. Write $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$. By Lemma 2.3, irreducible $\mathfrak{b}$-modules of this type are one dimensional, so it must be that $\dim F''_\alpha / F'_\alpha = 1$ for all $\alpha \in A$.

Suppose $Y \in \mathfrak{b}$. Since Y stabilizes the generalized flag

$$0 \subset M \subset \mathfrak{F} \subset M^\perp \subset V,$$

it follows from Lemma 3.1 that $Y \in M \otimes V + \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp + V \otimes M$. I will show that in fact $Y \in M \otimes V + V \otimes M$.

Now $Y = \sum_i v_i \otimes w_i + Z$, for some $v_i \in F''_{\alpha_i} \setminus F'_{\alpha_i}$ and $w_i \in (F'_{\alpha_i})^\perp \setminus M$ and $Z \in M \otimes V + V \otimes M$. One may safely assume that the set $\{v_i\}$ is linearly independent modulo M and modulo F'_β for all β.

Let $\sigma : V \otimes V \to V \otimes V$ denote the linear map which swaps the two factors. Since $Y \in \bigwedge^2 V$, we calculate $-Y = \sigma(Y) = \sum_i w_i \otimes v_i + \sigma(Z)$. Hence

$$\sum_i v_i \otimes w_i + w_i \otimes v_i = -Z - \sigma(Z). \tag{1}$$

Looking at the left hand side of (1), one can see that $\sum_i v_i \otimes w_i + w_i \otimes v_i$ is an element of $M^\perp \otimes M^\perp$, whereas the right hand side is an element of $M \otimes V + V \otimes M$. Hence the right hand side of (1) is an element of

$$(M^\perp \otimes M^\perp) \cap (M \otimes V + V \otimes M) = M \otimes M^\perp + M^\perp \otimes M.$$

For each i we have $w_i \in M^\perp \setminus M$, so there exists $\beta_i \in A$ such that $w_i \in F''_{\beta_i} \setminus F'_{\beta_i}$. Since $w_i \in F''_{\beta_i} \cap (F'_{\alpha_i})^\perp$, Lemma 5.1 implies that $\beta_i \geq \alpha_i$.

Assume, for the sake of a contradiction, that $\sum_i v_i \otimes w_i$ is nonzero. Let $\beta := \max_i\{\beta_i\}$, where this set is nonempty by hypothesis. So $\beta = \beta_1 = \cdots = \beta_k$ and $\beta > \beta_i$ for $i \neq 1, \ldots, k$. Meanwhile $\beta \geq \alpha_i$ for all i. By assumption $\{v_i\}$ is linearly independent modulo F'_β, so in fact $\alpha_i = \beta$ for at most one i.

1. First suppose that $\alpha_1 = \beta$. Then $\alpha_i < \beta$ for $i \neq 1$, i.e. $v_i \in F'_\beta$ for $i \neq 1$. Equation (1) yields

$$\begin{aligned} v_1 \otimes w_1 + w_1 \otimes v_1 &= -\sum_{i\neq 1}(v_i \otimes w_i + w_i \otimes v_i) - Z - \sigma(Z) \\ &\in F'_\beta \otimes M^\perp + M^\perp \otimes F'_\beta. \end{aligned}$$

   This contradicts the fact that $v_1, w_1 \in F''_\beta \setminus F'_\beta$.

2. Now suppose that $\alpha_i < \beta$ for all i. For $i = 1, \ldots, k$, there exist unique $b_i \in \mathbb{C}$ and $w'_i \in F'_\beta$ such that $w_i = b_i w_1 + w'_i$. Then equation (1) yields

$$\begin{aligned} & w_1 \otimes (b_1 v_1 + \cdots + b_k v_k) \\ = \;& -\sum_{i=1}^{k} w'_i \otimes v_i - \sum_{i\neq 1,\ldots,k} w_i \otimes v_i - \sum_i v_i \otimes w_i - Z - \sigma(Z) \\ \in \;& F'_\beta \otimes M^\perp + M^\perp \otimes M. \end{aligned}$$

   Since $w_1 \notin F'_\beta$, it follows that $b_1 v_1 + \cdots + b_k v_k \in M$. The fact that $b_1 = 1$ contradicts the assumption that the set $\{v_i\}$ is linearly independent modulo M.

Either case leads to a contradiction. Therefore $\sum_i v_i \otimes w_i = 0$, and $Y = Z \in M \otimes V + V \otimes M$.

Thus $Y \cdot M^\perp \subset M$. Since $Y \in \mathfrak{b}$ was arbitrary, indeed $\mathfrak{b} \cdot M^\perp \subset M$. Let L be any isotropic subspace containing M. Then $M \subset L \subset M^\perp$, so L is stable under $\mathfrak{b}$. Since M is a maximal $\mathfrak{b}$-stable isotropic subspace, $L = M$. Therefore M is a maximal isotropic subspace. ∎

Proposition 5.3. *There exists a maximal isotropic subspace $M \subset V$ which is stable under $\mathfrak{b}$. Furthermore, there exists a maximal chain $\mathcal{C}$ of closed $\mathfrak{b}$-stable subspaces in V containing M, with the additional property that $\mathcal{C}^\perp \subset \mathcal{C}$.*

Proof. As a corollary to Proposition 5.2, there exists a maximal isotropic subspace $M \subset V$ which is stable under $\mathfrak{b}$. (Observe that 0 is a $\mathfrak{b}$-stable isotropic subspace of V, and that the union of nested $\mathfrak{b}$-stable isotropic subspaces is a $\mathfrak{b}$-stable isotropic subspace. Hence there exists a subspace $M \subset V$ which is a maximal $\mathfrak{b}$-stable isotropic subspace. By Proposition 5.2, M is a maximal isotropic subspace of V.)

Suppose $\mathcal{C}$ is a chain of closed $\mathfrak{b}$-stable subspaces with $M \in \mathcal{C}$ and $\mathcal{C}^\perp \subset \mathcal{C}$. Suppose further that D is a closed $\mathfrak{b}$-stable subspace such that $\mathcal{C} \cup \{D\}$ is a chain. Then $\mathcal{D} := \mathcal{C} \cup \{D, D^\perp\}$ is a chain of closed $\mathfrak{b}$-stable subspaces with $M \in \mathcal{D}$ and such that $\mathcal{D}^\perp \subset \mathcal{D}$. To see that $\mathcal{D}$ is a chain, consider first the fact that since M and $M^\perp$ are elements of $\mathcal{C}$, the subspace D is either isotropic or coisotropic, i.e. either $D \subset D^\perp$ or $D^\perp \subset D$. It remains to show that for any $C \in \mathcal{C}$, either $C \subset D^\perp$ or $D^\perp \subset C$. But this follows immediately from the fact that either $D \subset C^\perp$ or $C^\perp \subset D$, since $C^\perp \in \mathcal{C}$ and C is closed. Hence a chain which is maximal with respect to chains $\mathcal{C}$ of closed $\mathfrak{b}$-stable subspaces containing M such that $\mathcal{C}^\perp \subset \mathcal{C}$ is in fact a maximal chain of closed $\mathfrak{b}$-stable subspaces. ∎

6. Isotropic subspaces in the standard representation of $\mathfrak{sp}_\infty$

In this section we assume $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{sp}(V)$. The propositions in this case are completely analogous to those in the previous section, but their proofs admit significant simplifications.

Lemma 6.1. *Suppose $M \subset V$ is a maximal $\mathfrak{b}$-stable isotropic subspace. If G' and G'' are $\mathfrak{b}$-stable subspaces with $M \subset G' \subset G'' \subset M^\perp$ and $\dim G''/G' = 1$, then $G'' \cap (G')^\perp = M$.*

Proof. Observe that $M \subset (G')^\perp \subset M^\perp$. Since $G' \cap (G')^\perp$ is $\mathfrak{b}$-stable and isotropic, and moreover $G' \cap (G')^\perp$ contains M, the maximality of M implies that $G' \cap (G')^\perp = M$. The inclusion $M = G' \cap (G')^\perp \subset G'' \cap (G')^\perp$ has codimension 0 or 1. Suppose, for the sake of a contradiction, that $G'' \cap (G')^\perp = M \oplus \mathbb{C}x$. Then $x \in M^\perp$, and $\langle x, x\rangle = 0$ since the pairing $\langle \cdot, \cdot \rangle$ is antisymmetric. Hence $\langle M \oplus \mathbb{C}x, M \oplus \mathbb{C}x\rangle = 0$, and $M \oplus \mathbb{C}x$ is isotropic. It is also $\mathfrak{b}$-stable. This contradicts the maximality of M. Hence $G'' \cap (G')^\perp = M$. ∎

Proposition 6.2. *A maximal $\mathfrak{b}$-stable isotropic subspace of V is maximal isotropic.*

Proof. Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_\alpha$ be defined in the same fashion as in the proof of Proposition 5.2. Suppose $Y \in \mathfrak{b}$. Again, $Y \in M \otimes V + \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp + V \otimes M$, so $Y = \sum_i v_i \otimes w_i + Z$, for some $v_i \in F''_{\alpha_i} \setminus F'_{\alpha_i}$ and $w_i \in (F'_{\alpha_i})^\perp \setminus M$ and $Z \in M \otimes V + V \otimes M$. One may safely assume that the set $\{v_i\}$ is linearly independent modulo M and modulo F'_β for all β.

Since $Y \in \mathrm{Sym}^2(V)$, we calculate $Y = \sigma(Y) = \sum_i w_i \otimes v_i + \sigma(Z)$. Hence

$$\sum_i v_i \otimes w_i - w_i \otimes v_i = Z - \sigma(Z). \tag{2}$$

As in the proof of Proposition 5.2, the right hand side of (2) is an element of $M \otimes M^{\perp} + M^{\perp} \otimes M$.

For each i we have $w_i \in F''_{\beta_i} \setminus F'_{\beta_i}$, where $M \subset F'_{\beta_i} \subset F''_{\beta_i} \subset M^{\perp}$. Since $w_i \in F''_{\beta_i} \cap (F'_{\alpha_i})^{\perp}$, we obtain from Lemma 6.1 that $\beta_i > \alpha_i$. The rest of the proof follows the same outline as the proof of Proposition 5.2, with the simplification that Case (1) has already been ruled out. ∎

Proposition 6.3. *There exists a maximal isotropic subspace $M \subset V$ which is stable under $\mathfrak{b}$. Furthermore, there exists a maximal chain $\mathcal{C}$ of closed $\mathfrak{b}$-stable subspaces in V containing M, with the additional property that $\mathcal{C}^{\perp} \subset \mathcal{C}$.*

The proof is identical to that of Proposition 5.3.

7. Maximal closed generalized flags

The following proposition is an improvement of Theorem 4.1 in the special cases of the infinite-dimensional simple root-reductive Lie algebras. The method of proof works also for Borel subalgebras of $\mathfrak{gl}(V, V_*)$.

Proposition 7.1. *Any Borel subalgebra of $\mathfrak{sl}(V, V_*)$ is the stabilizer of a maximal closed generalized flag in V. Any Borel subalgebra of $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ is the stabilizer of a maximal closed generalized flag $\mathfrak{F}$ in V with $\mathfrak{F} \cup \mathfrak{F}^{\perp} \cup \{M, M^{\perp}\}$ a chain for some maximal isotropic subspace $M \subset V$.*

Proof. If $\mathfrak{g} = \mathfrak{sl}(V, V_*)$, let $\mathcal{C}$ be a maximal chain of closed $\mathfrak{b}$-stable subspaces in V. If $\mathfrak{g} = \mathfrak{so}(V)$, let M be a $\mathfrak{b}$-stable maximal isotropic subspace in V, and let $\mathcal{C}$ be a maximal chain of closed $\mathfrak{b}$-stable subspaces in V, with $M \in \mathcal{C}$ and $\mathcal{C}^{\perp} \subset \mathcal{C}$, as in Proposition 5.3. If $\mathfrak{g} = \mathfrak{sp}(V)$, let M be a $\mathfrak{b}$-stable maximal isotropic subspace in V, and let $\mathcal{C}$ be a maximal chain of closed $\mathfrak{b}$-stable subspaces in V, with $M \in \mathcal{C}$ and $\mathcal{C}^{\perp} \subset \mathcal{C}$, as in Proposition 6.3.

Let $\mathfrak{F} := \mathrm{fl}(\mathcal{C})$, as in the proof of Theorem 4.1. Observe that if $\mathfrak{g}$ is one of $\mathfrak{so}(V)$ and $\mathfrak{sp}(V)$, then $\mathfrak{F} \cup \mathfrak{F}^{\perp} \cup \{M, M^{\perp}\}$ is a chain. That is, the maximality of $\mathcal{C}$ implies that $\mathfrak{F} \cup \mathcal{C}$ is a chain, and that $\mathfrak{F}^{\perp} \subset \mathcal{C}$. Since $M, M^{\perp} \in \mathcal{C}$, indeed $\mathfrak{F} \cup \mathfrak{F}^{\perp} \cup \{M, M^{\perp}\}$ is a chain.

We will show that $\mathfrak{F}$ is a maximal closed generalized flag. By the proof of Theorem 4.1, $\mathfrak{F}$ is a bivalent closed generalized flag, so it remains to show that every good pair of $\mathfrak{F}$ has codimension 1.

Suppose, for the sake of a contradiction, that there exists a good pair $F' \subset F''$ of $\mathfrak{F}$ with $\dim F''/F' = \infty$. Let $\mathcal{D}$ be a maximal chain of $\mathfrak{b}$-stable subspaces between F' and F'', and let $\mathfrak{G} := \mathrm{fl}(\mathcal{D})$. Consider $\mathfrak{G} = \{G'_\beta, G''_\beta\}_\beta$. It was seen in the proof of Theorem 4.1 that $\overline{G'_\beta} = F''$ for all β. That is, $(G'_\beta)^{\perp} = (F'')^{\perp}$ for all β.

Of course $\mathfrak{b}$ stabilizes the generalized flag $0 \subset F' \subset \mathfrak{G} \subset F'' \subset V$. Now consider $\mathfrak{g} \subset \mathfrak{gl}(V, V_*)$, where in the isotropic cases $V_* = V$. By Lemma 3.1, the stabilizer in $\mathfrak{gl}(V,V_*)$ of the generalized flag $0\subset F'\subset\mathfrak{G}\subset F''\subset V$ is

$$\begin{aligned}\mathrm{St}_{0\subset F'\subset\mathfrak{G}\subset F''\subset V} &= F'\otimes V_* + \sum_\beta G''_\beta\otimes(G'_\beta)^\perp + V\otimes(F'')^\perp\\ &= F'\otimes V_* + \sum_\beta G''_\beta\otimes(F'')^\perp + V\otimes(F'')^\perp\\ &= F'\otimes V_* + V\otimes(F'')^\perp.\end{aligned}$$

Hence $\mathfrak{b}\cdot F''\subset F'$, i.e. $\mathfrak{b}$ stabilizes any subspace between F' and F''. This contradicts the fact that there are no closed $\mathfrak{b}$-stable subspaces between F' and F''.

This concludes the proof that $\mathfrak{F}$ is a maximal closed generalized flag in V. It was previously noted that if $\mathfrak{g}$ is $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, then $\mathfrak{F}\cup\mathfrak{F}^\perp\cup\{M,M^\perp\}$ is a chain. Moreover, the proof of Theorem 4.1 gives that $\mathfrak{b}=\mathrm{St}_\mathfrak{F}$. ■

8. Borel subalgebras of $\mathfrak{sl}_\infty$

In this section it is shown that Borel subalgebras of $\mathfrak{sl}_\infty$ correspond to maximal closed generalized flags in the standard representation. Let $\mathfrak{b}\subset\mathfrak{sl}(V,V_*)$ be a Borel subalgebra. Here we denote by $\mathrm{St}_\mathfrak{F}$ the stabilizer in $\mathfrak{sl}(V,V_*)$ of any generalized flag $\mathfrak{F}$ in V.

Lemma 8.1. *Let $\mathfrak{F}$ be a maximal closed generalized flag in V. For any $u\in V$, one of the following cases occurs:*

$$\mathrm{St}_\mathfrak{F}\cdot u=\begin{cases}F'_u & \text{if } \overline{F'_u}=F''_u;\\ F'_u & \text{if } F'_u\subset F''_u \text{ is the only good pair of } \mathfrak{F};\\ F''_u & \text{otherwise.}\end{cases}$$

Proof. Fix $u\in V$. Consider $\mathfrak{F}=\{F'_\alpha,F''_\alpha\}_{\alpha\in A}$. There are three cases to consider.

Suppose first that there exists $\alpha\in A$ for which $(F'_\alpha)^\perp\cap u^\perp\not\subseteq(F''_\alpha)^\perp$. Then there exists $y\in(F'_\alpha)^\perp\cap u^\perp$ such that $y\notin(F''_\alpha)^\perp$. Hence there exists $x\in F''_\alpha$ such that $\langle x,y\rangle=1$. Then $\mathrm{St}_\mathfrak{F}=\mathrm{Span}_{\alpha\in A}\{v\otimes w-\langle v,w\rangle x\otimes y : v\in F''_\alpha, w\in(F'_\alpha)^\perp\}$. Let $v\in F''_\alpha$ and $w\in(F'_\alpha)^\perp$. Since $(v\otimes w-\langle v,w\rangle x\otimes y)\cdot u=\langle u,w\rangle v-\langle u,y\rangle x=\langle u,w\rangle v$, indeed $\mathrm{St}_\mathfrak{F}\cdot u=\bigcup_{u\notin\overline{F'_\alpha}}F''_\alpha$. It is easy to check that

$$\mathrm{St}_\mathfrak{F}\cdot u=\begin{cases}F'_u & \text{if } \overline{F'_u}=F''_u;\\ F''_u & \text{otherwise.}\end{cases}$$

Suppose second that $(F'_\alpha)^\perp=(F''_\alpha)^\perp$ for all $\alpha\in A$. Then the stabilizer of $\mathfrak{F}$ in $\mathfrak{gl}(V,V_*)$ is traceless because it is given by the formula $\sum_\alpha F''_\alpha\otimes(F'_\alpha)^\perp$. Let $v\in F''_\alpha$ and $w\in(F'_\alpha)^\perp$. Since $(v\otimes w)\cdot u=\langle u,w\rangle v$, indeed $\mathrm{St}_\mathfrak{F}\cdot u=\bigcup_{u\notin\overline{F'_\alpha}}F''_\alpha$. Again

$$\mathrm{St}_\mathfrak{F}\cdot u=\begin{cases}F'_u & \text{if } \overline{F'_u}=F''_u;\\ F''_u & \text{otherwise.}\end{cases}$$

Suppose third that $(F'_\alpha)^\perp \cap u^\perp \subset (F''_\alpha)^\perp$ for all $\alpha \in A$ and that there exists $\gamma \in A$ for which $(F'_\gamma)^\perp \neq (F''_\gamma)^\perp$. Then $(F'_\gamma)^\perp \cap u^\perp \subset (F''_\gamma)^\perp \subsetneq (F'_\gamma)^\perp$ implies that $(F'_\gamma)^\perp \cap u^\perp = (F''_\gamma)^\perp$. Thus $(F''_\gamma)^\perp \subset u^\perp$, and hence $u \in \overline{F''_\gamma} = F''_\gamma$. If $u \in F'_\gamma$, then $u^\perp \cap (F'_\gamma)^\perp = (F'_\gamma)^\perp$. Hence $u \in F''_\gamma \setminus F'_\gamma$. This argument implies that $\mathfrak{F}$ has exactly one good pair. One may check that

$$\mathrm{St}_{\mathfrak{F}} = (\sum_{\alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp) \cap \mathfrak{sl}(V, V_*) = \sum_{\gamma \neq \alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp + F''_\gamma \otimes (F''_\gamma)^\perp.$$

In this case, $\mathrm{St}_{\mathfrak{F}} \cdot u = F'_u$. ■

Lemma 8.2. *If $\mathfrak{F}$ and $\mathfrak{G}$ are maximal closed generalized flags in V with $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, then $\mathfrak{F} = \mathfrak{G}$.*

Proof. Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ and $\mathfrak{G} = \{G'_\beta, G''_\beta\}_{\beta \in B}$. For each $\alpha \in A$ choose $u_\alpha \in F''_\alpha \setminus F'_\alpha$. If there is exactly one $\gamma \in A$ such that $\overline{G'_\gamma} = G'_\gamma$, define $A' := A \setminus \{\gamma\}$. Otherwise, let $A' := A$.

Since $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, it follows that

$$\overline{\mathrm{St}_{\mathfrak{F}} \cdot u_\alpha} \subset \overline{\mathrm{St}_{\mathfrak{G}} \cdot u_\alpha}.$$

For any $\alpha \in A'$, Lemma 8.1 implies that $\overline{\mathrm{St}_{\mathfrak{F}} \cdot u_\alpha} = F''_\alpha$. Therefore for any $\alpha \in A'$, indeed $F''_\alpha = F''_{u_\alpha} = \overline{\mathrm{St}_{\mathfrak{F}} \cdot u_\alpha} \subset \overline{\mathrm{St}_{\mathfrak{G}} \cdot u_\alpha} \subset G''_{u_\alpha}$.

We will show that $F''_\alpha = G''_{u_\alpha}$ for all $\alpha \in A'$. There are two cases to consider.

1. Suppose $\overline{F'_\alpha} = F''_\alpha$. Then Lemma 8.1 implies that for any $u \notin F''_\alpha$, indeed $F''_\alpha \subset \mathrm{St}_{\mathfrak{F}} \cdot u$. Observe that G'_{u_α} is stable under $\mathrm{St}_{\mathfrak{F}}$ and $u_\alpha \notin G'_{u_\alpha}$. It follows that $G'_{u_\alpha} \subset F''_\alpha$. Thus $G'_{u_\alpha} \subset F''_\alpha \subset G''_{u_\alpha}$. Since $u_\alpha \in F''_\alpha$, it must be that $G'_{u_\alpha} \subsetneq F''_\alpha$. Since $\mathfrak{G}$ is a maximal closed generalized flag, necessarily $F''_\alpha = G''_{u_\alpha}$.

2. Suppose F'_α is closed. Then Lemma 8.1 implies that for any $u \notin F'_\alpha$, indeed $F''_\alpha \subset \mathrm{St}_{\mathfrak{F}} \cdot u$. Observe that G'_{u_α} is stable under $\mathrm{St}_{\mathfrak{F}}$ and $u_\alpha \notin G'_{u_\alpha}$. It follows that $G'_{u_\alpha} \subset F'_\alpha$. Thus $G'_{u_\alpha} \subset F'_\alpha \subset F''_\alpha \subset G''_{u_\alpha}$. Since $\mathfrak{G}$ is a maximal closed generalized flag, necessarily $F''_\alpha = G''_{u_\alpha}$.

If $A = A'$, then the proof is done, since a generalized flag is determined by its set of successors. Assume therefore that $A \neq A'$, in which case it remains to show that $F''_\gamma = G''_{u_\gamma}$. Observe first that $F'_\gamma = \bigcup_{\alpha < \gamma} F''_\alpha = \bigcup_{\alpha < \gamma} G''_{u_\alpha}$. Since $\dim F'_\alpha / F''_\gamma = \infty$ for any $\alpha > \gamma$, it must hold that

$$F''_\gamma = \bigcap_{\alpha > \gamma} F'_\alpha = \bigcap_{\alpha > \gamma} F''_\alpha = \bigcap_{\alpha > \gamma} G''_{u_\alpha}.$$

Since $\dim F''_\gamma / F'_\gamma = 1$, it follows that $F'_\gamma \subset F''_\gamma$ is a pair in $\mathfrak{G}$. Thus $F''_\gamma = G''_{u_\gamma}$, and consequently $\mathfrak{F} = \mathfrak{G}$. ■

The following result fully describes Borel subalgebras of $\mathfrak{sl}(V, V_*)$.

Theorem 8.3. *A subalgebra of $\mathfrak{sl}(V, V_*)$ is a Borel subalgebra if and only if it is the stabilizer of a maximal closed generalized flag in V. Furthermore, the map $\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$ is a bijection between maximal closed generalized flags in V and Borel subalgebras of $\mathfrak{sl}(V, V_*)$.*

Proof. Let $\mathfrak{F}$ be an arbitrary maximal closed generalized flag in V. Because $\mathrm{St}_{\mathfrak{F}}$ equals the stabilizer of any maximal generalized flag refining $\mathfrak{F}$, Proposition 3.2 yields that $\mathrm{St}_{\mathfrak{F}}$ is locally solvable. Hence there exists a Borel subalgebra $\mathfrak{b}$ with $\mathrm{St}_{\mathfrak{F}} \subset \mathfrak{b}$. By Proposition 7.1, there is a maximal closed generalized flag $\mathfrak{G}$ in V with $\mathfrak{b} = \mathrm{St}_{\mathfrak{G}}$. It follows from Lemma 8.2 that $\mathfrak{F} = \mathfrak{G}$. As a result, $\mathrm{St}_{\mathfrak{F}} = \mathfrak{b}$ is a Borel subalgebra. Hence $\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$ gives a map from maximal closed generalized flags in V to Borel subalgebras of $\mathfrak{sl}(V, V_*)$. Proposition 7.1 implies that the map is surjective, and Lemma 8.2 implies that it is injective. ■

9. Borel subalgebras of $\mathfrak{so}_\infty$

In this section it is shown that Borel subalgebras of $\mathfrak{so}_\infty$ almost correspond to maximal closed isotropic generalized flags in the standard representation. Let $\mathfrak{b} \subset \mathfrak{so}(V)$ be a Borel subalgebra. Here we denote by $\mathrm{St}_{\mathfrak{F}}$ the stabilizer in $\mathfrak{so}(V)$ of any generalized flag $\mathfrak{F}$ in V, and we denote by $\mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$ the stabilizer of $\mathfrak{F}$ in $\mathfrak{gl}(V, V)$. Of course, $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathfrak{F},\mathfrak{gl}} \cap \mathfrak{so}(V)$.

Definition 9.1. Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ and $\mathfrak{G} = \{G'_\beta, G''_\beta\}_{\beta \in B}$ be maximal closed isotropic generalized flags. We say that $\mathfrak{F}$ and $\mathfrak{G}$ are *twins* if A and B have maximal elements, denoted ∞, such that:

1. $\{F'_\alpha, F''_\alpha\}_{\alpha \in A \setminus \{\infty\}} = \{G'_\beta, G''_\beta\}_{\beta \in B \setminus \{\infty\}}$;
2. F'_∞ is closed and $\dim (F'_\infty)^\perp / F'_\infty = 2$; and
3. $F''_\infty \neq G''_\infty$ are the two maximal isotropic subspaces containing F'_∞.

Condition (1) of this definition forces $F'_\infty = G'_\infty$. As for condition (3), it makes sense after condition (2) because whenever $L \subset V$ is a closed isotropic subspace with $\dim L^\perp / L = 2$, there are exactly two maximal isotropic subspaces containing L. We say that $\mathfrak{F}$ *has a twin* if $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ is a maximal closed isotropic generalized flag with a maximal element ∞, such that F'_∞ is closed and $(F''_\infty)^\perp = F''_\infty$. If $\mathfrak{F}$ has a twin, let $\mathrm{tw}(\mathfrak{F})$ denote the twin of $\mathfrak{F}$. That is, $\mathrm{tw}(\mathfrak{F})$ is obtained from $\mathfrak{F}$ by replacing F''_∞ with the other maximal isotropic subspace containing F'_∞. Note that tw is an involution on the set of maximal closed isotropic generalized flags that have twins. Generalizing a phenomenon already present in the case of $\mathfrak{so}_{2n}$, the maximal closed isotropic generalized flags $\mathfrak{F}$ and $\mathrm{tw}(\mathfrak{F})$ have the same stabilizer in $\mathfrak{so}(V)$.

Lemma 9.2. *Let $\mathfrak{F}$ be a maximal generalized flag in V, and assume that $\mathfrak{F} \cup \mathfrak{F}^\perp \cup \{M, M^\perp\}$ is a chain for some maximal isotropic subspace $M \subset V$. Then $\mathrm{St}_{\mathfrak{F}_{iso}} = \mathrm{St}_{\mathfrak{F}}$.*

Proof. Clearly $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{F}_{iso}}$. Let $Z \in \mathrm{St}_{\mathfrak{F}_{iso}}$ be arbitrary.

Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$. One first shows that $\mathrm{St}_{\mathfrak{F}} = \sum_{\alpha \in A, F''_\alpha \subset M} F''_\alpha \wedge (F'_\alpha)^\perp$.

For any $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$, on the one hand $x \otimes y \in \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$, but on the other hand since $\mathfrak{F} \cup \mathfrak{F}^\perp$ is a chain, in fact $y \otimes x \in \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$. In detail, there exists $\beta \in A$ for which $y \in F''_\beta \setminus F'_\beta$. Since $\mathfrak{F} \cup \mathfrak{F}^\perp$ is a chain, and $y \in (F'_\alpha)^\perp$ and $y \notin F'_\beta$, it follows that $F'_\beta \subsetneq (F'_\alpha)^\perp$. Moreover since $\dim (F'_\alpha)^\perp/(F''_\alpha)^\perp \leq 1$, it must be that $F'_\beta \subset (F''_\alpha)^\perp$, and thus $F''_\alpha = (F''_\alpha)^{\perp\perp} \subset (F'_\beta)^\perp$. So $x \in (F'_\beta)^\perp$, and hence $y \otimes x \in F''_\beta \otimes (F'_\beta)^\perp \subset \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$. Thus the map of vector spaces (which is *not* a map of Lie algebras):

$$\begin{aligned} \varphi : \sum_{\alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp &\to \bigwedge^2 V \\ x \otimes y &\mapsto x \otimes y - y \otimes x \end{aligned}$$

in fact has its image in $\mathrm{St}_{\mathfrak{F}}$. As $\varphi|_{\mathrm{St}_{\mathfrak{F}}} = 2 \cdot \mathrm{Id}$, indeed φ maps surjectively onto $\mathrm{St}_{\mathfrak{F}}$. Because $\sum F''_\alpha \otimes (F'_\alpha)^\perp$ is spanned by elements of the form $x \otimes y$, with $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$ for some $\alpha \in A$, likewise $\mathrm{St}_{\mathfrak{F}}$ is spanned by elements of the form $x \otimes y - y \otimes x$, with $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$ for some $\alpha \in A$.

Suppose $M \neq M^\perp$. Observe that $M \subset M^\perp$ is a pair in the generalized flag $\mathfrak{F}$. In this case, $\mathrm{St}_{\mathfrak{F}}$ is in fact spanned by elements of the form $x \otimes y - y \otimes x$, with $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$ for $\alpha \in A$ which are not equal to the pair $M \subset M^\perp$. To see this, consider that the term in $\mathrm{St}_{\mathfrak{F}}$ corresponding to the pair $M \subset M^\perp$ is $M^\perp \otimes M^\perp$. Let $m \in M^\perp \setminus M$. Observe that

$$M^\perp \otimes M^\perp = M^\perp \otimes M + M \otimes M^\perp + \mathbb{C}(m \otimes m).$$

Now $M^\perp \otimes M + M \otimes M^\perp \subset \sum_{(M \subset M^\perp) \neq \alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp$ and $m \otimes m \in \mathrm{Sym}^2(V)$. Since σ fixes $M^\perp \otimes M + M \otimes M^\perp$ and $m \otimes m$, it follows that

$$\mathrm{St}_{\mathfrak{F}} = \Big(\sum_{(M \subset M^\perp) \neq \alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp \Big) \cap \bigwedge^2 V.$$

Moreover, the stabilizer $\mathrm{St}_{\mathfrak{F}}$ is spanned by elements of the form $x \otimes y - y \otimes x$ with $x \in F''_\alpha \subset M$ and $y \in (F'_\alpha)^\perp$. (Explicitly, if $M^\perp \subset F'_\alpha$, then one has $(F'_\alpha)^\perp \subset M^{\perp\perp} = M$.) This concludes the proof that $\mathrm{St}_{\mathfrak{F}} = \sum_{\alpha \in A, F''_\alpha \subset M} F''_\alpha \wedge (F'_\alpha)^\perp$.

Now

$$\mathrm{St}_{\mathfrak{F}_{iso},\mathfrak{gl}} = \sum_{F''_\alpha \subset M} F''_\alpha \otimes (F'_\alpha)^\perp + V \otimes M^\perp,$$

because it is the stabilizer of $\mathfrak{F}_{iso} \cup \{M \subset V\}$, which is a generalized flag in V. Then $Z = X + Y$ for some $X \in \sum_{F''_\alpha \subset M} F''_\alpha \otimes (F'_\alpha)^\perp$ and $Y \in V \otimes M^\perp$.

Note that $Z = -\sigma(Z)$, i.e. $X + Y = -\sigma(X) - \sigma(Y)$. A rearrangement yields $Y + \sigma(X) = -\sigma(Y) - X$, and the left hand side of this equation is clearly an element of $V \otimes M^\perp$, while the righthand side is clearly an element of $M^\perp \otimes V$. So $Y + \sigma(X) \in (V \otimes M^\perp) \cap (M^\perp \otimes V) = M^\perp \otimes M^\perp$. Now

$$\sigma(Y + \sigma(X)) = \sigma(Y) + X = -(Y + \sigma(X)),$$

and therefore $Y + \sigma(X) \in (M^\perp \otimes M^\perp) \cap \bigwedge^2(V) = \bigwedge^2 M^\perp$. Let $\eta := Y + \sigma(X)$, so $Z = X - \sigma(X) + \eta$. Clearly $X - \sigma(X) \in \mathrm{St}_{\mathfrak{F}}$. Observe that either $M \subset M^\perp$ is a pair in $\mathfrak{F}$, in which case $M^\perp \otimes M^\perp \subset \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$, or else $M = M^\perp$, in which case $M \otimes M \subset \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$. Since $\eta \in \bigwedge^2 M^\perp \subset \mathrm{St}_{\mathfrak{F}}$, it follows that $Z \in \mathrm{St}_{\mathfrak{F}}$. ∎

Lemma 9.3. *Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha\in A}$ be a maximal closed isotropic generalized flag in V. Then $\mathrm{St}_\mathfrak{F} = \sum_{\alpha\in A} F''_\alpha \wedge (F'_\alpha)^\perp$, and moreover $\mathrm{St}_\mathfrak{F}$ is locally solvable.*

Proof. Let M denote $\bigcup_{\alpha\in A} F''_\alpha$, which is a maximal isotropic subspace of V. Let $\mathcal{C}$ be any maximal chain in V containing $\mathfrak{F}\cup\mathfrak{F}^\perp$, and let $\mathfrak{H} := \mathrm{fl}(\mathcal{C})$. Note that $\mathfrak{H}_{iso}$ is a refinement of $\mathfrak{F}$. We will show that $\mathfrak{H}\cup\mathfrak{H}^\perp\cup\{M, M^\perp\}$ is a chain. Then Lemma 9.2 gives that $\mathrm{St}_\mathfrak{H} = \mathrm{St}_{\mathfrak{H}_{iso}}$. Since $\mathfrak{H}_{iso}$ is a refinement of $\mathfrak{F}$, and moreover since $\mathfrak{F}$ is a maximal closed isotropic generalized flag in M, it holds that $\mathrm{St}_{\mathfrak{H}_{iso}} = \mathrm{St}_\mathfrak{F}$. By Proposition 3.2, $\mathrm{St}_{\mathfrak{H},\mathfrak{gl}}$ is locally solvable. Hence $\mathrm{St}_\mathfrak{F} = \mathrm{St}_{\mathfrak{H}_{iso}} = \mathrm{St}_\mathfrak{H} \subset \mathrm{St}_{\mathfrak{H},\mathfrak{gl}}$ is locally solvable. The formula for $\mathrm{St}_\mathfrak{F}$ is seen in the proof of Lemma 9.2 to be a formula for $\mathrm{St}_\mathfrak{H} = \mathrm{St}_{\mathfrak{H}_{iso}}$.

It remains to show that $\mathfrak{H}\cup\mathfrak{H}^\perp\cup\{M, M^\perp\}$ is a chain. Clearly M and $M^\perp = \bigcap_{F\in\mathfrak{F}} F^\perp$ are automatically compatible with $\mathcal{C}$, and they remain compatible with $\mathfrak{H}$, and consequently also with $\mathfrak{H}^\perp$. Now suppose $H, I \in \mathfrak{H}$, and one must show that either $H^\perp \subset I$ or $I \subset H^\perp$. If H and I are both isotropic, then $I \subset M \subset M^\perp \subset H^\perp$. If H and I are both coisotropic, then $H^\perp \subset M \subset M^\perp \subset I$. It remains to deal with the cases

- $H \subset M \subset M^\perp \subset I$;
- $I \subset M \subset M^\perp \subset H$.

In the first case, $F' \subset H \subset F''$ for some immediate predecessor-successor pair $F' \subset F''$ in $\mathfrak{F}$. Thus $(F'')^\perp \subset H^\perp \subset (F')^\perp$. Since $\dim (F')^\perp/(F'')^\perp \le 1$, it must be the case that $H^\perp$ is either $(F')^\perp$ or $(F'')^\perp$. Since $\mathfrak{F}^\perp\cup\{I\}$ is a chain, $H^\perp$ either contains or is contained in I.

In the second case, $F' \subset I \subset F''$ for some immediate predecessor-successor pair $F' \subset F''$ in $\mathfrak{F}$. Since $\dim(F')^\perp/(F'')^\perp \le 1$, either $H \subset (F'')^\perp$ or $(F')^\perp \subset H$. If $H \subset (F'')^\perp$, then $I \subset F'' = \overline{F''} \subset H^\perp$, i.e. $I \subset H^\perp$.

Now assume that $(F')^\perp \subset H$. Suppose there exists $F \in \mathfrak{F}$ with $F^\perp \subset H$ and $F \subsetneq F'$. Then $H \subset \overline{F} \subset F' \subset I$, and there is nothing left to show. It remains to treat the case when $H \subset F^\perp$ for all $F \in \mathfrak{F}$ with $F \subsetneq F'$. Of course $F' = \bigcup_{F\subsetneq F'} F$. Hence $(F')^\perp = (\bigcup_{F\subsetneq F'} F)^\perp = \bigcap_{F\subsetneq F'} F^\perp$. So

$$H \subset \bigcap_{F\subsetneq F'} F^\perp = (F')^\perp \subset H.$$

Hence $H = (F')^\perp$, i.e. $H^\perp = \overline{F'}$ which is either F' or F''. Since $\mathfrak{F}\cup\{I\}$ is a chain, $H^\perp$ either contains or is contained in I. ■

Lemma 9.4. *Let $\mathfrak{F}$ be a maximal closed isotropic generalized flag in V. Suppose $u \in \bigcup_{F\in\mathfrak{F}} F$. Then*

$$\mathrm{St}_\mathfrak{F}\cdot u = \begin{cases} F''_u & \text{if } \overline{F'_u} = F'_u \\ F'_u & \text{if } \overline{F'_u} = F''_u. \end{cases}$$

Thus $\overline{\mathrm{St}_\mathfrak{F}\cdot u} = F''_u$.

Proof. Let $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$, and let $u \in \bigcup_\alpha F''_\alpha$. Lemma 9.3 states that $\mathrm{St}_{\mathfrak{F}} = \sum_\alpha F''_\alpha \wedge (F'_\alpha)^\perp$. Fix $\beta \in A$, and let $x \in F''_\beta$ and $y \in (F'_\beta)^\perp$. Then $(x \otimes y - y \otimes x) \cdot u = \langle u, y\rangle x - \langle u, x\rangle y = \langle u, y\rangle x$, since $\langle \bigcup_\alpha F''_\alpha, \bigcup_\alpha F''_\alpha\rangle = 0$. Hence $\mathrm{St}_{\mathfrak{F}} \cdot u = \bigcup_{u \notin \overline{F'_\beta}} F''_\beta$. The lemma follows easily. ∎

Lemma 9.5. *Suppose $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ and $\mathfrak{G} = \{G'_\beta, G''_\beta\}_{\beta \in B}$ are maximal closed isotropic generalized flags in V with $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$. If $\bigcup_{\alpha \in A} F''_\alpha$ is not equal to $\bigcup_{\beta \in B} G''_\beta$, then A and B have maximal elements ∞ with $F'_\infty = G'_\infty$ closed and $\dim (F'_\infty)^\perp / F'_\infty = 2$.*

Proof. Let $M := \bigcup_{F \in \mathfrak{F}} F$ and $N := \bigcup_{G \in \mathfrak{G}} G$, and suppose $M \neq N$. The maximality of $\mathfrak{F}$ and $\mathfrak{G}$ implies that both M and N are maximal isotropic subspaces. Thus neither M nor N contains the other, and there exist $m \in M \setminus N$ and $n \in N \setminus M$. There exists $\alpha \in A$ for which $m \in F''_\alpha \setminus F'_\alpha$. For any $y \in (F'_\alpha)^\perp$, it holds that $m \otimes y - y \otimes m \in \mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$. Since $m \notin N$, indeed $y - cm \in N$ for some $c \in \mathbb{C}$. Hence $(F'_\alpha)^\perp \subset N \oplus \mathbb{C}m$. As a result $M \subset M^\perp \subset (F'_\alpha)^\perp \subset N \oplus \mathbb{C}m$. Hence $M = (M \cap N) \oplus \mathbb{C}m$. Of course $M \cap N$ is necessarily closed, being the intersection of two closed subspaces of V.

Consider the chain $M \cap N \subset M \subset M^\perp \subset (M \cap N)^\perp$. Since $\dim M^\perp/M \leq 1$, and $\dim M/(M \cap N) = \dim (M \cap N)^\perp / M^\perp = 1$, it must be that the dimension of $(M \cap N)^\perp/(M \cap N)$ is either 2 or 3.

Also note that $n \in (M \cap N)^\perp$, since N is isotropic. Observe that

$$M \cap N \subset N \subset N^\perp \subset (M \cap N)^\perp,$$

and since $\dim N^\perp/N \leq 1$, and $\dim N/(M \cap N) = \dim (M \cap N)^\perp/N^\perp$, it must be the case that $\dim N/(M \cap N) = 1$. Thus $N = (M \cap N) \oplus \mathbb{C}n$.

Suppose, for the sake of a contradiction, that $\dim (M \cap N)^\perp/(M \cap N) = 3$. Then there exist $u \in M^\perp \setminus M$ and $v \in (M \cap N)^\perp \setminus M^\perp$ with the properties that $\langle u, u\rangle = \langle m, v\rangle = 1$ and $\langle u, v\rangle = \langle v, v\rangle = 0$. So $n = am + bu + cv + x$ for some $a, b, c \in \mathbb{C}$ and $x \in M \cap N$. Now $m \otimes u - u \otimes m \in \mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, and $\mathrm{St}_{\mathfrak{G}} \cdot N \subset N$, so

$$\begin{aligned}
(m \otimes u - u \otimes m) \cdot n &= (m \otimes u - u \otimes m) \cdot (am + bu + cv + x) \\
&= \langle am + bu + cv + x, u\rangle m - \langle am + bu + cv + x, m\rangle u \\
&= bm - cu \\
&\in N = (M \cap N) \oplus \mathbb{C}(am + bu + cv + x).
\end{aligned}$$

Therefore $(bm - cu) - \lambda(am + bu + cv) \in M \cap N$ for some $\lambda \in \mathbb{C}$. Equivalently one has $(b - \lambda a)m - (c + \lambda b)u - \lambda cv = 0$, which implies $b = c = 0$. It follows that $N = M$. This contradicts the hypothesis that $M \neq N$. It follows that $\dim (M \cap N)^\perp/(M \cap N) = 2$.

Since $n \in (M \cap N)^\perp \setminus M^\perp$, necessarily $\langle m, n\rangle \neq 0$. Assume without loss of generality that $\langle m, n\rangle = 1$. It remains to show that A and B have elements ∞ such that $F'_\infty = G'_\infty = M \cap N$.

We will show that if $F''_\alpha \not\subseteq M \cap N$, then $F''_\alpha = M$. Suppose there exists $z \in F''_\alpha \setminus M \cap N$. Rescaling z, it holds that $z = m + w$ for some

$w \in M \cap N$. For any $x \in M \cap N$, observe that $x \otimes n - n \otimes x \in \mathrm{St}_{\mathfrak{F}}$, and $(x \otimes n - n \otimes x) \cdot z = (x \otimes n - n \otimes x) \cdot (m + w) = \langle m + w, n\rangle x - \langle m + w, x\rangle n = x$. Hence $M \cap N \subset \mathrm{St}_{\mathfrak{F}} \cdot F''_\alpha \subset F''_\alpha \subset M$. Since $z \in F''_\alpha$ and $z \notin M \cap N$, it must be that $F''_\alpha = M$.

Likewise, if $G''_\beta \not\subseteq M \cap N$ then $G''_\beta = N$. Suppose there exists an element $z \in G''_\beta \setminus M \cap N$. The proof that $M \cap N \subset \mathrm{St}_{\mathfrak{F}} \cdot G''_\beta$ is analogous to the argument in the above paragraph. So $M \cap N \subset \mathrm{St}_{\mathfrak{F}} \cdot G''_\beta \subset \mathrm{St}_{\mathfrak{G}} \cdot G''_\beta \subset G''_\beta \subset N$. Since $z \in G''_\beta$ and $z \notin M \cap N$, it must be that $G''_\beta = N$.

Thus each of A and B has a maximal element ∞ with $F'_\infty = G'_\infty = M \cap N$. It was already observed that $M \cap N$ is closed, and it was also shown that $\dim(M \cap N)^\perp/(M \cap N) = 2$. ■

Lemma 9.6. *Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are maximal closed isotropic generalized flags in V with $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$. If $\mathfrak{F} \neq \mathfrak{G}$, then $\mathfrak{F}$ and $\mathfrak{G}$ are twins; in either case, $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathfrak{G}}$.*

Proof. Let $\mathfrak{F} = \{F''_\alpha, F'_\alpha\}_{\alpha \in A}$ and $\mathfrak{G} = \{G''_\beta, G''_\beta\}_{\beta \in B}$. If $\bigcup_{\alpha \in A} F''_\alpha \neq \bigcup_{\beta \in B} G''_\beta$, then let $\infty \in A, B$ be as in Lemma 9.5, and take $A' := A \setminus \{\infty\}$ and $B' := B \setminus \{\infty\}$. Otherwise, let $A' := A$ and $B' := B$. For each $\alpha \in A'$ choose $u_\alpha \in F''_\alpha \setminus F'_\alpha$.

Since $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, of course

$$\overline{\mathrm{St}_{\mathfrak{F}} \cdot u_\alpha} \subset \overline{\mathrm{St}_{\mathfrak{G}} \cdot u_\alpha}.$$

Lemma 9.5 implies that $u_\alpha \in \bigcup_{\beta \in B'} G''_\beta$, so by Lemma 9.4 that is $F''_\alpha = F''_{u_\alpha} \subset G''_{u_\alpha}$.

We will show that $F''_\alpha = G''_{u_\alpha}$ for all $\alpha \in A'$. There are two cases to consider.

1. Suppose $\overline{F'_\alpha} = F''_\alpha$. Then Lemma 8.1 implies that for any $u \notin F''_\alpha$, indeed $F''_\alpha \subset \mathrm{St}_{\mathfrak{F}} \cdot u$. Observe that G'_{u_α} is stable under $\mathrm{St}_{\mathfrak{F}}$ and $u_\alpha \notin G'_{u_\alpha}$. It follows that $G'_{u_\alpha} \subset F''_\alpha$. Thus $G'_{u_\alpha} \subset F''_\alpha \subset G''_{u_\alpha}$. Since $u_\alpha \in F''_\alpha$, it must be that $G'_{u_\alpha} \subsetneq F''_\alpha$. Since $\mathfrak{G}$ is a maximal closed generalized flag, necessarily $F''_\alpha = G''_{u_\alpha}$.

2. Suppose F'_α is closed. Then Lemma 8.1 implies that for any $u \notin F'_\alpha$, indeed $F''_\alpha \subset \mathrm{St}_{\mathfrak{F}} \cdot u$. Observe that G'_{u_α} is stable under $\mathrm{St}_{\mathfrak{F}}$ and $u_\alpha \notin G'_{u_\alpha}$. It follows that $G'_{u_\alpha} \subset F'_\alpha$. Thus $G'_{u_\alpha} \subset F'_\alpha \subset F''_\alpha \subset G''_{u_\alpha}$. Since $\mathfrak{G}$ is a maximal closed generalized flag, necessarily $F''_\alpha = G''_{u_\alpha}$.

On the one hand, if $\bigcup_{\alpha \in A} F''_\alpha = \bigcup_{\beta \in B} G''_\beta$, then $\mathfrak{F} = \mathfrak{G}$, since a generalized flag is determined by its successors. If, on the other hand, $\bigcup_{\alpha \in A} F''_\alpha \neq \bigcup_{\beta \in B} G''_\beta$, then we have shown that $\{F'_\alpha, F''_\alpha : \alpha \in A'\} = \{G'_\beta, G''_\beta : \beta \in B'\}$. Lemma 9.5 implies that $F'_\infty = G'_\infty$ is a closed isotropic subspace with $\dim(F'_\infty)^\perp/F'_\infty = 2$. There are precisely two maximal isotropic subspaces containing F'_∞, and they must be F''_∞ and G''_∞, respectively. Therefore $\mathfrak{G} = \mathrm{tw}(\mathfrak{F})$. We omit the proof of the fact that $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathrm{tw}(\mathfrak{F})}$. ■

The following result fully describes Borel subalgebras of $\mathfrak{so}(V)$.

Theorem 9.7. *A subalgebra of $\mathfrak{so}(V)$ is a Borel subalgebra if and only if it is the stabilizer of a maximal closed isotropic generalized flag in V. Furthermore, a fiber of the map*

$$\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$$

from maximal closed isotropic generalized flags in V to Borel subalgebras of $\mathfrak{so}(V)$ is either a single maximal closed isotropic generalized flag which has no twin, or a pair of twins.

Proof. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{so}(V)$. Proposition 7.1 states that $\mathfrak{b}$ is the stabilizer of a maximal closed generalized flag $\mathfrak{F}$ in V with $\mathfrak{F} \cup \mathfrak{F}^{\perp} \cup \{M, M^{\perp}\}$ being a chain for some maximal isotropic subspace $M \subset V$. By Lemma 9.2, $\mathfrak{b} = \mathrm{St}_{\mathfrak{F}_{iso}}$. Observe that $\mathfrak{F}_{iso}$ is a maximal closed isotropic generalized flag in V, since the union of the isotropic subspaces in $\mathfrak{F}$ must be M. Hence every Borel subalgebra of $\mathfrak{so}(V)$ is the stabilizer of a maximal closed isotropic generalized flag in V.

Let $\mathfrak{F}$ be an arbitrary maximal closed isotropic generalized flag in V. By Lemma 9.3, $\mathrm{St}_{\mathfrak{F}}$ is locally solvable. Hence there exists a Borel subalgebra $\mathfrak{b}$ with $\mathrm{St}_{\mathfrak{F}} \subset \mathfrak{b}$. We have seen that there is a maximal closed isotropic generalized flag $\mathfrak{G}$ with $\mathfrak{b} = \mathrm{St}_{\mathfrak{G}}$. It follows from Lemma 9.6 that $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathfrak{G}}$. This means that $\mathrm{St}_{\mathfrak{F}}$ is a Borel subalgebra. Hence $\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$ gives a map from maximal closed isotropic generalized flags in V to Borel subalgebras of $\mathfrak{so}(V)$. Proposition 7.1 implies that the map is surjective. Lemma 9.6 implies that if $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathfrak{G}}$, then either $\mathfrak{F} = \mathfrak{G}$, or $\mathfrak{F}$ and $\mathfrak{G}$ are twins. Since $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathrm{tw}(\mathfrak{F})}$ whenever $\mathfrak{F}$ has a twin, we have shown that a fiber of the map is either a single maximal closed isotropic generalized flag which has no twin, or a pair of twins. ∎

10. Borel subalgebras of $\mathfrak{sp}_{\infty}$

In this section it is shown that Borel subalgebras of $\mathfrak{sp}_{\infty}$ correspond to maximal closed isotropic generalized flags in the standard representation. Let $\mathfrak{b} \subset \mathfrak{sp}(V)$ be a Borel subalgebra. Here we denote by $\mathrm{St}_{\mathfrak{F}}$ the stabilizer in $\mathfrak{sp}(V)$ of any generalized flag $\mathfrak{F}$ in V, and we denote by $\mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$ the stabilizer of $\mathfrak{F}$ in $\mathfrak{gl}(V,V)$. Of course, $\mathrm{St}_{\mathfrak{F}} = \mathrm{St}_{\mathfrak{F},\mathfrak{gl}} \cap \mathfrak{sp}(V)$. If X and Y are subspaces of V, we denote their symmetrizer by $X \& Y := \{x \otimes y + y \otimes x : x \in Y, y \in Y\} \subset \mathrm{Sym}^2(V)$.

Lemma 10.1. *Let $\mathfrak{F}$ be a maximal generalized flag in V such that $\mathfrak{F} \cup \mathfrak{F}^{\perp} \cup \{M\}$ is a chain for some maximal isotropic subspace $M \subset V$. Then $\mathrm{St}_{\mathfrak{F}_{iso}} = \mathrm{St}_{\mathfrak{F}}$.*

Proof. Clearly $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{F}_{iso}}$. Let $Z \in \mathrm{St}_{\mathfrak{F}_{iso}}$ be arbitrary.

Let $\mathfrak{F} = \{F'_{\alpha}, F''_{\alpha}\}_{\alpha \in A}$. We first show that $\mathrm{St}_{\mathfrak{F}} = \sum_{\alpha \in A, F''_{\alpha} \subset M} F''_{\alpha} \& (F'_{\alpha})^{\perp}$.

For any $x \in F''_{\alpha}$ and $y \in (F'_{\alpha})^{\perp}$, we know on the one hand that $x \otimes y \in \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$, but on the other hand from the fact that $\mathfrak{F} \cup \mathfrak{F}^{\perp}$ is a chain, we find that also $y \otimes x \in \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$. In detail, we have $y \in F''_{\beta} \setminus F'_{\beta}$ for some $\beta \in A$. Since $\mathfrak{F} \cup \mathfrak{F}^{\perp}$ is a chain, and $y \in (F'_{\alpha})^{\perp}$ and $y \notin F'_{\beta}$, we have $F'_{\beta} \subsetneq (F'_{\alpha})^{\perp}$. Moreover since $\dim (F'_{\alpha})^{\perp}/(F''_{\alpha})^{\perp} \leq 1$, we have $F'_{\beta} \subset (F''_{\alpha})^{\perp}$, and thus $F''_{\alpha} \subset (F''_{\alpha})^{\perp\perp} \subset (F'_{\beta})^{\perp}$. So

$x \in (F'_\beta)^\perp$, and we see that $y \otimes x \in F''_\beta \otimes (F'_\beta)^\perp \subset \mathrm{St}_{\mathfrak{F},\mathfrak{gl}}$. Hence the map of vector spaces (which is *not* a map of Lie algebras):

$$\begin{aligned} \varphi : \sum_{\alpha \in A} F''_\alpha \otimes (F'_\alpha)^\perp &\rightarrow \mathrm{Sym}^2(V) \\ x \otimes y &\mapsto x \otimes y + y \otimes x \end{aligned}$$

in fact has its image in $\mathfrak{b}$. From the fact that $\varphi|_{\mathfrak{b}} = 2 \cdot \mathrm{Id}$, we find that φ maps surjectively onto $\mathfrak{b}$. Since $\sum F''_\alpha \otimes (F'_\alpha)^\perp$ is spanned by elements of the form $x \otimes y$, with $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$, we see that $\mathfrak{b}$ is spanned by elements of the form $x \otimes y + y \otimes x$, with $x \in F''_\alpha$ and $y \in (F'_\alpha)^\perp$.

In fact, $\mathfrak{b}$ is spanned by elements of the form $x \otimes y + y \otimes x$, with $x \in F''_\alpha \subset M$ and $y \in (F'_\alpha)^\perp$. (Explicitly, if $M \subset F'_\alpha$, then $y \in (F'_\alpha)^\perp \subset M^{\perp\perp} = M$.)

Now

$$\mathrm{St}_{\mathfrak{F}_{iso},\mathfrak{gl}} = \sum_{F''_\alpha \subset M} F''_\alpha \otimes (F'_\alpha)^\perp + V \otimes M,$$

because it is the stabilizer of $\mathfrak{F}_{iso} \cup \{M \subset V\}$, which is a generalized flag in V. Then $Z = X + Y$ for some $X \in \sum_{F''_\alpha \subset M} F''_\alpha \otimes (F'_\alpha)^\perp$ and $Y \in V \otimes M$.

Since $Z = \sigma(Z)$, we have $X + Y = \sigma(X) + \sigma(Y)$. A rearrangement yields $Y - \sigma(X) = \sigma(Y) - X$, and the left hand side of the equation is clearly an element of $V \otimes M$, while the righthand side is clearly an element of $M \otimes V$. So $Y - \sigma(X) \in (V \otimes M) \cap (M \otimes V) = M \otimes M$. Now

$$\sigma(Y - \sigma(X)) = \sigma(Y) - X = Y - \sigma(X),$$

and therefore $Y - \sigma(X) \in (M \otimes M) \cap \mathrm{Sym}^2(V) = \mathrm{Sym}^2(M)$. Let $\eta := Y - \sigma(X)$, so $Z = X + \sigma(X) + \eta$. Clearly $X + \sigma(X) \in \mathrm{St}_{\mathfrak{F}}$. Since $\eta \in \mathrm{Sym}^2(M) \subset \mathrm{St}_{\mathfrak{F}}$, we have $Z \in \mathrm{St}_{\mathfrak{F}}$. ■

Lemmas 10.2, 10.3, and 10.5 may be proved in the same manner as the analogous statements in Section 9, needing straightforward modifications only, so the proofs are omitted.

Lemma 10.2. *Let $\mathfrak{F} = \{F'_\beta, F''_\beta\}_{\beta \in B}$ be a maximal closed isotropic generalized flag in V. Then $\mathrm{St}_{\mathfrak{F}} = \sum_\beta F''_\beta \& (F'_\beta)^\perp$, and moreover $\mathrm{St}_{\mathfrak{F}}$ is locally solvable.*

Lemma 10.3. *Let $\mathfrak{F}$ be a maximal closed isotropic generalized flag in V. If $u \in \bigcup_{F \in \mathfrak{F}} F$, then*

$$\mathrm{St}_{\mathfrak{F}} \cdot u = \begin{cases} F''_u & \text{if } \overline{F'_u} = F'_u \\ F'_u & \text{if } \overline{F'_u} = F''_u. \end{cases}$$

Thus $\overline{\mathrm{St}_{\mathfrak{F}} \cdot u} = F''_u$.

Proposition 10.4. *If $\mathfrak{F}$ and $\mathfrak{G}$ are maximal closed isotropic generalized flags in V with $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, then $\bigcup_{F \in \mathfrak{F}} F = \bigcup_{G \in \mathfrak{G}} G$.*

Proof. Let $M := \bigcup_{F\in\mathfrak{F}} F$ and $N := \bigcup_{G\in\mathfrak{G}} G$. The maximality of $\mathfrak{F}$ and $\mathfrak{G}$ implies that both M and N are maximal isotropic subspaces. We will show that $\langle M, N\rangle = 0$. Suppose, for the sake of a contradiction, that there exist $m \in M$ and $n \in N$ such that $\langle m, n\rangle \neq 0$. Then $(m\otimes m)\cdot n = \langle n, m\rangle m$. Since $\mathrm{Sym}^2(M) \subset \mathrm{St}_{\mathfrak{F}}$, we have shown that $m \in \mathrm{St}_{\mathfrak{F}}\cdot N \subset \mathrm{St}_{\mathfrak{G}}\cdot N \subset N$. But $\langle m, n\rangle \neq 0$, which contradicts the fact that N is isotropic. Hence $\langle M, N\rangle = 0$, and $N \subset M^{\perp} = M$. By the maximality of N, we have $M = N$. ∎

Lemma 10.5. *If $\mathfrak{F}$ and $\mathfrak{G}$ are maximal closed isotropic generalized flags in V with $\mathrm{St}_{\mathfrak{F}} \subset \mathrm{St}_{\mathfrak{G}}$, then $\mathfrak{F} = \mathfrak{G}$.*

The following result fully describes Borel subalgebras of $\mathfrak{sp}(V)$.

Theorem 10.6. *A subalgebra of $\mathfrak{sp}(V)$ is a Borel subalgebra if and only if it is the stabilizer of a maximal closed isotropic generalized flag in V. Futhermore, the map from maximal closed isotropic generalized flags of V to Borel subalgebras of $\mathfrak{sp}(V)$*

$$\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$$

is bijective.

Proof. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{sp}(V)$. Proposition 7.1 states that $\mathfrak{b}$ is the stabilizer of a maximal closed generalized flag $\mathfrak{F}$ in V with $\mathfrak{F}\cup\mathfrak{F}^{\perp}\cup\{M\}$ being a chain for some maximal isotropic subspace $M \subset V$. By Lemma 9.2, $\mathfrak{b} = \mathrm{St}_{\mathfrak{F}_{iso}}$. Observe that $\mathfrak{F}_{iso}$ is a maximal closed isotropic generalized flag in V, since the union of the isotropic subspaces in $\mathfrak{F}$ must be M. Hence every Borel subalgebra of $\mathfrak{sp}(V)$ is the stabilizer of a maximal closed isotropic generalized flag in V.

Now let $\mathfrak{F}$ be an arbitrary maximal closed isotropic generalized flag in V. By Lemma 10.2, $\mathrm{St}_{\mathfrak{F}}$ is locally solvable. Hence there exists a Borel subalgebra $\mathfrak{b}$ with $\mathrm{St}_{\mathfrak{F}} \subset \mathfrak{b}$. We have seen that there is a maximal closed isotropic generalized flag $\mathfrak{G}$ with $\mathfrak{b} = \mathrm{St}_{\mathfrak{G}}$. It follows from Lemma 9.6 that $\mathfrak{F} = \mathfrak{G}$. As a result, $\mathrm{St}_{\mathfrak{F}} = \mathfrak{b}$ is a Borel subalgebra. Hence $\mathfrak{F} \mapsto \mathrm{St}_{\mathfrak{F}}$ gives a map from maximal closed isotropic generalized flags in V to Borel subalgebras of $\mathfrak{sp}(V)$. Proposition 7.1 gives that the map is surjective, and Lemma 10.5 implies that it is injective. ∎

11. Recapitulation of simple cases

One can find a nice kind of toral subalgebra in any Borel subalgebra $\mathfrak{b}$ of one of the simple infinite-dimensional root-reductive Lie algebras. In each case, there exist toral subalgebras $\mathfrak{t} \subset \mathfrak{b}$ such that $\mathfrak{b} = \mathfrak{t} + \mathfrak{n}$, where $\mathfrak{n}$ denotes the ad hoc nilradical of $\mathfrak{b}$. Hence irreducible representations of $\mathfrak{b}$ are given by characters of $\mathfrak{t}$. The relevant formulas are shown in Figure 1. A similar analysis is seen in the case of $\mathfrak{gl}_\infty$ in [3]. For more about toral subalgebras, see [1].

If $\mathfrak{b} \subset \mathfrak{sl}(V, V_*)$ is a Borel subalgebra, then $\mathfrak{b}$ is the stabilizer in $\mathfrak{sl}(V, V_*)$ of a unique maximal closed generalized flag $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha\in A}$ in V. It is also the stabilizer in $\mathfrak{sl}(V, V_*)$ of a unique maximal closed generalized flag $\mathfrak{G} = \{G'_\beta, G''_\beta\}_{\beta\in B}$ in V_*. Let C denote the good pairs of A, and we may also identify C with the subset of good pairs of B. There exist 1-dimensional subspaces $L_\gamma \subset V$ and

Figure 1: Formulas for the stabilizer $\mathfrak{b} \subset \mathfrak{g}$ of a maximal closed (isotropic) generalized flag $\{F'_\alpha, F''_\alpha\}$ in V, the ad hoc nilradical $\mathfrak{n}$ of $\mathfrak{b}$, and the toral subalgebra $\mathfrak{t}$ associated to lines L_γ and M_γ.

$\mathfrak{g}$	$\mathfrak{b}$	$\mathfrak{n}$	$\mathfrak{t}$
$\mathfrak{gl}(V,V_*)$	$\sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$	$\sum_\alpha F''_\alpha \otimes (F''_\alpha)^\perp$	$\bigoplus_{\gamma\in C} L_\gamma \otimes M_\gamma$
$\mathfrak{sl}(V,V_*)$	$\mathfrak{g} \cap \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$	$\sum_\alpha F''_\alpha \otimes (F''_\alpha)^\perp$	$\mathfrak{g} \cap \bigoplus_{\gamma\in C} L_\gamma \otimes M_\gamma$
$\mathfrak{so}(V)$	$\sum_\alpha F''_\alpha \wedge (F'_\alpha)^\perp$	$\sum_\alpha F''_\alpha \wedge (F''_\alpha)^\perp$	$\bigoplus_{\gamma\in C} L_\gamma \wedge M_\gamma$
$\mathfrak{sp}(V)$	$\sum_\alpha F''_\alpha \& (F'_\alpha)^\perp$	$\sum_\alpha F''_\alpha \& (F''_\alpha)^\perp$	$\bigoplus_{\gamma\in C} L_\gamma \& M_\gamma$

$M_\gamma \subset V_*$ for $\gamma \in C$ such that $\langle L_\gamma, M_c \rangle = \delta_{\gamma c}\mathbb{C}$, and such that $F''_\gamma = F'_\gamma \oplus L_\gamma$ and $(F'_\gamma)^\perp = (F''_\gamma)^\perp \oplus M_\gamma$. In fact, one can go so far as to require that there exist vector space complements X_α of F'_α in F''_α for $\alpha \in A \setminus C$, and vector space complements Y_β of G'_β in G''_β for $\beta \in B \setminus C$, such that $V = (\bigoplus_{\gamma\in C} L_\gamma) \oplus (\bigoplus_{\alpha\in A\setminus C} X_\alpha)$ and $V_* = (\bigoplus_{\gamma\in C} M_\gamma) \oplus (\bigoplus_{\beta\in B\setminus C} Y_\beta)$. The associated toral subalgebra of $\mathfrak{b}$ is

$$\mathfrak{t} = \Big(\bigoplus_{\gamma\in C} L_\gamma \otimes M_\gamma\Big) \cap \mathfrak{sl}(V,V_*).$$

The same construction produces toral subalgebras inside of Borel subalgebras of $\mathfrak{gl}(V,V_*)$ as well, and the formulas for $\mathfrak{gl}(V,V_*)$ are also given in Figure 1.

If $\mathfrak{b} \subset \mathfrak{so}(V)$ is a Borel subalgebra, then $\mathfrak{b}$ is the stabilizer in $\mathfrak{so}(V)$ of a maximal closed isotropic generalized flag $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha\in A}$ in V. Consider $\mathfrak{G} := \mathrm{fl}(\mathfrak{F}^\perp \cup \{V\}) = \{G'_\beta, G''_\beta\}_{\beta\in B}$. There exist 1-dimensional subspaces $L_\gamma \subset V$ and $M_\gamma \subset V$ for $\gamma \in C$ such that $\langle M_\gamma, M_c\rangle = 0$ and $\langle L_\gamma, M_c \rangle = \delta_{\gamma c}\mathbb{C}$, and such that $F''_\gamma = F'_\gamma \oplus L_\gamma$ and $(F'_\gamma)^\perp = (F''_\gamma)^\perp \oplus L_{-\gamma}$ for all $\gamma \in C$. In fact, one can go so far as to require that there exist vector space complements X_α of F'_α in F''_α for $\alpha \in A \setminus C$, and vector space complements Y_β of G'_β in G''_β for $\beta \in B \setminus C$, as well as a vector space complement S (necessarily of dimension 0 or 1) of $\bigcup_\alpha F''_\alpha$ in $(\bigcup_\alpha F''_\alpha)^\perp$, such that $V = (\bigoplus_{\gamma\in C} L_\gamma \oplus M_\gamma) \oplus (\bigoplus_{\alpha\in A\setminus C} X_\alpha) \oplus (\bigoplus_{\beta\in B\setminus C} Y_\beta) \oplus S$. The associated toral subalgebra is

$$\mathfrak{t} = \bigoplus_{\gamma\in C} L_\gamma \wedge M_\gamma.$$

If $\mathfrak{b} \subset \mathfrak{sp}(V)$ is a Borel subalgebra, then $\mathfrak{b}$ is the stabilizer in $\mathfrak{sp}(V)$ of a unique maximal closed isotropic generalized flag $\mathfrak{F} = \{F'_\alpha, F''_\alpha\}_{\alpha\in A}$ in V. Consider $\mathfrak{G} := \mathrm{fl}(\mathfrak{F}^\perp \cup \{V\}) = \{G'_\beta, G''_\beta\}_{\beta\in B}$. Let C denote the good pairs of A, and we may also consider C as a subset of B. There exist 1-dimensional subspaces $L_\gamma \subset V$ and $M_\gamma \subset V$ for $\gamma \in C$ such that $\langle L_\gamma, M_c \rangle = \delta_{\gamma c}\mathbb{C}$ and $\langle M_\gamma, M_c\rangle = 0$, and such that $F''_\gamma = F'_\gamma \oplus L_\gamma$ and $(F'_\gamma)^\perp = (F''_\gamma)^\perp \oplus M_\gamma$. In fact, one can go so far as to require that there exist vector space complements X_α of F'_α in F''_α for $\alpha \in A \setminus C$, and vector space complements Y_β of G'_β in G''_β for $\beta \in B \setminus C$, such that $V = (\bigoplus_{\gamma\in C} L_\gamma \oplus M_\gamma) \oplus (\bigoplus_{\alpha\in A\setminus C} X_\alpha) \oplus (\bigoplus_{\beta\in B\setminus C} Y_\beta)$. The associated toral subalgebra is

$$\mathfrak{t} = \bigoplus_{\gamma\in C} L_\gamma \& M_\gamma.$$

12. Three examples

Let V be the vector space with basis $\{x_i : i \in \mathbb{Z}_{\neq 0}\}$, and let V_* be the span of the elements $\{x_i^* \in V^* : i \in \mathbb{Z}_{\neq 0}\}$, where $\langle \cdot, \cdot \rangle : V \times V_* \to \mathbb{C}$ is defined by $\langle x_i, x_j^* \rangle := \delta_{ij}$. Consider for $i \in \mathbb{Z}_{\neq 0}$ the subspace $F_i := \mathrm{Span}\{x_j : j \leq i\} \subset V$. For each i the subspace $F_i \subset V$ is closed. The chain

$$\cdots \subset F_{-2} \subset F_{-1} \subset F_1 \subset F_2 \subset \cdots$$

is a maximal closed generalized flag in V, and let $\mathfrak{b}$ denote its stabilizer in $\mathfrak{sl}(V, V_*)$, which is a Borel subalgebra of $\mathfrak{sl}(V, V_*)$. This example arises naturally from the finite-dimensional situation, since $\mathfrak{b}$ is the union of Borel subalgebras of finite-dimensional subalgebras isomorphic to $\mathfrak{sl}_n$ exhausting $\mathfrak{sl}(V, V_*)$. Explicitly, let $V_n := \mathrm{Span}\{x_j : -n \leq j \leq n\} \subset V$ and $(V_n)_* := \mathrm{Span}\{x_j^* : -n \leq j \leq n\} \subset V_*$, and define $\mathfrak{g}_n := \mathfrak{sl}(V, V_*) \cap (V_n \otimes (V_n)_*)$. Then $\mathfrak{g}_n \cong \mathfrak{sl}_{2n}$, and one may check that $\mathfrak{b} \cap \mathfrak{g}_n$ is a Borel subalgebra of $\mathfrak{g}_n$.

For the second example, let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(V, V_*) \Subset \mathbb{C}X$, where the element X is taken to have the same commutation relations as the formal sum $\sum_{i>0} x_i \otimes (x_i^* + x_{-i}^*)$, in the notation of the first example. One may check that $\mathfrak{g}$ is a root-reductive Lie algebra. The Borel subalgebra $\mathfrak{b}$ of the first example is a locally solvable subalgebra of $\mathfrak{g}$. In fact $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. To check this claim, it suffices to show that $\mathfrak{b}$ is self-normalizing in $\mathfrak{g}$, in light of Proposition 13.1 below. Suppose $Y \in \mathfrak{sl}(V, V_*)$ and $a \in \mathbb{C}$ are such that $Y + aX \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$. Then $Y \in \mathfrak{g}_n$ for some n. Consider the element $Z := x_{n+1} \otimes x_{n+1}^* - x_{n+2} \otimes x_{n+2}^* \in \mathfrak{b}$, and compute

$$\begin{aligned} [Y + aX, Z] &= a[X, Z] \\ &= a[x_{n+1} \otimes (x_{n+1}^* + x_{-n-1}^*) + x_{n+2} \otimes (x_{n+2}^* + x_{-n-2}^*), Z] \\ &= a(-x_{n+1} \otimes x_{-n-1}^* + x_{n+2} \otimes x_{-n-2}^*). \end{aligned}$$

Since $-x_{n+1} \otimes x_{-n-1}^* + x_{n+2} \otimes x_{-n-2}^* \notin \mathfrak{b}$, it must be that $a = 0$. Hence $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \subset \mathfrak{sl}(V, V_*)$, so $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{b}$. Thus $\mathfrak{b}$ is an example of a Borel subalgebra of $\mathfrak{sl}(V, V_*)$ which remains maximal locally solvable when considered as a subalgebra of $\mathfrak{sl}(V, V_*) \Subset \mathbb{C}X$.

With the third example we will see that a Borel subalgebra of $\mathfrak{gl}_\infty$ may in fact be locally nilpotent. Suppose $\mathfrak{F}$ is a maximal closed generalized flag in the standard representation with no good pairs. Then $\overline{F'} = F''$, and consequently $(F')^\perp = (F'')^\perp$, for all immediate predecessor-successor pairs $F' \subset F''$ in $\mathfrak{F}$. It follows from the formulas in Figure 1 that the ad hoc nilradical $\mathfrak{n}$ of $\mathfrak{b} = \mathrm{St}_{\mathfrak{F}}$ is equal to $\mathfrak{b}$. Thus every element of $\mathfrak{b}$ is traceless, i.e. $\mathfrak{b} \subset \mathfrak{sl}_\infty$. Such an example is constructed explicitly in [3], as follows. Let V and V_* be the vector spaces with bases $\{v_q\}_{q \in \mathbb{Q}}$ and $\{w_q\}_{q \in \mathbb{Q}}$, respectively. A nondegenerate pairing $\langle \cdot, \cdot \rangle : V \times V_* \to \mathbb{C}$ is defined by

$$\langle v_r, w_s \rangle := \begin{cases} 1 & \text{if } r > s \\ 0 & \text{if } r \leq s. \end{cases}$$

Define $\mathfrak{F} = \{\mathrm{Span}\{v_r : r < q\}, \mathrm{Span}\{v_r : r \leq q\}\}_{q \in \mathbb{Q}}$. One may check that

$$\overline{\mathrm{Span}_{r<q}\{v_r\}} = \mathrm{Span}_{r \leq q}\{v_r\}$$

for all $q \in \mathbb{Q}$. Thus $\mathfrak{F}$ is a maximal closed generalized flag in V with no good pairs, and $\mathrm{St}_{\mathfrak{F}} = \mathrm{Span}\{v_r \otimes w_s : r \leq s\}$ is a locally nilpotent Borel subalgebra of $\mathfrak{gl}(V, V_*)$.

13. General case

Theorem 4.1 states that any Borel subalgebra of an infinite-dimensional indecomposable root-reductive Lie algebra is the simultaneous stabilizer of a Borel generalized flag in each of the standard representations. That is, if $\mathfrak{g}$ is an infinite-dimensional indecomposable root-reductive Lie algebra, the image of the map $\{\mathfrak{F}_m\} \mapsto \bigcap_m \mathrm{St}_{\mathfrak{F}_m}$ from families of Borel generalized flags in the standard representations of $\mathfrak{g}$ to subalgebras of $\mathfrak{g}$ contains the Borel subalgebras of $\mathfrak{g}$. At the same time, the image of the map $\{\mathfrak{F}_m\} \mapsto \bigcap_m \mathrm{St}_{\mathfrak{F}_m}$ from families of maximal closed generalized flags in the standard representations of $\mathfrak{g}$ to subalgebras of $\mathfrak{g}$ is contained in the Borel subalgebras of $\mathfrak{g}$. It is not the case that the simultaneous stabilizer of any family of Borel generalized flags in the standard representations is a Borel subalgebra. For instance, there exist Borel generalized flags in V which are not maximal closed, and the stabilizer of any such flag is not a Borel subalgebra of $\mathfrak{sl}(V, V_*)$.

We can calculate the intersection of a Borel subalgebra $\mathfrak{b}$ of an infinite-dimensional indecomposable root-reductive Lie algebra $\mathfrak{g}$ with any simple direct summand of $[\mathfrak{g}, \mathfrak{g}]$. Let $[\mathfrak{g}, \mathfrak{g}] \cong \bigoplus_m \mathfrak{s}_m$ be the decomposition into simple direct summands, and let V_m be the standard representations of $\mathfrak{g}$. Using Theorem 4.1, we know that for each m there exist a bivalent closed generalized flag $\mathfrak{F}_m$ in V_m and a Borel generalized flag $\mathfrak{G}_m$ refining $\mathfrak{F}_m$ such that $\mathfrak{b} = \bigcap_m \mathrm{St}_{\mathfrak{G}_m}$. Fix m, and consider $\mathfrak{F}_m = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$. Let $B \subset A$ denote the pairs α such that F'_α is closed and $\dim F''_\alpha / F'_\alpha = \infty$. Then one may check via a calculation similar to one in the proof of Proposition 7.1 that

$$\mathfrak{b} \cap \mathfrak{s}_m = \mathrm{St}_{\mathfrak{G}_m} \cap \mathfrak{s}_m = \Big(\sum_{\alpha \in A \setminus B} F''_\alpha \otimes (F'_\alpha)^\perp + \sum_{\beta \in B} F''_\beta \otimes (F''_\beta)^\perp \Big) \cap \mathfrak{s}_m.$$

Clearly if B is nonempty, then $\mathfrak{b} \cap \mathfrak{s}_m$ is not a Borel subalgebra of $\mathfrak{s}_m$.

Proposition 13.1. *If $\mathfrak{b} \subset [\mathfrak{g}, \mathfrak{g}]$ is a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, then the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ is the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$.*

Proof. Note that $\mathfrak{b} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \cap [\mathfrak{g}, \mathfrak{g}]$. For any $X \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \cap [\mathfrak{g}, \mathfrak{g}]$, it must be that $\mathfrak{b} + \mathbb{C}X$ is a locally solvable subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, since $[\mathfrak{b} + \mathbb{C}X, \mathfrak{b} + \mathbb{C}X] \subset \mathfrak{b}$. By the maximality of $\mathfrak{b}$, we know $\mathfrak{b} = \mathfrak{b} + \mathbb{C}X$, i.e. $X \in \mathfrak{b}$. Therefore $\mathfrak{b} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \cap [\mathfrak{g}, \mathfrak{g}]$.

Compute $[\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}), \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})] \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{b}$. As a result $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ is a locally solvable subalgebra of $\mathfrak{g}$.

Let $\mathfrak{b}'$ be any Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Then $\mathfrak{b}' \cap [\mathfrak{g}, \mathfrak{g}]$ is a locally solvable subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ containing $\mathfrak{b}$. By the maximality of $\mathfrak{b}$, it holds that $\mathfrak{b} = \mathfrak{b}' \cap [\mathfrak{g}, \mathfrak{g}]$. Therefore $[\mathfrak{b}', \mathfrak{b}] \subset [\mathfrak{b}', \mathfrak{b}'] \subset \mathfrak{b}' \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{b}$, i.e. $\mathfrak{b}' \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$. By the maximality of $\mathfrak{b}'$, we have $\mathfrak{b}' = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$. Thus $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ is the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. ■

As a corollary of Proposition 13.1, the simultaneous stabilizer in $\mathfrak{g}$ of a maximal closed (isotropic) generalized flag in each of the standard representations is independent of the choices made in defining the action of $\mathfrak{g}$ on its standard representations. Another easy conseqence is the following theorem.

Theorem 13.2. *Let $\mathfrak{g}$ be an arbitrary root-reductive Lie algebra. The map $\mathfrak{b} \mapsto \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ yields a bijection from the set of Borel subalgebras of $[\mathfrak{g}, \mathfrak{g}]$ to the set of Borel subalgebras of $\mathfrak{g}$ whose intersection with $[\mathfrak{g}, \mathfrak{g}]$ is a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$.*

Proof. Proposition 13.1 implies that the map $\mathfrak{b} \mapsto \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ from Borel subalgebras of $[\mathfrak{g}, \mathfrak{g}]$ to subalgebras of $\mathfrak{g}$ lands inside the set of Borel subalgebras of $\mathfrak{g}$. It was also seen in the proof that if $\mathfrak{b}$ is a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{b}$. That is, the composition of the first map with the map from Borel subalgebras of $\mathfrak{g}$ to subalgebras of $[\mathfrak{g}, \mathfrak{g}]$ given by intersecting, i.e. $\mathfrak{b} \mapsto \mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}]$, is the map $\mathfrak{b} \mapsto \mathfrak{b}$. The image of the map $\mathfrak{b} \mapsto \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b})$ is precisely the set of Borel subalgebras of $\mathfrak{g}$ which yield Borel subalgebras when intersected with $[\mathfrak{g}, \mathfrak{g}]$. ■

This yields a large class of Borel subalgebras of $\mathfrak{g}$ which are in bijection with the Borel subalgebras of $[\mathfrak{g}, \mathfrak{g}]$. Since $[\mathfrak{g}, \mathfrak{g}]$ decomposes into a direct sum of simple root-reductive Lie algebras, Borel subalgebras of $[\mathfrak{g}, \mathfrak{g}]$ can be understood as direct sums of Borel subalgebras of the simple direct summands of $[\mathfrak{g}, \mathfrak{g}]$. This is a good context in which to view the results of this paper on Borel subalgebras of $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$, the three infinite-dimensional simple root-reductive Lie algebras.

The question remains open whether there exists a root-reductive Lie algebra $\mathfrak{g}$ containing a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}]$ is not a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. If one could show that no such examples exist, then Theorem 13.2 would become a classification of the Borel subalgebras of root-reductive Lie algebras. This outcome would be nice in a way, yet it seems to me unlikely. I would conjecture that this phenomenon does occur. Such Borel subalgebras might seem pathological, but I do not see any simple way to preclude their existence. Indeed, the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is not as large in $\mathfrak{g}$ as one might think. As an illustration, a root-reductive Lie algebra $\mathfrak{g}$ and a maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{g}$ are constructed in [1] with $\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}] = 0$, a far cry from a maximal toral subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. In light of this, one might reasonably hope to construct explicitly an example of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}]$ is not a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. This last remaining gap in a basic understanding of Borel subalgebras of root-reductive Lie algebras would be closed by either producing such an example or proving that none exists.

References

[1] Dan-Cohen, E., I. Penkov, and N. Snyder, *Cartan subalgebras of root-reductive Lie algebras*, J. Algebra **308** (2007), 583–611.

[2] Dimitrov, I., and I. Penkov, *Weight modules of direct limit Lie algebras*, Int. Math. Res. Notes **5** (1999), 223–249.

[3] —, *Borel subalgebras of* $\mathfrak{gl}(\infty)$, Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo **6** (2004), 153–163.

[4] —, *Borel and Cartan subalgebras of* $\mathfrak{gl}(\infty)$, Manuscript.

[5] Mackey, G., *On infinite-dimensional linear spaces*, Trans. Amer. Math. Soc. **57** (1945), 155–207.

[6] Neeb, K.-H., and N. Stumme, *The classification of locally finite split simple Lie algebras*, J. Reine Angew. Math. **533** (2001), 25–53.

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