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Covariants and the No-name Lemma

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Abstract. A close connection between the no-name lemma (concerning algebraic groups acting on vector bundles) and the existence of sufficiently many independent rational covariants is pointed out. In particular, this leads to a new natural proof of the no-name lemma. For linearly reductive groups, the approach has a refined variant based on integral covariants. This yields a version of the no-name lemma that has a constructive nature.
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1. Introduction

The so-called rationality problem asks when $k(X)^G$, the field of invariant rational functions is purely transcendental over k (an algebraically closed base field), where G is a group acting linearly on the vector space X. One of the basic principles in studying this question is the following statement (for simplicity, we state it for the case $\mathrm{char}(k) = 0$): if the linear algebraic group G acts generically freely and morphically on the irreducible algebraic variety X, and W is a finite dimensional G-module, then $k(X \times W)^G$ is purely transcendental over $k(X)^G$. This statement (or some variant of it) has gone into the literature as the "no-name lemma" (see Remark 1.2 for references).

In this note we point out that the no-name lemma is closely related to the question about the number of generically independent (rational) covariants from X to W. In particular, we show that the no-name lemma follows from the following fact: if the stabilizer G_x of a general $x \in X$ acts trivially on W, then there are $\dim(W)$ generically independent rational covariants from X to W. This latter fact was proved by Reichstein [11] for generically free actions in characteristic zero; we extend it to positive characteristic and the above weaker and necessary condition on stabilizers. The picture is summarized in Theorem 1.1 below (see Section 2 for the terminology), stating the equivalence of various properties expressed in terms of stabilizers, rational covariants, birational isomorphisms, and generators

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of invariant fields, respectively (see Example 2.5 and Remark 2.6 for the role of the separability condition):

Theorem 1.1. *Let X be an irreducible generically separable G-variety, and W a d-dimensional G-module. Then the following are equivalent:*

(1) The stabilizer G_x acts trivially on W for a general $x \in X$.

(2) There exist d generically independent rational covariants from X to W.

(3) The G-varieties $X \times W$ and $X \times k^d$ (where G acts trivially on k^d) are birationally isomorphic over X.

(4) There exists a G-equivariant birational isomorphism $X \times W \to X \times k^d$ (where G acts trivially on k^d) of the form $(x, w) \mapsto (x, \Phi(x, w))$ which is linear on W (i.e. the map $\Phi(x, -) : W \to k^d$ is k-linear for a general $x \in X$).

(5) The invariant field $k(X \times W)^G$ is purely transcendental over $k(X)^G$ of transcendence degree d, generated by elements of the form $\sum_{j=1}^{d} \Phi_{ij}\varepsilon_j$, $(i = 1, \ldots, d)$, where $\Phi_{ij} \in k(X)$ and $\varepsilon_1, \ldots, \varepsilon_d$ is a basis in the dual space W^* of W.

Remark 1.2. The implication (1) $\Longrightarrow$ (3) (or some version of it) is called the "no-name lemma" (after [6]). The first published reference (in characteristic zero, for a generically free G-variety X) is [1], see [4] for a survey. A proof for arbitrary (not necessarily reductive) G is given in the recent paper [3]. A version for arbitrary base fields (using a concept of scheme theoretically free actions) is given in [12]. Although these references consider generically free G-varieties X, in a remark attributed to Kraft, it is mentioned in [6] that to make the conclusion (3), it is sufficient to assume that G_x acts trivially on W for a general $x \in X$.

As far as we know, the fact that conclusion (3) can be strengthened to (4) and (5) has not been emphasized before, except in the case of finite groups (see the proof of Proposition 1.1 in [7]). For finite groups, this is essentially Speiser's Lemma (asserting that a finite dimensional K-vector space endowed with a semilinear action of a finite group is spanned by invariant elements; see for example Lemma 2.3.8 in [8]) applied when K is a function field of a G-variety. So one may view the implications (1) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (5) as a generalization of Speiser's Lemma for linear algebraic groups.

The main point of our note is bringing (2) into the picture, and pointing out its equivalence to (4) and (5). Having this in mind, the implication (1) $\Longrightarrow$ (2) (due to Reichstein when X is generically free and $\mathrm{char}(k) = 0$; see Lemma 7.4 in [11]) appears to be a more fundamental principle than the no-name lemma: its statement and proof are rather natural, and yield the latter in the above explicit form as an immediate corollary.

The proof of Theorem 1.1 is presented in Section 2.

An additional benefit from paying attention to covariants is that for linearly reductive groups acting on affine varieties, it leads to a finer variant of the no-name lemma (see Proposition 3.1, Theorem 3.5). This is developed in Section 3, where we find exact conditions ensuring the supply of integral covariants to produce G-equivariant isomorphisms over invariant affine open sets defined by the non-vanishing of some relative invariant (see Propostition 3.3).

2. Rational covariants

Let G be a linear algebraic group over an algebraically closed field k of arbitrary characteristic, denote by G_0 the connected component of the identity. By a *G-variety* X we mean an algebraic variety with a G-action such that the action map $G\times X \to X$ is a morphism of algebraic varieties. Write $k[X]$ for the ring of regular functions on X, and when X is irreducible, write $k(X)$ for the field of rational functions on X. As usual, $k[X]^G$ is the subring of G-invariants, and $k(X)^G$ is the subfield of G-invariants. When X is affine, $k[X]$ is called the coordinate ring of X, and $k(X)$ is the field of fractions of $k[X]$ in this case. By a *G-module* we mean a G-variety V which is a finite dimensional vector space, on which G acts via linear transformations. We shall write k^d for the d-dimensional vector space endowed with the trivial G-action. We say that some property *holds for a general* $x \in X$ if it holds for all $x \in U$, where U is a Zariski dense open subset of X.

Let X be an irreducible G-variety and W a G-module. A *covariant* $F : X \to W$ is a G-equivariant morphism of algebraic varieties. Write $\mathrm{Cov}_G(X,W)$ for the set of covariants from X to W; this is contained in $\mathrm{RCov}_G(X,W)$, the set of *rational covariants* (i.e. rational G-equivariant maps). Note that $\mathrm{RCov}_G(X,W)$ is naturally a vector space over $k(X)^G$, whereas $\mathrm{Cov}_G(V,W)$ is a $k[X]^G$-submodule.

We say that the (rational) covariants $F_1,\dots,F_d$ are *generically independent*, if $F_1(x),\dots,F_d(x)$ are linearly independent vectors in W for some $x\in X$. Note that in this case $F_1(x),\dots,F_d(x)$ are linearly independent for a general $x \in X$. Obviously, the existence of $d=\dim(W)$ generically independent (rational) covariants from X to W is equivalent to the existence of a G-equivariant (rational) morphism $X \to GL(W)$, where G acts by left translation on $GL(W)$.

For $f\in k[X]$ denote by X_f the Zariski open subset $\{x\in X \mid f(x)\neq 0\}$. Recall that $f\in k[X]$ is a *relative invariant* if $g\cdot f = \theta(g)f$ for some character $\theta : G \to k^\times$. In this case X_f is a G-stable subset of X. Let us introduce the following ad hoc terminology: we say that the G-varieties $X\times Y$ and $X\times Z$ *are (birationally) isomorphic over* X if there is a G-equivariant (birational) isomorphism between them that commutes with the projection onto X.

We start with an elementary lemma:

Lemma 2.1. *Let X be an irreducible G-variety, W a d-dimensional G-module, and $F_1,\dots,F_d \in \mathrm{Cov}_G(X,W)$ generically independent covariants. Then there exists a non-zero relative invariant $f\in k[X]$ such that*

$$X_f = \{x\in X \mid F_1(x),\dots,F_d(x) \text{ are linearly independent}\}$$

and the G-varieties $X_f\times W$ and $X_f\times k^d$ (G acting trivially on the vector space k^d) are isomorphic over X_f (via an isomorphism constructed explicitly in the proof).

Proof. Suppose $F_1,\dots,F_d\in\mathrm{Cov}_G(X,W)$ are generically independent. Let $e_1,\dots,e_d$ be a basis in W, and $\varepsilon_1,\dots,\varepsilon_d$ the corresponding dual basis in W^*.

Then $F_j(x)=\sum_{i=1}^d F_{ij}(x)e_i$ for some $F_{ij}\in k[X]$. Write F for the $d\times d$ matrix whose (i,j)-entry is F_{ij}. The covariance of the F_j and multiplicativity of the determinant imply that $f:=\det(F_{ij})_{d\times d}$ is a relative invariant in $k[X]$ of weight $g\mapsto\det(g_W)^{-1}$, where g_W is the matrix of g acting on W with respect to the chosen basis. Moreover, X_f is the locus of $x\in X$ where the $F_1(x),\dots,F_d(x)$ span W. Define $\Phi=(\Phi_1,\dots,\Phi_d):X_f\times W\to k^d$ by

$$w=\sum_{i=1}^d \Phi_i(x,w)F_i(x). \tag{1}$$

Applying $g\in G$ to both sides, and taking into account the linearity of the action of G on W one gets that the maps $\Phi_i:X_f\times W\to k$ are G-invariant, so the morphism $(\mathrm{id}_{X_f}\times\Phi):X_f\times W\to X_f\times k^d$ is indeed G-equivariant.

In terms of coordinates, $\Phi_i=\sum_{j=1}^d(F^{-1})_{ij}\varepsilon_j$, where $(F^{-1})_{ij}$ is the (i,j)-entry of the inverse of F. This shows that $(\mathrm{id}_{X_f}\times\Phi):X_f\times W\to X_f\times k^d$ is a morphism of algebraic varieties, and $\Phi(x,-):W\to k^d$ is a k-linear isomorphism for all $x\in X_f$. Moreover, formula (1) shows that $\mathrm{id}_{X_f}\times\Phi$ is in fact an isomorphism, with inverse sending $(x,a)\in X_f\times k^d$ to $(x,\sum_{i=1}^d a_iF_i(x))\in X_f\times W$. ∎

Lemma 2.1 has the following converse:

Lemma 2.2. *Let Y be an irreducible G-variety, W a d-dimensional G-module, and suppose that the G-varieties $Y\times W$ and $Y\times k^d$ are isomorphic over Y. Then there is a G-equivariant isomorphism over Y between them which restricts to a k-linear isomorphism $\{x\}\times W\cong W\to k^d\cong\{x\}\times k^d$ for all $x\in Y$, and there exist d covariants $F_i:Y\to W$ $(i=1,\dots,d)$ such that $F_1(x),\dots,F_d(x)$ are linearly independent for all $x\in Y$.*

Proof. Let $(x,w)\mapsto(x,\Phi(x,w))$ be a G-isomorphism $Y\times W\to Y\times k^d$. Consider the coordinate functions Φ_i $(i=1,\dots,d)$ of Φ. Then $\Phi_i\in k[Y\times W]^G$. View $k[Y\times W]$ as a polynomial ring in the variables $\varepsilon_1,\dots,\varepsilon_d$ (a basis of W^*) with coefficients in $k[Y]$. The linear component of Φ_i is $\sum_{j=1}^d\Phi_{ij}\varepsilon_j$ with $\Phi_{ij}\in k[Y]$. Since the action of G is homogeneous, we have that $\sum_{j=1}^d\Phi_{ij}\varepsilon_j$ is G-invariant. Moreover, for all $x\in Y$, the matrix $(\Phi_{ij}(x))_{d\times d}$ is invertible, being the matrix of the differential at zero of the isomorphism $\Phi(x,-):W\to k^d$, so $\det(\Phi_{ij})_{d\times d}$ is a unit in $k[Y]$. The d desired covariants are $x\mapsto\sum_{i=1}^d F_{ij}(x)e_i$, $j=1,\dots,d$, where $F_{ij}(x)$ is the (i,j)-entry of the inverse of the matrix $(\Phi_{st}(x))_{d\times d}$, and $e_1,\dots,e_d$ is the basis dual to $\varepsilon_1,\dots,\varepsilon_d$. Indeed, the G-invariance of the Φ_i can be expressed as the matrix equality

$$(g\Phi_{ij})_{d\times d}=(\Phi_{ij})_{d\times d}\cdot g_W, \tag{2}$$

where g_W is the matrix of $g\in G$ acting on W with respect to a basis $e_1,\dots,e_d$ dual to ε_i. Formula (2) shows that $F_j:x\mapsto\sum_{i=1}^d F_{ij}(x)e_j$ is a covariant for $j=1,\dots,d$. Moreover, since $\det(F_{ij})_{d\times d}$ is a unit in $k[Y]$, the $F_j(x)$ $(j=1,\dots,d)$ are linearly independent for all $x\in Y$. ∎

It is a bit less obvious, but turns out from Theorem 1.1 that under mild technical conditions, if the G-varieties $X\times W$ and $X\times k^{\dim(W)}$ are birationally isomorphic over X, then there is a G-equivariant birational isomorphism over X between them which is *linear on* W, i.e. for a general $x \in X$, the restriction $\{x\} \times W \cong W \to k^d \cong \{x\} \times k^d$ is k-linear.

The G-variety X is *generically free* if the stabilizer G_x of a general $x \in X$ is trivial. The action of G on X is *generically separable* if for a general $x \in X$ the orbit morphism $G_0 \to G_0 x$, $g \mapsto gx$ is separable (i.e. its differential is surjective; this holds automatically when $\mathrm{char}(k) = 0$). Note that if X is a generically free G-variety, then generic separability is equivalent to the following: the map $G \times X \to X \times X$, $(g, x) \mapsto (x, gx)$ is birational between $G \times X$ and the graph $\{(x, gx) \mid g \in G, x \in X\}$ of the action. We refer to Chapter AG in [2] for the definition and basic properties of separability.

Proof of Theorem 1.1. The implication (4) $\implies$ (3) is trivial.

(3) $\implies$ (1): The G-equivariant birational isomorphism $X \times W \to X \times k^d$ restricts to a G_x-equivariant birational isomorphism $\Phi(x, -) : W \to k^d$ for a general $x \in X$; since G_x acts trivially on k^d, it acts trivially on W.

(2) $\implies$ (4): Suppose $F_1, \ldots, F_d \in \mathrm{RCov}_G(X, W)$ are generically independent. Denote by U the subset in X where $F_1(x), \ldots, F_d(x)$ are all defined, it is a G-stable dense open subset in X. Apply Lemma 2.1 for the G-variety U.

(4) $\implies$ (5) is straightforward: Suppose $(x, w) \mapsto (x, \Phi(x, w))$ is a G-equivariant birational isomorphism $X \times W \to X \times k^d$, which is linear on W. Then $\Phi(x, w) = (\Phi_1(x, w), \ldots, \Phi_d(x, w))$ with $\Phi_i = \sum_{j=1}^d \Phi_{ij}\varepsilon_j \in k(X \times W)^G$, where $\Phi_{ij} \in k(X)$. Moreover, since $k(X \times k^d)^G$ is obviously generated over $k(X)^G$ by the coordinate functions on k^d, we get that $k(X \times W)^G$ is generated over $k(X)^G$ by the d algebraically independent elements Φ_i.

(5) $\implies$ (2): Suppose that $k(X \times W)^G$ is purely transcendental over $k(X)^G$ generated by $\Phi_i = \sum_{j=1}^d \Phi_{ij}\varepsilon_j$, $i = 1, \ldots, d$, where $\varepsilon_1, \ldots, \varepsilon_d$ is a basis of W^*. Write g_W for the matrix of $g \in G$ acting on W with respect to the basis $e_1, \ldots, e_d$ dual to ε_i. The G-invariance of the Φ_i can be expressed as the matrix equality

$$(g\Phi_{ij})_{d\times d} = (\Phi_{ij})_{d\times d} \cdot g_W. \tag{3}$$

We claim that $\det(\Phi_{ij})_{d\times d} \neq 0 \in k(X)$. Indeed, assume on the contrary that the rows of $(\Phi_{ij})_{d\times d}$ are linearly dependent over $k(X)$. After a possible reordering of the Φ_i, we may assume that the first r rows are linearly independent, whereas

$$(\Phi_{r+1,1}, \ldots, \Phi_{r+1,d}) = \sum_{i=1}^{r} f_i \cdot (\Phi_{i1}, \ldots, \Phi_{id}) \tag{4}$$

with $f_i \in k(X)$. Next we show that all the f_i are G-invariant: apply $g \in G$ to (4); Taking into account (3) and multiplying both sides of the resulting vector equality by g_W^{-1} we obtain

$$(\Phi_{r+1,1}, \ldots, \Phi_{r+1,d}) = \sum_{i=1}^{r} (gf_j)(\Phi_{i1}, \ldots, \Phi_{id}). \tag{5}$$

Take the difference of (4) and (5):

$$(0, \ldots, 0) = \sum_{i=1}^{r} (f_j - gf_j)(\Phi_{i1}, \ldots, \Phi_{id}).$$

Since the first r rows of the matrix $(\Phi_{ij})_{d\times d}$ are linearly independent over $k(X)$, it follows that $f_i = gf_i$ for $i = 1,\dots,r$. This holds for all $g \in G$, hence $f_1,\dots,f_r \in k(X)^G$. Consequently, we have

$$\Phi_{r+1} = \sum_{i=1}^{r} f_i\Phi_i \in k(X \times W)^G,$$

contradicting the assumption that $\Phi_1,\dots,\Phi_d$ are algebraically independent over $k(X)^G$.

Thus we proved that the matrix $(\Phi_{ij}) \in k(X)^{d\times d}$ is invertible; denote by $F_{ij} \in k(X)$ the (i,j)-entry of its inverse. Then formula (3) shows that $F_j : x \mapsto \sum_{i=1}^{d} F_{ij}(x)e_j$ is a covariant for $j = 1,\dots,d$. Moreover, these covariants are generically independent, since $\det(F_{ij})_{d\times d} = \det(\Phi_{ij})^{-1}_{d\times d} \neq 0$.

(1) $\implies$ (2): Our first step is to reduce to the case when the action of G on X is generically free, and W is a faithful G-module. Denote by N the kernel of the action of G on W. Then N is a closed normal subgroup of G, and the stabilizer G_x is contained in N for a general $x \in X$, and G/N acts faithfully on W. In fact we may assume $G_x \leq N$ for all x by omitting a proper closed G-stable subset of X. Let $\pi : X \to X/N$ be a rational quotient; i.e., X/N is a model (defined up to birational isomorphism) of $k(X)^N$, and π the dominant rational map corresponding to the field inclusion $k(X)^N \to k(X)$. There is a unique rational G-action (factoring through G/N) on X/N such that π is G-equivariant (see for example Theorem 5 in [13]). By a theorem of Weil [15] (see [13] for the case when G is not connected) we may assume that G/N acts morphically (not just rationally) on X/N.

Now observe that the action of G/N on X/N is generically free. Indeed, let U be an N-stable dense open subset of X such that $\pi|_U : U \to \pi(U) = U/N$ is a geometric quotient morphism (i.e. it is an open morphism whose fibers are N-orbits); such an U exists by Rosenlicht's Theorem [13]. It is easy to see that $\pi|_U$ extends to a G-equivariant morphism $\pi : \cup_{g\in G} gU \to \cup_{g\in G} g\pi(U)$, which is a geometric quotient with respect to the action of N. In other words, we may assume that U is G-stable, hence passing from X to a G-stable dense open subset if necessary, we may assume that the G-equivariant morphism $\pi : X \to X/N$ is a geometric quotient with respect to the action of N. Suppose $g \in G_{\pi(x)}$ for some $x \in X$. Then $gx = nx$ for some $n \in N$, hence $g^{-1}n \in G_x \leq N$, implying that $g \in N$. So for all $y \in X/N$ we have $G_y = N$.

By Lemma 2.3 below, the action of G/N on X/N is generically separable.

Note that if $F_1,\dots,F_d \in \mathrm{RCov}_{G/N}(X/N, W)$ are generically independent, then $F_1 \circ \pi,\dots,F_d \circ \pi$ are generically independent rational covariants from X to W. Therefore it is sufficient to deal with the case when G acts generically freely and separably on X, and faithfully on W. From now on we assume that this is the case. Replacing X by a G-stable dense open subset, we may assume that a geometric quotient $\pi : X \to X/G =: Y$ exists. It is explained in Section 2.5 in [10] that π has a quasisection. That is, there exists a finite rational Galois covering $\alpha : Z \to Y$ (with Galois group Γ) and a rational map $\sigma : Z \to X$ with $\pi \circ \sigma = \alpha$ (the characteristic is assumed to be zero in [10], but the argument given there remains valid if one replaces "algebraic closure" by "separable closure"). Consider the fibre product $\tilde{X} := X \times_Y Z$, and the pullback $\tilde{\pi} : \tilde{X} \to Z$ of π. Then the map $Z \to \tilde{X}$, $z \mapsto [\sigma(z), z]$ is a rational section for $\tilde{\pi}$. Therefore by Lemma 2.4 below, the map $(g, z) \mapsto [g\sigma(z), z]$ is a G-equivariant birational isomorphism $G \times Z \cong \tilde{X}$ (G acts trivially on Z and by left multiplication on itself). Identify $\tilde{X}$ and $G \times Z$ accordingly. Then the action of Γ on $G \times Z$ is given by $\gamma(g, z) = (gc_\gamma(z)^{-1}, \gamma(z))$, where $c_\gamma : Z \to G$ is a rational map characterized by the equality $\sigma(\gamma(z)) = c_\gamma(z)\sigma(z)$. This equality shows that the 1-cocycle condition holds: $c_{\gamma\rho} = (c_\gamma \circ \rho)c_\rho$ for all $\gamma, \rho \in \Gamma$. Set $d_\gamma := R \circ c_\gamma : Z \to GL(W)$, where $R : G \to GL(W)$ is the given representation of G on W. Since the Galois cohomology $H^1(\Gamma, GL(W \otimes k(Z)))$ is trivial (see for example the section on Galois descent in [8]), there is a rational map $A : Z \to GL(W)$ with $d_\gamma(z) = A(\gamma(z))^{-1}A(z)$. Now consider the G-equivariant rational map $\tilde{X} = G \times Z \to GL(W)$, $(g, z) \mapsto R(g)A(z)^{-1}$. One checks easily that this map is Γ-invariant, hence factors through a rational G-equivariant map $F : X \to GL(W)$. ∎

The following two Lemmas were used in the preceding proof. These technical statements relating mainly separability must be well known. We include their proof, since we did not find a convenient reference.

Lemma 2.3. *Let X be an irreducible G-variety, N a closed normal subgroup of G such that $G_x \leq N$ for all $x \in X$, and let $\pi : X \to Y$ be a morphism of G-varieties, which is a rational quotient with respect to the action of N. If the G-variety X is generically separable, then the G/N-variety Y is generically separable.*

Proof. Write $\overline{G} := G/N$. We may assume that the fibers of π are N-orbits (see the proof of (1) $\implies$ (2) in Theorem 1.1), and so $\overline{G}$ acts freely on Y. The field $k(X)$ is a separable extension of $k(X)^N$ (see for example [2]), so the differential $\mathrm{d}_x\pi : T_xX \to T_{\pi(x)}Y$ is surjective for a general $x \in X$, hence

$$\dim(\ker(\mathrm{d}_x\pi)) = \dim(X) - \dim(Y) = \dim(Nx).$$

On the other hand, $\ker(\mathrm{d}_x\pi) \supseteq \ker(\mathrm{d}_x\pi|_{T_xGx}) \supseteq T_x(Nx)$, and $\dim(T_xNx) = \dim(Nx)$. It follows that

$$\ker(\mathrm{d}_x(\pi)) = T_x(Nx) = \ker(\mathrm{d}_x\pi|_{T_xGx}) = \ker(\mathrm{d}_x(\pi|_{Gx})).$$

Hence (for a general $x \in X$) we have

$$\dim(\mathrm{Im}(\mathrm{d}_x\pi|_{Gx})) = \dim(Gx) - \dim(Nx) = \dim(\overline{G}\pi(x))$$

(in the last equality we use that the stabilizer of $\pi(x)$ in G is N). So $\mathrm{d}_x(\pi|_{Gx}) : T_xGx \to T_{\pi(x)}\overline{G}\pi(x)$ is surjective. The differential of the morphism $G \to Gx$, $g \mapsto gx$ is also surjective (by the assumption on generic separability of the action of G on X), hence the differential of $G \to \overline{G}\pi(x)$, $g \mapsto g\pi(x)$ is surjective at $1 \in G$. This morphism is the composition of the natural surjection $G \to \overline{G}$ and the morphism $\overline{G} \to \overline{G}\pi(x)$, $g \mapsto g\pi(x)$, implying that the differential at $1 \in \overline{G}$ of the latter is surjective. This holds for a general point $\pi(x) \in Y$, thus the action of $\overline{G}$ on Y is indeed generically separable. ∎

Lemma 2.4. *Let X be a a generically free and separable G-variety, $\pi : X \to Y$ a morphism whose fibres are G-orbits, and $\sigma : Y \to X$ is a morphism with $\pi \circ \sigma = \mathrm{id}_Y$. Then $G \times \sigma(Y) \to X$, $(g,s) \mapsto gs$ is a birational isomorphism.*

Proof. Set $S := \sigma(Y)$, and $\mu : G \times S \to X$, $(g,s) \mapsto gs$. A routine calculation shows that μ is a separable morphism. The generic freeness of the action implies that μ is bijective. It follows that μ is birational, see eg. Chapter AG in [2]. ∎

Example 2.5. The equivalence of $(2),(4),(5)$, and the implications $(4) \implies (3) \implies (1)$ in Theorem 1.1 obviously hold without the generic separability assumption. On the other hand, the following example shows that the assumption on generic separability is necessary to have the implication $(1) \implies (3)$. Indeed, assume that the characteristic of k is $p > 0$. Let G be the multiplicative group of the base field k, acting on $X = k$ by $g \cdot x = g^p x$. Then X is a generically free G-variety. However, consider $W = k$ with G acting by scalar multiplication $g \cdot w = gw$. Then $k(X \times W)^G = k(s/t^p)$, where s,t denote coordinates on X and W, respectively. Obviously, $k(s, s/t^p)$ is a proper subfield of $k(X \times W)$. Consequently, a birational isomorphism $X \times W \to X \times k$ of the form $(x,w) \mapsto (x, \Phi(x,w))$ with $\Phi \in k(X \times W)^G$ does not exist.

Remark 2.6. E. B Vinberg has communicated to me the following generalization of Theorem 1.1, not assuming that the action of G on X is generically separable: Each of the assertions $(2),(3),(4),(5)$ in Theorem 1.1 is equivalent to

(1') The scheme theoretic stabilizer of a general point of X acts on W trivially.

To prove $(1') \implies (2)$ one first reduces to the case when the scheme theoretic stabilizer of a general point in X is trivial, i.e. the action is generically free and separable (one should replace N in the first step of the proof $(1) \implies (2)$ by the scheme theoretic kernel of the action of G on W). The implication $(3) \implies (1')$ is obvious: if the action of G on W is trivial, then the action on W of any scheme theoretic subgroup is also trivial.

3. Linearly reductive groups

For linearly reductive groups acting on affine varieties, the results of Theorem 1.1 have a counterpart using integral covariants.

Proposition 3.1. *Let G be a linearly reductive group, X an irreducible affine G-variety, and W a G-module. If for some $x \in X$ having closed orbit, the stabilizer G_x acts trivially on W, then there exists a relative invariant $f \in k[X]$ such that $f(x) \neq 0$ and the G-varieties $X_f \times W \to X_f \times k^{\dim(W)}$ (G acting trivially on $k^{\dim(W)}$) are isomorphic over X.*

Proof. By Theorem 1 from [9] (see also Proposition 4.2.5 in [5]), our assumptions imply the existence of $d = \dim_k(W)$ covariants $F_1, \dots, F_d \in \mathrm{Cov}_G(X,W)$ such that $F_1(x), \dots, F_d(x)$ are linearly independent over k. Now apply Lemma 2.1. ∎

Remark 3.2. When X is a G-module, Proposition 3.1 has a constructive nature in the following sense: if the group G and the G-modules X and W are given as in section 4.2.3 in [5], then there is an algorithm that either produces a non-zero relative invariant $f \in k[X]$ and an explicit isomorphism $X_f \times W \to X_f \times k^d$, or proves that there are no d generically independent covariants from X to W (and hence the assumptions of Proposition 3.1 do not hold). Indeed, an algorithm to compute $k[X]^G$-module generators of $\mathrm{Cov}_G(X, W)$ is explained on page 157 of [5]. The output of the algorithm is a collection F_{ij} of polynomials in $k[X]$ ($i = 1, \ldots, d$, $j = 1, \ldots, m$), such that $F_j : x \mapsto \sum_{i=1}^d F_{ij}(x)e_i$ ($j = 1, \ldots, m$) is a $k[X]^G$-module generating system of $\mathrm{Cov}_G(X, W)$ (here $e_1, \ldots, e_d$ is a basis of W). If the $d \times m$ matrix (F_{ij}) has a $d \times d$ minor with non-zero determinant $f \in k[X]$, then the covariants corresponding to the d columns of this minor are generically independent. Using these covariants one obtains an explicit isomorphism $X_f \times W \to X_f \times k^d$ as in Lemma 2.1. If $m < d$ or all the $d \times d$ minors of (F_{ij}) have determinant zero in $k[X]$, then clearly there are no d generically independent integral covariants from X to W.

The equivalent conditions in Theorem 1.1 have the following analogues in this context:

Proposition 3.3. *Let G be a linearly reductive group, X an irreducible affine G-variety, and W a G-module. The following are equivalent:*

(1) There is a non-zero relative invariant $f \in k[X]$ such that G_x acts trivially on W for all $x \in X_f$.

(2) There is a non-zero relative invariant $f \in k[X]$ and an $x \in X_f$ whose G-orbit is closed in X_f, and G_x acts trivially on W.

(3) There is a non-zero relative invariant $f \in k[X]$ such that there are $d := \dim(W)$ covariants $F_i : X_f \to W$ ($i = 1, \ldots, d$) with $F_1(x), \ldots, F_d(x)$ linearly independent for some $x \in X_f$.

(4) There is a non-zero relative invariant $f \in k[X]$ such that the G-varieties $X_f \times W$ and $X_f \times k^{\dim(W)}$ are isomorphic.

Proof. (1) $\implies$ (2): The G-variety X_f contains a point x whose G-orbit is closed.

(2) $\implies$ (3): Apply Theorem 1 of [9] for the affine G-variety X_f.

(3) $\implies$ (4): Apply Lemma 2.1.

(4) $\implies$ (1): The G-equivariant isomorphism $X_f \times W \to X_f \times k^d$ restricts for any $x \in X_f$ to a G_x-equivariant isomorphism $\{x\} \times W \to \{x\} \times k^d$. Now G_x acts trivially on the latter, hence G_x acts trivially on W. ■

Remark 3.4. In general in an affine G-variety X there are more closed G'-orbits than G-orbits, where G' is the commutator subgroup of G. When $\mathrm{char}(k) = 0$ and G is connected reductive, by Theorem 1 in [14], $G'x$ is closed in X if and only if there is a relative invariant b on X such that $b(x) \neq 0$ and the G-orbit of x is closed in the affine open set X_b. Therefore in this case (1), (2), (3), and (4) in Proposition 3.3 are equivalent also to the following:

(5) There is a point $x \in X$ whose G'-orbit is closed, and the stabilizer G_x of x in G acts trivially on W.

The considerations above can be applied to generically free actions of linearly reductive groups on factorial affine varieties: (An affine variety X is called *factorial* if $k[X]$ is a unique factorization domain.)

Theorem 3.5. *Suppose* $\mathrm{char}(k) = 0$, *G is reductive, acting generically freely on the factorial affine variety X. Then for any G-module W, there is a non-zero relative invariant f on X such that the G-varieties $X_f \times W$ and $X_f \times k^d$ are isomorphic over X_f.*

Proof. There is a non-empty affine open G-stable subset U in X such that the generic G-orbit is closed in U (see Theorem 2.18 in [4]). Clearly, U is of the form X_b for some non-zero relative invariant b (see Theorem 3.1 in [10]). Now apply Proposition 3.1 for the affine G-variety U and W. ∎

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