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The Tits Construction and Some Simple Lie Superalgebras in Characteristic 3

Alberto Elduque[*]

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Abstract. Some simple Lie superalgebras, specific of characteristic 3, defined by S. Bouarroudj and D. Leites, will be related to the simple alternative and commutative superalgebras discovered by I. P. Shestakov.
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Throughout the paper, the ground field k will always be assumed to be of characteristic $\neq 2$.

1. Tits construction

In 1966 [17], Tits gave a unified construction of the exceptional simple classical Lie algebras by means of two ingredients: a unital composition algebra and a degree three simple Jordan algebra. The approach used by Benkart and Zelmanov in [3] will be followed here (see also [10]) to review this construction.

Let C be a unital composition algebra over the ground field k with norm n. Thus, C is a finite dimensional unital k-algebra, with the nondegenerate quadratic form $n : C \to k$ such that $n(ab) = n(a)n(b)$ for any $a, b \in C$. Then, each element satisfies the degree 2 equation

$$a^2 - t(a)a + n(a)1 = 0, \tag{1}$$

where $t(a) = n(a, 1)\ \bigl(= n(a + 1) - n(a) - 1\bigr)$ is called the *trace*. The subspace of trace zero elements will be denoted by C^0.

Moreover, for any $a, b \in C$, the linear map $D_{a,b} : C \to C$ given by

$$D_{a,b}(c) = [[a, b], c] + 3(a, c, b) \tag{2}$$

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where $[a,b]=ab-ba$ is the commutator, and $(a,c,b)=(ac)b-a(cb)$ the associator, is a derivation: the *inner derivation* determined by the elements a,b (see [14, Chapter III, §8]). These derivations satisfy

$$D_{a,b}=-D_{b,a},\quad D_{ab,c}+D_{bc,a}+D_{ca,b}=0, \tag{3}$$

for any $a,b,c\in C$. The linear span of these derivations will be denoted by $\mathfrak{inder}\,C$. It is an ideal of the whole Lie algebra of derivations $\mathfrak{der}\,C$ and, if the characteristic is $\neq 3$, it is the whole $\mathfrak{der}\,C$.

The dimension of C is restricted to 1, 2, 4 (quaternion algebras) and 8 (Cayley algebras). If the ground field k is algebraically closed, the only unital composition algebra are, up to isomorphism, the ground field k, the cartesian product of two copies of the ground field $k\times k$, the algebra of two by two matrices $\mathrm{Mat}_2(k)$, and the split Cayley algebra $C(k)$. (See, for instance, [19, Chapter 2].)

Now, let J be a unital Jordan algebra with a *normalized trace* $t_J: J\to k$. That is, t_J is a linear map such that $t_J(1)=1$ and $t_J\bigl((xy)z\bigr)=t_J\bigl(x(yz)\bigr)$ for any $x,y,z\in J$. Then $J=k1\oplus J^0$, where $J^0=\{x\in J: t_J(x)=0\}$. For any $x,y\in J^0$, the product xy splits as

$$xy=t_J(xy)1+x*y, \tag{4}$$

with $x*y\in J^0$. Then $x*y=xy-t_J(xy)1$ gives a commutative multiplication on J^0. The linear map $d_{x,y}: J\to J$ defined by

$$d_{x,y}(z)=x(yz)-y(xz), \tag{5}$$

is the inner derivation of J determined by the elements x and y. Since $d_{1,x}=0$ for any x, it is enough to deal with the inner derivations $d_{x,y}$, with $x,y\in J^0$. The linear span of these derivations will be denoted by $\mathfrak{inder}\,J$, which is an ideal of the whole Lie algebra of derivations $\mathfrak{der}\,J$.

Given C and J as before, consider the space

$$\mathcal{T}(C,J)=\mathfrak{inder}\,C\oplus\bigl(C^0\otimes J^0\bigr)\oplus\mathfrak{inder}\,J \tag{6}$$

(unadorned tensor products are always considered over k), with the anticommutative multiplication $[.,.]$ specified by:

$$\left.\begin{array}{l}
\bullet\ \mathfrak{inder}\,C \text{ and } \mathfrak{inder}\,J \text{ are Lie subalgebras,}\\
\bullet\ [\mathfrak{inder}\,C,\mathfrak{inder}\,J]=0,\\
\bullet\ [D,a\otimes x]=D(a)\otimes x,\ [d,a\otimes x]=a\otimes d(x),\\
\bullet\ [a\otimes x,b\otimes y]=t_J(xy)D_{a,b}+\bigl([a,b]\otimes x*y\bigr)+2t(ab)d_{x,y},
\end{array}\right. \tag{7}$$

for all $D\in\mathfrak{inder}\,C$, $d\in\mathfrak{inder}\,J$, $a,b\in C^0$, and $x,y\in J^0$.

The conditions for $\mathcal{T}(C,J)$ to be a Lie algebra are the following:

$$\begin{array}{ll}
\text{(i)} & \displaystyle\sum_{\circlearrowleft} t\bigl([a_1,a_2]a_3\bigr)\,d_{(x_1*x_2),x_3}=0,\\[2ex]
\text{(ii)} & \displaystyle\sum_{\circlearrowleft} t_J\bigl((x_1*x_2)x_3\bigr)\,D_{[a_1,a_2],a_3}=0,\\[2ex]
\text{(iii)} & \displaystyle\sum_{\circlearrowleft}\Bigl(D_{a_1,a_2}(a_3)\otimes t_J(x_1x_2)x_3+[[a_1,a_2],a_3]\otimes(x_1*x_2)*x_3\\
& \qquad\qquad +2t(a_1a_2)a_3\otimes d_{x_1,x_2}(x_3)\Bigr)=0
\end{array} \tag{8}$$

for any $a_1, a_2, a_3 \in C^0$ and any $x_1, x_2, x_3 \in J^0$. The notation "$\sum_{\circlearrowleft}$" indicates summation over the cyclic permutation of the variables.

These conditions appear in [2, Proposition 1.5], but there they are stated in the more general setting of superalgebras, a setting we will deal with later on. In particular, over fields of characteristic $\neq 3$, these conditions are fulfilled if J is a separable Jordan algebra of degree three over k and $t_J = \frac{1}{3}T$, where T denotes the generic trace of J (see for instance [12]).

Over an algebraically closed field k of characteristic $\neq 3$, the degree 3 simple Jordan algebras are, up to isomorphism, the algebras of 3×3 hermitian matrices over a unital composition algebra: $H_3(C')$ (see [12]). By varying C and C', $\mathcal{T}(C, H_3(C'))$ is a classical simple Lie algebra, and Freudenthal's Magic Square (Table 1) is obtained.

	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

Table 1: Freudenthal's Magic Square

Let us have a look at the rows in the Tits construction of Freudenthal's Magic Square.

First row: Here $C = k$, so $C^0 = 0$ and $\mathfrak{inder}\, C = 0$. Thus, $\mathcal{T}(C, J)$ is just $\mathfrak{inder}\, J$. In particular, $\mathcal{T}(k, J)$ makes sense and is a Lie algebra for any Jordan algebra J.

Second row: Here $C = k \times k$, so C^0 consists of the scalar multiples of $(1, -1)$, and thus $\mathcal{T}(C, J)$ can be identified with $J^0 \oplus \mathfrak{inder}\, J$. The elements in J^0 multiply as $[x, y] = 4d_{x,y}$ because $t\big((1,-1)^2\big) = t\big((1,1)\big) = 2$. Given any Jordan algebra J, its *Lie multiplication algebra* $\mathcal{L}(J)$ (see [14]) is the Lie subalgebra of the general linear Lie algebra $\mathfrak{gl}(J)$ generated by $l_J = \{l_x : x \in J\}$, where $l_x : y \mapsto xy$ denotes the left multiplication by x. Then $\mathcal{L}(J) = l_J \oplus \mathfrak{inder}\, J$. The map

$$\begin{gathered} \mathcal{T}(C, J) \to \mathcal{L}(J) \\ (1, -1) \otimes x + d \mapsto 2l_x + d, \end{gathered}$$

is a monomorphism. Its image is $\mathcal{L}^0(J) = l_{J^0} \oplus \mathfrak{inder}\, J$. Again this shows that $\mathcal{T}(k \times k, J)$ makes sense and is a Lie algebra for any Jordan algebra with a normalized trace. Given any separable Jordan algebra of degree 3 over a field k of characteristic $\neq 3$, $\mathcal{L}^0(J)$ is precisely the derived algebra $[\mathcal{L}(J), \mathcal{L}(J)]$. This latter Lie algebra makes sense for any Jordan algebra over any field. (Recall that the characteristic is assumed to be $\neq 2$ throughout.)

Third row: Here $C = \mathrm{Mat}_2(k)$ or, if the ground field is not assumed to be algebraically closed, C is any quaternion algebra Q. Under these circumstances, Q^0 is a simple three-dimensional Lie algebra under the commutator ($[a, b] =$

$ab-ba$), and any simple three-dimensional Lie algebra appears in this way. Besides, for any $a, b \in Q^0$, the inner derivation $D_{a,b}$ is just $\mathrm{ad}_{[a,b]}$, since Q is associative. Hence, $\mathfrak{inder}\, Q$ can be identified with Q^0, and $\mathcal{T}(Q,J)$ with

$$Q^0 \oplus \big(Q^0 \otimes J^0\big) \oplus \mathfrak{inder}\, J \simeq \big(Q^0 \otimes (k1 \oplus J^0)\big) \oplus \mathfrak{inder}\, J \simeq \big(Q^0 \otimes J\big) \oplus \mathfrak{inder}\, J,$$

and the Lie bracket (7) in $\mathcal{T}(Q,J)$ becomes the bracket in $\big(Q^0 \otimes J\big) \oplus \mathfrak{inder}\, J$ given by

- $\mathfrak{inder}\, J$ is a Lie subalgebra,
- $[d, a \otimes x] = a \otimes d(x)$,
- $[a \otimes x, b \otimes y] = \big([a,b] \otimes xy\big) + 2t(ab)d_{x,y}$,

for any $a, b \in Q^0$, $x, y \in J$, and $d \in \mathfrak{inder}\, J$, since $t_J(xy)1 + x * y = xy$ for any $x, y \in J$. This bracket makes sense for any Jordan algebra (not necessarily endowed with a normalized trace), it goes back to [16] and, in the split case $Q = \mathrm{Mat}_2(k)$, the resulting Lie algebra is the well-known Tits-Kantor-Koecher Lie algebra of the Jordan algebra J.

Fourth row: In the last row, C is a Cayley algebra over k. If the characteristic of the ground field k is $\neq 3$, the Lie algebra $\mathfrak{der}\, C = \mathfrak{inder}\, C$ is a simple Lie algebra of type G_2 (dimension 14), and C^0 is its unique seven dimensional irreducible module. In particular, over any algebraically closed field of characteristic $\neq 3$, the Lie algebra $\mathcal{T}\big(C(k), J\big)$ is a Lie algebra graded over the root system G_2. These G_2-graded Lie algebras contain a simple subalgebra isomorphic to $\mathfrak{der}\, C(k)$ such that, as modules for this subalgebra, they are direct sums of copies of modules of three types: adjoint, the irreducible seven dimensional module, and the trivial one dimensional module. These Lie algebras have been determined in [3] and the possible Jordan algebras involved are essentially the degree 3 Jordan algebras.

In characteristic 3, however, the situation is completely different. To begin with, given a Cayley algebra C over a field k of characteristic 3, its Lie algebra of derivations $\mathfrak{der}\, C$ is no longer simple (see [1]) but contains a unique minimal ideal, which is precisely $\mathfrak{inder}\, C = \mathrm{ad}_{C^0}$ (note that $D_{a,b} = \mathrm{ad}_{[a,b]}$ in characteristic 3 because of (2)), which is isomorphic to the Lie(!) algebra $\big(C^0, [.,.]\big)$. This latter Lie algebra is a form of the projective special linear Lie algebra $\mathfrak{psl}_3(k)$ (and any form of $\mathfrak{psl}_3(k)$ appears in this way [8, §4]). Moreover, in [1] it is shown that the quotient $\mathfrak{der}\, C/\mathfrak{inder}\, C = \mathfrak{der}\, C/\mathrm{ad}_{C^0}$ is isomorphic too, as a Lie algebra, to $\big(C^0, [.,.]\big)$. Therefore, the algebra $\mathcal{T}(C,J)$ in (6) can be identified in this case with

$$\begin{aligned} C^0 \oplus \big(C^0 \otimes J^0\big) \oplus \mathfrak{inder}\, J &\simeq \big(C^0 \otimes (k1 \oplus J)\big) \oplus \mathfrak{inder}\, J \\ &\simeq (C^0 \otimes J) \oplus \mathfrak{inder}\, J, \end{aligned}$$

and hence, as a module for $\mathfrak{inder}\, C \simeq C^0$, it is a direct sum of copies of the adjoint module and of the trivial module. The Lie algebras and superalgebras with these properties will be determined in this paper.

2. A family of Lie algebras

Throughout this section, the ground field k will be always assumed to be of characteristic 3. Let C be a Cayley algebra, that is, an eight dimensional unital composition algebra over k, and let C^0 denote its subspace of trace zero elements.

For any $a, b, c \in C$, a simple computation using that the associator is skew symmetric on its arguments since C is alternative gives:

$$\begin{aligned}
&[[a, b], c] + [[b, c], a] + [[c, a], b] \\
&\qquad = (a, b, c) - (b, a, c) + (b, c, a) - (c, b, a) + (c, a, b) - (a, c, b) \\
&\qquad = 6(a, b, c) = 0.
\end{aligned}$$

Hence C is a Lie algebra under the bracket $[a, b] = ab - ba$, and C^0 is an ideal of C. Besides, for any $a, b \in C^0$, since the subalgebra generated by any two elements in C is associative (Artin's Theorem, see [14]), one obtains

$$\begin{aligned}
[[a, b], b] &= ab^2 + b^2a - 2bab \\
&= ab^2 + b(ba + ab) \qquad (\text{as } 2 = -1) \\
&= -n(b)a - n(a, b)b \quad (\text{as } a^2 = -n(a)1 \text{ for any } a \in C^0)
\end{aligned}$$

Thus, for any $a, b \in C^0$,

$$[[a, b], b] = n(b, b)a - n(a, b)b. \tag{9}$$

If C is split, then it contains a basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ with multiplication given by (see [14, 19]):

$$\begin{aligned}
&e_i^2 = e_i,\ i = 1, 2, \quad e_1e_2 = e_2e_1 = 0, \\
&e_1u_j = u_j = u_je_2,\ e_2v_j = v_j = v_je_1,\ j = 1, 2, 3, \\
&e_2u_j = u_je_1 = 0 = e_1v_j = v_je_2,\ j = 1, 2, 3, \\
&u_iv_j = -\delta_{ij}e_1,\ v_iu_j = -\delta_{ij}e_2,\ i, j = 1, 2, 3\ (\delta_{ij} \text{ is } 1 \text{ for } i = j,\ 0 \text{ otherwise}) \\
&u_iu_j = \epsilon_{ijk}v_k,\ v_iv_j = \epsilon_{ijk}u_k,\ (\epsilon_{ijk} \text{ skew symmetric with } \epsilon_{123} = 1).
\end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned}
&n(e_i) = 0 = n(u_j, u_k) = n(v_j, v_k),\ i = 1, 2,\ j, k = 1, 2, 3, \\
&n(e_1, e_2) = 1,\ n(u_j, v_k) = \delta_{jk},\ j, k = 1, 2, 3.
\end{aligned} \tag{11}$$

Then, with $h = e_1 - e_2$, C^0 is the linear span of $\{h, u_1, u_2, u_3, v_1, v_2, v_3\}$, and these elements multiply as:

$$\begin{aligned}
&[h, u_j] = 2u_j,\ [h, v_j] = -2v_j,\ j = 1, 2, 3, \\
&[u_i, u_j] = 2\epsilon_{ijk}v_k,\ [v_i, v_j] = 2\epsilon_{ijk}u_k,\ i, j, k = 1, 2, 3, \\
&[u_j, v_k] = -\delta_{jk}h,\ j, k = 1, 2, 3.
\end{aligned} \tag{12}$$

Denote by $\mathfrak{s}$ the Lie algebra $\big(C^0, [.,.]\big)$. It is easy to check that the $\mathfrak{s}$-module $\mathfrak{s} \otimes \mathfrak{s}$ is generated by $u_1 \otimes v_1$, which is an eigenvector for ad_h with eigenvalue 0. From this fact, it follows at once that the dimension of the space of invariant linear maps $\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{s} \otimes \mathfrak{s}, \mathfrak{s})$ is 1, being this space generated by the Lie bracket. Also, $\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{s} \otimes \mathfrak{s}, k)$ is one dimensional, spanned by the bilinear map induced by the norm $n(.,.)$. By extending scalars, this is shown to be valid for any Cayley algebra, not necessarily split.

Thus, let C be any Cayley algebra over k and let $\mathfrak{s}$ be the Lie algebra $\big(C^0, [.,.]\big)$. Let $\mathfrak{g}$ be a Lie algebra endowed with an action of $\mathfrak{s}$ on $\mathfrak{g}$ by derivations:

$\rho : \mathfrak{s} \to \mathfrak{Der}\,\mathfrak{g}$, such that, as a module for $\mathfrak{s}$, $\mathfrak{g}$ is a direct sum of copies of the adjoint module and the one dimensional trivial module. Gathering together the copies of the adjoint module and the copies of the trivial module, $\mathfrak{g}$ can be identified with

$$\mathfrak{g} = \bigl(\mathfrak{s} \otimes A\bigr) \oplus \mathfrak{d}, \tag{13}$$

where $\mathfrak{d}$ is the sum of the trivial $\mathfrak{s}$-modules and A is a vector space. As $\mathfrak{d}$ equals $\{d \in \mathfrak{g} : \rho(s)(d) = 0\ \forall s \in \mathfrak{s}\}$, it is a subalgebra of $\mathfrak{g}$. Now, the invariance of the bracket in $\mathfrak{g}$ under the action of $\mathfrak{s}$, together with the fact that $\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{s} \otimes \mathfrak{s}, \mathfrak{s})$ (respectively $\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{s} \otimes \mathfrak{s}, k)$) is spanned by the Lie bracket (respectively, the bilinear form induced by the norm) shows that there are bilinear maps

$$\begin{aligned}
&\mathfrak{d} \times A \to A, && (d,a) \mapsto d(a),\\
&A \times A \to A, && (a_1,a_2) \mapsto a_1a_2,\\
&A \times A \to \mathfrak{d}, && (a_1,a_2) \mapsto d_{a_1,a_2},
\end{aligned} \tag{14}$$

such that the Lie bracket on $\mathfrak{g} = \bigl(\mathfrak{s} \otimes A\bigr) \oplus \mathfrak{d}$ is given by:

$$\begin{aligned}
&\bullet\ [d, s \otimes a] = s \otimes d(a) = -[s \otimes a, d],\\
&\bullet\ [s_1 \otimes a_1, s_2 \otimes a_2] = [s_1,s_2] \otimes a_1a_2 + n(s_1,s_2)d_{a_1,a_2},\\
&\bullet\ [d_1,d_2] \text{ is the product in the subalgebra } \mathfrak{d},
\end{aligned} \tag{15}$$

for any $d, d_1, d_2 \in \mathfrak{d}$, $s, s_1, s_2 \in \mathfrak{s}$, and $a, a_1, a_2 \in A$. The skew symmetry of the Lie bracket forces the product a_1a_2 on A to be commutative, and the bilinear map d_{a_1,a_2} to be skew symmetric.

Now, let us consider the Jacobi identity $J(z_1,z_2,z_3) = 0$ on $\mathfrak{g}$, where $J(z_1,z_2,z_3) = \sum_{\circlearrowleft} [[z_1,z_2]z_3]$:

- With $z_1 = d_1$, $z_2 = d_2$ in $\mathfrak{d}$ and $z_3 = s \otimes a$, $s \in \mathfrak{s}$, $a \in A$, this gives $[d_1,d_2](a) = d_1(d_2(a)) - d_2(d_1(a))$. That is, the linear map $\Phi : \mathfrak{d} \to \mathfrak{gl}(A)$, $\Phi(d) : a \mapsto d(a)$, is a representation of the Lie algebra $\mathfrak{d}$.

- With $z_1 = d$, $z_2 = s_1 \otimes a_1$ and $z_3 = s_2 \otimes a_2$, $d \in \mathfrak{d}$, $s_1, s_2 \in \mathfrak{s}$ and $a_1, a_2 \in A$, the Jacobi identity gives:

$$\begin{aligned}
d(a_1a_2) &= d(a_1)a_2 + a_1d(a_2),\\
[d, d_{a_1,a_2}] &= d_{d(a_1),a_2} + d_{a_1,d(a_2)},
\end{aligned}$$

  for any $a_1, a_2 \in A$. That is, $\Phi(\mathfrak{d}) \subseteq \mathfrak{Der}\,A$ holds and $d : A \times A \to \mathfrak{d}$ is a $\mathfrak{d}$-invariant bilinear map.

- With $z_i = s_i \otimes a_i$, $i = 1,2,3$, $s_i \in \mathfrak{s}$, $a_i \in A$, the Jacobi identity gives:

$$\sum_{\circlearrowleft} n\bigl([s_1,s_2],s_3\bigr)d_{a_1a_2,a_3} = 0 \tag{16a}$$

$$\sum_{\circlearrowleft} \Bigl(\bigl([[s_1,s_2],s_3] \otimes (a_1a_2)a_3\bigr) + \bigl(n(s_1,s_2)s_3 \otimes d_{a_1,a_2}(a_3)\bigr)\Bigr) = 0 \tag{16b}$$

Extending scalars, it can be assumed that C is split, so that a basis of C^0 as in (12) is available. With $s_i = u_i$, $i = 1, 2, 3$, $n([u_1, u_2], u_3) = 2n(v_3, u_3) = 2$ and cyclically. (Note that, because of the invariance under derivations of the norm, $n([s_1, s_2], s_3) = n(s_1, [s_2, s_3]) = n([s_2, s_3], s_1)$.) Hence (16a) is equivalent to

$$d_{a_1a_2,a_3} + d_{a_2a_3,a_1} + d_{a_3a_1,a_2} = 0 \tag{17}$$

for any $a_1, a_2, a_3 \in A$.

Also, $[[u_1, u_2], u_3] = 2[v_3, u_3] = 2h$ and cyclically, and $n(u_j, u_k) = 0$ for any j, k. Thus (16b) gives

$$(a_1a_2)a_3 + (a_2a_3)a_1 + (a_3a_1)a_2 = 0 \tag{18}$$

for any $a_1, a_2, a_3 \in A$.

With $s_1 = u_1$, $s_2 = v_1$ and $s_3 = h$, $[[s_1, s_2], s_3] = 0$, $[[s_2, s_3], s_1] = 2h$, and $[[s_3, s_1], s_2] = -2h$, while $n(s_1, s_2) = 1$, and $n(s_2, s_3) = 0 = n(s_3, s_1)$, so (16b) gives (note that $2 = -1$ in k):

$$(a_3a_1)a_2 - (a_2a_3)a_1 + d_{a_1,a_2}(a_3) = 0,$$

which, by the commutativity of the product on A, is equivalent to:

$$d_{a_1,a_2}(a) = a_1(a_2a) - a_2(a_1a), \tag{19}$$

for any $a_1, a_2, a \in A$.

Lemma 2.1. *Let k be a field of characteristic 3. The commutative algebras over k satisfying (18) are precisely the commutative alternative algebras. Moreover, given any such algebra A, for any a_1, a_2 consider the linear map $d_{a_1,a_2} = [l_{a_1}, l_{a_2}]$, where l_a denotes the multiplication by a. Then d_{a_1,a_2} is a derivation of A and equation (17) is satisfied.*

Proof. By commutativity, (18) is equivalent to $2(a_1a_2)a_2 + a_1a_2^2 = 0$, or $(2 = -1)$ to $a_1a_2^2 = (a_1a_2)a_2$ for any a_1, a_2, which is the right alternative law. Because of the commutativity, this is equivalent to the left alternative law, and hence the algebra is alternative. Now, any commutative alternative algebra is a Jordan algebra, since the Jordan identity is $(x^2, y, x) = 0$ for any x, y, which is satisfied because any two elements in an alternative algebra generate an associative subalgebra (Artin's Theorem, see [14]). Hence $d_{a_1,a_2} = [l_{a_1}, l_{a_2}]$ is a derivation of A. Finally, equation (17) becomes the linearization of $[l_{x^2}, l_x] = 0$. ∎

Conversely, let C be a Cayley algebra over k and let $\mathfrak{s}$ be the Lie algebra $\big(C^0, [.,.]\big)$. Let A be a commutative alternative algebra, and let $\mathfrak{d}$ be a Lie algebra endowed with a Lie algebra homomorphism $\Phi : \mathfrak{d} \to \mathfrak{Der}\, A$ (thus, in particular, A is a module for $\mathfrak{d}$), and a $\mathfrak{d}$-invariant bilinear map $d : A \times A \to \mathfrak{d}$, $(a_1, a_2) \mapsto d_{a_1,a_2}$, such that $\Phi(d_{a_1,a_2}) = [l_{a_1}, l_{a_2}]$ and $\sum_{\circlearrowleft} d_{a_1a_2,a_3} = 0$ for any $a_1, a_2, a_3 \in A$. Then equation (16a) holds trivially by the invariance of the norm in C, and equation (16b) holds too, as it is equivalent to

$$\begin{aligned}0&=\sum_{\circlearrowleft}\Bigl([[s_1,s_2,],s_3]\otimes(a_1a_2)a_3+n(s_1,s_2)s_3\otimes\bigl(a_1(a_2a_3)-a_2(a_1a_3)\bigr)\Bigr)\\&=\sum_{\circlearrowleft}\Bigl(\bigl([[s_1,s_2],s_3]-n(s_2,s_3)s_1+n(s_3,s_1)s_2\bigr)\otimes(a_1a_2)a_3\Bigr).\end{aligned}$$

But $(a_1a_2)a_3=-(a_2a_3)a_1-(a_3a_1)a_2$, so (16) holds if

$$[[s_1,s_2],s_3]-n(s_2,s_3)s_1+n(s_3,s_1)s_2=[[s_2,s_3],s_1]-n(s_3,s_1)s_2+n(s_1,s_2)s_3$$

for any $s_1,s_2,s_3\in\mathfrak{s}$, or

$$[[s_1,s_2],s_3]+[[s_2,s_3],s_1]=2n(s_1,s_3)s_2-n(s_2,s_3)s_1-n(s_2,s_1)s_3,$$

which is equivalent to (9).

Therefore:

Theorem 2.2. *Let C be a Cayley algebra over a field k of characteristic 3, let $\mathfrak{s}$ be the Lie algebra $\bigl(C^0,[.,.]\bigr)$. Let $\mathfrak{g}$ be a Lie algebra with an action of $\mathfrak{s}$ by derivations such that, as a module for $\mathfrak{s}$, $\mathfrak{g}$ is a direct sum of irreducible modules of two types: the adjoint and the trivial one-dimensional modules. Then there is a commutative alternative algebra A over k and a Lie algebra $\mathfrak{d}$ over k, endowed with a Lie algebra homomorphism $\Phi:\mathfrak{d}\to\mathfrak{Der}\,A$ and a $\mathfrak{d}$-invariant skewsymmetric bilinear map $d:A\times A\to\mathfrak{d}$, $(a_1,a_2)\mapsto d_{a_1,a_2}$ with $\Phi(d_{a_1,a_2})=[l_{a_1},l_{a_2}]$ for any $a_1,a_2\in A$, such that $\mathfrak{g}$ is isomorphic to the Lie algebra*

$$\bigl(\mathfrak{s}\otimes A)\oplus\mathfrak{d}$$

with Lie bracket given by

$$\begin{aligned}&\bullet\ \mathfrak{d}\ \textit{is a Lie subalgebra,}\\&\bullet\ [d,s\otimes a]=s\otimes d(a),\ \textit{for}\ s\in\mathfrak{s},\ a\in A,\ d\in\mathfrak{d},\\&\bullet\ [s_1\otimes a_1,s_2\otimes a_2]=[s_1,s_2]\otimes a_1a_2+n(s_1,s_2)d_{a_1,a_2},\\&\qquad\qquad \textit{for}\ s_1,s_2\in\mathfrak{s},\ \textit{and}\ a_1,a_2\in A,.\end{aligned}\tag{20}$$

Conversely, the formulas in (20) define a Lie algebra on the vector space $\bigl(\mathfrak{s}\otimes A\bigr)\oplus\mathfrak{d}$, which is endowed with an action of $\mathfrak{s}$ by derivations: $\rho:\mathfrak{s}\to\mathfrak{Der}\bigl((\mathfrak{s}\otimes A)\oplus\mathfrak{d}\bigr)$, such that $\rho(s)(s'\otimes a)=[s,s']\otimes a$, $\rho(s)(d)=0$, for any $s,s'\in S$, $a\in A$ and $d\in\mathfrak{d}$.

Remark 2.3. Over fields of characteristic $\neq 2,3$, any commutative alternative algebra is associative, because for any x,y,z,

$$\begin{aligned}3(x,y,z)&=(x,y,z)+(y,z,x)+(z,x,y)\\&=(xy)z-x(yz)+(yz)x-y(zx)+(zx)y-z(xy)\\&=[xy,z]+[yz,x]+[zx,y]=0.\end{aligned}$$

Remark 2.4. Let A be a unital commutative alternative algebra over a field k of characteristic 3 such that $1 \notin (A, A, A)$ (this is trivially the case for the unital commutative associative algebras). Then A is a Jordan algebra with a normalized trace, because if A^0 is any codimension 1 subspace of A containing (A, A, A) but not containing 1, then $A = k1 \oplus A^0$, and the linear form $t : A \to k$, such that $t(1) = 1$ and $t(A^0) = 0$ is a normalized trace. Let C be a Cayley algebra over k and define $\mathcal{T}(C, A)$ as in (6):

$$\mathcal{T}(C, A) = \mathfrak{inder}\, C \oplus \bigl(C^0 \otimes A^0\bigr) \oplus \mathfrak{inder}\, A.$$

Then $\mathcal{T}(C, A)$ is a Lie algebra (bracket as in (7)), isomorphic to $\bigl(\mathfrak{s} \otimes A\bigr) \oplus \mathfrak{d}$, with $\mathfrak{d} = \mathfrak{inder}\, A = [l_A, l_A]$, $\mathfrak{s} = C^0$, and bracket as in Theorem 2.2 (with Φ the natural inclusion).

This gives the natural extension of the fourth row in Tits construction to characteristic 3. The Jordan algebras that appear have nothing to do with the separable degree 3 Jordan algebras.

It must be remarked that the simple commutative alternative algebras are just the fields (see [19, p. 143]), but there are prime commutative alternative and not associative algebras in characteristic 3 (see [13][2]). Recall that an algebra is simple if its multiplication is not trivial and it contains no proper ideal, while it is prime if the product of any two nonzero ideals is again nonzero.

3. Superalgebras

All the arguments used in the proofs of Lemma 2.1 and Theorem 2.2 are valid in the setting of superalgebras, if parity signs are added suitably. The super version of Theorem 2.2 is:

Theorem 3.1. *Let C be a Cayley algebra over a field k of characteristic 3, let $\mathfrak{s}$ be the Lie algebra $\bigl(C^0, [.,.]\bigr)$. Let $\mathfrak{g}$ be a Lie superalgebra with an action of $\mathfrak{s}$ by (even) derivations such that, as a module for $\mathfrak{s}$, $\mathfrak{g}$ is a direct sum of irreducible modules of two types: adjoint and trivial. Then there is a commutative alternative superalgebra A over k and a Lie superalgebra $\mathfrak{d}$ over k, endowed with a homomorphism of Lie superalgebras $\Phi : \mathfrak{d} \to \mathfrak{Der}\, A$ and an even $\mathfrak{d}$-invariant (relative to Φ) super skewsymmetric bilinear map $d : A \times A \to \mathfrak{d}$, $(a_1, a_2) \mapsto d_{a_1,a_2}$ with $\Phi(d_{a_1,a_2}) = [l_{a_1}, l_{a_2}]$ for any $a_1, a_2 \in A$, such that $\mathfrak{g}$ is isomorphic to the Lie superalgebra*

$$\bigl(\mathfrak{s} \otimes A\bigr) \oplus \mathfrak{d}$$

with Lie bracket given by

$$\begin{aligned} &\bullet\ \mathfrak{d} \text{ is a Lie subalgebra},\\ &\bullet\ [d, s \otimes a] = s \otimes d(a), \text{ for } s \in \mathfrak{s},\ a \in A,\ d \in \mathfrak{d},\\ &\bullet\ [s_1 \otimes a_1, s_2 \otimes a_2] = [s_1, s_2] \otimes a_1 a_2 + n(s_1, s_2) d_{a_1,a_2},\\ &\qquad\qquad \text{for } s_1, s_2 \in \mathfrak{s}, \text{ and } a_1, a_2 \in A. \end{aligned} \tag{21}$$

Conversely, the formulas in (21) define a Lie superalgebra on the vector superspace $\bigl(\mathfrak{s} \otimes A\bigr) \oplus \mathfrak{d}$ (the even part is $\bigl(\mathfrak{s} \otimes A_{\bar 0}\bigr) \oplus \mathfrak{d}_{\bar 0}$ and the odd part is

[2] The author is indebted to the referee for pointing out this reference.

$\big(\mathfrak{s}\otimes A_{\bar 1}\big)\oplus\mathfrak{d}_{\bar 1}$*), which is endowed with an action of* $\mathfrak{s}$ *by (even) derivations:* $\rho:\mathfrak{s}\to\mathfrak{Der}\big((\mathfrak{s}\otimes A)\oplus\mathfrak{d}\big)$*, such that* $\rho(s)(s'\otimes a)=[s,s']\otimes a$*,* $\rho(s)(d)=0$*, for any* $s,s'\in\mathfrak{s}$*,* $a\in A$ *and* $d\in\mathfrak{d}$*.*

Recall that given a superalgebra A, its Lie superalgebra of derivations is the Lie superalgebra $\mathfrak{Der}\,A=(\mathfrak{Der}\,A)_{\bar 0}\oplus(\mathfrak{Der}\,A)_{\bar 1}$ (a subalgebra of the general linear Lie superalgebra $\mathfrak{gl}(A)$), where for any homogeneous $d\in\mathfrak{Der}\,A$ and homogeneous $a_1,a_2\in A$:

$$d(a_1a_2)=d(a_1)a_2+(-1)^{da_1}a_1d(a_2),$$

where, as usual, $(-1)^{da_1}$ is -1 if both d and a_1 are odd, and $(-1)^{da_1}$ is 1 otherwise. The Lie bracket of homogeneous elements in $\mathfrak{gl}(A)$ is given by $[f,g]=fg-(-1)^{fg}gf$. The fact that d in the Theorem above is *even* means that d_{A_i,A_j} is contained in $\mathfrak{d}_{i+j}$ for any $i,j\in\{\bar 0,\bar 1\}$; and the invariance of d relative to Φ means that

$$[f,d_{a_1,a_2}]=d_{\Phi(f)(a_1),a_2}+(-1)^{fa_1}d_{a_1,\Phi(f)(a_2)}$$

for any homogeneous elements $f\in\mathfrak{d}$ and $a_1,a_2\in A$.

The importance of Theorem 3.1 lies in the fact that there do exist interesting examples of commutative alternative simple superalgebras in characteristic 3. Besides:

Proposition 3.2. *Let* A *be a nonzero commutative alternative superalgebra over a field* k *of characteristic* 3*, and let* $\mathfrak{d}$ *be a Lie superalgebra endowed with a homomorphism of Lie superalgebras* $\Phi:\mathfrak{d}\to\mathfrak{Der}\,A$ *and an invariant (relative to* Φ*) super skewsymmetric bilinear map* $d:A\times A\to\mathfrak{d}$ *with* $\Phi(d_{a_1,a_2})=[l_{a_1},l_{a_2}]$ *for any* $a_1,a_2\in A$*. Let* $\mathfrak{g}=\big(\mathfrak{s}\otimes A\big)\oplus\mathfrak{d}$ *be the Lie superalgebra constructed by means of* (21)*. Then* $\mathfrak{g}$ *is simple if and only if the following conditions are fulfilled:*

(i) Φ *is one-to-one,*

(ii) $\mathfrak{d}=d_{A,A}$ $(=$ *span* $\{d_{a_1,a_2}:a_1,a_2\in A\})$*,*

(iii) A *is simple.*

Proof. Assume first that $\mathfrak{g}$ is simple. Since $\ker\Phi$ is an ideal, not only of $\mathfrak{d}$, but of the whole $\mathfrak{g}$, it follows that Φ is one-to-one. Also, $\big(\mathfrak{s}\otimes A\big)\oplus d_{A,A}$ is an ideal of $\mathfrak{g}$, so $d_{A,A}=\mathfrak{d}$. Finally, if I is a nonzero ideal of A, then I is invariant under $d_{A,A}$ because $\Phi(d_{A,A})=[l_A,l_A]$ is contained in the Lie multiplication algebra of A. Hence $\big(\mathfrak{s}\otimes I\big)\oplus d_{I,A}$ is an ideal of $\mathfrak{g}$, and it follows that $I=A$. Hence A has no proper ideals, so it is either simple or $\dim A=1$ and $A^2=0$. In the latter case $\Phi(\mathfrak{d})=\Phi(d_{A,A})=[l_A,l_A]$ would be 0, and $\mathfrak{g}=\mathfrak{s}\otimes A$ would be a trivial Lie superalgebra ($[\mathfrak{g},\mathfrak{g}]=0$), a contradiction to the simplicity of $\mathfrak{g}$.

Conversely, if condtions (i)–(iii) are satisfied, A is unital [15], and hence $\mathfrak{s}\,(\simeq\mathfrak{s}\otimes 1)$ is a subalgebra of $\mathfrak{g}$. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, the invariance of $\mathfrak{a}$ under the adjoint action of $\mathfrak{s}$ shows that $\mathfrak{a}=\big(\mathfrak{s}\otimes I\big)\oplus\mathfrak{e}$ for an ideal I of A and an ideal $\mathfrak{e}$ of $\mathfrak{d}$. Now, the simplicity of A forces that either $I=0$, but then $\mathfrak{e}\subseteq\ker\Phi=0$ and $\mathfrak{a}=0$, or $I=A$ and then $\mathfrak{s}\otimes A$ is contained in $\mathfrak{a}$, so $d_{A,A}=\mathfrak{d}$ is contained in $\mathfrak{a}$ too and $\mathfrak{a}=\mathfrak{g}$. Hence $\mathfrak{g}$ is simple. ∎

Shestakov's classification [15] of the simple alternative superalgebras over k (characteristic 3) shows that any central simple commutative alternative superalgebra is, up to isomorphism, either:

(i) the ground field k,

(ii) the three dimensional composition superalgebra $B(1,2)$, with even part $B(1,2)_{\bar 0} = k1$, and odd part $B(1,2)_{\bar 1} = ku + kv$, with 1 the unity element and $u^2 = 0 = v^2$, $uv = -vu = 1$. This is the Jordan superalgebra of a superform on a vector odd space of dimension 2,

(iii) an algebra $B = B(\Gamma, D, 0)$, where Γ is a commutative associative algebra, $D \in \mathfrak{der}\,\Gamma$ is a derivation such that Γ has no proper ideal invariant under D, $B_{\bar 0} = \Gamma$, $B_{\bar 1} = \Gamma u$ (a copy of Γ) and the multiplication is given by:

- the multiplication in Γ,
- $a(bu) = (ab)u = (au)b$ for any $a, b \in \Gamma$,
- $(au)(bu) = aD(b) - D(a)b$, for any $a, b \in \Gamma$.

Given a form $\mathfrak{s}$ of $\mathfrak{psl}_3(k)$ and the commutative alternative superalgebra $A = B(1,2)$, its Lie superalgebra of derivations is naturally isomorphic to $\mathfrak{sl}(A_{\bar 1}) \simeq \mathfrak{sl}_2(k)$, and the simple Lie superalgebra $\mathfrak{g} = \big(\mathfrak{s} \otimes A\big) \oplus \mathfrak{d}_{A,A}$ in Theorem 3.1 has even and odd parts given by:

$$\begin{aligned}\mathfrak{g}_{\bar 0} &= \big(\mathfrak{s} \otimes A_{\bar 0}\big) \oplus (\mathfrak{d}_{A,A})_{\bar 0} \simeq \mathfrak{s} \oplus \mathfrak{sl}_2(k),\\ \mathfrak{g}_{\bar 1} &= \big(\mathfrak{s} \otimes A_{\bar 1}\big) \oplus (\mathfrak{d}_{A,A})_{\bar 1} = \mathfrak{s} \otimes A_{\bar 1}.\end{aligned}$$

For an algebraically closed field k, this coincides with the Lie superalgebra that appears in [9, Theorem 4.22(i)], and also with the derived subalgebra of the Lie superalgebra $\mathfrak{g}(S_2, S_{1,2})$ in [6, 7].

Also, assuming that k is algebraically closed, according to [4] or [18] any finite dimensional commutative associative algebra Γ over k endowed with a derivation D satisfying that Γ is D-simple (that is, there is no proper ideal invariant under D) is isomorphic to a truncated polynomial algebra $k[t_1, \dots, t_n : t_i^3 = 0,\ i = 1, \dots, n]$ which, in turn, is isomorphic to the divided power algebra $\mathcal{O}(1;n)$, which is the k-algebra spanned by the symbols $t^{(r)}$, $0 \le r < 3^n - 1$, with $t^{(0)} = 1$ and multiplication given by $t^{(r)}t^{(s)} = \binom{r+s}{r} t^{(r+s)}$. The isomorphism takes t_i to $t^{(3^{i-1})}$, $i = 1, \dots, n$. The simplest $D \in \mathfrak{der}\,\mathcal{O}(1;n)$ for which $\mathcal{O}(1;n)$ is D-simple is the derivation given by $D : t^{(r)} \mapsto t^{(r-1)}$ for any r.

4. Bouarroudj-Leites superalgebras

Recently, S. Bouarroudj and D. Leites [5] have constructed an interesting family of finite dimensional simple Lie superalgebras in characteristic 3 by means of the so called Cartan-Tanaka-Shchepochkina prolongs. These superalgebras are denoted by bj, of dimension 24, and $\mathrm{Bj}(1;N|7)$ (N an arbitrary natural number), of dimension $2^4 \times 3^N$. All these algebras are consistently $\mathbb{Z}$-graded: $\mathfrak{g} = \oplus_{i=-2}^{2\cdot 3^N - 1} \mathfrak{g}_i$ and $\mathfrak{g}_{\bar 0}$ (respectively $\mathfrak{g}_{\bar 1}$) is the sum of the even (resp. odd) homogeneous components.

Besides, $\mathfrak{g}_0$ is the direct sum of a one dimensional center and an ideal isomorphic to $\mathfrak{psl}_3(k)$, $\dim \mathfrak{g}_{-2} = 1$, $\mathfrak{g}_{-1}$ is an adjoint module for the ideal isomorphic to $\mathfrak{psl}_3(k)$ in $\mathfrak{g}_0$. The positive homogeneous components are all either a trivial one dimensional module or an adjoint module for $\mathfrak{psl}_3(k)$, or a direct sum of both. Therefore, these Lie superalgebras fit in the setting of the previous section.

The Lie superalgebra bj satisfies that its even part is isomorphic to $\mathfrak{sl}_2(k) \oplus \mathfrak{psl}_3(k)$, while its odd part is, as a module for the even part, the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and the adjoint module for $\mathfrak{psl}_3(k)$. Therefore, it coincides with the Lie superalgebra obtained in the previous section for $A = B(1,2)$, which appeared first in [9].

Over an algebraically closed field k of characteristic 3, by dimension count, the Lie superalgebra $\mathrm{Bj}(1; N|7)$ must be necessarily isomorphic to a Lie superalgebra as in Theorem 3.1 for $\Gamma = \mathcal{O}(1; N)$ and a suitable derivation D.

Let $\Gamma = \mathcal{O}(1; N)$ and let D be the derivation $D : t^{(r)} \mapsto t^{(r-1)}$ for any r. Then the commutative alternative superalgebra $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$ is consistently $\mathbb{Z}$-graded with $\deg t^{(r)} = 2r$, $\deg u = -1$. In this way, $B = \oplus_{i=-1}^{2(3^N-1)} B_i$, and $\dim B_i = 1$ for any $i = -1, \ldots, 2(3^N - 1)$. Then $\mathfrak{d} = d_{B,B}$ becomes a $\mathbb{Z}$-graded Lie superalgebra too with $\deg D = -2$. For any $a, b \in \Gamma$, $d_{a,b} = [l_a, l_b] = 0$, so $\mathfrak{d}_{\bar 0} = [l_{B_{\bar 1}}, l_{B_{\bar 1}}]$.

But for any $a, b, c \in \Gamma$:

$$\begin{aligned}
[l_{au}, l_{bu}](c) &= (au)(bcu) + (bu)(acu) \\
&= D(a)bc - aD(b)c - abD(c) + D(b)ac - bD(a)c - baD(c) \\
&= -2abD(c) = abD(c), \\
[l_{au}, l_{bu}](cu) &= (au)\big(D(b)c - bD(c)\big) + (bu)\big(D(a)c - aD(c)\big) \\
&= \big(aD(b)c - abD(c) + D(a)bc - abD(c)\big)u \\
&= D(abc)u.
\end{aligned}$$

Thus $\mathfrak{d}_{\bar 0} = \operatorname{span}\{d_x : x \in \Gamma\}$, with $d_x|_\Gamma = xD$, $d_x(yu) = D(xy)u$ for any $x, y \in \Gamma$. The degree of d_x is $\deg x - 2$.

Also, $\mathfrak{d}_{\bar 1} = [l_{B_{\bar 0}}, l_{B_{\bar 1}}]$, and for any $a, b, c \in \Gamma$:

$$\begin{aligned}
[l_a, l_{bu}](c) &= a((bc)u) - (bu)(ac) = 0 \\
[l_a, l_{bu}](cu) &= a\big(D(b)c - bD(c)\big) - (bu)(acu) \\
&= aD(b)c - abD(c) - D(b)ac + bD(ac) \\
&= (D(a)b)c.
\end{aligned}$$

Thus $\mathfrak{d}_{\bar 1} = \operatorname{span}\{\delta_x : x \in \Gamma\}$, with $\delta_x|_\Gamma = 0$, $\delta_x(yu) = xy$ for any $x, y \in \Gamma$. The degree of δ_x is $\deg x + 1$.

Hence $\mathfrak{d}$ has dimension 2×3^N, and then the simple Lie superalgebra $\big(\mathfrak{psl}_3(k) \otimes B\big) \oplus \mathfrak{d}$, with $B = B(\Gamma, D, 0)$ has dimension $7 \times 2 \times 3^N + 2 \times 3^N = 2^4 \times 3^N$. Moreover, $\mathfrak{d}$ is consistently $\mathbb{Z}$-graded:

$$\mathfrak{d} = \mathfrak{d}_{-2} \oplus \mathfrak{d}_0 \oplus \mathfrak{d}_1 \oplus \mathfrak{d}_2 \oplus \cdots \oplus \mathfrak{d}_{2\cdot 3^N - 4} \oplus \mathfrak{d}_{2\cdot 3^N - 3} \oplus \mathfrak{d}_{2\cdot 3^N - 1}.$$

Observe that $\mathfrak{d}_{-1} = 0 = \mathfrak{d}_{2\cdot 3^N - 2}$, while $\mathfrak{d}_i$ has dimension 1 for any other i with $-2 \le i \le 2 \cdot 3^N - 1$.

Now, with $\deg g = 0$ for any $g \in \mathfrak{psl}_3(k)$, the simple Lie superalgebra $\mathfrak{g} = \big(\mathfrak{psl}_3(k) \otimes B\big) \oplus \mathfrak{d}$ is consistently $\mathbb{Z}$-graded too:

$$\mathfrak{g} = \oplus_{i=-2}^{2\cdot 3^N-1} \mathfrak{g}_i,$$

with $\mathfrak{g}_{-2} = \mathfrak{d}_{-2} = kD$, $\mathfrak{g}_{-1} = \mathfrak{psl}_3(k) \otimes B_{-1} = \mathfrak{psl}_3(k) \otimes u$, $\mathfrak{g}_0 = \big(\mathfrak{psl}_3(k) \otimes B_0\big) \oplus \mathfrak{d}_0 = \big(\mathfrak{psl}_3(k) \otimes 1\big) \oplus k(t^{(1)}D)$, and each $\mathfrak{g}_i$, $0 < i < 2 \cdot 3^N - 2$ is the direct sum of $\mathfrak{psl}_3(k) \otimes B_i$ and $\mathfrak{d}_i$, that is, as a a module for $\mathfrak{psl}_3(k)$, it is the direct sum of a copy of the adjoint module and a copy of the one dimensional trivial module. Finally, $\mathfrak{g}_{2\cdot 3^N-2}$ is just $\mathfrak{psl}_3(k) \otimes B_{2\cdot 3^N-2} = \mathfrak{psl}_3(k) \otimes t^{(3^N-1)}$ (just a copy of the adjoint module), and $\mathfrak{g}_{2\cdot 3^N-1}$ is just $\mathfrak{d}_{2\cdot 3^N-1} = k\delta_{t^{(3^N-1)}}$ (a copy of the trivial module). This is exactly the way $\mathrm{Bj}(1;N|7)$ is graded, and this is no coincidence:

Theorem 4.1. *Let k be an algebraically closed field of characteristic 3, let N be a natural number, and let Γ be the algebra of divided powers $\mathcal{O}(1;N)$. Consider the derivation D of Γ given by $D(t^{(r)}) = t^{(r-1)}$ for any r and the simple commutative alternative superalgebra $B = B(\Gamma, D, 0)$. Then the simple Lie superalgebra $\mathfrak{g} = \big(\mathfrak{psl}_3(k) \otimes B\big) \oplus d_{B,B}$ in Theorem 3.1 is isomorphic to the Lie superalgebra of Bouarroudj and Leites* $\mathrm{Bj}(1;N|7)$.

Proof. Both $\mathfrak{g}$ and $\mathrm{Bj}(1;N|7)$ share the same negative part $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, and hence both of them embed in the universal graded Lie algebra $U(\mathfrak{g}_-)$ (see [11]), which is contained in the Lie algebra of special derivations of the tensor product of the divided power algebra $\mathcal{O}(1)$ and the Grassmann superalgebra $\Lambda(7)$ on a vector space of dimension 7 ($\mathcal{O}(1)$ is the span of $t^{(r)}$ for any $r \geq 0$, with $t^{(r)}t^{(s)} = \binom{r+s}{r} t^{(r+s)}$). Consider both $\mathfrak{g}$ and $\mathrm{Bj}(1;N|7)$ as subalgebras of $U = U(\mathfrak{g}_-)$. Actually, Bouarroudj and Leites consider an infinite dimensional Lie superalgebra $\mathrm{Bj}(1|7)$ which is contained in $U = U(\mathfrak{g}_-)$. The superalgebra $\mathrm{Bj}(1;N|7)$ is just the intersection of $\mathrm{Bj}(1|7)$ with the Lie superalgebra of special derivations of $\mathcal{O}(1;N) \otimes \Lambda(7)$.

Since $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-2}$ is determined by its action on $\mathfrak{g}_{-1}$. By transitivity, U_0 embeds in $\mathrm{End}(\mathfrak{g}_{-1}) \simeq \mathrm{End}(\mathfrak{psl}_3(k))$, and both $\mathfrak{g}_0$ and $\mathrm{Bj}(1;N|7)_0$ act on $\mathfrak{psl}_3(k)$ in the same way (the adjoint action of $\mathfrak{psl}_3(k)$ and the one dimensional center acting as a nonzero scalar). Hence, as homogeneous subalgebras of U, $\mathfrak{g}_0 = \mathrm{Bj}(1;N|7)_0 = \mathrm{Bj}(1|7)_0$. Write $\mathrm{Bj} = \mathrm{Bj}(1|7)$. For any $i > 0$ Bj_i is defined recursively as

$$\{x \in U_i : [x, \mathfrak{g}_{-2}] \subseteq \mathrm{Bj}_{i-2},\ [x, \mathfrak{g}_{-1}] \subseteq \mathrm{Bj}_{i-1}\},$$

so it follows that $\mathfrak{g}_i \subseteq \mathrm{Bj}_i$. But by dimension count, it follows that $\mathfrak{g}_i = \mathrm{Bj}(1;N|7)_i = \mathrm{Bj}(1|7)_i$ for any $0 < i < 2 \cdot 3^N - 2$, while both $\mathfrak{g}_{2\cdot 3^N-2}$ and $\mathrm{Bj}(1;N|7)_{2\cdot 3^N-2}$ are the unique copy of the adjoint module for $\mathfrak{psl}_3(k) \subseteq U_0$ in $\mathrm{Bj}(1|7)_{2\cdot 3^N-2}$, and both $\mathfrak{g}_{2\cdot 3^N-1}$ and $\mathrm{Bj}(1;N|7)_{2\cdot 3^N-1}$ are the unique copy of the one-dimensional trivial module for $\mathfrak{psl}_3(K)$ in $\mathrm{Bj}(1|7)_{2\cdot 3^N-1}$. Therefore, $\mathfrak{g} = \mathrm{Bj}(1;N|7)$ (as homogeneous subalgebras of U). ■

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Alberto Elduque
Departamento de Matemáticas e
Instituto Universitario de
Matemáticas y Aplicaciones
Universidad de Zaragoza
50009 Zaragoza, Spain
elduque@unizar.es

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