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Generalized Dolbeault Sequences in Parabolic Geometry

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Abstract. In this paper, we show the existence of a sequence of invariant differential operators on a particular homogeneous model G/P of a Cartan geometry. The first operator in this sequence is closely related to the Dirac operator in k Clifford variables, $D = (D_1, \dots, D_k)$, where $D_i = \sum_j e_j \cdot \partial_{ij} : C^\infty((\mathbb{R}^n)^k, \mathbb{S}) \to C^\infty((\mathbb{R}^n)^k, \mathbb{S})$. We describe the structure of these sequences in case the dimension n is odd. It follows from the construction that all these operators are invariant with respect to the action of the group G.

These results are obtained by constructing homomorphisms of generalized Verma modules, which are purely algebraic objects.
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1. Motivation

There are two basic generalizations of the space of holomorphic functions to higher dimensions. One of them is the notion of holomorphic functions in several variables, $f : \mathbb{R}^{2k} \simeq \mathbb{C}^k \to \mathbb{C}$, $\bar{\partial}_j f = 0$ for $j = 1, \dots, k$. The second possible generalization deals with s.c. *monogenic functions*, which are defined on $\mathbb{R}^n$ with values in the *Clifford algebra* or the *space of spinors* and solve the *Dirac equation* $\sum_j e_j \cdot \partial_j f = 0$. They have similar nice properties as holomorphic functions and coincide with them for $n = 2$ ([10]).

Recently, many variations and generalizations of the classical Dirac operator appeared. While mathematical physicists study its spectra on different Riemannian spin-manifolds and others construct its analogs in non-riemannian geometries (see e.g. [18, 19]), we may define the *Dirac operator in several Clifford variables* by $D : C^\infty((\mathbb{R}^n)^k, \mathbb{S}) \to C^\infty((\mathbb{R}^n)^k, \mathbb{C}^k \otimes \mathbb{S})$, $D = (D_1, \dots, D_k)$ (after identifying elements of the image with k spinor valued functions), $D_i = \sum_j e_j \cdot \partial_{ij}$ where $\mathbb{S}$ is the (usually complex) spinor space, x_{uv} the standard coordinates on $(\mathbb{R}^n)^k, u = 1, \dots, k, v = 1, \dots, n$, and $\cdot$ the Clifford multiplication $\mathbb{R}^n \times \mathbb{S} \to \mathbb{S}$.

This is a common generalization of the space of holomorphic functions in

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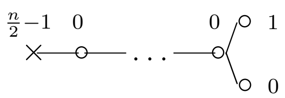
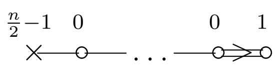
several complex variables ($n = 2$, k arbitrary) and the classical Dirac operator ($k = 1$).

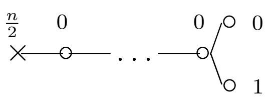
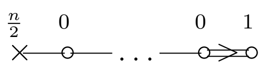
Many problems can be studied using a resolution of D, i.e. a (locally) exact complex of PDE's starting with the operator D. In the case of holomorphic functions in several complex variables, D being the Cauchy–Riemann operator ($n = 2$, k arbitrary), this is just the Dolbeault sequence. For $k = 2$, n even, the problem was studied in [12, 17]. However, for arbitrary n, k, the form of this resolution is not known yet, except of some special cases (see [4, 7, 8, 22]).

In this paper, the problem is treated in the framework of *parabolic geometry* and some particular results are obtained for n odd, k arbitrary. We construct sequences of differential operators starting with the Dirac operator D that are good candidates for being a resolution (the proof that they indeed form a resolution is still in progress). Our sequences contain all operators that are *invariant* with respect to the action of a quite large group and continue the Dirac operator.

Because the space of spinors arises naturally as a fundamental representation of the Lie group $\mathrm{Spin}(n)$, it is natural to consider the Dirac operator as acting not only on $C^\infty(\mathbb{R}^n, \mathbb{S})$ but rather on more general sections of a spinor bundle over a spin manifold M (see [9]). The simplest spin structure on the sphere S^n is the bundle $\mathrm{Spin}(n+1) \to \mathrm{Spin}(n+1)/\mathrm{Spin}(n) \simeq S^n$ and the associated spinor bundle is $\mathrm{Spin}(n+1) \times_{\mathrm{Spin}(n)} \mathbb{S}$. The usual Dirac operator acts between sections of this bundle and is invariant with respect to the group $\mathrm{Spin}(n+1)$ (the sections $\Gamma(G \times_H \mathbb{V})$ can be naturally identified with invariant functions $C^\infty(G, \mathbb{V})^H$ and the action of G is $g \cdot f(x) := f(g^{-1}x)$). However, Dirac operator has a larger group of invariance. Whereas $\mathrm{Spin}(n+1)$ acts on the sphere by rotations, it is well known that Dirac operator is invariant with respect to all Möbius transformations. This is reflected by the fact that the bundle $\mathrm{Spin}(n+1) \to \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$ is a reduction of a larger bundle $\mathrm{Spin}(n+1,1)/P$, where $\mathrm{Spin}(n+1,1)$ acts on the null-cone of a form g of signature $(n+1,1)$ that defines the group $\mathrm{Spin}(n+1,1)$. The projectivisation of this null-cone is homeomorphic to the sphere S^n and P is the stabilizer of one line. It was shown in [11] that considering $\mathbb{S}_1$ as a representation of P with highest weight

 resp. 

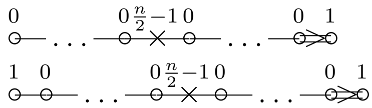
and $\mathbb{S}_2$ a representation of P with highest weight

 resp.  ,

the Dirac operator is a $\mathrm{Spin}(n+1,1)$-invariant differential operator $D : \Gamma(\mathrm{Spin}(n+1,1) \times_P \mathbb{S}_1) \to \Gamma(\mathrm{Spin}(n+1,1) \times_P \mathbb{S}_2)$. In this sense, the Dirac operator is conformally invariant, as $\mathrm{Spin}(n+1,1)$ (or, more exactly, its connected component) is the double-cover of the group of all Möbious transformations.

The subalgebra P is a *parabolic subalgebra* of $G = \mathrm{Spin}(n+1,1)$, i.e. its Lie algebra $\mathfrak{p}$ contains a Borel algebra $\mathfrak{b}$ of $\mathfrak{g}$, the Lie algebra of G. The bundle $G \to G/P$ together with the Maurer-Cartan form on $T(G)$ is an example of a s.c. *parabolic geometry* (see [5, 20]).

In [11], an analogous construction is described for the group $G = \mathrm{Spin}(n+k,k)$ and P being a parabolic subgroup fixing a maximal vector subspace of the null cone of the metric of signature $(n+k,k)$ defining $\mathrm{Spin}(n+k,k)$. The reductive part of P is isomorphic to $\mathrm{GL}(k) \times \mathrm{Spin}(n)$. The Lie algebra $\mathfrak{p}$ of P determines a gradation of the Lie algebra $\mathfrak{g}$ of G so that $\mathfrak{g} = \oplus_{j=-2}^{2} \mathfrak{g}_j$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Again, choosing proper irreducible P-modules $\mathbb{V}_1$ resp. $\mathbb{V}_2$ with highest weights



resp.

(and similar for n even), we showed in [11, 13] that there exists a G-invariant differential operator $D : \Gamma(G \times_P \mathbb{V}_1) \to \Gamma(G \times_P \mathbb{V}_2)$ and, identifying local sections in the neighborhood of eP with $\mathbb{V}_i$-valued functions on the vector space $\mathfrak{g}_- = \oplus_{j<0} \mathfrak{g}_j$ in a natural way and restricting to functions that are constant in $\mathfrak{g}_{-2} \subset \mathfrak{g}_-$, this operator coincides with the Dirac operator in k Clifford variables (identifying $\mathfrak{g}_{-1} \simeq (\mathbb{R}^n)^k$ as the adjoint representation of $\mathfrak{g}_0 \simeq \mathfrak{gl}(k) \times \mathfrak{so}(n)$).

The question is, whether we can find sequences of G-invariant differential operators extending the operator D. In case of the Dirac operator in one variable ($k = 1$), this is not possible. We showed in [12] that for $k = 2$, there exist two further G-invariant differential operators so that they form a complex together with the first one.

In general, for any semisimple Lie group G, a parabolic subgroup P and some P-modules $\mathbb{V}_1, \mathbb{V}_2$, the G-invariant differential operators between sections of vector bundles $D : \Gamma(G \times_P \mathbb{V}_1) \to \Gamma(G \times_P \mathbb{V}_2)$ are in $1-1$ correspondence with the $\mathfrak{g}$-homomorphisms of generalized Verma modules $M_{\mathfrak{p}}(\mathbb{V}_2^*) \to M_{\mathfrak{p}}(\mathbb{V}_1^*)$ induced by dual representations $\mathbb{V}_2^*$ and $\mathbb{V}_1^*$ (see [6]). Therefore, the generalized Verma modules and their homomorphisms will be studied in the rest of this paper.

2. Basics on generalized Verma modules

2.1. Bruhat ordering.

Let as assume that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, i.e. a subalgebra containing a Borel subalgebra $\mathfrak{b}$. This induces a gradation $\oplus_{j=-k}^{k} \mathfrak{g}_j$ of $\mathfrak{g}$ so that $\mathfrak{p} = \sum_{j \geq 0} \mathfrak{g}_j$. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$, Φ^+ a set of positive roots of $\mathfrak{g}$ (and also of $\mathfrak{p}$) and Δ the set of simple roots, compatible with Φ^+. There is a $1-1$ correspondence between subsets Σ of Δ and parabolic subalgebras $\mathfrak{p}_\Sigma \subset \mathfrak{g}$, where $\mathfrak{p}_\Sigma$ contains the Cartan subalgebra, all positive root spaces and all those negative root spaces $\mathfrak{g}_{-\beta}$, such that β can be expressed as a sum of simple roots from $\Delta - \Sigma$. These roots form the set of simple roots of the algebra $\mathfrak{g}_0$ from the associated grading $\oplus_{j=-k}^{k} \mathfrak{g}_j$. In the Dynkin diagram, we draw the simple roots in Σ as crossed ($\times$).

For any pair $(\mathfrak{g}, \mathfrak{p})$ there exists a unique element $E \in \mathfrak{g}$ called *grading element* so that $\mathrm{ad}(E)(X) = jX$ for any $X \in \mathfrak{g}_j$, $j = -k, \ldots, k$.

For each $\beta \in \Phi^+$, the *root reflection* s_β is a reflection in $\mathfrak{h}^*$ fixing the hyperplane orhogonal to β in the Killing metric. In coordinates, $s_\beta(\gamma) = \gamma - \gamma(H_\beta)\beta$ where H_β is the β-coroot (see e.g. [15]). The choice of Δ determines the length $l(w)$ of any element w of the Weyl group W of $\mathfrak{g}$. It is the minimal number k such that $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_k}}$, $\alpha_{i_j} \in \Delta$, s_{α_i} being simple root reflections. This defines the Bruhat ordering on W in the following way: $w \leq w'$ if and only if there exist $w = w_0 \to w_1 \to w_2 \to \dots \to w_l = w'$, where $w_i \to w_{i+1}$ means that $w_{i+1} = s_{\beta_i} w_i$ for some $\beta_i \in \Phi^+$ and the length $l(w_{i+1}) = l(w_i) + 1$.

2.2. Generalized Verma modules (GVM).

Let $\mathbb{V}$ be a (usually finite dimensional) irreducible $\mathfrak{p}$-module with highest weight λ. The generalized Verma module (further GVM), introduced by Lepowsky ([16]) is defined by $M_{\mathfrak{p}}(\mathbb{V}) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}$, where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, considered as a left $\mathcal{U}(\mathfrak{g})$ and a right $\mathcal{U}(\mathfrak{p})$-module. $M_{\mathfrak{p}}(\mathbb{V})$ is a highest weight module with highest weight λ and highest weight vector $1 \otimes v_\lambda$, where v_λ is a highest weight vector in $\mathbb{V}$. As a $\mathfrak{g}_-$-module and $\mathfrak{g}_0$-module, $M_{\mathfrak{p}}(\mathbb{V}) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{V}$. The GVM is uniquely determined by its highest weight λ, therefore we will sometimes denote the GVM with highest weight λ by $M_{\mathfrak{p}}(\lambda+\delta)$, where $\delta = \frac{1}{2}\sum_{\beta\in\Phi^+} \beta$. Assuming that $\mathbb{V}$ is finite dimensional, the set of GVM's is isomorphic to the set of $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral weights (this means weights λ such that $\lambda(H_\alpha)$ is non-negative and integral for each $\alpha \in \Delta - \Sigma$). The set of these weights will in the sequel be denoted by $P_{\mathfrak{p}}^{++}$.

If $\mathfrak{p} = \mathfrak{b} = \mathfrak{h} \oplus_{\beta\in\Phi^+} \mathfrak{g}_\beta$ is the Borel subalgebra of $\mathfrak{g}$, the GVM $M_{\mathfrak{b}}(\mathbb{V})$ is called true Verma module, or simply Verma module (in this case, $\mathbb{V}$ is a one-dimensional representation of $\mathfrak{b}$ and its weight can be any $\lambda \in \mathfrak{h}^*$). Each highest weight module with highest weight λ is isomorphic to a quotient of the Verma module $M_{\mathfrak{b}}(\lambda + \delta)$.

2.3. Duality between GVM homomorphisms and invariant differential operators. A G-invariant differential operator $D : \Gamma(G \times_P \mathbb{V}) \to \Gamma(G \times_P \mathbb{W})$ is completely determined by the values $Ds(eP)$ on sections ($e \in G$ is the identity element). If the operator is of order k, the value $Ds(eP)$ depends only on the k-jet $J^k_{eP}s$ of a section s in eP. So, the operator D is determined by a map $\tilde{D} : J^k_{eP}(G\times_P\mathbb{V}) \to \mathbb{W}$ that evaluates the image of a section s in eP, identifying the fiber over eP with $\mathbb{W}$ in a natural way. More precisely, $D(s)(eP) = [e, \tilde{D}(j^k_{eP}s))]_P$. Because D is G-invariant, $\tilde{D}$ has to be P-equivariant, the action of P on the jets being the action on representatives.

The P-module $J^k_{eP}(G \times_P \mathbb{V})$ of k-jets of sections is naturally isomorphic to the space of k-jets of P-invariant functions $J^k_e(C^\infty(G, \mathbb{V})^P)$ (the action of P here being $(p \cdot f)(x) = f(p^{-1}x)$). It can be shown that this is dual, as a P-module, to $\mathcal{U}_k(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}^*$ (where $\mathcal{U}_k(\mathfrak{g})$ is the k-th filtration of $\mathcal{U}(\mathfrak{g})$) and the duality is given by

$$(Y_1 \dots Y_l \otimes_{\mathcal{U}(\mathfrak{p})} A)(j^k_e f) := A((L_{Y_1} \dots L_{Y_l} f)(e)) \tag{1}$$

for $l \leq k$, $A \in \mathbb{V}^*$, $j^k_e f$ the k-jet of f in e, $Y_j \in \mathfrak{g}$ and L_{Y_j} the derivation with respect to the left invariant vector fields on G induced by Y_j (see [6] for details).

Any P-homomorphism $\tilde{D} : J^k_e(C^\infty(G, \mathbb{V})^P) \to \mathbb{W}$ is determined by its dual map $\tilde{D}^* : \mathbb{W}^* \to J^k_e(C^\infty(G, \mathbb{V})^P)^*$ and we see from (1) that the right hand side can be identified with a P-submodule of $M_{\mathfrak{p}}(\mathbb{V}^*)$. There is a bijective correspondence between P-homomorphisms $\mathbb{W}^* \to M_{\mathfrak{p}}(\mathbb{V}^*)$ and $(\mathfrak{g}, P)$-homomorphisms $M_{\mathfrak{p}}(\mathbb{W}^*) \to M_{\mathfrak{p}}(\mathbb{V}^*)$ called Frobenius reciprocity. In our case, a P-homomorphism

$\tilde{D}^* : \mathbb{W}^* \to M_{\mathfrak{p}}(\mathbb{V}^*)$ exists if and only if there exists a $(\mathfrak{g}, P)$-homomorphism $M_{\mathfrak{p}}(\mathbb{W}^*) \to M_{\mathfrak{p}}(\mathbb{V}^*)$ of GVM's.

It follows that there is a duality between invariant linear differential operators $D : \Gamma(G \times_P \mathbb{V}) \to \Gamma(G \times_P \mathbb{W})$ of any finite order and $(\mathfrak{g}, P)$-homomorphisms of GVM's $M_{\mathfrak{p}}(\mathbb{W}^*) \to M_{\mathfrak{p}}(\mathbb{V}^*)$. If the inducing representations $\mathbb{V}$ and $\mathbb{W}$ are both finite dimensional P-modules, then $M_{\mathfrak{p}}(\mathbb{V})$ and $M_{\mathfrak{p}}(\mathbb{W})$ are $(\mathfrak{g}, P)$-modules and if P is connected, each $\mathfrak{g}$-homomorphism $M_{\mathfrak{p}}(\mathbb{V}) \to M_{\mathfrak{p}}(\mathbb{W})$ is a $(\mathfrak{g}, P)$-homomorphism as well.

Finally, note that if the Lie groups (G, P) are real but the representation spaces $\mathbb{V}, \mathbb{W}$ are complex representations of P, then the real GVM $M_{\mathfrak{p}}(\mathbb{V})$ is ($\mathfrak{g}$-) isomorphic to the complex GVM induced by $\mathbb{V}$, considered as a complex representation of the complexified Lie algebra $\mathfrak{p}^{\mathbb{C}}$. Therefore, we may restrict to GVM's associated to complex Lie algebras $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}})$.

2.4. Homomorphisms of GVM's. The GVM's are highest weight modules, therefore they admit central characters. As each $\mathfrak{g}$-homomorphism of highest weight modules must preserve the central character, it follows from Harris-Chandra theorem (see, e.g. [15]) that a $\mathfrak{g}$-homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ may exist only if μ and λ are on the same orbit of the Weyl group W of the Lie algebra $\mathfrak{g}$. (Recall that the highest weights of these modules are $\mu - \delta$ and $\lambda - \delta$.) For $\lambda \in P_{\mathfrak{p}}^{++} + \delta$, there exist only a finite number of weights $\mu \in P_{\mathfrak{p}}^{++} + \delta$ on the same orbit of the Weyl grup.

In the case of true Verma modules, there is a classification of their homomorphisms, done by Verma and Bernstein-Gelfand-Gelfand ([1, 2, 23]), summarized in the following statements:

Theorem 2.1. *Let $\mu, \lambda \in \mathfrak{h}^*$. Each homomorphism $M_{\mathfrak{b}}(\mu) \to M_{\mathfrak{b}}(\lambda)$ is injective and $\dim(\mathrm{Hom}(M_{\mathfrak{b}}(\mu), M_{\mathfrak{b}}(\lambda))) \leq 1$. Therefore, we can write $M_{\mathfrak{b}}(\mu) \subset M_{\mathfrak{b}}(\lambda)$ in such case.*

A nonzero homomorphism of Verma modules $M_{\mathfrak{b}}(\mu) \to M_{\mathfrak{b}}(\lambda)$ exists if and only if there exist weights $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ so that $\lambda_{i+1} = s_{\beta_i}\lambda_i$ for some positive roots β_i and $\lambda_i(H_{\beta_i}) \in \mathbb{N}$ for all i ($s_\beta \in W$ is the β-root reflection). Equivalently, $\lambda_i - \lambda_{i-1}$ is a positive integral multiple of some positive root for all i.

Let $\lambda \in P_{\mathfrak{g}}^{++} + \delta$ (i.e. $\lambda - \delta$ is $\mathfrak{g}$-dominant and $\mathfrak{g}$-integral). Then there exists a nonzero homomorphism $M_{\mathfrak{b}}(w'\lambda) \to M_{\mathfrak{b}}(w\lambda)$ if and only if $w \leq w'$ in the Bruhat ordering.

If λ is only $\mathfrak{g}$-dominant ($\lambda(H_\beta) > 0$ for all $\beta \in \Phi^+$), then the existence of a nonzero homomorphism $M_{\mathfrak{b}}(w'\lambda) \to M_{\mathfrak{b}}(w\lambda)$ still implies $w \leq w'$ in the Bruhat ordering (but not conversely).

Because $M_{\mathfrak{p}}(\lambda)$ is a highest weight module, it is isomorphic to a quotient of true Verma module $M_{\mathfrak{b}}(\lambda)/M$. It was proved by Lepowsky that $M \simeq \sum_{\alpha \in \Delta - \Sigma} M_{\mathfrak{b}}(s_\alpha\lambda)$ ($\Sigma \subset \Delta$ determines the parabolic subalgebra $\mathfrak{p}$ and all the modules $M_{\mathfrak{b}}(s_\alpha\lambda)$ are considered as submodules of $M_{\mathfrak{b}}(\lambda)$). A homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is called standard, if it is induced by a quotient of a true Verma module homomorphism $M_{\mathfrak{b}}(\mu) \to M_{\mathfrak{b}}(\lambda)$. Up to multiple, there exists at most one standard homomorphism from $M_{\mathfrak{p}}(\mu)$ to $M_{\mathfrak{p}}(\lambda)$. The following is known about standard homomorphisms of GVM's ([16]):

Theorem 2.2. *Let $\mu, \lambda \in P_{\mathfrak{p}}^{++} + \delta$, $i : M_{\mathfrak{b}}(\mu) \to M_{\mathfrak{b}}(\lambda)$ be a homomorphism of Verma modules. Then the standard homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is zero if and only if there exists $\alpha \in \Delta - \Sigma$ so that $i(M_{\mathfrak{b}}(\mu)) \subset M_{\mathfrak{b}}(s_\alpha \lambda)$ (identifying $M_{\mathfrak{b}}(s_\alpha\lambda)$ with a submodule of $M_{\mathfrak{b}}(\lambda)$).*

Let us denote by $W_{\mathfrak{p}}$ the subgroup of W generated by simple root reflections $\{s_\alpha,\ \alpha \in \Delta - \Sigma\}$ and $W^{\mathfrak{p}}$ the subset of W consisting of those $w \in W$ so that $w\tilde{\lambda}$ is $\mathfrak{p}$-dominant for each $\mathfrak{g}$-dominant weight $\tilde{\lambda}$. Any $w \in W$ can be uniquely decomposed $w = w_p w^p$ where $w_p \in W_{\mathfrak{p}}$ and $w^p \in W^{\mathfrak{p}}$ and the length $l(w) = l(w_p) + l(w^p)$. We define the *parabolic Hasse graph* for $(\mathfrak{g}, \mathfrak{p})$ to be the set $W^{\mathfrak{p}}$ of vertices with arrows $w \to w'$ if and only if $w \to w'$ in W.

The following two properties of the parabolic Hasse graph will be used later (for the proof, see [3]):

Lemma 2.3. *(1) If $w' = s_\gamma w$, then either $w \le w'$ or $w' \le w$ in the Bruhat ordering.*

(2) Let $w, w' \in W^{\mathfrak{p}}$ and $w \le w'$ in the Bruhat ordering. Then there exists a path $w \to w_1 \to \ldots \to w_n \to w'$ so that all w_i are in $W^{\mathfrak{p}}$.

The following theorem can be used to prove the existence of a standard GVM homomorphism:

Theorem 2.4. *Let $\tilde{\lambda}$ be a strictly dominant weight (i.e. $\tilde{\lambda}(H_\beta) > 0$ for $\beta \in \Phi^+$), $w, w' \in W^{\mathfrak{p}}$, $w \to w'$ in the parabolic Hasse graph for $(\mathfrak{g}, \mathfrak{p})$ and assume that $w\tilde{\lambda}$, $w'\tilde{\lambda} \in P_{\mathfrak{p}}^{++} + \delta$. Further, suppose that there exists a nonzero homomorphism of true Verma modules $M_{\mathfrak{b}}(w'\tilde{\lambda}) \to M_{\mathfrak{b}}(w\tilde{\lambda})$. Then the standard homomorphism $M_{\mathfrak{p}}(w'\tilde{\lambda}) \to M_{\mathfrak{p}}(w\tilde{\lambda})$ is nonzero.*

Remark 2.5. In [16], the theorem is formulated only for $\tilde{\lambda} \in P^{++} + \delta$, but the proof works for non-integral $\tilde{\lambda}$ as well. Note, that for non-integral (and neither $\mathfrak{g}$-, nor $\mathfrak{p}$-dominant) $\tilde{\lambda} - \delta$, the weights $w\tilde{\lambda} - \delta$ and $w'\tilde{\lambda} - \delta$ may still be $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral.

Proof. Assume that the standard homomorphism is zero. It follows from lemma 2.2 that there exists $\alpha \in \Delta - \Sigma$ so that $M_{\mathfrak{b}}(w'\tilde{\lambda}) \subset M_{\mathfrak{b}}(s_\alpha w\tilde{\lambda})$. The last statement of theorem 2.1 implies that $w' > s_\alpha w$ in the Bruhat ordering. But, because $w\tilde{\lambda} \in P_{\mathfrak{p}}^{++} + \delta$ and $\alpha \in \Delta - \Sigma$, we have $(w\tilde{\lambda})(H_\alpha) \in \mathbb{N}$ and it follows from 2.1 that $M_{\mathfrak{b}}(s_\alpha w\tilde{\lambda}) \subset M_{\mathfrak{b}}(w\tilde{\lambda})$ and $l(s_\alpha w) = l(w) + 1$. So we have $l(w') > l(s_\alpha w) > l(w)$ which contradicts $l(w') = l(w) + 1$. ∎

For any weight λ, there always exists a dominant weight $\tilde{\lambda}$ (i.e. $\tilde{\lambda}(H_\beta) \ge 0$ for $\beta \in \Phi^+$) on the same orbit of the Weyl group. If there exists some β so that $\tilde{\lambda}(H_\beta) = 0$, we say that the generalized Verma modules $M_{\mathfrak{p}}(w\tilde{\lambda})$ have *singular character* and the weights $w\tilde{\lambda}$ are called *singular*. Theorem 2.4 cannot be generalized to singular weights, because for singular $\tilde{\lambda}$, the weight $w\tilde{\lambda}$ doesn't determine w uniquely. (However, there are indications that a similar theorem may be true, if we admit non-standard GVM homomorphisms.)

The following lemma will be used for comparing lengths of two elements in $W^{\mathfrak{p}}$:

Lemma 2.6. *Let E be the grading element for the pair $(\mathfrak{g},\mathfrak{p})$ and let $w,w'\in W^{\mathfrak{p}}$, $w'=s_\gamma w$ and $l(w')>l(w)$. Then $w\delta(E)-w'\delta(E)\in\mathbb{N}$.*

Proof. Because $w\in W^{\mathfrak{p}}$ and $w'=s_\gamma w\in W^{\mathfrak{p}}$, the uniqueness of the decomposition $W=W_{\mathfrak{p}}W^{\mathfrak{p}}$ yields $s_\gamma\notin W_{\mathfrak{p}}$. From the definition, $W_{\mathfrak{p}}=W_{\mathfrak{g}_0}$, the Weyl group of $\mathfrak{g}_0$, so the root γ cannot be expressed as sum of simple roots in $\Delta-\Sigma$. The definition of the grading $\oplus_j\mathfrak{g}_j$ of $\mathfrak{g}$, associated to the pair $(\mathfrak{g},\mathfrak{p})$ implies that the γ-root space generator $X_\gamma\in\mathfrak{g}_i$ for some $i>0$, so $\gamma(E)=i\in\mathbb{N}$. We obtain $w'\delta(E)=(s_\gamma w\delta)(E)=(w\delta-w\delta(H_\gamma)\gamma)(E)=w\delta(E)-iw\delta(H_\gamma)$. Because δ is dominant and $l(w')>l(w)$, we have $w\delta(H_\gamma)>0$. The weight δ is also integral, because $\delta(H_\alpha)=1$ for each $\alpha\in\Delta$. So the difference $(w\delta-w'\delta)(E)=iw\delta(H_\gamma)$ is a product of two positive integers. ■

2.5. Order of the differential operator dual to a GVM homomorphism. The following theorem is an important tool for determining the order of an operator, dual to a homomorphism of generalized Verma modules, if the highest weights of the inducing representations are known.

Theorem 2.7. *Let μ,λ be the highest weights of some irreducible finite-dimensional P-modules $\mathbb{V}_\mu$, $\mathbb{V}_\lambda$ and $\phi: M_{\mathfrak{p}}(\mathbb{V}_\mu)\to M_{\mathfrak{p}}(\mathbb{V}_\lambda)$ be a nonzero homomorphism of generalized Verma modules. Let E be the grading element for $(\mathfrak{g},\mathfrak{p})$ and let $o:=(\lambda-\mu)(E)$. Then o is an integer larger or equal to the order of the dual differential operator $\Gamma(G\times_P\mathbb{V}_\lambda^*)\to\Gamma(G\times_P\mathbb{V}_\mu^*)$. Further, if $o\in\{1,2\}$, then o is the order of the operator.*

Proof. Let v_μ be the highest weight vector of $\mathbb{V}_\mu$ and $\phi(1\otimes v_\mu)=\sum_j y_j\otimes v_j$, $y_j\in\mathcal{U}(\mathfrak{g}_-)$, $v_j\in\mathbb{V}_\lambda$ ($M_{\mathfrak{p}}(\lambda)\simeq\mathcal{U}(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$ as vector space). Let k be the maximal integer so that $y_i\in\mathcal{U}_k(\mathfrak{g}_-)-\mathcal{U}_{k-1}(\mathfrak{g}_-)$ for some y_i and let $0\neq g_0\in\mathcal{U}(\mathfrak{g}_0)$. Simple commutation relations show that ϕ maps $1\otimes g_0\cdot v_\mu$ into $\mathcal{U}_k(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$ but not to $\mathcal{U}_{k-1}(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$. $\mathbb{V}_\mu$ is an irreducible $\mathfrak{p}$-module and $\mathfrak{g}_0$ is the reductive part of $\mathfrak{p}$, so $\mathcal{U}(\mathfrak{g}_0)v_\mu=\mathbb{V}_\mu$ and ϕ maps $1\otimes\mathbb{V}_\mu$ into $\mathcal{U}_k(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$. Let $v\in\mathbb{V}_\mu$, $\phi(1\otimes v)=\sum_j\tilde{y}_j\otimes\tilde{v}_j$, $\tilde{v}_j\in\mathbb{V}_\lambda$, $\tilde{y}_j\in\mathcal{U}_k(\mathfrak{g}_-)$ and $\tilde{y}_i\notin\mathcal{U}_{k-1}(\mathfrak{g}_-)$ for some i. Let $\tilde{y}_j=y_1^{(j)}\dots y_{l(j)}^{(j)}$ for some $y_u^{(j)}\in\mathfrak{g}_-$, $l(j)\leq k$ and $l(i)=k$.

Applying the duality (1), the differential operator D satisfies

$$v((Df)(0))=\sum_j\tilde{v}_j(L_{y_1^{(j)}}\dots L_{y_{l(j)}^{(j)}}(f)(0)),$$

where $L_{y_u^{(j)}}$ are the left invariant vector fields generated by $y_u^{(j)}\in\mathfrak{g}_-$. So, the operator D dual to the homomorphism is of order k.

Let us suppose that the operator has order k, i.e. ϕ maps $1\otimes v_\mu$ into $\mathcal{U}_k(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$ but not into $\mathcal{U}_{k-1}(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$. Let $\{y_1,\dots y_n\}$ be an ordered basis of $\mathfrak{g}_-$ that consists of generators of negative root spaces in $\mathfrak{g}_-$.

Let $\phi(1\otimes v_\mu)=\sum_j\tilde{y}_j\otimes v_j$ and assume that all the v_j's are weight vectors in $\mathbb{V}_\lambda$ and $\tilde{y}_j$ is a product of the y_j's (it follows from the PBW theorem that such expression is always possible). Then all $\tilde{y}_j\otimes v_j$ are weight vectors and, because their sum is a weight vector of weight μ, each $\tilde{y}_j\otimes v_j$ is a weight vector of weight μ as well.

Because $\phi(1\otimes v_\mu)\notin\mathcal{U}_{k-1}(\mathfrak{g}_-)\otimes\mathbb{V}_\lambda$, there exists i such that $\tilde{y}_i=y_{i_1}\dots y_{i_k}$ is a product of k elements. Let $u_j\in\mathbb{N}$ be defined by $y_{i_j}\in\mathfrak{g}_{-u_j}$. The action of the grading element on $y_{i_1}\dots y_{i_k}\otimes v_i$ is

$$\begin{aligned}&E\cdot(y_{i_1}\dots y_{i_k}\otimes v_i)=Ey_{i_1}\dots y_{i_k}\otimes_{\mathcal{U}(\mathfrak{p})}v_i=\\&=(y_{i_1}E+[E,y_{i_1}])y_{i_2}\dots y_{i_k}\otimes_{\mathcal{U}(\mathfrak{p})}v_i=\dots=\\&=y_{i_1}\dots y_{i_k}(\lambda(E)-u_1-\dots-u_k)\otimes v_i\end{aligned}$$

But $y_{i_1}\dots y_{i_k}\otimes v_i$ is a weight vector of weight μ, so the left hand side equals $\mu(E)(y_{i_1}\dots y_{i_k}\otimes v_i)$. It follows

$$(\lambda-\mu)(E)=\sum_j u_j\geq k \tag{2}$$

because $u_j\geq 1$ for all j. So, we see that $(\lambda-\mu)(E)$ is always an integer larger or equal to the order of the operator.

It follows immediately that $(\lambda-\mu)(E)=1$ implies that the operator is of first order. To finish the proof, it remains to show that for a first order operator, $(\lambda-\mu)(E)$ is 1.

Assume that D is an operator of first order. This means that $\phi(1\otimes v_\mu)=\sum_j y_j\otimes v_j$ for $y_j\in\mathcal{U}_1(\mathfrak{g}_-)$ and again, assume that y_j are either constants or generators of negative root spaces and v_i are weight vectors. All the terms $y_j\otimes v_j$ are of weight μ, and therefore,

$$\mu(E)(y_j\otimes v_j)=E(y_j\otimes v_j)=(\lambda(E)+[E,y_j])(y_j\otimes v_j)$$

so $[E,y_j]=(\mu-\lambda)(E)$ for all j and it follows that all the y_j's are from the same graded components of $\mathfrak{g}$. If $y_j\in\mathfrak{g}_{-1}$, so $(\lambda-\mu)(E)=1$ and we are done. Assume, for contradiction, that $y_j\in\mathfrak{g}_{-k}$ for $k>1$.

Because $\sum_j y_j\otimes v_j\in\mathfrak{g}_{-k}\otimes\mathbb{V}_\lambda$, choosing a basis $\{\tilde{v}_1,\dots,\tilde{v}_m\}$ of $\mathbb{V}_\lambda$, $\sum_j y_j\otimes v_j$ can be uniquely expressed as $\sum_{j=1}^m\tilde{y}_j\otimes\tilde{v}_j$ for some $\tilde{y}_j\in\mathfrak{g}_{-k}$. Because it is a homomorphic image of a highest weight vector in $M_\mathfrak{p}(\mu)$, it must be annihilated by all positive root spaces in $\mathfrak{g}$, in particular, by any generator x of a root space in $\mathfrak{g}_1$:

$$\begin{aligned}x\cdot(\sum_j\tilde{y}_j\otimes\tilde{v}_j)&=\sum_j x\tilde{y}_j\otimes_{\mathcal{U}(\mathfrak{p})}\tilde{v}_j=\sum_j(\tilde{y}_jx+[x,\tilde{y}_j])\otimes_{\mathcal{U}(\mathfrak{p})}\tilde{v}_j=\\&=\sum_j(\tilde{y}_j\otimes_{\mathcal{U}(\mathfrak{p})}x\cdot\tilde{v}_j+[x,\tilde{y}_j]\otimes_{\mathcal{U}(\mathfrak{p})}\tilde{v}_j)=\sum_j[x,\tilde{y}_j]\otimes\tilde{v}_j=0\end{aligned}$$

because $[x,\tilde{y}_j]\in\mathfrak{g}_{-k+1}\subset\mathfrak{g}_-$ and $x\cdot\tilde{v}_\lambda=0$. Because $\tilde{v}_j$ forms a basis of $\mathbb{V}_\mu$, it follows that for each j, $[x,\tilde{y}_j]=0$ for all $x\in\mathfrak{g}_1$. The grading fulfills that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$ and $\mathfrak{g}_1$ generates $\mathfrak{p}^+=\sum_{i\geq1}\mathfrak{g}_i$. The Jacobi identity implies that if $\tilde{y}_j$ commutes with $\mathfrak{g}_1$, it commutes with all the $\mathfrak{p}^+$ as well. Let $\tilde{y}_j=\sum_i a_iy_{-\phi_i}$ where $y_{-\phi_i}$ is a generator of the $-\phi_i$-root space. Define $x:=\sum_i a_ix_{\phi_i}$, where x_{ϕ_i} is a generator of the ϕ-root space. We see that $x\in\mathfrak{g}_k$ and $[x,\tilde{y}_j]=\sum_i a_i^2[x_\phi,y_{-\phi}]\neq0$ and we have a contradiction.

So, if $o=(\lambda-\mu)(E)=2$, we know that the order of the differential operator is at most two, but it cannot be one because in that case $(\lambda-\mu)(E)=1$. So, the operator must be of second order. ■

3. The orbits associated with the Dirac operator

3.1. Existence of the homomorphisms.

Let as suppose that n is odd, $\mathfrak{g} = B_{k+(n-1)/2} = \mathfrak{so}(n+2k,\mathbb{C})$, $\mathfrak{p}$ its parabolic subalgebra corresponding to

$$\circ\!\!-\!\!-\ldots-\!\!-\!\!\circ\!\!-\!\!-\!\!\times\!\!-\!\!-\!\!\circ\!\!-\!\!-\ldots-\!\!-\!\!\circ\!\!\Rightarrow\!\!\circ$$

where the k-th node is crossed ($\Sigma = \{\alpha_k\}$). Let as represent the elements of $\mathfrak{g}$ as matrices antisymmetric with respect to the anti-diagonal, choose the Cartan algebra $\mathfrak{h}$ to be the algebra of diagonal matrices in $\mathfrak{g}$ and a natural basis $\{\epsilon_i\}$ of $\mathfrak{h}^*$ defined by

$$\epsilon_i(\mathrm{diag}(\mathrm{a}_1,\ldots,\mathrm{a}_{\mathrm{k}+(\mathrm{n}-1)/2},0,-\mathrm{a}_{\mathrm{k}+(\mathrm{n}-1)/2},\ldots,-\mathrm{a}_1)) := \mathrm{a}_\mathrm{i}$$

(see e.g. [14] for details).

The subalgebra $\mathfrak{p}$ induces the 2-gradation $\mathfrak{g} = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right)$, where $\mathfrak{g}_0$ consists of blocks of dimension $k\times k$, $n\times n$ and $k\times k$. The corresponding grading element is $E = \mathrm{diag}(1,\ldots,1,0,\ldots,0,-1\ldots,-1)$ and the action of a weight $[a_1,\ldots,a_k|b_1,\ldots,b_{(n-1)/2}]$ on E is $\sum_i a_i$.

In this section, we will try to describe the structure of GVM homomorphisms on the Weyl orbit of the weight

$$\lambda = \overset{0}{\circ}\!-\ldots-\!\overset{0}{\circ}\!-\!\overset{-\frac{n}{2}}{\times}\!-\!\overset{0}{\circ}\!-\ldots-\!\overset{0}{\circ}\!\Rightarrow\!\overset{1}{\circ} + \delta.$$

It was shown in [11, 12] that there exists a GVM homomorphism $M_\mathfrak{p}(\mu)\to M_\mathfrak{p}(\lambda)$ so that the dual differential operator is closely related to the Dirac operator in various Clifford variables, as noticed in the introduction (choosing the real Lie groups $G = \mathrm{Spin}(n+k,k)$ and P the parabolic subgroup so that its complexified Lie algebra is $\mathfrak{p}$).

In the ϵ_i-basis, $\delta = [\ldots,5/2,3/2,1/2]$, $\mathfrak{g}$-dominant weights are those $[a_1,\ldots,a_{k+(n-1)/2}]$ such that $a_1\ge a_2\ge\ldots\ge a_{k+(n-1)/2}\ge 0$ and $\mathfrak{p}$-dominant weights must fulfill $a_1\ge a_2\ge\ldots\ge a_k$ and $a_{k+1}\ge\ldots\ge a_{k+(n-1)/2}\ge 0$. A weight is $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral, if, moreover, $a_i - a_j\in\mathbb{Z}$ for $i,j\le k$ and $a_l\in\mathbb{Z}/2$ for $l>k$. Positive roots are all $[0,\ldots,0,1,0,\ldots,-1,\ldots]$, $[\ldots,1,\ldots,1,\ldots]$ and $[\ldots,0,1,0,\ldots]$. The corresponding root reflections map the weight $[\ldots,a_i,\ldots,a_j,\ldots]$ to $[\ldots,a_j,\ldots,a_i,\ldots]$ (transpositions), or to $[\ldots,-a_j,\ldots,a_i,\ldots]$ (sign-transpositions) or to $[\ldots,-a_i,\ldots,a_j,\ldots]$ (sign-change).

The weight λ we consider can be writen in the ϵ_i-basis as

$$\lambda = [(2k-1)/2,\ldots,3/2,1/2|\ldots,3,2,1].$$

Lemma 3.1. *Let $k=2$. Then there exist three nonzero weights $\mu,\nu,\xi\in P_\mathfrak{p}^{++}$ on the orbit of λ and nonzero standard homomorphisms*

$$M_\mathfrak{p}(\xi)\to M_\mathfrak{p}(\nu)\to M_\mathfrak{p}(\mu)\to M_\mathfrak{p}(\lambda),$$

where the weights are described by the following diagram:

Proof. The existence of true Verma module homomorphisms $M_{\mathfrak{b}}(\xi) \to \ldots \to M_{\mathfrak{b}}(\lambda)$ follows easily from Theorem 2.1. All the weights are from $P_{\mathfrak{p}}^{++} + \delta$ and they are on the orbit of the $\mathfrak{g}$-dominant weight $\tilde{\lambda} = [\ldots, 4, 3, 2, 3/2, 1, 1/2]$. This weight is nonsingular, because its coefficients are strictly decreasing and the last one is strictly positive.

Let w resp. w', w'', w''' be the elements of W that takes $\tilde{\lambda}$ to λ resp. μ, ν, ξ. Easy calculation shows that w can be characterized by $w\delta = [5/2, 1/2| \ldots, 9/2, 7/2, 3/2]$ and $w'\delta = [5/2, -1/2| \ldots, 9/2, 7/2, 3/2]$. Because w' and w are connected by a root reflection, lemma 2.3 states that either $w \leq w'$ ot $w' \leq w$ in the Bruhat ordering and there exists a sequence $w = w_0 \to w_1 \to \ldots \to w_{n-1} \to w_n = w'$, $w_i \in W^{\mathfrak{p}}$. Lemma 2.6 states $(w_i\delta - w_{i+1}\delta)(E) \in \mathbb{N}$ for all i, where E is the grading element. But we compute $(w\delta - w'\delta)(E) = (5/2+1/2) - (5/2-1/2) = 1$, so the only possibility is $n = 1$ and $w \to w'$. Applying 2.4, we see that the standard map $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is nonzero.

The element w'' takes δ to $[1/2, -5/2| \ldots, 9/2, 7/2, 3/2]$ and

$$(w''\delta - w'\delta)(E) = (5/2 - 1/2) - (1/2 - 5/2) = 4.$$

The lenth difference $l(w'') - l(w')$ must be odd, because $w'' = s_\gamma w'$ for $\gamma = [1, 1|0, \ldots, 0]$, and a root reflection has negative determinant. So either $w' \to w''$, or $w' \to w_1 \to w_2 \to w''$. In the first case, we apply theorem 2.4 as before. Suppose $w' \to w_1 \to w_2 \to w''$ and suppose, for contradiction, that the standard homomorpism $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ is zero. Theorem 2.2 says that the true Verma modules

$$M_{\mathfrak{b}}(\nu) \subset M_{\mathfrak{b}}(s_\alpha \mu) \tag{3}$$

for some simple root $\alpha \neq \alpha_2$. We know that for such α, $s_\alpha \in W_{\mathfrak{p}}$ and, because μ is $\mathfrak{p}$-dominant, $M_{\mathfrak{b}}(s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(\mu)$. The weight $s_\alpha\mu$ is one of the following types:

1. $[-1/2, 3/2| \ldots, 3, 2, 1]$ if $\alpha = \alpha_1$
2. $[3/2, -1/2|(n-1)/2, \ldots, l-1, l, \ldots, 2, 1]$ if $\alpha = \alpha_i$, $2 < i < k + (n-1)/2$
3. $[3/2, -1/2| \ldots, 3, 2, -1]$ if $\alpha = \alpha_{k+(n-1)/2}$

First we show that $\alpha \neq \alpha_1$. If $\alpha = \alpha_1$, (3) implies that $s_{\alpha_1}\mu - \nu = [-1, 3|0, \ldots, 0]$ is a sum of positive roots, but this is not possible, as no positive root is of the form $[-1, \text{something}]$.

Now assume that $s_\alpha\mu$ is of type (2). Because

$$M_{\mathfrak{b}}(w''\tilde{\lambda}) = M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(\mu) = M_{\mathfrak{b}}(w'\tilde{\lambda}),$$

$l(w') - l(w) = 3$ and ν is not connected with $s_\alpha\mu$ by any root reflection, it follows from Theorem 2.1 that there must be β_1, β_2 so that

$$M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\beta_1}\nu) = M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(s_\alpha\mu). \tag{4}$$

Note, that the weights are $s_\alpha\mu = [3/2, -1/2| \ldots, l-1, l, \ldots, 2, 1]$ and $\nu = s_{\beta_1}s_{\beta_2}s_\alpha\mu = [1/2, -3/2| \ldots, 2, 1]$. In coordinates, s_{β_j} cannot be a (sign)-transposition interchanging an integer and a half-integer, because of the conditions $s_\alpha\mu(H_{\beta_2}) \in \mathbb{N}$ and $s_{\beta_2}s_\alpha\mu(H_{\beta_1}) \in \mathbb{N}$. So, exactly one of these reflections interchanges $(3/2, -1/2)$ to $(1/2, -3/2)$ and the other one interchanges

$(l-1,l)$ to $(l,l-1)$. So either $s_{\beta_2}s_\alpha\mu=[1/2,-3/2|\ldots,l-1,l\ldots]$ or $s_{\beta_2}s_\alpha\mu=[3/2,-1/2|\ldots,l,l-1,\ldots]$. In the first case, $M_{\mathfrak b}(s_{\beta_2}s_\alpha\mu)=M_{\mathfrak b}(s_\alpha\nu)\subsetneq M_{\mathfrak b}(\nu)$ (ν is $\mathfrak p$-dominant) which contradicts (4). In the second case, $M_{\mathfrak b}(s_{\beta_2}s_\alpha\mu)=M_{\mathfrak b}(\mu)\subsetneq M_{\mathfrak b}(s_\alpha\mu)$ by (4), which contradicts the fact that $M_{\mathfrak b}(s_\alpha\mu)\subsetneq M_{\mathfrak b}(\mu)$. So $s_\alpha\mu$ cannot be of type (2).

Similarly, we can show that $s_\alpha(\mu)$ cannot be of type (3). But this means that (3) does not hold and the standard map $M_{\mathfrak p}(\nu)\to M_{\mathfrak p}(\mu)$ is nonzero.

Finally, note that $w'''\delta=[-1/2,-5/2|\ldots]$, so $(w''\delta-w''\delta)(E)=(1/2-5/2)-(-1/2-5/2)=1$, therefore $w''\to w'''$ and the standard homomorphism $M_{\mathfrak p}(\xi)\to M_{\mathfrak p}(\nu)$ is nonzero. ■

If $n\neq 5$, there are no other weights from $P_{\mathfrak p}^{++}+\delta$ on the orbit of $\tilde\lambda$. In case $n=5$, there are other weights $[2,1|3/2,1/2]$, $[2,-1|3/2,1/2]$, $[1,-2|3/2,1/2]$ and $[-1,-2|3/2,1/2]$ on this orbit, but there is no nonzero homomorphism from the GVM's in the last theorem to any of these and vice versa.

Theorem 3.2. *The sequence of homomorphisms $M_{\mathfrak p}(\xi)\to M_{\mathfrak p}(\nu)\to M_{\mathfrak p}(\mu)\to M_{\mathfrak p}(\lambda)$ is a complex.*

Proof. We want to show that the standard homomorphism $M_{\mathfrak p}(\nu)\to M_{\mathfrak p}(\lambda)$ is zero. This can be using theorem 2.2 and the facts that

$$M_{\mathfrak b}([\frac{1}{2},-\frac{3}{2}|\ldots,2,1])\subset M_{\mathfrak b}([\frac{1}{2},\frac{3}{2}|\ldots,2,1])=M_{\mathfrak b}(s_{\alpha_1}[\frac{3}{2},\frac{1}{2}|\ldots,2,1]).$$

Similarly, one shows that $M_{\mathfrak p}(\xi)\to M_{\mathfrak p}(\mu)$ is zero. ■

Definition 3.3. Let as define an oriented graph S_k in the following way: S_1 has 2 vertices connected by an arrow ($\bullet\to\bullet$), S_2 contains 4 vertices connected linearly by arrows ($\bullet\to\bullet\to\bullet\to\bullet$). For $k\geq 3$, S_k contains 2 disjoint subsets S^1 and S^2 of vertices so that the subgraphs $S^1$and S^2 are both isomorphic to S_{k-1}, where S^1 contains the "first" vertex and S^2 the "last" one. Similarly, S^1 contains 2 copies of S_{k-2}, denote them by $S^{1,1}$ and $S^{1,2}$ and S^2 contains 2 copies of S_{k-2}, denote them by $S^{2,1}$ and $S^{2,2}$. Let ϕ resp. ψ be the isomorphism $S_{k-2}\to S^{1,2}$ resp. $S_{k-2}\to S^{2,1}$. Then for each vertex $x\in S_{k-2}$ there is an arrow $\phi(x)\to\psi(x)$ in S_k. For completeness, define S_0 to be a one-point graph.

Graphically, S_k has the following structure:

We draw the graphs S_k for $k=3,4$:

Theorem 3.4. *Let $(\mathfrak g,\mathfrak p)$ and λ be like at the beginning of this section and let $k\neq(n-1)/2$. There are 2^k weights from $(P_{\mathfrak p}^{++}+\delta)\cap W\lambda$ and they can be assigned to the vertices of the graph S_k so that for each arrow $\mu\to\nu$ in this graph there exists a nonzero standard homomorphism $M_{\mathfrak p}(\nu)\to M_{\mathfrak p}(\mu)$ and each nonzero standard homomorphism between GVM's with highest weights from $((P_{\mathfrak p}^{++}+\delta)\cap W\lambda)-\delta$ is a composition of these. The weight λ itself is assigned to the minimal vertex in S_k.*

Proof. The condition on a weight $\nu = [a_1, \dots, a_k | b_1, \dots, b_{(n-1)/2}]$ to be from $P_{\mathfrak{p}}^{++} + \delta$ is $a_1 > \dots > a_k$, $b_1 > \dots > b_{(n-1)/2} > 0$, $a_i - a_j \in \mathbb{Z}$, $b_i - b_j \in \mathbb{Z}$ and the b_i's are all integers or all half-integers. Simple combinatorics implies that, if $\nu \in P_{\mathfrak{p}}^{++} + \delta$ is on the orbit of λ and $k \neq (n-1)/2$, the only possibility is $\nu = [a_1, \dots, a_k | (n-1)/2, \dots, 2, 1]$, where $(a_1, \dots, a_k)$ is some strictly decreasing sign-permutation of $((2k-1)/2, \dots, 3/2, 1/2)$.

These conditions imply that there is either $(2k-1)/2$ on the first position, or $-(2k-1)/2$ on the k-th position and the remaining of the first k positions contains a decreasing sign-permutation of $((2k-3)/2, \dots, 1/2)$. This proves that there are 2^k such weights. Define R_k to be the set of these weights, R_k^1 to be the set of weights with $(2k-1)/2$ on the first position and R_k^2 to be the set of weights with $-(2k-1)/2$ on the k-th position.

We will prove that the map $i : R_{k-1} \to R_k^1$ given by $([a_1, \dots, a_{k-1} | \dots]) \mapsto ([(2k-1)/2, a_1, \dots, a_{k-1} | \dots])$ preserves the existence of nonzero standard GVM homomorphisms (i.e. there exists a nonzero standard $M_{\mathfrak{p}_{k-1,n}}(\nu) \to M_{\mathfrak{p}_{k-1,n}}(\mu)$ if and only if there exists a nonzero standard $M_{\mathfrak{p}_{k,n}}(i(\nu)) \to M_{\mathfrak{p}_{k,n}}(i(\mu))$, the subscripts k, n means that the rank of the Lie algebra is $k + (n-1)/2$).

We start with the Borel case $\mathfrak{p} = \mathfrak{b}$. Let $M_{\mathfrak{b}_{k-1,n}}(\nu) \to M_{\mathfrak{b}_{k-1,n}}(\mu)$ be a true Verma module homomorphism. Let as denote by $\tilde{i}$ the map $\mathfrak{h}_{k-1,n}^* \to \mathfrak{h}_{k,n}^*$ defined by $[a_1, \dots, a_{k-1} | b_1, \dots, b_{(n-1)/2}] \mapsto [0, a_1, \dots, a_{k-1} | b_1, \dots, b_{(n-1)/2}]$. According to 2.1, there exists a nonzero homomorphism $M_{\mathfrak{b}_{k-1,n}}(\nu) \to M_{\mathfrak{b}_{k-1,n}}(\mu)$ if and only if there exists a sequence $\mu = \mu_0, \mu_1, \dots, \mu_l = \nu$ of weights connected by root reflections so that $\mu_j - \mu_{j-1}$ is a positive integral multiple of a positive root from $\Phi_{k-1,n}^+$ (this is the set of positive roots of $\mathfrak{g} = \mathfrak{so}(2(k-1)+n)$) for all j. In this case, the sequence $i(\mu) = i(\mu_0), i(\mu_1), \dots, i(\mu_l) = i(\nu)$ has similar properties, because $\mu_j = s_\gamma \mu_{j-1}$ implies $i(\mu_j) = s_{\tilde{i}(\gamma)} i(\mu_{j-1})$ and for each $\gamma \in \Phi_{k-1,n}^+$, $\tilde{i}(\gamma) \in \Phi_{k,n}^+$. So, there exists a nonzero homomorphism $M_{\mathfrak{b}_{k,n}}(i(\nu)) \to M_{\mathfrak{b}_{k,n}}(i(\mu))$. On the other hand, if there exists a nonzero homomorphism $M_{\mathfrak{b}_{k,n}}(i(\nu)) \to M_{\mathfrak{b}_{k,n}}(i(\mu))$, it follows that there is a sequence $i(\mu) = [(2k-1)/2, \text{something}] = i(\mu_0), i(\mu_1), \dots, i(\mu_l) = [(2k-1)/2, \text{something}]$, $i(\mu_j) = s_{\gamma_j} i(\mu_{j-1})$, so that $i(\mu_j) - i(\mu_{j-1})$ is a positive multiple of a positive root. Therefore, the coefficient on the first position is not increasing in this sequence: so, it is constant $(2k-1)/2$. This means that the root reflections γ_j don't interchange the first coordinate with some other and the roots γ_j have zeros on first positions. So, there exist $\tilde{\gamma_j} \in \Phi_{k-1,n}^+$ so that $\tilde{i}\tilde{\gamma_j} = \gamma_j$ and we obtain that there exists a nonzero homomorphism $M_{\mathfrak{b}_{k-1,n}}(\nu) \to M_{\mathfrak{b}_{k-1,n}}(\mu)$.

It follows from Theorem 2.2 that the standard homomorphism $M_{\mathfrak{p}_{k-1,n}}(\nu) \to M_{\mathfrak{p}_{k-1,n}}(\mu)$ is zero if and only if $M_{\mathfrak{b}_{k-1,n}}(\nu) \subset M_{\mathfrak{b}_{k-1,n}}(s_{\alpha_j}\mu)$ for some simple root $\alpha_j \neq \alpha_{k-1}$. Then $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\tilde{i}(\alpha_j)} i(\mu))$ follows from the previous paragraph, $\tilde{i}(\alpha_j) \neq \alpha_k$ and the standard homomorphism $M_{\mathfrak{p}}(i(\nu)) \to M_{\mathfrak{p}}(i(\mu))$ is zero as well. On the other hand, if $M_{\mathfrak{p}}(i(\nu)) \to M_{\mathfrak{p}}(i(\mu))$ is zero, then $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\alpha_i} i(\mu))$ for some simple root $\alpha_i \neq \alpha_k$. If $i = 1$, then $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\alpha_1} i(\mu))$ implies $s_{\alpha_1}(i(\mu)) - i(\nu)$ is a sum of positive roots. But $i(\nu)$ contains $(2k-1)/2$ on the first position and $s_{\alpha_1}(i(\mu))$ contains a number strictly smaller then $(2k-1)/2$ on the first position, which is a contradiction. Therefore, $i > 1$ and there is a $\alpha \in \Delta_{k-1,n}$, $\alpha \neq \alpha_{k-1}$ so that $\tilde{i}(\alpha) = \alpha_i$. Then

$M_{\mathfrak{b}_{k-1,n}}(\nu) \subset M_{\mathfrak{b}_{k-1,n}}(s_\alpha\mu)$, and the map $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ is zero as well.

We see that for any $\mu, \nu \in R_{k-1}$, there exists a nonzero standard GVM homomorphism $M_{\mathfrak{p}_{k-1,n}}(\nu) \to M_{\mathfrak{p}_{k-1,n}}(\mu)$ if and only if there exists a nonzero standard homomorphism $M_{\mathfrak{p}_{k,n}}(i(\nu)) \to M_{\mathfrak{p}_{k,n}}(i(\mu))$. Similarly, we can define the map $j : R_{k-1} \to R_k^2$ by $[a_1, \ldots, a_{k-1}|\ldots] \mapsto [a_1, \ldots, a_{k-1}, -(2k-1)/2|\ldots]$ and prove that there exists a nonzero standard GVM homomorphism $M_{\mathfrak{p}_{k-1,n}}(\nu) \to M_{\mathfrak{p}_{k-1,n}}(\mu)$ if and only if there exists a nonzero standard homomorphism

$$M_{\mathfrak{p}_{k,n}}(j(\nu)) \to M_{\mathfrak{p}_{k,n}}(j(\mu)).$$

Let as now denote the maps i and j described before by i_k and j_k, specifying the dimension of the (resulting) weights. It remains to prove that for each $x \in R_{k-2}$ there exists a nonzero standard GVM homomorphism $M_{\mathfrak{p}_{k,n}}(j_k i_{k-1}(x)) \to M_{\mathfrak{p}_{k,n}}(i_k j_{k-1}(x))$. In other words, we want to show that for any decreasing sign-permutation $(a_2, \ldots, a_{k-1})$ of $((2k-5)/2, \ldots, 1/2)$, there exists a nonzero standard homomorphism $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$, where

$$\nu = [\frac{2k-3}{2}, a_2, \ldots, a_{k-1}, -\frac{2k-1}{2}|\ldots, 2, 1],$$

$$\mu = [\frac{2k-1}{2}, a_2, \ldots, a_{k-1}, -\frac{2k-3}{2}|\ldots, 2, 1].$$

It follows from 2.1 that there is a homomorphism of the corresponding true Verma modules (the weights are connected by the root reflection with respect to $[1, 0, \ldots, 0, 1|0, \ldots, 0]$.)

There is a unique $\mathfrak{g}$-dominant weight $\tilde{\lambda}$ on the orbit of λ: $\tilde{\lambda} = [(n-1)/2, (n-1)/2-1, \ldots, k, k-1/2, k-1, k-3/2, \ldots, 3/2, 1, 1/2]$ in case $(n-1)/2 \geq k$ and $\tilde{\lambda} = [k-1/2, k-3/2, \ldots, n/2, n/2-1/2, n/2-1, \ldots, 1, 1/2]$ in case $(n-1)/2 < k$.

Let w resp. w' be the Weyl group element taking $\tilde{\lambda}$ to μ resp. ν. Simple computations show that, if $(n-1)/2 \geq k-1$, then w takes $\delta = \frac{1}{2}[\ldots, 5, 3, 1]$ to $\frac{1}{2}[4k-3, b_2, \ldots, b_{k-1}, -(4k-7)|\ldots]$ where $(b_2, \ldots, b_{k-1})$ is some decreasing sign-permutation of $((4k-11)/2, \ldots, 5/2, 1/2)$ and w' takes δ to $\frac{1}{2}[4k-7, b_2, \ldots, b_{k-1}, -(4k-3)|\ldots]$. The difference of the grading element evaluation is then $(w\delta - w'\delta)(E) = \frac{1}{2}((4k-3)-(4k-7)+\sum_j b_j) - \frac{1}{2}((4k-7)-(4k-3)+\sum_j b_j) = 4$. If $(n-1)/2 < k-1$, then w takes $\delta(E)$ to $[k+n/2-1, \ldots, -(k+n/2-2)|\ldots]$, w' takes δ to $[k+n/2-2, \ldots, -(k+n/2-1)|\ldots]$ and $(w\delta - w'\delta)(E) = 2$ in this case.

So, in either case, $(w\delta - w'\delta)(E) \leq 4$ and, similarly as in the proof of lemma 3.1, either $w \to w'$ or $w \to w_1 \to w_2 \to w'$. If $w \to w'$, we apply Theorem 2.4 and see that there is a nonzero standard homomorphism $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$.

Let $w \to w_1 \to w_2 \to w'$ and assume, for the sake of contradiction, that the standard map $M_{\mathfrak{p}}(w'\tilde{\lambda}) \to M_{\mathfrak{p}}(w\tilde{\lambda})$ is zero. Therefore,

$$M_{\mathfrak{b}}(\nu) = M_{\mathfrak{b}}(w'\tilde{\lambda}) \subset M_{\mathfrak{b}}(s_\alpha w\tilde{\lambda}) = M_{\mathfrak{b}}(s_\alpha\mu) \tag{5}$$

for some simple root $\alpha \neq \alpha_k$.

The weight $s_\alpha(\mu)$ is one of the following types:

1. $[a_2,(2k-1)/2,\dots,a_{k-1},-(2k-3)/2|\dots,3,2,1]$ if $\alpha=\alpha_1$
2. $[(2k-1)/2,\dots,a_l,a_{l-1},\dots,-(2k-3)/2|\dots]$ if $\alpha=\alpha_j$, $1<j<k-1$
3. $[(2k-1)/2,\dots,-(2k-3)/2,a_{k-1}|\dots]$ if $\alpha=\alpha_{k-1}$
4. $[(2k-1)/2,\dots,-(2k-3)/2|(n-1)/2,\dots,l-1,l,\dots,2,1]$ if $\alpha=\alpha_j$, $k<j<k+(n-1)/2$
5. $[(2k-1)/2,\dots,-(2k-3)/2|\dots,3,2,-1]$ if $\alpha=\alpha_{k+(n-1)/2}$

First we show that it is not of type 1. If $\alpha=\alpha_1$, (5) implies $s_{\alpha_1}(\mu)-\nu$ is a sum of positive roots, i.e.

$$[a_2,(2k-1)/2,\dots,-(2k-3)/2|\dots]-[(2k-3)/2,a_2,\dots,-(2k-1)/2|\dots]\in\mathbb{N}\Phi^+,$$

where $a_2\le(2k-5)/2$. But the difference cannot be obtained as a sum of positive roots, because it contains a negative number $a_2-(2k-3)/2$ on the first position.

Now assume that $s_\alpha(\mu)$ is of type $2-5$. Because

$$M_{\mathfrak{b}}(w'\tilde\lambda)=M_{\mathfrak{b}}(\nu)\subsetneq M_{\mathfrak{b}}(s_\alpha\mu)\subsetneq M_{\mathfrak{b}}(\mu)=M_{\mathfrak{b}}(w\tilde\lambda),$$

$l(w')-l(w)=3$ and ν is not connected to $s_\alpha(\mu)$ by any root reflection, it follows from Theorem 2.1 that there must be β_1,β_2 so that

$$M_{\mathfrak{b}}(\nu)\subsetneq M_{\mathfrak{b}}(s_{\beta_1}\nu)=M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu)\subsetneq M_{\mathfrak{b}}(s_\alpha\mu)\tag{6}$$

Similarly as in the proof of lemma 3.1, we will show that α cannot be of type $2-5$. Let α be of type 2, i.e.

$$\begin{aligned}s_\alpha(\mu)&=[(2k-1)/2,\dots,a_l,a_{l-1},\dots,-(2k-3)/2|\dots],\\ \nu&=[(2k-3)/2,\dots,a_{l-1},a_l,\dots,-(2k-1)/2)|\dots].\end{aligned}$$

The root reflections s_{β_1} and s_{β_2} cannot interchange an integer with a half-integer, because of the integrality conditions $s_\alpha(\mu)(H_{\beta_2})\in\mathbb{N}$ and $s_{\beta_2}s_\alpha(\mu)(H_{\beta_1})\in\mathbb{N}$. There are two possibilities: either s_{β_2} interchanges a_l with a_{l-1} and s_{β_1} sign-interchanges $((2k-1)/2,-(2k-3)/2)$ with $((2k-3)/2,-(2k-1)/2)$ on the particular positions, or s_{β_2} sign-interchanges $((2k-1)/2,-(2k-3)/2)$ with $((2k-3)/2,-(2k-1)/2)$ and s_{β_1} interchanges a_l with a_{l-1}. In the first case, $\beta_2=\alpha$ and (6) implies $M_{\mathfrak{b}}(\mu)\subsetneq M_{\mathfrak{b}}(s_\alpha\mu)$, which contradicts the fact that $M(s_\alpha\mu)\subsetneq M(\mu)$ for a simple root $\alpha\ne\alpha_k$ and $\mu\in P_{\mathfrak{p}}^{++}+\delta$. In the second case, $\beta_1=\alpha$ and (6) implies $M_{\mathfrak{b}}(\nu)\subsetneq M_{\mathfrak{b}}(s_\alpha\nu)$, which also contradicts $M_{\mathfrak{b}}(s_\alpha\nu)\subsetneq M_{\mathfrak{b}}(\nu)$.

Let α be of type 3, i.e.

$$\begin{aligned}s_\alpha(\mu)&=[(2k-1)/2,\dots,-(2k-3)/2,a_{k-1}|\dots],\\ \nu&=[(2k-3)/2,\dots,a_{k-1},-(2k-1)/2)|\dots]\end{aligned}$$

If either $\beta_1=\alpha$ or $\beta_2=\alpha$, we get contradiction similarly as in case (2). But there is no other possibility, because the a_{k-1} on the k-th position has to move somehow to the $(k-1)$-th position: if β_2 would fix it, then $\beta_1=\alpha$, if β_2 would take it to the $(k-1)$-th position, then $\beta_2=\alpha$ and if β_2 would take it (possibly with a minus sign) to the l-th position for $l \neq k, k-1, 1$, then β_1 has to (sign-) interchange the l-th and $(k-1)$-th position, so $s_{\beta_1} s_{\beta_2}$ would fix the $(2k-1)/2$ on the first position, which is impossible. The last possibility is $l = 1$: this would mean that β_2 takes a_{k-1} to the first position (possibly with a minus sign), but $|a_{k-1}| < (2k-3)/2$ implies that $s_{\beta_2} s_\alpha(\mu)$ has a smaller number on the first position as ν and $s_{\beta_2} s_\alpha(\mu) - \nu$ is not expressible as a sum of positive roots. This contradicts $M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\beta_2} s_\alpha \mu)$.

In case 4, we have

$$\begin{aligned} s_\alpha(\mu) &= [(2k-1)/2, \ldots, -(2k-3)/2|(n-1)/2, \ldots, l-1, l, \ldots, 2, 1] \\ \nu &= [(2k-3)/2, \ldots, -(2k-1)/2)|(n-1)/2, \ldots, l, l-1, \ldots, 2, 1] \end{aligned}$$

Because the reflections with respect to β_1, β_2 cannot interchange an integer and a half-integer, it follows that one of them interchanges l with $l-1$, so either $\beta_1 = \alpha$ or $\beta_2 = \alpha$ and we get a contradiction as in case 2. The same happens in case 5.

In either case, we get a contradiction, so the standard map $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ is nonzero.

So, we can assign weights from R_k to the vertices of the graph S_k so that we assign the weights from R^1 to S^1, the weights from R^2 to S^2 and the proof follows by induction.

Finally, it is easy to check that any possible nonzero standard GVM homomorphisms on the orbit is a composition of the homomorphisms described above by reducing this problem to true Verma module homomorphisms and considering theorem 2.1. ■

In case $k = (n-1)/2$, all the GVM homomorphisms described in the last theorem exist as well, but the whole orbit contains also weights of type $[\ldots, 2, 1|(2k-1)/2, \ldots, 3/2, 1/2]$. There is no nonzero GVM homomorphism $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ where μ is of such type and ν of the type $[\ldots, 3/2, 1/2|\ldots, 2, 1]$ (or opposite).

3.2. Orders of the operators.

Theorem 3.5. *All the operators dual to the homomorphisms described in theorem 3.4 have order* 1 *or* 2*. For any* k*, the connecting operators* $\phi(x) \to \psi(x)$ *(described in definition 3.3) have order* 2 *and the graph homomorphisms* $S_{k-1} \to S_k^1$ *and* $S_{k-1} \to S_k^2$ *respect orders. This determines, by induction, all the order of all the operators.*

If we draw a line for first order operators and a double-line for second order operators in the diagrams, we obtain the following pictures:

Proof. Recall that the action of a weight on the grading element is

$$[a_1, \ldots, a_k|b_1, \ldots, b_{(n-1)/2}](E) = \sum_j a_j.$$

Applying theorem 2.7 and the knowledge of the highest weights of the particular representations, we see that

$$[(\frac{2k-1}{2}, a_2, \ldots, a_{k-1}, -\frac{2k-3}{2}|\ldots](E) - [\frac{2k-3}{2}, a_2, \ldots, -\frac{2k-1}{2}|\ldots](E) =$$
$$= (\frac{2k-1}{2} - \frac{2k-3}{2}) - (\frac{2k-3}{2} - \frac{2k-1}{2}) = 2,$$

so the "connecting" operators are of second order. The other operators are of first order, because

$$[a_1, \ldots, a_{j-1}, \frac{1}{2}, a_{j+1}, \ldots|\ldots](E) - [a_1, \ldots, a_{j-1}, -\frac{1}{2}, a_{j+1} \ldots|\ldots](E) =$$
$$\frac{1}{2} - (-\frac{1}{2}) = 1.$$

■

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