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Discrete Cocompact Subgroups of the Generic Filiform Nilpotent Lie Groups

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Abstract. The discrete cocompact subgroups of the generic filiform nilpotent Lie group with an arbitrary dimension are determined up to isomorphism. We close this paper with two examples in which we determine explicitly the discrete cocompact subgroups of the four dimensional and the five dimensional generic filiform nilpotent Lie groups.
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1. Introduction

For $n \geq 2$, let $\mathcal{L}_n$ be the $(n+1)$-dimensional real nilpotent Lie algebra with a basis $\mathscr{B} = \{X_1, \ldots, X_{n+1}\}$ and non-trivial Lie brackets

$$[X_{n+1}, X_j] = X_{j-1}, \quad 2 \leq j \leq n. \tag{1}$$

$\mathcal{L}_n$ is called the generic filiform Lie algebra or threadlike nilpotent Lie algebra (see [12]). Let $L_n = \exp(\mathcal{L}_n)$ be the associated connected and simply connected nilpotent Lie group. The group L_n is a semi-product $\mathbb{R}^n \rtimes \mathbb{R}$ with the pointwise multiplication given by

$$\Big((x_1, \ldots, x_n), x\Big) \cdot \Big((y_1, \ldots, y_n), y\Big) = \Big((x_1, \ldots, x_n) + \eta(x)(y_1, \ldots, y_n), x + y\Big) \tag{2}$$

for which $\eta : \mathbb{R} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ is given by $\eta(x) = [a_{ij}(x)]_{1 \leq i,j \leq n}$ where

$$a_{ij}(x) = \begin{cases} 0, & \text{if } i > j \\ \frac{x^{j-i}}{(j-i)!}, & \text{if } i \leq j. \end{cases}$$

Note that L_2 is the Heisenberg Lie group and L_n is n-step nilpotent.

A discrete subgroup Γ of G such that G/Γ is compact is called a uniform subgroup of G. Uniform subgroups in a simply connected nilpotent Lie group G

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are not guaranteed to exist (see [2, Example 5.1.13], [11, Remark 2.14] and [4]). Malcev [5] has shown that a simply connected nilpotent Lie group G admits a uniform subgroup if and only if its Lie algebra $\mathfrak{g}$ has a basis with rational structure constants. It follows immediately from (1) that every generic filiform group L_n admits a uniform subgroup. It is of interest to classify the uniform subgroups of L_n. The motivation comes originally, first from [3], where the discrete cocompact subgroup of the $(2n+1)$-dimensional Heisenberg group are classified. In particular, we obtain all discrete cocompacte subgroups of L_2 (see also [1]). In [8], the authors have studied the group L_3 more closely, determining the isomorphism classes of all its discrete cocompact subgroups. These are given by three integer parameters p_1, p_2, p_3 that satisfy certain conditions. Furthermore, in [12] the authors have studied, for each L_n, the infinite dimensional simple quotients of the group C^*-algebra of one of its discrete cocompact subgroups (see also [9], [10] and [7]). The aim of this paper is to determine, for each n, the discrete cocompact subgroups of the generic filiform Lie group L_n. To present the results of the paper we need some notations. Let

$$\mathbf{B} = \begin{pmatrix} 0 & \mathbf{I}_{n-1} \\ 0 & 0 \end{pmatrix}$$

where $\mathbf{I}_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix, and let

$$\mathbf{A} = e^{\mathbf{B}} = [a_{ij}]_{1 \leq i,j \leq n}$$

where

$$a_{ij} = \begin{cases} 0 & \text{if } i > j \\ \frac{1}{(j-i)!} & \text{if } i \leq j. \end{cases}$$

We denote by $\mathscr{D}$ the subset of $\mathrm{M}(n, \mathbb{Z})$ consisting of all matrices of the form

$$[\![D, m]\!] = \begin{pmatrix} D & 0 \\ 0 & m \end{pmatrix} \tag{3}$$

satisfying

$$[\![D, m]\!]^{-1} \mathbf{A} [\![D, m]\!] \in \mathrm{SL}(n, \mathbb{Z}) \tag{4}$$

and

$$\gcd((m, x_{ij}), 1 \leq i, j \leq n-1) = 1 \tag{5}$$

where $m \in \mathbb{N}^*$, the block matrix $D = (x_{ij}; 1 \leq i, j \leq n-1)$ is an upper-triangular integer invertible matrix and $\mathrm{SL}(n, \mathbb{Z})$ is the set of all integer matrices with determinant 1. Finally, let $\mathrm{GL}(n, \mathbb{Z})$ denotes the group of integer $n \times n$ matrices with determinant ± 1:

$$\mathrm{GL}(n, \mathbb{Z}) = \{M \in \mathrm{M}(n, \mathbb{Z}) : \det(M) = \pm 1\}. \tag{6}$$

We are now ready to formulate our main results.

Theorem 1.1. *Let G be the generic filiform Lie group L_n such that $n \geq 3$.*

(1) *If $[\![D,m]\!] \in \mathscr{D}$, then*

$$\Gamma_{[\![D,m]\!]} = \exp(\mathbb{Z}e_1)\cdots\exp(\mathbb{Z}e_{n+1}) \tag{7}$$

where the vectors e_j $(1 \leq j \leq n)$ are the columns of $[\![D,m]\!]$ in the basis $(X_1,\ldots,X_n)$, and $e_{n+1} = X_{n+1}$, is a discrete uniform subgroup of G.

(2) *If Γ is a discrete uniform subgroup of G, then there exist $\Phi \in \mathrm{Aut}(G)$ and $[\![D,m]\!] \in \mathscr{D}$ such that $\Phi(\Gamma) = \Gamma_{[\![D,m]\!]}$.*

(3) *For $[\![D_1,m_1]\!]$ and $[\![D_2,m_2]\!]$ in $\mathscr{D}$, $\Gamma_{[\![D_1,m_1]\!]}$ and $\Gamma_{[\![D_2,m_2]\!]}$ are isomorphic groups if and only if there exists $T \in GL(n,\mathbb{R})$ such that $T\mathbf{B} = \mathbf{B}T$ and $[\![D_2,m_2]\!]^{-1}T[\![D_1,m_1]\!] \in GL(n,\mathbb{Z})$.*

2. Notation and basic facts

The purpose of this section is to recall some facts about rational structures of connected and simply connected nilpotent Lie groups, to be used below.

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. Let $\mathrm{Log} : G \longrightarrow \mathfrak{g}$ denote the inverse of $\exp$.

Let X be an element of $\mathfrak{g}$. The linear mapping $Y \longmapsto [X,Y]$ of $\mathfrak{g}$ into $\mathfrak{g}$ is called the adjoint linear mapping of X and is denoted by $\mathrm{ad}\,X$. The conjugation in G is given by

$$\exp(X)\exp(Y)\exp(X)^{-1} = \exp\left(e^{\mathrm{ad}\,X}(Y)\right)$$

for all $X, Y \in \mathfrak{g}$.

2.1. Rational structures and uniform subgroups. Let G be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $\{X_1,\ldots,X_n\}$ with rational structure constants.

Let $\mathfrak{g}$ have a fixed rational structure given by $\mathfrak{g}_{\mathbb{Q}}$ and let $\mathfrak{h}$ be an $\mathbb{R}$-subspace of $\mathfrak{g}$. Define $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$. We say that $\mathfrak{h}$ is *rational* if $\mathfrak{h} = \mathbb{R}\text{-span}(\mathfrak{h}_{\mathbb{Q}})$, and that a connected, closed subgroup H of G is *rational* if its Lie algebra $\mathfrak{h}$ is rational. The elements of $\mathfrak{g}_{\mathbb{Q}}$ (or $G_{\mathbb{Q}} = \exp(\mathfrak{g}_{\mathbb{Q}})$) are called *rational elements* (or *rational points*) of $\mathfrak{g}$ (or G).

A discrete subgroup Γ is called *uniform* in G if the quotient space G/Γ is compact. The homogeneous space G/Γ is called a *compact nilmanifold.* If G has a uniform subgroup Γ, then $\mathfrak{g}$ (hence G) has a rational structure such that $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\log\Gamma)$. Conversely, if $\mathfrak{g}$ has a rational structure given by some $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, then G has a uniform subgroup Γ such that $\log\Gamma \subset \mathfrak{g}_{\mathbb{Q}}$ (see [2] and [5]). If Γ_1 and Γ_2 are uniform subgroups in G, then $\mathbb{Q}\text{-span}(\log\Gamma_1) = \mathbb{Q}\text{-span}(\log\Gamma_2)$

if and only if Γ_1 and Γ_2 are commensurable; that is, $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. If we endow G with the rational structure induced by a uniform subgroup Γ and if H is a Lie subgroup of G, then H is rational if and only if $H \cap \Gamma$ is a uniform subgroup of H. Note that the notion of rational depends on Γ.

Let Γ be a uniform subgroup of G. A strong Malcev (or Jordan-Hölder) basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ is said to be *strongly based on* Γ if

$$\Gamma = \exp(\mathbb{Z}X_1) \cdots \exp(\mathbb{Z}X_n). \tag{8}$$

Such a basis always exists (see [2], [6]).

The *lower central series* (or the *descending central series*) of $\mathfrak{g}$ is the decreasing sequence of characteristic ideals of $\mathfrak{g}$ defined inductively as follows

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}; \qquad \mathcal{C}^{p+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^p(\mathfrak{g})] \ \ (p \geq 1).$$

The characteristic ideal $\mathcal{C}^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is called the derived ideal of the Lie algebra $\mathfrak{g}$ and denoted by $\mathcal{D}(\mathfrak{g})$. Let $\mathrm{D(G)} = \exp(\mathcal{D}(\mathfrak{g}))$, observe that we have $\mathrm{D(G)} = [G, G]$.

The Lie algebra $\mathfrak{g}$ is called *k-step nilpotent Lie algebra* if there is an integer k such that

$$\mathcal{C}^{k+1}(\mathfrak{g}) = \{0\}; \qquad \mathcal{C}^k(\mathfrak{g}) \neq \{0\}.$$

We denote the center of G by $Z(G)$ and the center of $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{g})$. Note that if $\mathfrak{g}$ is k-step nilpotent, then $\mathcal{C}^k(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g})$.

Proposition 2.1. *[[6, Theorem 3], [2, Corollary 5.2.2]] If $\mathfrak{g}$ has rational structure, all the algebras in the descending central series are rational.*

A proof of the next result can be found in Proposition 5.3.2 of [2].

Proposition 2.2. *Let Γ be uniform subgroup in a nilpotent Lie group G, and let $H_1 \subsetneqq H_2 \subsetneqq \cdots \subsetneqq H_k = G$ be rational normal subgroups of G. Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}, \mathfrak{h}_k = \mathfrak{g}$ be the corresponding Lie algebras. Then there exists a strong Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}$.*

A rational structure on $\mathfrak{g}$ induces a rational structure on the dual space $\mathfrak{g}^*$ (for further details, see [2, Chap. 5]). If $\mathfrak{g}$ has a rational structure given by the uniform subgroup Γ, a real linear functional $f \in \mathfrak{g}^*$ is rational ($f \in \mathfrak{g}^*_{\mathbb{Q}}, \mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-span$(\mathrm{Log}(\Gamma))$) if $\langle f, \mathfrak{g}_{\mathbb{Q}} \rangle \subset \mathbb{Q}$, or equivalently $\langle f, \mathrm{Log}(\Gamma) \rangle \subset \mathbb{Q}$.

Let $\mathrm{Aut}(G)$ (respectively $\mathrm{Aut}(\mathfrak{g})$) denote the group of automorphism of G (respectively $\mathfrak{g}$). If $\varphi \in \mathrm{Aut}(G)$, φ_* will denote the derivative of φ at identity. The mapping $\mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\mathfrak{g})$, $\varphi \longmapsto \varphi_*$ is a groups isomorphism (since G is simply connected).

3. Proof of Theorem 1.1

Throughout this section, let $G = L_n$ be the generic filiform Lie group of dimension $n+1$ with Lie algebra $\mathfrak{g} = \mathfrak{L}_n$. First, we fix some notation for the duration of this section. Let

$$\mathfrak{m} = \mathbb{R}\text{-span}\{X_1, \dots, X_n\}$$

and

$$\mathscr{B}_{\mathfrak{m}} = \{X_1, \dots, X_n\}.$$

It is clear that $\mathfrak{m}$ is a one-codimensional abelian ideal of $\mathfrak{g}$ and $\mathscr{B}_{\mathfrak{m}}$ is a Jordan-Hölder basis of $\mathfrak{m}$. Note that

$$\mathbf{B} = \mathrm{Mat}(\mathrm{ad}\, X_{n+1}|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) \quad \text{and} \quad \mathbf{A} = \mathrm{Mat}(e^{\mathrm{ad}\, X_{n+1}}|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}).$$

Proposition 3.1. *If $n \geq 3$ then $M = exp\,(\mathfrak{m})$ is the unique one-codimensional abelian normal subgroup of G. In particular, $\mathfrak{m}$ is stable under every Lie algebra automorphism of $\mathfrak{g}$.*

Proof. We suppose that there exists another abelian normal subgroup $M' = \exp(\mathfrak{m}')$ of codimension one of G with Lie algebra $\mathfrak{m}'$ such that $M \neq M'$. Then there exist $\alpha \in \mathbb{R}^*$ and $v \in \mathfrak{m}$ such that the vector $\alpha X_{n+1} + v$ belongs to $\mathfrak{m}'$. On the other hand, we have $\dim(\mathfrak{m} \cap \mathfrak{m}') = n - 1$. Then $n \geq 3$ implies that $\dim(\mathfrak{m} \cap \mathfrak{m}') \geq 2$ from which it follows that $\mathfrak{m} \cap \mathfrak{m}' \neq \mathfrak{z}(\mathfrak{g})$. Consequently, we have $[\alpha X_{n+1} + v, \mathfrak{m} \cap \mathfrak{m}'] \neq \{0\}$. This contradicts the assumption that $\mathfrak{m}'$ is abelian. The rest of proof is clear. ■

Proof. [Proof of Theorem 1.1] For the first part, we observe that the set $\Gamma_{[\![D,m]\!]}$ is a subgroup of G if and only if for all $i = 1, \dots, n$, we have

$$\exp(e_{n+1}) \exp(e_i) \exp(-e_{n+1}) \in \Gamma_{[\![D,m]\!]}$$

and

$$\exp(-e_{n+1}) \exp(e_i) \exp(e_{n+1}) \in \Gamma_{[\![D,m]\!]}.$$

Equivalently, for $i = 1, \dots, n$, we have

$$\exp\left(e^{\mathrm{ad}\, e_{n+1}} e_i\right) \in \Gamma_{[\![D,m]\!]} \quad \text{and} \quad \exp\left(e^{-\mathrm{ad}\, e_{n+1}} e_i\right) \in \Gamma_{[\![D,m]\!]}.$$

But this is equivalent to

$$e^{\mathrm{ad}\, e_{n+1}} e_i \in \mathbb{Z}\text{-span}\{e_1, \dots, e_i\} \quad \text{and} \quad e^{-\mathrm{ad}\, e_{n+1}} e_i \in \mathbb{Z}\text{-span}\{e_1, \dots, e_i\}$$

for every $i, 1 \leq i \leq n$. We see that the conditions boil down to the matrix equations

$$\mathbf{A}[\![D,m]\!] = [\![D,m]\!] B_1 \quad \text{and} \quad \mathbf{A}^{-1}[\![D,m]\!] = [\![D,m]\!] B_2 \tag{9}$$

where $B_1, B_2 \in \mathrm{M}(n, \mathbb{Z})$ and $[\![D,m]\!]$ is the matrix with column vectors $e_1, \dots, e_n$ expressed via the basis $\mathscr{B}_{\mathfrak{m}}$. Moreover, since

$$\det([\![D,m]\!]^{-1} \mathbf{A} [\![D,m]\!]) = 1,$$

then the relation (9) is equivalent to

$$[\![D, m]\!]^{-1}\mathbf{A}[\![D, m]\!] \in \mathrm{SL}(n, \mathbb{Z}). \tag{10}$$

This establishes the first part of the theorem. The proof of the second part of the theorem will be achieved through a sequence of partial results. As a first step, we show that if we give G the rational structure induced by a uniform subgroup Γ then the subalgebra $\mathfrak{m}$ is rational in $\mathfrak{g}$.

Lemma 3.2. *Let G be the generic filiform Lie group L_n. Let Γ be a discrete uniform subgroup of G. If $n \geq 3$ then the subgroup $M = exp(\mathfrak{m})$ is rational in G.*

Proof. We will use the fact that if $l \in \mathfrak{g}_{\mathbb{Q}}^*$ ($\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-span(Log($\Gamma$))) and $\mathfrak{k}$ is a rational subspace of $\mathfrak{g}$, then the annihilator $\mathfrak{k}^l = \{X \in \mathfrak{g} : B_l(X, \mathfrak{k}) = \langle l, [X, \mathfrak{k}]\rangle = \{0\}\}$ of $\mathfrak{k}$ with respect to the bilinear form B_l is rational, where B_l is the skew-symmetric bilinear form on $\mathfrak{g}$ defined by $B_l(X, Y) = \langle l, [X, Y]\rangle$ (see [2, Proposition 5.2.7]). Let

$$\Omega = \left\{ l = \sum_{i=1}^{n+1} l_i X_i^* \in \mathfrak{g}^* : \ l_1 \neq 0 \right\}$$

be the layer of the generic coadjoint orbits (see [2, Chapter 3]). As $\mathfrak{g}_{\mathbb{Q}}^*$ is dense in $\mathfrak{g}^*$ and Ω is a non-empty Zariski open set in $\mathfrak{g}^*$ then $\mathfrak{g}_{\mathbb{Q}}^* \cap \Omega \neq \emptyset$. Let $l \in \mathfrak{g}_{\mathbb{Q}}^* \cap \Omega$ (l is called be *a rational linear functional in general position*). The annihilator $\mathcal{D}(\mathfrak{g})^l$ of $\mathcal{D}(\mathfrak{g})$ with respect to the bilinear form B_l is $\mathfrak{m}$. In fact, since $\mathcal{D}(\mathfrak{g}) \subset \mathfrak{m}$, then $\mathfrak{m}^l \subset \mathcal{D}(\mathfrak{g})^l$. Or $\mathfrak{m}$ is a maximal totally isotropic subspace for the form B_l, then $\mathfrak{m}^l = \mathfrak{m}$. On the other hand, it is clear that $\mathcal{D}(\mathfrak{g}) \not\subset \mathfrak{g}(l)$ ($X_2 \notin \mathfrak{g}(l)$ and since $n \geq 3$ then $X_2 \in \mathcal{D}(\mathfrak{g})$). Then $\mathcal{D}(\mathfrak{g})^l \neq \mathfrak{g}$ and therefore $\mathcal{D}(\mathfrak{g})^l = \mathfrak{m}$. Finally, by Proposition 2.1, the ideal $\mathcal{D}(\mathfrak{g})$ is rational and therefore $\mathfrak{m}$ is rational too. This completes the proof. ∎

The next example shows that the Lemma 3.2 is not true if $n = 2$.

Example 3.3. Let $G = L_2$ and let

$$\Gamma = \exp(\mathbb{Z}X_1)\exp\left(\mathbb{Z}(\sqrt{2}X_2 + X_3)\right)\exp\left(\mathbb{Z}(\sqrt{2}X_3)\right).$$

It is clear that Γ is a uniform subgroup of G. An easy calculation shows that $M \cap \Gamma = \exp(\mathbb{Z}X_1)$. This implies that $M \cap \Gamma$ is not uniform in M and hence M is not rational in G.

Lemma 3.4. *Let G be the generic filiform Lie group L_n ($n \geq 3$). Let Γ be a discrete uniform subgroup of G. Then there exist $\Psi \in \mathrm{Aut}(G)$ and $u_1, \ldots, u_{n-1}$ a linearly independent set of vectors of $\mathfrak{g}$ such that*

(1) $u_j = \sum_{i=1}^{j} t_{ij} X_i$ *with* $t_{ij} \in \mathbb{R}$ *and* $t_{jj} > 0$;

(2) $\Psi(\Gamma) = \exp(\mathbb{Z}u_1) \cdots \exp(\mathbb{Z}u_{n-1})\exp(\mathbb{Z}X_n)\exp(\mathbb{Z}X_{n+1})$.

Proof. For $j = 1, \ldots, n$, the ideal $\mathfrak{a}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$ is rational. In fact, if $j = n$, then $\mathfrak{a}_n = \mathfrak{m}$ and the aim follows from Lemma 3.2. If $j \neq n$, we observe that $\mathfrak{a}_j = \mathcal{C}^{n-j}(\mathfrak{g})$ and hence the rationality of $\mathfrak{a}_j$ follows from Proposition 2.1. Note also that by Proposition 5.3.2 of [2], there exists a strong Malcev basis $\{Y_1, \ldots, Y_{n+1}\}$ for $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1}$ and $\mathfrak{a}_n = \mathfrak{m}$. We note $Y_{n+1} = aX_{n+1} + v$ where $v \in \mathfrak{m}$ and $a \in \mathbb{R}^*$ (since $Y_{n+1} \notin \mathfrak{m}$). The mapping $\Psi_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $\Psi_*(Y_{n+1}) = X_{n+1}$, $\Psi_*(Y_n) = X_n$ and $\Psi_*(\operatorname{ad}^j Y_{n+1}(Y_n)) = X_{n-j}$ if $1 \leq j \leq n-1$, is a Lie algebra automorphism. For $j = 1, \ldots, n-1$, let $u_j = \Psi_*(Y_j)$. Then

$$\Psi(\Gamma) = \exp(\mathbb{Z}u_1) \cdots \exp(\mathbb{Z}u_{n-1}) \exp(\mathbb{Z}X_n) \exp(\mathbb{Z}X_{n+1}).$$

It remains to verify that $u_j = \sum_{i=1}^{j} t_{ij} X_i$ such that $t_{jj} > 0$. Given $j = 1, \ldots, n-1$. We begin by observing that

$$\operatorname{ad}^{n-j} Y_{n+1}(Y_n) \in \mathbb{R}^* X_j \oplus \mathbb{R}\text{-span}\{X_1, \ldots, X_{j-1}\}.$$

From this we conclude that

$$Y_j \in \mathbb{R}\text{-span}\left\{\operatorname{ad}^{n-i} Y_{n+1}(Y_n) : \ 1 \leq i \leq j\right\}.$$

Write $Y_j = \sum_{i=1}^{j} t_{ij} \operatorname{ad}^{n-i} Y_{n+1}(Y_n)$ $(t_{ij} \in \mathbb{R})$. Then

$$\begin{aligned} u_j &= \Psi_*(Y_j) \\ &= \sum_{i=1}^{j} t_{ij} \Psi_*(\operatorname{ad}^{n-i} Y_{n+1}(Y_n)) \\ &= \sum_{i=1}^{j} t_{ij} X_i. \end{aligned}$$

On the other hand, since the basis $\{Y_1, \ldots, Y_{n+1}\}$ passing through $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, then $Y_j \in \mathfrak{a}_j \setminus \mathfrak{a}_{j-1}$ and hence $t_{jj} \neq 0$. We can assume that $t_{jj} > 0$, if not, we replace u_j by $-u_j$. This finishes the proof of the lemma. ∎

Lemma 3.5. *With the notations of Lemma 3.4 we have $t_{ij} \in \mathbb{Q}$ for any j and i $(1 \leq i \leq j \leq n-1)$.*

Proof. As

$$\exp(X_{n+1}) \exp(X_n) \exp(-X_{n+1}) \in \Psi(\Gamma),$$

then there exist $\alpha_{1,n}, \ldots, \alpha_{n-1,n} \in \mathbb{Z}$ such that

$$\exp(X_{n+1}) \exp(X_n) \exp(-X_{n+1}) = \exp(\alpha_{1,n} u_1) \ldots \exp(\alpha_{n-1,n} u_{n-1}) \exp(X_n),$$

and therefore

$$e^{\operatorname{ad} X_{n+1}}(X_n) = \alpha_{1,n} u_1 + \ldots + \alpha_{n-1,n} u_{n-1} + X_n. \tag{11}$$

Similarly, for $j = 1, \dots, n-1$, the element $\exp(X_{n+1})\exp(u_j)\exp(-X_{n+1})$ belongs to $\Psi(\Gamma)$, this implies that there exist $\alpha_{1,j}, \dots, \alpha_{j-1,j} \in \mathbb{Z}$ such that

$$\exp(X_{n+1})\exp(u_j)\exp(-X_{n+1}) = \exp(\alpha_{1,j}u_1)\dots\exp(\alpha_{j-1,j}u_{j-1})\exp(u_j).$$

Which implies that

$$e^{\operatorname{ad} X_{n+1}}(u_j) = \alpha_{1,j}u_1 + \dots + \alpha_{j-1,j}u_{j-1} + u_j. \tag{12}$$

Let us consider the following linear system in $\frac{1}{2}n(n-1)$ variables t_{ij} $(1 \leq j \leq n-1,\ 1 \leq i \leq j)$ and $\frac{1}{2}n(n-1)$ equations

$$\begin{cases} e^{\operatorname{ad} X_{n+1}}(X_n) = \alpha_{1,n}u_1 + \ldots + \alpha_{n-1,n}u_{n-1} + X_n; \\ e^{\operatorname{ad} X_{n+1}}(u_j) = \alpha_{1,j}u_1 + \dots + \alpha_{j-1,j}u_{j-1} + u_j; \\ u_j = \displaystyle\sum_{i=1}^{j} t_{ij}X_i; \\ 1 \leq j \leq n-1. \end{cases} \tag{13}$$

The matrix notation of the linear system (13) is

$$\mathbf{A}\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,n-1} & 0 \\ 0 & t_{22} & \cdots & t_{2,n-1} & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & t_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,n-1} & 0 \\ 0 & t_{22} & \cdots & t_{2,n-1} & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & t_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_{1,2} & \cdots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & 1 & \cdots & \alpha_{2,n-1} & \alpha_{2,n} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{14}$$

If we note $\alpha_{jj} = 1 (1 \leq j \leq n)$ and $t_{in} = \begin{cases} 0, & \text{if } i \leq n-1 \\ 1, & \text{if } i = n \end{cases}$, then the relation (14) is equivalent to

$$\sum_{i \leq k \leq j} a_{ik}t_{kj} = \sum_{i \leq k \leq j} t_{ik}\alpha_{kj} \qquad (1 \leq i < j \leq n) \tag{15}$$

But since $a_{jj} = \alpha_{jj} = 1$ $(1 \leq j \leq n)$, we obtain

$$\sum_{i < k \leq j} a_{ik}t_{kj} = \sum_{i \leq k < j} t_{ik}\alpha_{kj} \qquad (1 \leq i < j \leq n) \tag{16}$$

Therefore, if we consider the variables t_{ij} in the following order

$$t_{1,1}, \dots t_{1,n-1}, t_{2,2}, \dots, t_{2,n-1}, \dots, t_{n-1,n-1},$$

then the matrix T of the system (13) has the following form

$$T = \begin{pmatrix} \begin{pmatrix} \alpha_{1,2} & 0 & \\ & \ddots & \\ * & & \alpha_{n-1,n} \end{pmatrix} & & * & \\ & 0 & \begin{pmatrix} \alpha_{2,3} & 0 & \\ & \ddots & \\ * & & \alpha_{n-1,n} \end{pmatrix} & \\ & & & \ddots \\ & & & & \alpha_{n-1,n} \end{pmatrix}.$$

We can conclude from this, that

$$\det(T) = \prod_{j=2}^{n} \alpha_{j-1,j}^{j-1}.$$

On the other hand, the relations (11) and (12) imply that $\alpha_{j-1,j} \neq 0$ for all $j, 1 \leq j \leq n$. It follows that $\det(T) \neq 0$ and hence the system (13) is a Cramer system with rational coefficients, then the solutions (t_{ij}) are all rational too. This proves Lemma 3.5. ∎

Lemma 3.6. *Let G be the generic filiform Lie group L_n $(n \geq 3)$. Let Γ be a discrete uniform subgroup of G. Then there exist $\Phi \in \mathrm{Aut}(G)$, $e_1, \ldots, e_{n-1}$ a linearly independent set of vectors of $\mathfrak{g}$ and $m \in \mathbb{N}^*$ such that*

(1) $\forall\ j = 1, \ldots, n-1 :\ e_j = \displaystyle\sum_{i=1}^{j} x_{ij} X_i$ *with* $x_{ij} \in \mathbb{Z}$ *and* $x_{jj} \neq 0$;

(2) $\gcd(m, x_{ij};\ 1 \leq i \leq j \leq n-1) = 1$;

(3) $\Phi(\Gamma) = exp\,(\mathbb{Z}e_1) \cdots exp\,(\mathbb{Z}e_{n-1})\, exp\,(m\mathbb{Z}X_n)\, exp\,(\mathbb{Z}X_{n+1})$.

Proof. Following Lemma 3.5, let m' be the least common denominator of t_{ij} $(1 \leq j \leq n-1,\ 1 \leq i \leq j)$. We define $x'_{ij} = m't_{ij}$ and $d = \gcd(m', x'_{ij}; 1 \leq i \leq j \leq n-1)$. Let $m = \frac{m'}{d}$. The mapping $\Phi_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $\Phi_*(X_{n+1}) = X_{n+1}$ and for each $1 \leq j \leq n-1$, $\Phi_*(X_j) = mX_j$ is a Lie algebra automorphism. We note, for $j = 1, \ldots, n-1$, $e_j = \Phi_*(u_j) = mu_j = \displaystyle\sum_{i=1}^{j} x_{ij} X_i$. It is clear that $x_{ij} \in \mathbb{Z}$ for any $i, j, 1 \leq i \leq j \leq n-1$ and $\gcd(m, x_{ij};\ 1 \leq i \leq j \leq n-1) = 1$. Then $\Phi, e_1, \ldots, e_{n-1}$ and m having properties (1), (2) and (3). ∎

Now we can complete the proof of 2. Let $[\![D, m]\!]$ be the matrix with column vectors $e_1, \ldots, e_n$ expressed in the basis $\mathscr{B}_{\mathfrak{m}}$ where $e_1, \ldots, e_{n-1}, m$ as in Lemma 3.6 and $e_n = mX_n$. By Lemma 3.6 we have $[\![D, m]\!] \in \mathscr{D}$ and $\Phi(\Gamma) = \Gamma_{[\![D,m]\!]}$. This completes the proof of the second part of the theorem. Finally, we achieve with the proof of 3. We show both directions. Let $T \in \mathrm{GL}(n, \mathbb{R})$ such that $\mathbf{B}T = T\mathbf{B}$ and $[\![D_2, m_2]\!]^{-1} T [\![D_1, m_1]\!] \in \mathrm{GL}(n, \mathbb{Z})$. Let the linear function $\phi_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$\mathrm{Mat}(\phi_*, \mathscr{B}) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

We recall that $\mathbf{B} = \mathrm{Mat}(\mathrm{ad}\, X_{n+1}|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}})$, then the condition $\mathbf{B}T = T\mathbf{B}$ implies that $\phi_* \in \mathrm{Aut}(\mathfrak{g})$. On the other hand, let $(e_1, \ldots, e_n)$ be the column vectors for $[\![D_2, m_2]\!]$ expressed in the basis $\mathscr{B}_{\mathfrak{m}}$, then the second condition $[\![D_2, m_2]\!]^{-1} T [\![D_1, m_1]\!] \in \mathrm{GL}(n, \mathbb{Z})$ implies that

$$\mathbb{Z}\text{-span}\,\{e_1, \ldots, e_n\} = \mathbb{Z}\text{-span}\,\{\phi_*(e_1), \ldots, \phi_*(e_n)\}\,.$$

It follows that

$$\phi(\Gamma_{[\![D_1, m_1]\!]}) = \Gamma_{[\![D_2, m_2]\!]}.$$

Conversely, if $\Gamma_{[\![D_1, m_1]\!]} \simeq \Gamma_{[\![D_2, m_2]\!]}$ then there exists $\phi \in \mathrm{Aut}(G)$ such that $\phi(\Gamma_{[\![D_1, m_1]\!]}) = \Gamma_{[\![D_2, m_2]\!]}$. We note that it is possible to choose ϕ_* to satisfy $\phi_*(X_{n+1}) = X_{n+1}$. Moreover, it follows from Proposition 3.1 that $\phi_*(\mathfrak{m}) = \mathfrak{m}$. Then

$$\langle \phi_*, [X_{n+1}, X_i] \rangle = [X_{n+1}, \langle \phi_*, X_i \rangle];\ \forall i = 1, \ldots, n. \tag{17}$$

The description (17) can be expressed in matrix form

$$\mathrm{Mat}(\phi_*|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) \mathrm{Mat}(\mathrm{ad}\, X_{n+1}|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) X_i = \mathrm{Mat}(\mathrm{ad}\, X_{n+1}|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) \mathrm{Mat}(\phi_*|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) X_i$$

for any $i, 1 \leq i \leq n$. Equivalently

$$\mathrm{Mat}(\phi_*|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}})\ \mathbf{B} = \mathbf{B}\ \mathrm{Mat}(\phi_*|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}).$$

Finally, let $(e_1, \ldots, e_n)$ (resp. $(e'_1, \ldots, e'_n)$) be the column vectors for $[\![D_1, m_1]\!]$ (resp. $[\![D_2, m_2]\!]$) expressed in the basis $\mathscr{B}_{\mathfrak{m}}$. Then the vectors $\phi_*(e_1), \ldots, \phi_*(e_n)$ form a basis of the lattice $\mathbb{Z}\text{-span}\,\{e'_1, \ldots, e'_n\}$. It follows that there exists a matrix $T \in \mathrm{GL}(n, \mathbb{Z})$ such that $[e'_1, \ldots, e'_n] = [\phi_*(e_1), \ldots, \phi_*(e_n)]T$. Therefore, we have evidently

$$[\![D_2, m_2]\!] = \mathrm{Mat}(\phi_*|_{\mathfrak{m}}, \mathscr{B}_{\mathfrak{m}}) [\![D_1, m_1]\!] T,$$

completing the proof of our lemma. ∎

Remark 3.7. Theorem 2.4 of [3] shows that Theorem 1.1 remains valid if $n = 2$.

The next proposition presents some simple properties of the elements of $\mathscr{D}$.

Proposition 3.8. *Let $[\![D, m]\!] \in \mathscr{D}$ where $D = (x_{ij})_{1 \leq i \leq j \leq n-1}$. Then we have*

(1) $(n-1)!$ *divides* m*;*

(2) $x_{n-1,n-1}$ *divides* m*;*

(3) x_{ii} *divides* $x_{i+1,i+1}$ $(i = 1, \ldots, n-2)$.

Proof. We preserve the notation of Theorem 1.1. Let $\Gamma = \Gamma_{[\![D,m]\!]}$. Following Theorem 1.1, Γ is a uniform subgroup of G. As

$$\exp\left(X_{n+1}\right)\exp\left(mX_n\right)\exp\left(-X_{n+1}\right) \in \Gamma,$$

then there exist $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{Z}$ such that

$$\exp\left(X_{n+1}\right)\exp\left(mX_n\right)\exp\left(-X_{n+1}\right) = \exp\left(\alpha_1 e_1\right)\ldots\exp\left(\alpha_{n-1}e_{n-1}\right)\exp\left(mX_n\right).$$

Which implies that

$$e^{\operatorname{ad} X_{n+1}}(mX_n) = \alpha_1 e_1 + \cdots + \alpha_{n-1}e_{n-1} + mX_n.$$

Hence, we obtain that

$$mX_n + mX_{n-1} + \frac{m}{2!}X_{n-2} + \ldots + \frac{m}{(n-1)!}X_1 = \alpha_1 e_1 + \cdots + \alpha_{n-1}e_{n-1} + mX_n$$

and in particular, we have

$$\frac{m}{(n-1)!} = \alpha_1 x_{11} + \ldots + \alpha_{n-1}x_{1,n-1} \in \mathbb{Z},$$

and

$$m = \alpha_{n-1}x_{n-1,n-1}.$$

Which shows that $(n-1)!$ divides m and $x_{n-1,n-1}$ divides m.

Similarly, for $i = 1, \ldots, n-1$, we have $\exp\left(X_{n+1}\right)\exp\left(e_i\right)\exp\left(-X_{n+1}\right) \in \Gamma$. Hence $e^{\operatorname{ad} X_{n+1}}(e_i) \in \mathbb{Z}\text{-span}\left\{e_1, \ldots, e_i\right\}$ and therefore

$$[X_{n+1}, e_i] + \frac{1}{2}[X_{n+1}, [X_{n+1}, e_i]] + \ldots \in \mathbb{Z}\text{-span}\left\{e_1, \ldots, e_{i-1}\right\}.$$

For which it follows that $x_{i,i} \in (x_{i-1,i-1})\mathbb{Z}$. ■

4. Examples

As an application of Theorem 1.1, we have

Proposition 4.1. *Every uniform subgroup Γ of L_3 has the following form*

$$\Gamma \simeq \exp(\mathbb{Z}X_1)\exp(p_1\mathbb{Z}X_2)\exp(\mathbb{Z}(p_1p_2X_3 - \frac{p_3}{2}X_2))\exp(\mathbb{Z}X_4)$$

where p_1, p_2, p_3 are integers satisfying $p_1 > 0$, $p_2 > 0$, $p_1p_2 + p_3 \in 2\mathbb{Z}$ and $0 \leq \frac{p_3}{2} < p_1$. Furthermore, different choices for the p's give non isomorphic subgroups.

Proof. Let Γ be a uniform subgroup of L_3, then by Theorem 1.1 we have

$$\Gamma \simeq \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}e_2)\exp(\mathbb{Z}e_3))\exp(\mathbb{Z}e_4)$$

where $e_1 = aX_1, e_2 = cX_2 + bX_1, e_3 = mX_3, e_4 = X_4$ and the integers a, b, c, m satisfying

$$\begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & m \end{pmatrix} \in \mathscr{D}.$$

This implies that a divides c, c divides m and

$$\frac{m}{2a} - \frac{bm}{ac} \in \mathbb{Z} \tag{18}$$

Write $c = p_1 a$ and $m = p_2 c = p_1 p_2 a$. Then the relation (18) implies that

$$\frac{p_1 p_2}{2} - \frac{bp_2}{a} \in \mathbb{Z}. \tag{19}$$

Next, the mapping $\Theta_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by $\Theta_*(X_4) = X_4, \Theta_*(e_3) = \frac{1}{a}e_3 - \frac{bp_2}{a}X_2, \Theta_*(e_2) = p_1 X_2$ and $\Theta_*(e_1) = X_1$ is a Lie algebra automorphism. It follows that

$$\Gamma \simeq \exp(\mathbb{Z}X_1)\exp(p_1\mathbb{Z}X_2)\exp\left(\mathbb{Z}(p_1p_2X_3 - \frac{bp_2}{a}X_2)\right)\exp(\mathbb{Z}X_4).$$

Let $r = \frac{2bp_2}{a}$, we deduce from the relation (19) that r is an integer and has the same parity as p_1p_2. On the other hand, let $r = q(2p_1) + p_3$ with $q \in \mathbb{Z}$ and $0 \le p_3 < 2p_1$. It is easily verified that r and p_3 have the same parity and

$$\Gamma \simeq \Gamma_1 = \exp(\mathbb{Z}X_1)\exp(p_1\mathbb{Z}X_2)\exp\left(\mathbb{Z}(p_1p_2X_3 - \frac{p_3}{2}X_2)\right)\exp(\mathbb{Z}X_4).$$

Let p_1', p_2' and p_3' such that $p_1' > 0, p_2' > 0, p_1'p_2' + p_3' \in 2\mathbb{Z}$, $0 \le \frac{p_3'}{2} < p_1'$ and

$$\Gamma_1 \simeq \Gamma_2 = \exp(\mathbb{Z}X_1)\exp(p_1'\mathbb{Z}X_2)\exp\left(\mathbb{Z}(p_1'p_2'X_3 - \frac{p_3'}{2}X_2)\right)\exp(\mathbb{Z}X_4). \tag{20}$$

Let $\phi \in \mathrm{Aut}(G)$ which establishes the isomorphism (20) such that $\phi_*(X_4) = X_4$. It is clear that $\phi_*(X_1) = X_1$. On the other hand, since $\phi_* \in \mathrm{Aut}(\mathfrak{g})$, then there exist $x, y \in \mathbb{R}$ such that $\phi_*(X_2) = X_2 + xX_1$ and $\phi_*(X_3) = X_3 + xX_2 + yX_1$. Moreover, since $\mathrm{D(G)}$ is stable under ϕ then $\Gamma_1 \cap \mathrm{D(G)} = \Gamma_2 \cap \mathrm{D(G)}$ and hence

$$\exp(\mathbb{Z}X_1)\exp(p_1\mathbb{Z}X_2) = \exp(\mathbb{Z}X_1)\exp(p_1'\mathbb{Z}X_2).$$

Therefore, we obtain $p_1 = p_1'$. Similarly, replacing $\mathrm{D(G)}$ by M, we can show that $p_1p_2 = p_1'p_2'$ and hence $p_2 = p_2'$. It remains to show that $p_3 = p_3'$. For this, we observe that there exist $\alpha, \beta \in \mathbb{Z}$ satisfy

$$\phi_*(p_1p_2X_3 - \frac{p_3}{2}X_2) = (p_1p_2X_3 - \frac{p_3'}{2}X_2) + \alpha p_1 X_2 + \beta X_1.$$

In particular, we have $xp_1p_2 - \frac{p_3}{2} = -\frac{p_3'}{2} + \alpha p_1$ and therefore $\frac{p_3'}{2} - \frac{p_3}{2} = p_1(\alpha - xp_2)$. We now use the facts that $0 \le \frac{p_3}{2} < p_1$ and $0 \le \frac{p_3'}{2} < p_1$ we deduce that $p_3 = p_3'$. ■

The next remark gives an isomorphisms between the uniform subgroups obtained in Proposition 4.1 and those obtained in Theorem 1 of [8].

Remark 4.2. In [8], the generic filiform nilpotent Lie algebra $\mathfrak{L}_3$ is spanned by the strong Malcev basis $\{e_1, \ldots, e_4\}$ such that

$$e_1 = X_1, \quad e_2 = X_2 + \frac{1}{2}X_1, \quad e_3 = X_3 - \frac{1}{2}X_1, \quad e_4 = X_4.$$

Let Γ be a uniform subgroup of L_3. It follows from [8, Theorem 1] that there exist integers q_1, q_2 and q_3 satisfying $q_2, q_3 > 0$ and $0 \leq q_1 \leq \frac{1}{2}\gcd\{q_2, q_3\}$ such that

$$\begin{aligned} \Gamma &\simeq \mathbf{H}_4(q_1, q_2, q_3) && (21)\\ &\simeq \{\exp(je_1)\exp((q_3k + q_1m)e_2)\exp((q_2q_3m)e_3)\exp(ne_4) : \ j, k, m, n \in \mathbb{Z}\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see [8, page 230])} \\ &= \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}(q_3e_2))\exp(\mathbb{Z}(q_2q_3e_3 + q_1e_2))\exp(\mathbb{Z}e_4). && (22)\end{aligned}$$

Next, using the notation introduced in the proofs of Lemma 3.4, Lemma 3.6 and Proposition 4.1, we remark that the mapping $\mathrm{T} = \phi \circ \Theta \circ \Phi \circ \Psi$ is an isomorphism between

$$\exp(\mathbb{Z}e_1)\exp(\mathbb{Z}(q_3e_2))\exp(\mathbb{Z}(q_2q_3e_3 + q_1e_2))\exp(\mathbb{Z}e_4)$$

and

$$\Gamma(p_1, p_2, p_3) = \exp(\mathbb{Z}X_1)\exp(p_1\mathbb{Z}X_2)\exp\Big(\mathbb{Z}(p_1p_2X_3 - \frac{p_3}{2}X_2)\Big)\exp(\mathbb{Z}X_4)$$

for some integers p_1, p_2, p_3 as in Proposition 4.1. Finally, we compose the isomorphism T with the isomorphism given in [8, page 230] between (21) and (22), we obtain an explicit isomorphism between $\mathbf{H}_4(q_1, q_2, q_3)$ and $\Gamma(p_1, p_2, p_3)$.

Proposition 4.3. *Every uniform subgroup Γ of L_4 has the following form*

$$\begin{aligned}\Gamma \simeq\ & exp(\mathbb{Z}X_1)\, exp(p_1\mathbb{Z}X_2)\, exp\Big(\mathbb{Z}(p_1p_2X_3 + \frac{\alpha}{2}X_2)\Big) \\ & exp\Big(\mathbb{Z}(p_1p_2p_3X_4 + \frac{\beta}{2}X_3 + \frac{\gamma}{12}X_2)\Big)\, exp(\mathbb{Z}X_5) \qquad (23)\end{aligned}$$

where $p_1, p_2, p_3, \alpha, \beta, \gamma$ are integers satisfying $p_1 > 0, p_2 > 0, p_3 > 0$ and

$$\begin{cases} \alpha + p_1p_2 \in 2\mathbb{Z} \\ \gamma + 3\beta + 2p_1p_2p_3 \in 12\mathbb{Z} \\ \beta - \alpha p_3 + p_1p_2p_3 \in 2p_1\mathbb{Z}. \end{cases} \qquad (24)$$

Furthermore, if

$$\begin{cases} 0 \leq \alpha < 2p_1 \\ 0 \leq \gamma < 12p_1 \\ 0 \leq \beta < 2p_1p_2, \end{cases} \qquad (25)$$

then different choices for $p_1, p_2, p_3, \alpha, \beta$ and γ give non isomorphic groups.

Proof. Let Γ be a uniform subgroup of L_4, then by Theorem 1.1 we have

$$\Gamma \simeq \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}e_2)\exp(\mathbb{Z}e_3))\exp(\mathbb{Z}e_4)\exp(\mathbb{Z}e_5)$$

where

$$\begin{array}{rcl} e_1 & = & a_{11}X_1 \\ e_2 & = & a_{12}X_1 + a_{22}X_2 \\ e_3 & = & a_{13}X_1 + a_{23}X_2 + a_{33}X_3 \\ e_4 & = & mX_4 \\ e_5 & = & X_5 \end{array}$$

and the integers m, a_{ij} $(1 \leq j \leq 3, 1 \leq i \leq j)$ satisfying

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \in \mathscr{D}$$

This implies that, a_{11} divides a_{22}, a_{22} divides a_{33}, a_{33} divides m and

$$\begin{cases} \dfrac{a_{23}}{a_{11}} + \dfrac{a_{33}}{2a_{11}} - \dfrac{a_{12}a_{33}}{a_{11}a_{22}} \in \mathbb{Z} \\ \dfrac{m}{6a_{11}} - \dfrac{ma_{12}}{2a_{11}a_{22}} + m\dfrac{a_{12}a_{23} - a_{22}a_{13}}{a_{11}a_{22}a_{33}} \in \mathbb{Z} \\ \dfrac{m}{2a_{22}} - \dfrac{ma_{23}}{a_{22}a_{33}} \in \mathbb{Z} \end{cases} \tag{26}$$

Let $b = \dfrac{a_{22}a_{13} - a_{12}a_{23}}{a_{11}a_{22}}$ and let $\Phi_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$\begin{array}{rcl} \Phi_*(X_1) & = & \dfrac{1}{a_{11}}X_1 \\ \Phi_*(X_2) & = & \dfrac{1}{a_{11}}X_2 - \dfrac{a_{12}}{a_{11}a_{22}}X_1 \\ \Phi_*(X_3) & = & \dfrac{1}{a_{11}}X_3 - \dfrac{a_{12}}{a_{11}a_{22}}X_2 - \dfrac{b}{a_{33}}X_1 \\ \Phi_*(X_4) & = & \dfrac{1}{a_{11}}X_4 - \dfrac{a_{12}}{a_{11}a_{22}}X_3 - \dfrac{b}{a_{33}}X_2 \\ \Phi_*(X_5) & = & X_5 \end{array}$$

We note that it is clear that $\Phi_* \in \mathrm{Aut}(\mathfrak{g})$ and an easy computation shows that

$$\begin{aligned}
\Phi_*(e_1) &= X_1 \\
\Phi_*(e_2) &= \frac{a_{22}}{a_{11}} X_2 \\
\Phi_*(e_3) &= \frac{a_{33}}{a_{11}} X_3 + \Big(\frac{a_{22}a_{23} - a_{12}a_{33}}{a_{11}a_{22}}\Big) X_2 \\
\Phi_*(e_4) &= \frac{m}{a_{11}} X_4 - m\frac{a_{12}}{a_{11}a_{22}} X_3 - \frac{bm}{a_{33}} X_2 \\
\Phi_*(e_5) &= X_5.
\end{aligned}$$

Therefore, if we put

$$\begin{aligned}
a_{22} &= p_1 a_{11} \\
a_{33} &= p_2 a_{22} = p_2 p_1 a_{11} \\
m &= p_3 a_{33} = p_3 p_2 p_1 a_{11} \\
\alpha &= 2\Big(\frac{a_{22}a_{23} - a_{12}a_{33}}{a_{11}a_{22}}\Big) \\
\beta &= -2m\frac{a_{12}}{a_{11}a_{22}} \\
\gamma &= -12\frac{bm}{a_{33}},
\end{aligned}$$

where p_1, p_2 and p_3 belong to $\mathbb{N}^*$, then

$$\begin{aligned}
\Phi_*(e_1) &= X_1 \\
\Phi_*(e_2) &= p_1 X_2 \\
\Phi_*(e_3) &= p_1 p_2 X_3 + \frac{1}{2}\alpha X_2 \\
\Phi_*(e_4) &= p_1 p_2 p_3 X_4 + \frac{1}{2}\beta X_3 + \frac{1}{12}\gamma X_2 \\
\Phi_*(e_5) &= X_5
\end{aligned}$$

and the condition (26) is equivalent to

$$\begin{cases}
\alpha + p_1 p_2 \in 2\mathbb{Z} \\
\frac{1}{12}\gamma + \frac{\beta}{4} + \frac{1}{6} p_1 p_2 p_3 \in \mathbb{Z} \\
\frac{\beta}{2p_1} + \frac{1}{2} p_2 p_3 - \frac{\alpha p_3}{2p_1} \in \mathbb{Z}.
\end{cases} \tag{27}$$

We deduce from (27) that α, β and γ are integers. This proves the first part of the proposition. By an argument similar to the proof of Proposition 4.1, we can prove that the condition (25) proves the second part of the proposition. This completes the proof. ∎

References

[1] Auslander L., L. Green, and F. Hahn, "Flows on homogeneous spaces," Ann. of Math. Studies **53**, Princeton Univ. Press, 1963.

[2] Corwin L., and F. P. Greenleaf, "Representations of nilpotent Lie groups and their applications, Part I," Cambridge Studies in Adv. Math. **18**, Cambridge University Press, Cambridge, New York, 1989.

[3] Gordon C. S., E. N. Wilson, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Mich. Math. J. **33** (1986), 253–271.

[4] Eberlein, P., *Geometry of* 2*-step nilpotent Lie groups*, Modern Dynamical Systems and Applications, Cambridge University Press, (2004), 67–101.

[5] Malcev, A. I., "On a class of homogeneous spaces," Amer. Math. Soc. Transl. **39**, 1951.

[6] Matsushima Y., *On the discrete subgroups and homogeneous spaces of nilpotent Lie groups*, Nagoya Math. J. **2** (1951), 95–110.

[7] Milnes, P., *Groups and* C^**-algebras from a* 4*-dimensional Anzai flow*, Math. Nachr. **235** (2002), 129–141.

[8] Milnes P., and S. Walters, *Discrete cocompact subgroups of the four-dimensional nilpotent connected Lie group and their group* C^**-algebras*, J. Math. Anal. Appl. **253** (2001), 224–242.

[9] —, *Simple quotients of the group* C^**-algebra of a discrete* 4*-dimensional nilpotent group*, Houston J. Math. **19** (1993), 615–636.

[10] —, *Simple infinite dimensional quotients of* $C^*(G)$ *for discrete* 5*-dimensional nilpotent groups* G, Illinois J. Math. **41** (1997), 315–340.

[11] Raghunathan, M. S., "Discrete subgroups of Lie groups," Ergebnisse der Mathematik Band **68**, Springer Berlin, New York, etc., 1972.

[12] Reihani K., and P. Milnes, Analysis on discrete cocompact subgroups of the generic filiform Lie groups, Acta Math. Hungarica **112** (2006), 157–179.

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