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A Central Element in the Universal Enveloping Algebra of Type $\mathrm{D_n}$ via Minor Summation Formula of Pfaffians

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Abstract. It is well known that the universal enveloping algebra $U(\mathfrak{o}_{2n})$ of the orthogonal Lie algebra $\mathfrak{o}_{2n}$ of size even has a central element given explicitly in terms of Pfaffian of a certain matrix which is alternating along the anti-diagonal (we call such a matrix *anti-alternating* for short) whose entries are in $U(\mathfrak{o}_{2n})$. In this paper, we establish a minor summation formula of Pfaffian of the noncommutative anti-alternating matrix, as well as of commutative anti-alternating matrix. As an application, we show that the eigenvalues of the Pfaffian-type central element on the highest weight representations of $\mathfrak{o}_{2n}$ can be easily computed.
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0. Introduction

Let us denote by $\mathfrak{o}_{2n}$ the complex orthogonal Lie algebra defined by

$$\mathfrak{o}_{2n} = \{X \in \mathrm{Mat}_{2n}(\mathbb{C}); {}^tX J_{2n} + J_{2n}X = O\},$$

where J_{2n} is the nondegenerate symmetric matrix with 1's on the anti-diagonal and 0's elsewhere. Then a matrix X is in $\mathfrak{o}_{2n}$ if and only if it is alternating along the anti-diagonal, which we call *anti-alternating* for short in this paper. We remark that the subspaces of $\mathfrak{o}_{2n}$ consisting of all diagonal matrices, and of all upper triangular matrices form Cartan subalgebra, and nilpotent subalgebra spanned by positive root vectors, respectively, which we denote by $\mathfrak{h}$ and $\mathfrak{n}$.

It is known that the center $ZU(\mathfrak{o}_{2n})$ of the universal enveloping algebra $U(\mathfrak{o}_{2n})$ of the Lie algebra $\mathfrak{o}_{2n}$ is generated by elements that are given in terms of column (minor) determinants and Pfaffian of certain matrices $\mathbf{\Phi} = (\Phi_{i,j})_{i,j=1,\dots,2n}$ whose diagonal and upper triangular entries are in $U(\mathfrak{h})$ and $\mathfrak{n} \subset U(\mathfrak{n})$, respectively (see below for the Pfaffian type; one must shift diagonal entries for the determinant type). Here, for a matrix $\mathbf{\Phi} = (\Phi_{i,j})_{i,j}$ with $\Phi_{i,j}$ in a noncommutative associative algebra in general, the column determinant of $\mathbf{\Phi}$, which we denote

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by $\det(\mathbf{\Phi})$, is defined to be

$$\det(\mathbf{\Phi}) = \sum_{\sigma\in\mathfrak{S}_{2n}} \operatorname{sgn}(\sigma)\Phi_{\sigma(1),1}\Phi_{\sigma(2),2}\cdots\Phi_{\sigma(2n),2n}. \tag{0.1}$$

Column minor determinants are defined similarly.

If, moreover, $\mathbf{\Phi}$ is anti-alternating, i.e. $\mathbf{\Phi}J_{2n}$ is alternating as above, Pfaffian of $\mathbf{\Phi}J_{2n}$, which we simply denote by $\operatorname{Pf}(\mathbf{\Phi})$, is defined to be

$$\operatorname{Pf}(\mathbf{\Phi}) = \frac{1}{2^n n!}\sum_{\sigma\in\mathfrak{S}_{2n}} \operatorname{sgn}(\sigma)\,\widetilde{\Phi}_{\sigma(1),\sigma(2)}\widetilde{\Phi}_{\sigma(3),\sigma(4)}\cdots\widetilde{\Phi}_{\sigma(2n-1),\sigma(2n)}, \tag{0.2}$$

where $\widetilde{\Phi}_{i,j}$ denotes the (i,j)th entry of $\mathbf{\Phi}J_{2n}$.

It is easy to compute the eigenvalues of the central elements given by column determinants on the highest weight representations of $\mathfrak{o}_{2n}$ with highest weight λ; if we apply $\det(\mathbf{\Phi})$ to the highest vector, the only term that survives in the sum (0.1) is the one that corresponds to $\sigma = 1$ since $\Phi_{i,j}$ is in $\mathfrak{n}$ if $i<j$ as we remarked above (cf. [4]).

The same principle does not work for the Pfaffian-type element.

Let us first consider the commutative case. Write an anti-alternating matrix $X \in \mathfrak{o}_{2n}$ as

$$X = \begin{bmatrix} a & b \\ c & -J_n\,{}^t a J_n \end{bmatrix}, \tag{0.3}$$

with a, b, c all of size $n\times n$ that are parametrized as follows:

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \quad b = \begin{bmatrix} b_{1,n} & \cdots & b_{1,2} & 0 \\ \vdots & \iddots & 0 & -b_{1,2} \\ b_{n-1,n} & 0 & \iddots & \vdots \\ 0 & -b_{n-1,n} & \cdots & -b_{1,n} \end{bmatrix},$$
$$c = \begin{bmatrix} c_{1,n} & \cdots & c_{n-1,n} & 0 \\ \vdots & \iddots & 0 & -c_{n-1,n} \\ c_{1,2} & 0 & \iddots & \vdots \\ 0 & -c_{1,2} & \cdots & -c_{1,n} \end{bmatrix}. \tag{0.4}$$

Then we find that the Pfaffian of XJ_{2n}, which we denote by $\operatorname{Pf}(X)$ as above, is expanded as follows (see (1.11)):

$$\operatorname{Pf}(X) = \sum_{k=0}^{\lfloor n/2\rfloor} \sum_{\substack{I,J\subset[n] \\ |I|=|J|=2k}} \operatorname{sgn}(\bar{I}, I)\operatorname{sgn}(\bar{J}, J)\det(a^{\bar{I}}_{\bar{J}})\operatorname{Pf}(b_I)\operatorname{Pf}(c_J). \tag{0.5}$$

Here the second sum is over all index sets I and J, both of cardinality $2k$ and contained in $[n] := \{1, 2, \ldots, n\}$, with $\bar{I}$ and $\bar{J}$ denoting the complements of I and J in $[n]$ respectively, $a^{\bar{I}}_{\bar{J}}$ submatrix of a whose row and column indices are in

$\bar{I}$ and $\bar{J}$ respectively, and b_I, c_I submatrices of b, c whose row and column indices are both in I (see Section 1 for details).

Furthermore, the minor summation formula of Pfaffian (0.5) holds true for a rectangular submatrix a. More precisely, for positive integers p, q with $p+q=2n$, the formula holds true if we write an anti-alternating matrix $X \in \mathfrak{o}_{2n}$ as (0.3), but with a, b, c being of size $p \times q, p \times p, q \times q$, respectively, and $(2,2)$-block replaced by $-J_q\, {}^t a J_p$ (Theorem 1.1). It is immediate to see that the minor summation formula given in [6, Theorem 3.5] corresponds to ours with p and q both even.

Returning to the noncommutative case, let $X_{i,j} := E_{i,j} - E_{-j,-i} \in \mathfrak{o}_{2n} \subset U(\mathfrak{o}_{2n})$, where $E_{i,j}$ is the matrix unit with 1 in the (i,j)th entry and 0 elsewhere, and $-i$ stands for $2n+1-i$. Define an anti-alternating $2n \times 2n$ matrix $\boldsymbol{X}$ with entries in $U(\mathfrak{o}_{2n})$ by

$$\boldsymbol{X} := \begin{bmatrix} a & b \\ c & -J_n\, {}^t a J_n \end{bmatrix}, \tag{0.6a}$$

where

$$a_{i,j} = X_{i,j}, \quad b_{i,j} = X_{i,-j}, \quad c_{i,j} = X_{-j,i} \tag{0.6b}$$

for $i, j \in [n]$. One of the main purpose of this paper is to show that the following expansion formula of the noncommutative Pfaffian $\mathrm{Pf}\,(\boldsymbol{X})$ holds:

Theorem A. *Let $\boldsymbol{X}$ be the anti-alternating matrix defined by* (0.6). *Then the noncommutative Pfaffian* $\mathrm{Pf}\,(\boldsymbol{X})$ *is expanded as follows:*

$$\mathrm{Pf}\,(\boldsymbol{X}) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{I,J \subset [n] \\ |I|=|J|=2k}} \mathrm{sgn}\,(\bar{I}, I)\, \mathrm{sgn}\,(\bar{J}, J) \det\Big(a_{\bar{J}}^{\bar{I}} + \mathbf{1}_{\bar{J}}^{\bar{I}}\, \boldsymbol{\rho}(|\bar{J}|)\Big)\, \mathrm{Pf}\,(c_J)\, \mathrm{Pf}\,(b_I)\,, \tag{0.7}$$

where $\boldsymbol{\rho}(j)$ denotes the diagonal matrix $\mathrm{diag}(j-1, j-2, \ldots, 1, 0)$.

Using the formula, one can easily compute the eigenvalues of $\mathrm{Pf}\,(\boldsymbol{X})$ on the highest weight representations of $\mathfrak{o}_{2n}$ as in the case of the determinant-type elements.

Example. If $n=2$, then a matrix $X \in \mathfrak{o}_{2n}$ written as (0.3) looks like

$$X = \begin{bmatrix} a_{1,1} & a_{1,2} & b_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 & -b_{1,2} \\ c_{1,2} & 0 & -a_{2,2} & -a_{1,2} \\ 0 & -c_{1,2} & -a_{2,1} & -a_{1,1} \end{bmatrix}, \tag{0.8}$$

and the Pfaffian $\mathrm{Pf}\,(X)$ is given by

$$\mathrm{Pf}\,(X) = a_{1,1}a_{2,2} - a_{2,1}a_{1,2} + c_{1,2}b_{1,2}, \tag{0.9}$$

of which the sum of the first and the second terms corresponds to $I = J = \varnothing$, and the last to $I = J = \{1,2\}$ in the right-hand side of (0.5), respectively.

Passing to the noncommutative case, let us write the matrix $\boldsymbol{X}$ as (0.8). Since, by definition, symmetrization of the right-hand side of (0.9) provides $\mathrm{Pf}\,(\boldsymbol{X})$, we obtain

$$\begin{aligned} \mathrm{Pf}\,(\boldsymbol{X}) &= \frac{1}{2}\Big(a_{1,1}a_{2,2} - a_{2,1}a_{1,2} + c_{1,2}b_{1,2} + a_{2,2}a_{1,1} - a_{1,2}a_{2,1} + b_{1,2}c_{1,2}\Big) \\ &= (a_{1,1}+1)\, a_{2,2} - a_{2,1}a_{1,2} + c_{1,2}b_{1,2}, \end{aligned}$$

where we used the commutation relations (2.6) below.

The paper is organized as follows: Section 1 is devoted to the proof of the commutative minor summation formula of Pfaffian for $X \in \mathfrak{o}_{2n}$ parametrized as in (0.3) with a rectangular submatrix of size $p \times q$ $(p+q=2n)$. We give the proof by induction on $p+q$, the size of X, which reveals that iteration of the row/column expansion formula of Pfaffian yields our formula, though in the Appendix, we will give another proof of the commutative minor summation formulae using the exterior calculus. In Section 2, using the exterior calculus with noncommutative coefficient, we prove our main theorem, i.e., the noncommutative minor summation formula of the Pfaffian $\mathrm{Pf}\,(\boldsymbol{X})$ for the matrix $\boldsymbol{X}$. As an application, we show that the eigenvalues of the central element $\mathrm{Pf}\,(\boldsymbol{X})$ on the highest weight modules can be easily computed.

Convention. For a positive integer n, let us denote by $[n]$ the set $\{1, 2, \dots, n\}$, by $[-n]$ the set $\{-n, \dots, -2, -1\}$, and by $[\pm n]$ the union $[n] \cup [-n]$. For a pair of index sets $I \subset J$, its complement $\bar{I}$ is always taken in J unless otherwise mentioned. The symbol $\sqcup$ denotes the disjoint union. For index sets $I = \{i_1 < \cdots < i_r\}$, $J = \{j_1 < \cdots < j_s\}$ and $K = \{k_1 < \cdots < k_{r+s}\}$ such that $K = I \sqcup J$, let us denote by $\mathrm{sgn}\begin{pmatrix} K \\ I, J \end{pmatrix}$ the signature of the permutation

$$\begin{pmatrix} k_1 & \dots & k_r & k_{r+1} & \dots & k_{r+s} \\ i_1 & \dots & i_r & j_1 & \dots & j_s \end{pmatrix}.$$

When dealing with Pfaffian of an anti-alternating matrix of size $2n \times 2n$, it is convenient to use the signed index. Namely, for any index $i \in [2n]$, we shall agree that $-i$ stands for $2n+1-i$. Finally, for a real number x, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

1. Commutative Minor Summation Formula

In this section, we prove minor summation formulae of Pfaffian of anti-alternating matrices whose entries are in a commutative algebra, by iterating the row/column expansion formula of Pfaffian.

For a positive integer N, let us denote by J_N the $N \times N$ matrix with 1's on the anti-diagonal and 0's elsewhere:

$$J_N = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Recall that a matrix X is in $\mathfrak{o}_{2n}$ if and only if it is anti-alternating, i.e. alternating along the anti-diagonal, so that we can define the Pfaffian of XJ_{2n}, which we denote by $\mathrm{Pf}\,(X)$ simply, as in the previous section.

For positive integers p, q with $p+q=2n$, we write a matrix $X \in \mathfrak{o}_{2n}$ as

$$X = \begin{bmatrix} a & b \\ c & -J_q {}^t a J_p \end{bmatrix}, \tag{1.1}$$

where a, b, c are of size $p \times q, p \times p, q \times q$, respectively, that are parametrized as

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,q} \\ \vdots & & \vdots \\ a_{p,1} & \cdots & a_{p,q} \end{bmatrix}, \quad b = \begin{bmatrix} b_{1,p} & \cdots & b_{1,2} & 0 \\ \vdots & \iddots & 0 & -b_{1,2} \\ b_{p-1,p} & 0 & \iddots & \vdots \\ 0 & -b_{p-1,p} & \cdots & -b_{1,p} \end{bmatrix},$$
$$c = \begin{bmatrix} c_{1,q} & \cdots & c_{q-1,q} & 0 \\ \vdots & \iddots & 0 & -c_{q-1,q} \\ c_{1,2} & 0 & \iddots & \vdots \\ 0 & -c_{1,2} & \cdots & -c_{1,q} \end{bmatrix}. \tag{1.2}$$

Define their submatrices by

$$a_J^I := (a_{i,j})_{i \in I, j \in J}, \quad b_I := (b_{i,j})_{i,j \in I}, \quad c_J := (c_{i,j})_{i,j \in J}$$

for $I \subset [p], J \subset [q]$. Note that b_I and c_J are still anti-alternating.

Theorem 1.1. *Let $r = \min(p,q)$ and ϵ the parity of p, i.e., is equal to 0 or 1 according as p is even or odd. If $X = \begin{bmatrix} a & b \\ c & -J_q {}^t a J_p \end{bmatrix}$ is a matrix in $\mathfrak{o}_{2n}$ with a, b, c parametrized as (1.2), then the Pfaffian* $\mathrm{Pf}\,(X)$ *is expanded as follows:*

$$\mathrm{Pf}\,(X) = \sum_{k=0}^{\lfloor r/2 \rfloor} \sum_{\substack{I \subset [p], J \subset [q] \\ |\bar{I}| = |\bar{J}| = 2k+\epsilon}} \mathrm{sgn}\left(\bar{I}, I\right) \mathrm{sgn}\left(\bar{J}, J\right) \det(a_{\bar{J}}^{\bar{I}})\, \mathrm{Pf}\,(b_I)\, \mathrm{Pf}\,(c_J), \tag{1.3}$$

where $\mathrm{sgn}\left(\bar{I}, I\right) = \mathrm{sgn}\begin{pmatrix} 1 \ldots p \\ \bar{I}, I \end{pmatrix}$ *and* $\mathrm{sgn}\left(\bar{J}, J\right) = \mathrm{sgn}\begin{pmatrix} 1 \ldots q \\ \bar{J}, J \end{pmatrix}$.

Proof. First let us recall the co-Pfaffian expansion (see e.g. [2]). For an alternating matrix $A = (\alpha_{i,j})_{i,j \in [2n]}$ and for a subset of indices $I \subset [2n]$, denote by A_I the alternating submatrix $(\alpha_{i,j})_{i,j \in I}$, whose row and column indices are both in I. Then the (i,j)th cofactor Pfaffian $\gamma_{i,j}(A)$ is defined to be

$$\gamma_{i,j}(A) = \begin{cases} (-1)^{i+j-1}\, \mathrm{Pf}\left(A_{[1..\widehat{\imath}..\widehat{\jmath}..2n]}\right) & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{i+j}\, \mathrm{Pf}\left(A_{[1..\widehat{\jmath}..\widehat{\imath}..2n]}\right) & \text{if } i > j, \end{cases}$$

for $i, j \in [2n]$, where $\widehat{\imath}$ means omitting i. As in the case of determinant, the following expansion formula holds:

$$\delta_{i,j}\, \mathrm{Pf}\,(A) = \sum_{k=1}^{2n} \alpha_{i,k}\, \gamma_{j,k}(A). \tag{1.4}$$

We will give the proof of the theorem by induction on $p+q$, since it reveals that iteration of the co-Pfaffian expansion (1.4) of Pfaffian yields our minor summation formula. It suffices to prove when $p \leqslant q$. The case where $p=q=1$ is trivial. Now expanding $\mathrm{Pf}\,(X)$ of the matrix X given by (1.1) and (1.2) with respect to the first row, we obtain

$$\mathrm{Pf}\,(X) = \sum_{j=1}^{q} a_{1,j}\gamma_{1,-j} + \sum_{j=2}^{p} b_{1,j}\gamma_{1,j}, \tag{1.5}$$

where we put $\gamma_{i,j} = \gamma_{i,j}(XJ_{2n})$ for brevity. Note that $\gamma_{i,j}$ is given, up to the sign, by Pfaffian of the submatrix obtained by deleting the ith and jth rows, and $-i$th and $-j$th columns from X, which is anti-alternating.

By inductive hypothesis, one obtains that

$$\gamma_{i,-j} = (-)^{1+j} \sum_{k=0}^{r-1} \sum_{\substack{I\subset[2...p] \\ J\subset[1..\widehat{j}..q] \\ |\bar{I}|=|\bar{J}|=2k+\epsilon_1}} \mathrm{sgn}\begin{pmatrix} 2...p \\ I,\bar{I}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} 1..\widehat{j}..q \\ J,\bar{J}\end{pmatrix} \det(a^{\bar{I}}_{\bar{J}})\,\mathrm{Pf}\,(b_I)\,\mathrm{Pf}\,(c_J)\,, \tag{1.6}$$

$$\gamma_{i,j} = (-)^{j} \sum_{k=0}^{r-1} \sum_{\substack{I\subset[2..\widehat{j}..p] \\ J\subset[q] \\ |\bar{I}|=|\bar{J}|=2k+\epsilon}} \mathrm{sgn}\begin{pmatrix} 2..\widehat{j}..p \\ I,\bar{I}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} 1...q \\ J,\bar{J}\end{pmatrix} \det(a^{\bar{I}}_{\bar{J}})\,\mathrm{Pf}\,(b_I)\,\mathrm{Pf}\,(c_J)\,, \tag{1.7}$$

where ϵ_1 is the parity of $p-1$, i.e., $\epsilon_1 = 1-\epsilon$.

In the right-hand side of (1.5), we denote the term of degree $2k+\epsilon$ in the variables $a_{i,j}$ by T_k $(k=0,1,\dots,r)$. Then it follows from (1.6) and (1.7) that

$$\begin{aligned} T_k = &\sum_{j=1}^{q}(-)^{1+j} \sum_{\substack{I\subset[2...p] \\ J\subset[1..\widehat{j}..q] \\ |\bar{I}|=|\bar{J}|=2k-\epsilon_1}} \mathrm{sgn}\begin{pmatrix} 2...p \\ I,\bar{I}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} 1..\widehat{j}..q \\ J,\bar{J}\end{pmatrix} a_{1,j}\det(a^{\bar{I}}_{\bar{J}})\,\mathrm{Pf}\,(b_I)\,\mathrm{Pf}\,(c_J) \\ &+ \sum_{j=2}^{p}(-)^{j} \sum_{\substack{I\subset[2..\widehat{j}..p] \\ J\subset[q] \\ |\bar{I}|=|\bar{J}|=2k+\epsilon}} \mathrm{sgn}\begin{pmatrix} 2..\widehat{j}..p \\ I,\bar{I}\end{pmatrix} \mathrm{sgn}\begin{pmatrix} 1...q \\ J,\bar{J}\end{pmatrix} \det(a^{\bar{I}}_{\bar{J}})\, b_{1,j}\,\mathrm{Pf}\,(b_I)\,\mathrm{Pf}\,(c_J)\,. \end{aligned} \tag{1.8}$$

Let us rewrite the first sum in the right-hand side of (1.8) as

$$\sum_{\substack{I\subset[2...p] \\ |\bar{I}|=2k-\epsilon_1}} \mathrm{sgn}\begin{pmatrix} 2...q \\ I,\bar{I}\end{pmatrix} \mathrm{Pf}\,(b_I) \left(\sum_{j=1}^{q} \sum_{\substack{J\subset[1..\widehat{j}..q] \\ |\bar{J}|=2k-\epsilon_1}} (-)^{j+1}\,\mathrm{sgn}\begin{pmatrix} 1..\widehat{j}..q \\ J,\bar{J}\end{pmatrix} a_{1,j}\det(a^{\bar{I}}_{\bar{J}})\,\mathrm{Pf}\,(c_J) \right).$$

In the round brackets of the equation above, fixing an index set $J_1 \subset [q]$ of length $q-1-(2k-\epsilon_1) = q-2k-\epsilon$ and denoting its complement by $\bar{J}_1$, one sees that the coefficient of $\mathrm{Pf}\,(c_{J_1})$ equals

$$\begin{aligned}&\sum_{j\in\bar{J}_1}(-)^{j+1}\operatorname{sgn}\begin{pmatrix}1..\widehat{j}..q\\ \bar{J}_1\smallsetminus\{j\},J_1\end{pmatrix}a_{1,j}\det(a^{\bar{I}}_{\bar{J}_1\smallsetminus\{j\}})\\ &=\operatorname{sgn}\begin{pmatrix}1...q\\ \bar{J}_1\,J_1\end{pmatrix}\det(a^{\bar{I}_1}_{\bar{J}_1})\end{aligned}$$

by the expansion formula of the determinant, where $\bar{I}_1 := \{1\}\sqcup\bar{I}$. Hence, one obtains that the first sum of T_k equals

$$\sum_{\substack{1\notin I_1\subset[p]\\ |\bar{I}_1|=2k+\epsilon}}\sum_{\substack{J_1\subset[q]\\ |\bar{J}_1|=2k+\epsilon}}\operatorname{sgn}\big(I_1,\bar{I}_1\big)\operatorname{sgn}\big(J_1,\bar{J}_1\big)\det(a^{\bar{I}_1}_{\bar{J}_1})\,\mathrm{Pf}\,(b_{I_1})\,\mathrm{Pf}\,(c_{J_1})\,. \tag{1.9}$$

Now let us turn to the second sum of T_k. It can be written as

$$\sum_{\substack{J\subset[q]\\ |\bar{J}|=2k+\epsilon}}\operatorname{sgn}\begin{pmatrix}1...q\\ J,\bar{J}\end{pmatrix}\mathrm{Pf}\,(c_J)\left(\sum_{j=2}^{p}\sum_{\substack{I\subset[2..\widehat{j}..p]\\ |\bar{I}|=2k+\epsilon}}(-)^j\operatorname{sgn}\begin{pmatrix}2..\widehat{j}..p\\ I,\bar{I}\end{pmatrix}\det(a^{\bar{I}}_{\bar{J}})b_{1,j}\,\mathrm{Pf}\,(b_I)\right).$$

In the round brackets of the equation above, similarly to the case of the first sum, fixing an index set $\bar{I}_1\subset[2...p]$ of length $p-2-2k-\epsilon$, and denoting its complement in $[p]$ by I_1, one sees that the coefficient of $\det(a^{\bar{I}_1}_{\bar{J}})$ equals

$$\operatorname{sgn}\begin{pmatrix}1...p\\ I_1,\bar{I}_1\end{pmatrix}\mathrm{Pf}\,(b_{I_1})\,,$$

whence, one obtains that the second sum of T_k equals

$$\sum_{\substack{J\subset[q]\\ |\bar{J}|=2k+\epsilon}}\sum_{\substack{1\in I_1\subset[p]\\ |\bar{I}_1|=2k+\epsilon}}\operatorname{sgn}\big(I_1,\bar{I}_1\big)\operatorname{sgn}\big(J,\bar{J}\big)\det(a^{\bar{I}_1}_{\bar{J}})\,\mathrm{Pf}\,(b_{I_1})\,\mathrm{Pf}\,(c_J) \tag{1.10}$$

by the expansion formula of Pfaffian.

It follows from (1.9) and (1.10) that T_k equals

$$\begin{aligned}&\left(\sum_{\substack{1\notin I_1\subset[p]\\ |\bar{I}_1|=2k+\epsilon}}\sum_{\substack{J_1\subset[q]\\ |\bar{J}_1|=2k+\epsilon}}+\sum_{\substack{J_1\subset[q]\\ |\bar{J}_1|=2k+\epsilon}}\sum_{\substack{1\in I_1\subset[p]\\ |\bar{I}_1|=2k+\epsilon}}\right)\operatorname{sgn}\big(I_1,\bar{I}_1\big)\operatorname{sgn}\big(J_1,\bar{J}_1\big)\det(a^{\bar{I}_1}_{\bar{J}_1})\,\mathrm{Pf}\,(b_{I_1})\,\mathrm{Pf}\,(c_{J_1})\\ &=\sum_{\substack{I_1\subset[p]\\ |\bar{I}_1|=2k+\epsilon}}\sum_{\substack{J_1\subset[q]\\ |\bar{J}_1|=2k+\epsilon}}\operatorname{sgn}\big(I_1,\bar{I}_1\big)\operatorname{sgn}\big(J_1,\bar{J}_1\big)\det(a^{\bar{I}_1}_{\bar{J}_1})\,\mathrm{Pf}\,(b_{I_1})\,\mathrm{Pf}\,(c_{J_1})\,.\end{aligned}$$

This completes the proof. ∎

In particular, if we take $p=q=n$ in Theorem 1.1, we obtain the following formula:

$$\mathrm{Pf}\,(X)=\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{\substack{I,J\subset[n]\\ |I|=|J|=2k}}\operatorname{sgn}\big(\bar{I},I\big)\operatorname{sgn}\big(\bar{J},J\big)\det(a^{\bar{I}}_{\bar{J}})\,\mathrm{Pf}\,(b_I)\,\mathrm{Pf}\,(c_J)\,. \tag{1.11}$$

In the next section, we will consider a noncommutative version of (1.11), i.e., the version for a matrix whose entries are elements in the universal enveloping algebra $U(\mathfrak{o}_{2n})$.

Remark 1.2. The co-Pfaffian matrix of A is, by definition, an alternating matrix whose (i,j)th entries are given by $\gamma_{i,j}(A)$ for $i,j \in [2n]$, which we denote by $\widehat{A}$. Then one can verify that

$$\frac{\mathrm{Pf}\,(A_I)}{\mathrm{Pf}\,(A)} = \mathrm{sgn}\left(I,\bar{I}\right) \mathrm{Pf}\left((\widehat{A}/\mathrm{Pf}\,(A))_{\bar{I}}\right) \tag{1.12}$$

if A is invertible. Note that (1.12) makes sense only if $|I|$ is even. Using the relation (1.12), it is immediate to see that the minor summation formula given in [6, Theorem 3.5] coincides with our (1.3) with p and q both even.

2. Noncommutative Minor Summation Formula

Now we turn to the noncommutative case.

Let us consider the following $2n \times 2n$-matrix whose entries are in the universal enveloping algebra $U(\mathfrak{o}_{2n})$:

$$\boldsymbol{X} = (X_{i,j})_{i,j\in[\pm n]} \quad \text{with} \quad X_{i,j} := E_{i,j} - E_{-j,-i} \in \mathfrak{o}_{2n} \subset U(\mathfrak{o}_{2n}). \tag{2.1}$$

The commutation relations among $X_{i,j}$'s are given by

$$[X_{i,j}, X_{k,l}] = \delta_{j,k} X_{i,l} + \delta_{i,l} X_{-j,-k} - \delta_{j,-l} X_{i,-k} - \delta_{i,-k} X_{-j,l} \tag{2.2}$$

for $i,j,k,l \in [\pm n]$.

Since $\boldsymbol{X}$ is anti-alternating by definition, one can define Pfaffian of $\boldsymbol{X} J_{2n}$, which we denote by $\mathrm{Pf}\,(\boldsymbol{X})$ simply. Then it is well known that $\mathrm{Pf}\,(\boldsymbol{X})$ belongs to the center $ZU(\mathfrak{o}_{2n})$ of $U(\mathfrak{o}_{2n})$.

Given the matrix $\boldsymbol{X}$ in (2.1), we introduce a 2-form with coefficient in $U(\mathfrak{o}_{2n})$:

$$\Omega = \sum_{i,j\in[\pm n]} e_i e_{-j} X_{i,j} \in \bigwedge^2 \mathbb{C}^{2n} \otimes U(\mathfrak{o}_{2n}). \tag{2.3}$$

Lemma 2.1. *The relation between the Pfaffian* $\mathrm{Pf}\,(\boldsymbol{X})$ *and* Ω *is given by*

$$\Omega^n = e_1 \cdots e_n e_{-n} \cdots e_{-1} 2^n n!\, \mathrm{Pf}\,(\boldsymbol{X}). \tag{2.4}$$

Proof. If we set $\widetilde{\boldsymbol{X}} := \boldsymbol{X} J_{2n}$, its (i,j)th entry $\widetilde{X}_{i,j}$ is given by $X_{i,-j}$ for $i,j \in [2n]$. Hence we obtain

$$\begin{aligned}
\Omega^n &= \sum_{i_1,j_1,\ldots,i_n j_n\in[\pm n]} e_{i_1} e_{j_1} \cdots e_{i_n} e_{j_n} X_{i_1,-j_1} \cdots X_{i_n,-j_n} \\
&= \sum_{i_1,j_1,\ldots,i_n j_n\in[2n]} e_{i_1} e_{j_1} \cdots e_{i_n} e_{j_n} \widetilde{X}_{i_1,j_1} \cdots \widetilde{X}_{i_n,j_n} \\
&= e_1 e_2 \cdots e_{2n-1} e_{2n} \sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}\,(\sigma) \widetilde{X}_{\sigma(1),\sigma(2)} \cdots \widetilde{X}_{\sigma(2n-1),\sigma(2n)} \\
&= e_1 e_2 \cdots e_n e_{n+1} \cdots e_{2n-1} e_{2n}\, 2^n n!\, \mathrm{Pf}\left(\widetilde{\boldsymbol{X}}\right) \\
&= e_1 e_2 \cdots e_n e_{-n} \cdots e_{-2} e_{-1}\, 2^n n!\, \mathrm{Pf}\,(\boldsymbol{X})
\end{aligned}$$

by (0.2), where $-j$ stands for $2n+1-j$ for $j \in [2n]$ by our convention. ∎

As in (0.6) in the introduction, we write the matrix $\boldsymbol{X}$ as

$$\boldsymbol{X} = \begin{bmatrix} a & b \\ c & -J_n {}^t a J_n \end{bmatrix},$$

with

$$a_{i,j} = X_{i,j}, \quad b_{i,j} = X_{i,-j}, \quad c_{i,j} = X_{-j,i} \tag{2.5}$$

for $i,j \in [n]$. By (2.2), the commutation relations among $a_{i,j}, b_{i,j}, c_{i,j}$ are given by

$$\begin{aligned}
[a_{i,j}, a_{k,l}] &= \delta_{j,k} a_{i,l} - \delta_{l,j} a_{k,j}, \\
[a_{i,j}, b_{k,l}] &= \delta_{j,k} b_{i,l} - \delta_{j,l} b_{i,k}, \\
[a_{i,j}, c_{k,l}] &= \delta_{i,k} c_{l,j} - \delta_{i,l} c_{k,j}, \\
[b_{i,j}, c_{k,l}] &= \delta_{j,l} a_{i,k} + \delta_{i,k} a_{j,l} - \delta_{i,l} a_{j,k} - \delta_{j,k} a_{i,l}, \\
[b_{i,j}, b_{k,l}] &= [c_{i,j}, c_{k,l}] = 0
\end{aligned} \tag{2.6}$$

for $i,j,k,l \in [n]$.

Corresponding to the parametrization of the matrix $\boldsymbol{X}$, we set

$$\Xi = \sum_{i,j \in [n]} e_i e_{-j} \, a_{i,j}, \quad \Theta = \sum_{i,j \in [n]} e_i e_j \, b_{i,j}, \quad \Theta' = \sum_{i,j \in [n]} e_{-j} e_{-i} \, c_{i,j}.$$

Obviously, we see that

$$\Omega = \Theta' + 2\Xi + \Theta.$$

Lemma 2.2. *The following commutation relations hold:*

$$[\Theta, \Theta'] = 4\tau\Xi, \quad [\Theta, \Xi] = 2\tau\Theta, \quad [\Theta', \Xi] = -2\tau\Theta',$$

where $\tau = \sum_{i \in [n]} e_i e_{-i}$.

Proof. It is straightforward to show that these relations follow from (2.6). For example, one sees that

$$\begin{aligned}
[\Xi, \Theta] &= \sum_{i,j,k,l} e_i e_{-j} e_k e_l [a_{i,j}, b_{k,l}] \\
&= \sum_{i,j,k,l} e_i e_{-j} e_k e_l (\delta_{j,k} b_{i,l} - \delta_{j,l} b_{i,k}) \\
&= \sum_{i,j,l} e_i e_{-j} e_j e_l b_{i,l} - \sum_{i,j,k} e_i e_{-j} e_k e_j b_{i,k} \\
&= -\sum_{i,j,l} e_{-j} e_j e_i e_l b_{i,l} - \sum_{i,j,k} e_{-j} e_j e_i e_k b_{i,k} \\
&= -2\tau\Theta.
\end{aligned}$$

The other relations follow similarly. ∎

For a parameter $u \in \mathbb{C}$ and for a nonnegative integer $r = 0, 1, 2, \dots$, set

$$\Xi(u) := \Xi + u\tau \quad \text{and} \quad \Xi^{(r)}(u) := \Xi(u)\Xi(u-1)\cdots\Xi(u-r+1). \tag{2.7}$$

The following propositions are due to [5, Lemma 4.5 and Proposition 2.6].

Proposition 2.3. *For $m = 0, 1, \ldots, n$, we have*

$$\Omega^m = \sum_{\substack{p,q,r \geqslant 0 \\ p+q+r=m}} \frac{m!}{p!q!r!} 2^r \, \Xi^{(r)}(q-p+r-1)\Theta'^p \Theta^q.$$

Proof. By Lemma 2.2, our 2-forms Ξ, Θ, Θ' satisfy the same commutation relations as those given in [5, Lemma 4.1]. Therefore, exactly the same argument therein implies the proposition (see [5, Lemma 4.5] for details). ∎

For $j \in [n]$ and $u \in \mathbb{C}$, set

$$\eta_j(u) = \sum_{i \in [n]} e_i a_{i,j}(u) \tag{2.8}$$

with $a_{i,j}(u) = a_{i,j} + u\delta_{i,j}$. Then they are anti-commutative *when the parameter shift taken into account*, i.e., they satisfy

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = 0 \tag{2.9}$$

for $i, j \in [n]$ (cf. [5, Lemma 2.1]). Note then that

$$\Xi(u) = \sum_{j \in [n]} \eta_j(u) e_{-j}. \tag{2.10}$$

Given $I, J \subset [n]$, define submatrices of a, b, c by

$$a^I_J := (a_{i,j})_{i \in I, j \in J}, \quad b_I := (b_{i,j})_{i,j \in I}, \quad c_J := (c_{i,j})_{i,j \in J}$$

as in the commutative case. Furthermore, we will use the following notations for economy:

$$e_I := e_{i_1} e_{i_2} \cdots e_{i_r} \quad \text{and} \quad e_{-J} := e_{-j_s} e_{-j_{s-1}} \cdots e_{-j_1}$$

if $I = \{i_1 < i_2 < \cdots < i_r\}$ and $J = \{j_1 < j_2 < \cdots j_s\}$.

Proposition 2.4. *For $r = 0, 1, \ldots, n$ and $u \in \mathbb{C}$, we have*

$$\Xi^{(r)}(u+r-1) = r! \sum_{\substack{I, J \subset [n] \\ |I|=|J|=r}} e_I e_{-J} \det\left(a^I_J + \mathbf{1}^I_J \operatorname{diag}(u+r-1, u+r-2, \ldots, u)\right),$$

where det *denotes the column determinant and* $\mathbf{1}$ *the identity matrix of size $n \times n$.*

Proof. First, note that the column determinant in the sum is explicitly written as

$$\det \begin{bmatrix} a_{i_1,j_1}(u+r-1) & a_{i_1,j_2}(u+r-2) & \cdots & a_{i_1,j_r}(u) \\ a_{i_2,j_1}(u+r-1) & a_{i_2,j_2}(u+r-2) & \cdots & a_{i_2,j_r}(u) \\ \vdots & \vdots & & \vdots \\ a_{i_r,j_1}(u+r-1) & a_{i_r,j_2}(u+r-2) & \cdots & a_{i_r,j_r}(u) \end{bmatrix}$$

$$= \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) a_{i_{\sigma(1)},j_1}(u+r-1) a_{i_{\sigma(2)},j_2}(u+r-2) \cdots a_{i_{\sigma(r)},j_r}(u) \tag{2.11}$$

if $I=\{i_1<i_2<\cdots<i_r\}$ and $J=\{j_1<j_2<\cdots<j_r\}$.

On the other hand, it follows from (2.7), (2.8), (2.9) and (2.10) that

$$
\begin{aligned}
&\Xi^{(r)}(u+r-1)\\
&=(-)^{r(r-1)/2}\sum_{\alpha_1,\alpha_2,\dots,\alpha_r}\eta_{\alpha_1}(u+r-1)\eta_{\alpha_2}(u+r-2)\cdots\eta_{\alpha_r}(u)e_{-\alpha_1}e_{-\alpha_2}\cdots e_{-\alpha_r}\\
&=(-)^{r(r-1)/2}\sum_{j_1<j_2<\cdots<j_r}\sum_{\sigma\in\mathfrak{S}_r}\eta_{j_{\sigma(1)}}(u+r-1)\eta_{j_{\sigma(2)}}(u+r-2)\cdots\eta_{j_{\sigma(r)}}(u)\\
&\qquad\qquad\times e_{-j_{\sigma(1)}}\cdots e_{-j_{\sigma(r)}}\\
&=r!\sum_{j_1<j_2<\cdots<j_r}\eta_{j_1}(u+r-1)\eta_{j_2}(u+r-2)\cdots\eta_{j_r}(u)e_{-j_r}\cdots e_{-j_1},\\
&=r!\sum_{j_1<j_2<\cdots<j_r}\sum_{\beta_1,\beta_2,\dots,\beta_r}e_{\beta_1}e_{\beta_2}\cdots e_{\beta_r}e_{-j_r}\cdots e_{-j_1}\\
&\qquad\qquad\times a_{\beta_1,j_1}(u+r-1)a_{\beta_2,j_2}(u+r-2)\cdots a_{\beta_r,j_r}(u)\\
&=r!\sum_{j_1<j_2<\cdots<j_r}\sum_{i_1<i_2<\cdots<i_r}e_{i_1}e_{i_2}\cdots e_{i_r}e_{-j_r}\cdots e_{-j_1}\\
&\qquad\qquad\times\operatorname{sgn}(\sigma)\,a_{i_{\sigma(1)},j_1}(u+r-1)a_{i_{\sigma(2)},j_2}(u+r-2)\cdots a_{i_{\sigma(r)},j_r}(u).
\end{aligned}
$$

This completes the proof. ∎

Lemma 2.5. *For $p,q=0,1,\dots,$ we have*

$$
\begin{aligned}
\Theta^p&=2^p p!\sum_{I\subset[n],|I|=2p}e_I\operatorname{Pf}(b_I),\\
\Theta'^q&=2^q q!\sum_{J\subset[n],|J|=2q}e_{-J}\operatorname{Pf}(c_J).
\end{aligned}
$$

Proof. It is a direct calculation to show these formulae, as in the proof of Lemma 2.1. In fact,

$$
\begin{aligned}
\Theta^p&=\sum_{\alpha_1,\beta_1,\dots,\alpha_p,\beta_p}e_{\alpha_1}e_{\beta_1}e_{\alpha_2}e_{\beta_2}\cdots e_{\alpha_p}e_{\beta_p}\,b_{\alpha_1,\beta_1}b_{\alpha_2,\beta_2}\cdots b_{\alpha_p,\beta_p}\\
&=\sum_{i_1<i_2<\cdots<i_{2p}}e_{i_1}e_{i_2}\cdots e_{i_{2p}}\sum_{\sigma\in\mathfrak{S}_{2p}}\operatorname{sgn}(\sigma)\,b_{i_{\sigma(1)},i_{\sigma(2)}}\cdots b_{i_{\sigma(2p-1)},i_{\sigma(2p)}}\\
&=2^p p!\sum_{I\subset[n],|I|=2p}e_I\operatorname{Pf}(b_I).
\end{aligned}
$$

The other formula can be shown in a similar way. ∎

Now we are ready to prove our main theorem.

Proof. (*Proof of Theorem A in the introduction*). By Propositions 2.3, 2.4 and Lemma 2.5, one obtains that

$$
\begin{aligned}
\Omega^n=2^n n!&\sum_{\substack{p,q,r\geqslant 0\\ p+q+r=n}}\sum_{\substack{I_1,J_1\subset[n]\\ |I_1|=|J_1|=r}}\sum_{\substack{I,J\subset[n]\\ |I|=2p,|J|=2q}}e_{I_1}e_Ie_{-J_1}e_{-J}\\
&\times\det\left(a^{I_1}_{J_1}+\mathbf{1}^{I_1}_{J_1}\operatorname{diag}(u+r-1,u+r-2,\dots,u)\right)\operatorname{Pf}(c_J)\operatorname{Pf}(b_I)
\end{aligned}
$$

with $u = q - p$. Since Ω^n is of top degree, the terms corresponding to I_1, I, J_1, J in the sum vanish unless $I_1 \sqcup I = J_1 \sqcup J = [n]$, in particular, unless $p = q$. Thus

$$\Omega^n = 2^n n! \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{I,J \subset [n] \\ |I|=|J|=2k}} e_{\bar{I}} e_I e_{-\bar{J}} e_{-J} \det\left(a^{\bar{I}}_{\bar{J}} + \mathbf{1}^{\bar{I}}_{\bar{J}}\, \boldsymbol{\rho}(n-2k)\right) \operatorname{Pf}(c_J) \operatorname{Pf}(b_I)\,.$$

Now the theorem follows if one compares this with the right-hand side of (2.4). ∎

Using the theorem one can easily compute the eigenvalues of the Pfaffian $\operatorname{Pf}(\boldsymbol{X})$ on any highest weight representations of $\mathfrak{o}_{2n}$.

Corollary 2.6. *The eigenvalue of the central element* $\operatorname{Pf}(\boldsymbol{X}) \in ZU(\mathfrak{o}_{2n})$ *on the highest weight representation with highest weight* $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ *is given by* $\prod_{i=1}^n (\lambda_i + n - i)$.

Proof. Use the notation (2.5). Take $\mathfrak{h} := \bigoplus_{i\in[n]} \mathbb{C} a_{i,i}$ as Cartan subalgebra of $\mathfrak{o}_{2n}$. Denote the linear functional on $\mathfrak{h}$ sending $\operatorname{diag}(h_1, \ldots, h_n, -h_n, \ldots, -h_1)$ to h_i by ϵ_i. Then, the positive root vectors are $a_{i,j}$ for $1 \leqslant i < j \leqslant n$, and $b_{i,j}$ for $1 \leqslant i, j \leqslant n$ with root $\epsilon_i - \epsilon_j$, and $\epsilon_i + \epsilon_j$, respectively (see [1, 7]).

Applying $\operatorname{Pf}(\boldsymbol{X})$ to the highest weight vector, say v_λ, one sees that the only term that survives in the sum of (0.7) is the one corresponding to $I = J = \varnothing$ since $b_{i,j}$ is a positive root vector. Thus, by (2.11), one obtains that

$$\begin{aligned} \operatorname{Pf}(\boldsymbol{X})\, v_\lambda &= \det(a + \boldsymbol{\rho}(n)) v_\lambda \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1}(n-1) a_{\sigma(2),2}(n-2) \cdots a_{\sigma(n),n}(0) v_\lambda \\ &= (\lambda_1 + n - 1)(\lambda_2 + n - 2) \cdots \lambda_n\, v_\lambda, \end{aligned}$$

since $a_{i,j}$ is also a positive root vector if $i < j$. ∎

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A. Proof of Commutative Minor Summation Formula via Exterior Calculus

In this appendix, we give another proof of commutative minor summation formulae (1.3) in Theorem 1.1, using the exterior calculus.

Let $X = (x_{i,j})_{i,j} \in \mathfrak{o}_{2n}$ be an anti-alternating matrix with commutative entries:

$$X = \left[\begin{array}{cccc|ccc} x_{1,1} & x_{1,2} & \cdots & x_{1,q} & x_{1,-p} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \iddots & \vdots \\ x_{p,1} & x_{p,2} & \cdots & x_{p,q} & 0 & \cdots & x_{p,-1} \\ \hline x_{-q,1} & x_{-q,2} & \cdots & 0 & x_{-q,-p} & \cdots & x_{-q,-1} \\ \vdots & \vdots & \iddots & \vdots & \vdots & & \vdots \\ x_{-2,1} & 0 & \cdots & x_{-2,q} & x_{-2,-p} & \cdots & x_{-2,-1} \\ 0 & x_{-1,2} & \cdots & x_{-1,q} & x_{-1,-p} & \cdots & x_{-1,-1} \end{array}\right],$$

where $x_{-j,-i} = -x_{i,j}$ for all i, j. Define 2-forms Ω by

$$\Omega := \sum_{i,-j \,\in\, [p] \sqcup [-q]} e_i e_{-j} x_{i,j}$$

for X. By the same argument as in the proof of Lemma 2.1 one can show that

$$\Omega^n = e_1 \cdots e_p e_{-q} \cdots e_{-1} 2^n n! \operatorname{Pf}(X).$$

Note that by our convention $-q$ stands for $2n+1-q = p+1$ and so on. Parametrizing X as in (1.1) and (1.2), we define 2-forms Ξ, Θ, Θ' by

$$\Xi = \sum_{i \in [p], j \in [q]} e_i e_{-j}\, a_{i,j}, \quad \Theta = \sum_{i,j \in [p]} e_i e_j\, b_{i,j}, \quad \Theta' = \sum_{i,j \in [q]} e_{-j} e_i\, c_{i,j}.$$

It is clear that

$$\Omega = \Theta' + 2\Xi + \Theta.$$

Furthermore, the trinomial expansion formula in Proposition 2.3 holds without parameter-shift in the commutative case:

$$\Omega^m = \sum_{\substack{h,s,t \geqslant 0 \\ h+s+t=m}} \frac{m!}{h!s!t!} 2^h\, \Xi^h \Theta^s \Theta'^t \tag{A.1}$$

for $m = 0, 1, \ldots$. The same calculation as in Proposition 2.4 and Lemma 2.5 yields

$$\begin{aligned}
\Xi^h &= h! \sum_{\substack{I \subset [p], J \subset [q] \\ |I|=|J|=h}} e_I e_{-J} \det(a^I_J), \\
\Theta^s &= 2^s s! \sum_{I \subset [p], |I|=2s} e_I \operatorname{Pf}(b_I), \\
\Theta'^t &= 2^t t! \sum_{J \subset [q], |J|=2t} e_{-J} \operatorname{Pf}(c_J)
\end{aligned}$$

for $h, s, t = 0, 1, \ldots$. Substituting these into (A.1) with $m = n$, we see that

$$\Omega^n = 2^n n! \sum_{\substack{h,s,t \geqslant 0 \\ h+s+t=n}} \sum_{\substack{I_1, I \subset [p], J_1, J \subset [q] \\ |I_1|=|J_1|=h \\ |I|=2s, |J|=2t}} e_{I_1} e_I e_{-J_1} e_{-J} \det(a^{I_1}_{J_1}) \operatorname{Pf}(b_I) \operatorname{Pf}(c_J). \tag{A.2}$$

Since Ω^n is of top degree, the terms that survive in the sum (A.2) are those corresponding to I_1, I, J_1, J satisfying $I_1 \sqcup I = [p]$ and $J_1 \sqcup J = [q]$. In particular, $h + 2s = p$ and $h + 2t = q$, where $|I_1| = |J_1| = h, |I| = 2s, |J| = 2t$. Now setting $h = 2k + \epsilon$ with ϵ the parity of p (=the parity of q), we obtain the formula.

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